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Combinatorics and Representations of Finite Groups

Jørn B. Olsson

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Contents

Introduction

These notes contain an expanded version of some lectures given during my stay in Essen in the first half of 1993. The main purpose is to give a description of the interplay between combinatorics (especially the study of partitions and related objects) and irreducible representations of some classes of finite groups (especially symmetric groups and their coverings and some linear groups). The origin was the study of (numerical) properties of the representations of symmetric groups including the degrees and the distribution into blocks. Of course the symmetric groups are very special in several respects, but as it has turned out some of the combinatorial analysis applied there may be modified to give results for other groups.

In the first chapter we present the combinatorial concepts and results needed for our study of irreducible representations and blocks. Thereby an attempt has been made to make analogies very clear. Thus for instance cores and quotients, which are well known for partitions, may also be defined for bar partitions and symbols. Their definitions are inspired and even helped by the properties we want them to have to be able to apply them to study degrees of irreducible characters and to enumerate characters in blocks. Chapter II is devoted to the analysis of character degrees, especially the description of power of a given prime dividing a given character degree. This is important in chapter III, where we consider characters in blocks. It should be stressed that the combinatorial methods may be used to obtain much more information about representations than described here.

I have been interested in this subject since 1975 and substantial parts of these notes are slightly modified excerpts from some of my papers written partly in collaborations. Some of the sections were written from scratch. In the later parts of the notes some proofs are only sketched or even omitted so that the last sections have the nature of surveys. Also some items in the notes classified as "remarks" are really up-to-date surveys over related problems and topics which could not be treated here.

These notes were prepared during a relative short period of five months and some lapses of notation and inconsistencies may have escaped my attention. For these I apologize and offer the quotation below.

I wish to thank Professor Michler for suggesting to write these notes and the "Institute for Experimental Mathematics" in Essen for the hospitality during a very productive stay. Thanks are also due to Ms. Sabine van Ackern for her quick and competent preparation of the manuscript.

"So eine Arbeit wird eigentlich nie fertig,
man muß sie für fertig erklären,
wenn man nach Zeit und Umständen
das möglichste getan hat." (J.W. Goethe)

Copenhagen, November 1993

J.B. Olsson

I. Combinatorics with Partitions

1 Partitions, hooks, β-sets

Let $n \in \mathbb{N}$. A partition λ of n is a decreasing sequence of integers

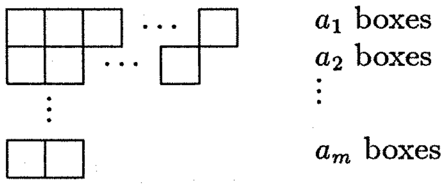
$$\lambda = (a_1, a_2, \dots, a_m)$$

satisfying $a_1 \geq a_2 \geq \dots \geq a_m > 0$ and $a_1 + a_2 + \dots + a_m = n$. The integers a_i are called the *parts* of λ and $m = l(\lambda)$ the *length* of λ. The set of partitions of n is denoted by $\mathcal{P}(n)$ and $p(n) = |\mathcal{P}(n)|$. We put $\mathcal{P}(0) = \{(0)\}$ and $p(0) = 1$. If $\lambda \in \mathcal{P}(n)$ we also write $|\lambda| = n$.

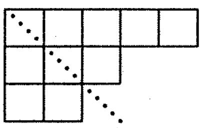
When $\lambda = (a_1, a_2, \dots, a_m)$ is a partition we define

$$\mathcal{Y}(\lambda) = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m,\ 1 \leq j \leq a_i\}.$$

Thus $|\mathcal{Y}(\lambda)| = |\lambda|$ and the set $\mathcal{Y}(\lambda)$ is visualized in the *Young diagram* (also called the Ferrer's diagram) of λ containing n boxes arranged as follows:

$Y(\lambda):$ 

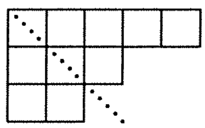
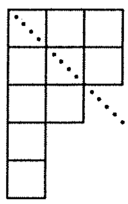
Thus

$Y((5,3,2)):$ 

The *conjugate partition* λ^0 of λ is the partition satisfying

$$\mathcal{Y}(\lambda^0) = \{(j,i) \in \mathbb{N} \times \mathbb{N} \mid (i,j) \in \mathcal{Y}(\lambda)\}$$

Thus the Young diagram $Y(\lambda^0)$ is obtained from $Y(\lambda)$ by reflecting it in the diagonal:

Reflecting  yields 

whence $(5,3,2)^0 = (3^2, 2, 1^2)$. (Here, as elsewhere, the exponents denote the multiplicities of the parts.)

For $(i,j) \in \mathcal{Y}(\lambda)$, λ a partition, we put

$$\mathfrak{H}_{ij}(\lambda) = \left\{ (i',j') \in \mathcal{Y}(\lambda) \;\middle|\; \begin{array}{lllc} i = i' & \text{and} & j' \geq j & \text{or} \\ j = j' & \text{and} & i' > i & \end{array} \right\},$$

the (i,j)-*hook* in λ. It is easy to visualize the hooks of λ in the Young diagram $Y(\lambda)$. The boxes with $\times$ show the $(2,3)$-hook of $\lambda = (8,7,5,4,1)$

		×	×	×	×	×	
		×					
		×					

The (i,j)-*hooklength* of λ is $h_{ij}(\lambda) = |\mathfrak{H}_{ij}(\lambda)|$. If $\lambda^0 = (b_1, b_2, \dots, b_l)$ is the partition conjugate to λ then obviously

$$h_{ij}(\lambda) = (a_i - j) + (b_j - i) + 1.$$

We put

$$\begin{array}{rcll} a_{ij}(\lambda) & = & a_i - j & \text{the } (i,j)\text{-}\textit{armlength of } \lambda \\ b_{ij}(\lambda) & = & b_j - i & \text{the } (i,j)\text{-}\textit{leglength of } \lambda \end{array}$$

so that

$$h_{ij}(\lambda) = a_{ij}(\lambda) + b_{ij}(\lambda) + 1.$$

(In the above example with $\lambda = (8,7,5,4,1)$ we have $h_{23}(\lambda) = 7$, $a_{23}(\lambda) = 4$, $b_{23}(\lambda) = 2$.)

We may form the hookdiagram of λ, Hook (λ), by inserting the (i,j)-hooklength of λ in the (i,j)-box of $Y(\lambda)$.

Example. Hookdiagram of $\lambda = (8,7,5,4,1)$

12	10	9	8	6	4	3	1
10	8	7	6	4	2	1	
7	5	4	3	1			
5	3	2	1				
1							

= Hook (8,7,5,4,1)

We put $\mathcal{H}_i(\lambda) = \{h_{i1}(\lambda), h_{i2}(\lambda), \dots, h_{ia_i}(\lambda)\}$ and $\mathcal{H}(\lambda) = \dot{\cup}_i \mathcal{H}_i(\lambda)$. $\mathcal{H}(\lambda)$ is considered as a "multiset", i.e. the elements are counted with their multiplicities. Thus

Hook (3,3)=

4	3	2
3	2	1

$$\mathcal{H}(3,3) \quad = \{4,3,3,2,2,1\}$$

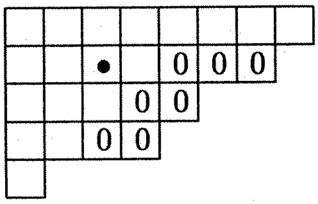
It is also useful to consider the *rim* of λ

$$\mathcal{R}(\lambda) = \{(i',j') \in \mathcal{Y}(\lambda) \;|\; (i'+1,\ j'+1) \notin \mathcal{Y}(\lambda)\}$$

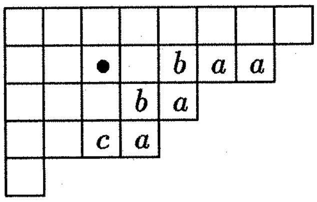
and for $(i,j) \in \mathcal{Y}(\lambda)$

$$\mathcal{R}_{ij}(\lambda) = \{(i',j') \in \mathcal{R}(\lambda) \mid i' \geq i \quad \text{and} \quad j' \geq j\}.$$

The boxes with 0 show the (2,3)-rim of $\lambda = (8,7,5,4,1)$:



Notice that if $\lambda^0 = (b_1, \dots, b_l)$ then $(b_j, j) \in \mathfrak{H}_{ij}(\lambda) \cap \mathcal{R}_{ij}(\lambda)$. Write a c in this box of $\mathcal{R}_{ij}(\lambda)$. In the remaining boxes (i',j') of $\mathcal{R}_{ij}(\lambda)$ we write an a, if $(i'+1,j') \notin \mathcal{R}(\lambda)$ and a b if $(i'+1,j') \in \mathcal{R}(\lambda)$. In the above example we get



It is clear that the boxes marked with an a in $\mathcal{R}_{ij}(\lambda)$ are exactly those in the positions $(b_{j'}, j')$, $j < j' \leq a_i$. Moreover, as is easily seen, the boxes marked b in $R_{ij}(\lambda)$ are exactly those in the positions $(i', a_{i'+1})$, $i \leq i' < b_j$. Thus $a_{ij}(\lambda)$ boxes of $\mathcal{R}_{ij}(\lambda)$ are marked a and $b_{ij}(\lambda)$ boxes of $\mathcal{R}_{ij}(\lambda)$ are marked b. We have proved

Lemma (1.1). For $(i,j) \in \mathcal{Y}(\lambda)$

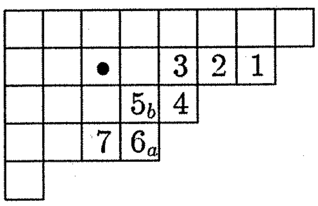
$$|\mathcal{R}_{ij}(\lambda)| = h_{ij}(\lambda).$$

The argument involving the marking of boxes in $\mathcal{R}_{ij}(\lambda)$ by a's and b's is also helpful in the proof of the following

Proposition (1.2) (Frame, Robinson, Thrall). For $(i,j) \in \mathcal{Y}(\lambda)$ we have in the above notation

$$\{h_{ij'}(\lambda) \mid j' \geq j\} \cup \{h_{ij}(\lambda) - h_{i'j}(\lambda) \mid i' > i\} = \{1, 2, \dots, h_{ij}(\lambda)\}.$$

Proof. Write the integers $1, 2, \dots, h_{ij}(\lambda)$ in the boxes of $\mathcal{R}_{ij}(\lambda)$ starting at the top and going from the right to the left. For example



Then in the boxes marked a by the above procedure we have the integers $h_{ij'}(\lambda)$, $j' > j$ (by (1.1)!) and in the boxes marked b we have the integers $h_{ij}(\lambda) - h_{i'j}(\lambda)$, $i' > i$. The box marked c is of course numbered $h_{ij}(\lambda)$.

(In the above example the 5 in the boxed marked b is 7-2 where $7 = h_{23}(\lambda)$ and $2 = h_{43}(\lambda)$ and the 6 in the box marked a equals $h_{24}(\lambda)$.)

We next describe the removal of the (i,j)-hook of the partition λ, when $(i,j) \in \mathcal{Y}(\lambda)$: The partition $\lambda \backslash H_{ij}(\lambda)$, defined by

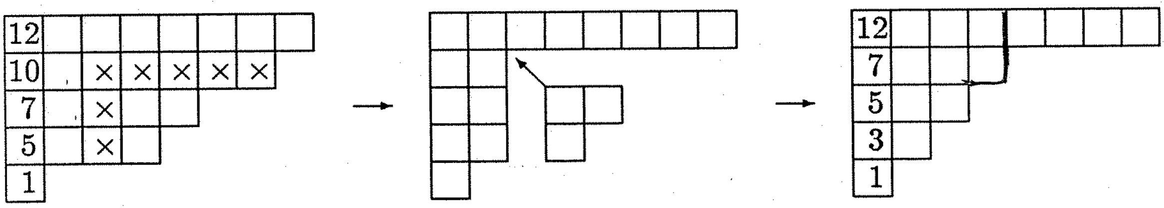
$$\mathcal{Y}(\lambda \backslash H_{ij}(\lambda)) = \mathcal{Y}(\lambda) \backslash \mathcal{R}_{ij}(\lambda)$$

is a partition of $n - h_{ij}(\lambda)$, when $\lambda \in \mathcal{P}(n)$. Then $\lambda \backslash H_{ij}(\lambda)$ is said to be obtained from λ *by removing the* (i,j)*-hook*. We also often write $\lambda \backslash H_{ij}$ or simply $\lambda \backslash H$ for such a partition.

Thus if $\lambda = (8,7,5,4,3)$ then $\lambda \backslash H_{23} = (8,4,3,2,1)$.

Instead of removing $\mathcal{R}_{ij}(\lambda)$ from $\mathcal{Y}(\lambda)$ we may also remove the boxes corresponding to $\mathfrak{H}_{ij}(\lambda)$ from the Young diagram $Y(\lambda)$. This gives us two diagrams (possibly empty) which pushed together form $Y(\lambda \backslash H_{ij})$.

Example.



In the above example we have included the hooklengths $h_{i1}(\lambda)$ and $h_{i1}(\lambda \backslash H_{23})$. They are 12, $\underline{10}$, 7,5,1 for λ and 12,7,5, $\underline{3}$, 1 for $\lambda \backslash H_{23}$. The only difference between the hooklengths is that 10 (for λ) has been replaced by $3 = 10 - \underline{7}$ for $\lambda \backslash H_{23}$. Moreover $7 = h_{23}(\lambda)$. This suggests that the removal of a hook in λ may be described by a subtraction operation on a set of non-negative integers related to λ. This is indeed the case:

A β*-set* is a finite subset

$$X = \{h_1, h_2, \ldots, h_t\} \quad (h_1 > h_2 > \ldots > h_t \geq 0)$$

of $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $s \in \mathbb{N}_0$ and X is as above we define a new β-set X^{+s} as follows:

$$X^{+s} = \{h_1 + s, h_2 + s, \ldots, h_t + s, s-1, s-2, \ldots, 0\}.$$

The partition $P^*(X)$ associated to the β-set X as above is defined as

$$P^*(X) = (h_1 - (t-1), h_2 - (t-2), \ldots, h_t) = \lambda$$

where we neglect the parts 0. We call X a β*-set for the partition* λ if $P^*(X) = \lambda$.

An easy calculation shows the following:

Proposition (1.3). Let $\lambda = (a_1, a_2, \dots, a_m)$ be a partition. Put $X_\lambda = \{h_{11}(\lambda), h_{21}(\lambda), \dots, h_{m1}(\lambda)\}$ where $h_{i1}(\lambda) = h_i + m - i$. Thus X_λ consists of the "first column hooklengths" of λ. Then the β-sets for λ are exactly the sets X_λ^{+s}, $s \in \mathbb{N}_0$.

Suppose that $X = \{h_1, \dots, h_t\}$ is a β-set for the partition $\lambda = (a_1, \dots, a_m)$. We put

$$\mathcal{H}_i(X) = \{1, 2, \dots, h_i\} \setminus \{h_i - h_j \mid j > i\}$$

for $1 \leq i \leq t$. It follows from the definitions that $|\mathcal{H}_i(X)| = a_i$ for $1 \leq i \leq m$ and $\mathcal{H}_i(X) = \emptyset$ for $i > m$. Notice also that $\mathcal{H}_i(X^{+s}) = \mathcal{H}_i(X)$ for all $s \geq 0$.

Proposition (1.4). If X is a β-set for λ, then

$$\mathcal{H}_i(X) = \mathcal{H}_i(\lambda) \quad \text{for all (relevant)} \quad i.$$

Proof. By the above remarks it suffices to show the statement, when $X = X_\lambda$ is as in (1.3). But in that case the result follows immediately from the definition of $\mathcal{H}_i(X_\lambda)$ and (1.2) for $j = 1$.

Corollary (1.5). If $X = \{h_1, \dots h_t\}$ is a β-set for λ and $h \in \mathbb{N}$ then

$$h \in \mathcal{H}_i(\lambda) \Leftrightarrow h_i - h \geq 0 \quad \text{and} \quad h_i - h \notin X.$$

Proof. By (1.4) $\mathcal{H}_i(\lambda) = \mathcal{H}_i(X)$. Assume $h_i - h \geq 0$. Then $h_i - h \in X \Leftrightarrow$ (There exists a j, s.t. $h_i - h = h_j$) $\Leftrightarrow$ (There exist a j, s.t. $h = h_i - h_j$) $\Leftrightarrow h \notin \mathcal{H}_i(X)$.

As before $\mathcal{H}(\lambda)$ is the multiset of hooklengths of the partition λ.

Corollary (1.6) (Nakayama). Let $h \in \mathcal{H}(\lambda)$, λ a partition. If $e \in \mathbb{N}$, $e|h$, then $e \in \mathcal{H}(\lambda)$.

Example. Since $12 \in \mathcal{H}(\lambda)$, $\lambda = (8, 7, 5, 3, 1)$ we must also have $2, 3, 6 \in \mathcal{H}(\lambda)$. Indeed, $6 = h_{15}(\lambda)$, $3 = h_{17}(\lambda)$, $2 = h_{26}(\lambda)$.

Proof. Let $X = \{h_1, \dots, h_t\}$ be a β-set for λ. Assume that $h = ef \in \mathcal{H}_i(\lambda) = \mathcal{H}_i(X)$. By (1.5) $h_i = h_i - e0 \in X$ and $h_i - h = h_i - ef \notin X$. Thus there exists an l, $0 \leq l < f$ such that $h_i - el \in X$ and $h_i - e(l+1) \notin X$. If then $h_i - el = h_j$ we get that $e \in H_j(\lambda)$, by (1.5).

A related result to (1.6) can be proved using (1.2) directly. We omit the proof which is not difficult.

Corollary (1.7). Suppose that $h_{ij}(\lambda) = ef$, where $e, f \in \mathbb{N}$. Then exactly f of the hooklengths

$$h_{i'j'}(\lambda), \quad (i', j') \in \mathfrak{H}_{ij}(\lambda)$$

are divisible by e.

The next result is of fundamental importance for our investigations:

Proposition (1.8). Let $X = \{h_1, h_2, \dots, h_t\}$ be a β-set for $\lambda = (a_1, a_2, \dots, a_m)$. Let $(i, j) \in \mathcal{Y}(\lambda)$. Then $h_i - h_{ij}(\lambda) \geq 0$, $h_i - h_{ij}(\lambda) \notin X$ and

$$Y = X \cup \{h_i - h_{ij}(\lambda)\} \setminus \{h_i\}$$

is a β-set for $\lambda\backslash H_{ij}$.

Proof. The first part of the statement was proved in (1.5). We may assume that $X = X_\lambda$, so that $t = m$. Let $a_{ij}(\lambda) = a$, $b_{ij}(\lambda) = b$. Then the removal of the (i,j)-hook from λ only affects the parts $a_i, a_{i+1}, \ldots, a_{i+b}$ of λ. Considering $Y(\lambda\backslash H_{ij})$ as being obtained by removing the boxes corresponding to $\mathfrak{H}_{ij}(\lambda)$ in $Y(\lambda)$ and pushing the obtained diagrams together shows that

$$\lambda\backslash H_{ij} = (a_1, a_2, \ldots, a_{i-1}, a_{i+1} - 1, a_{i+2} - 1, \ldots a_{i+b} - 1, a_i - (a+1), a_{i+b+1}, \ldots, a_k).$$

Now it is easy to compute a β-set for $\lambda\backslash H_{ij}$ with cardinality m explicitly. This β-set turns out to be Y.

Remark (1.9). In the notation Proposition of (1.8) and its proof we have: If we order the elements of Y in decreasing order, then the element $h_i - h_{ij}(\lambda)$ is numbered $i + b_{ij}(\lambda)$. This follows from the explicit description of $\lambda\backslash H_{ij}$.

This remark will also be of importance later.

2 Partition sequences and Frobenius symbols

When Frobenius [22] determined the irreducible characters of S_n for the first time in 1900 he was aware that they may be indexed canonically by the partitions of n. However, he preferred a "more useful" indexation of the characters by what he called "characteristics". He also considered β-sets for partitions which arose naturally in his construction of the characters. In this section we describe a theory of cuts in partition sequences which associates to a given partition λ an infinite sequence of symbols including all the β-sets for λ and also Frobenius' "characteristic".

This section gives a different viewpoint to the results of section 1 and it is also of particular importance for the study of "bar partitions" and thus for the study of properties of spin characters. (The material below was taken from [48].)

As in section 1 we use the following notation. Let

$$\lambda = (a_1, a_2, \ldots, a_m), \quad a_1 \geqq a_2 \geqq \ldots \geqq a_m > 0$$

be a partition. The *dual* (or *conjugate*) partition of λ will be denoted by

$$\lambda^0 = (b_1, b_2, \ldots, b_l), \quad b_1 \geqq b_2 \geqq \ldots \geqq b_l > 0$$

The set of *first column hook lengths* of λ is

$$X_\lambda = \{a_i + (m - i) \mid i = 1, 2, \ldots, m\}$$

so $X_\lambda \subseteq N_0$ $(= N \cup \{0\})$. If $X \subseteq N_0$ and $r \in N_0$ we define

$$X^{+r} = \{x + r \mid x \in X\} \cup \{r-1, r-2, \ldots, 1, 0\}.$$

Then the β-sets for λ are by (1.3) exactly the sets X_λ^{+r}, $r \in N_0$.

A *partition sequence* Λ is a double infinite sequence of zeros and ones, such that if we consider the sequence going from the left to the right we have:

(i) All entries to the left of a certain point are zeros.

(ii) All entries to the right of a certain point are ones.

Thus for example

$$\ldots\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \ldots \tag{1}$$

is a partition sequence, where the dots on the left and the right represent infinite sequences of zeros and ones, respectively. For simplicity we may symbolize infinite sequences of zeros and ones by $\underline{0}$ and $\underline{1}$, so that (1) may be written

$$\underline{0}\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ \underline{1} \tag{2}$$

In a partition sequence Λ, the zeros will be numbered $1, 2, 3, 4, \ldots$ in the order they occur moving from the right to the left in Λ, and similarly the ones will be numbered $1, 2, 3, 4, \ldots$ in the order they occur moving from the left to the right in Λ. We call this the *natural numbering* of the zeros and ones in Λ.

In the above example

	...		5	4			3		2	1			Natural numbering of the zeros.
(3)	$\underline{0}$	1	0	0	1	1	0	1	0	0		$\underline{1}$	
		1			2	3		4			...		Natural numbering of the ones.

If the number of ones to the left of the i-th zero (in the natural numbering of Λ) is a_i, we write

$$P(\Lambda) = (a_1, a_2, \ldots, a_m) \tag{4}$$

if $a_m \neq 0$, $a_{m+1} = 0$, so that $P(\Lambda)$ is in fact a partition.

Example. If Λ is the partition sequence in (2) then

$$P(\Lambda) = (4, 4, 3, 1, 1) = (4^2, 3, 1^2).$$

This is seen from (3).

It is obvious from the definition that

Lemma (2.1). The map P is a bijection between the set of all partition sequences and the set of all partitions of nonnegative integers.

Note. The sequence $\Lambda = \underline{01}$ is mapped onto the empty partition (0) of 0.

If $P(\Lambda) = \lambda$, we call Λ the *partition sequence* of λ. The partition sequence Λ of the partition λ incorporates in a natural way all the β-sets for λ as we shall see. This is one of the reasons why we formally consider infinite sequences.

A *β-numbering* of a partition sequence Λ is obtained by numbering some entry in Λ occurring before the first entry one (or the first entry one itself) as 0 and then the following entries as they occur going from the left to the right by $1, 2, 3, \ldots$. In such a β-numbering the numbers of zero entries form a finite subset of N_0, i.e. a β-set. If $P(\Lambda) = \lambda$, this β-set is in fact X_λ^{+r}, where r is the number of first entry 1 in the β-numbering.

Example. Let Λ be as in (1) and (2). A β-numbering of Λ is

$$\begin{array}{cccccccccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & \ldots & & \beta\text{-numbering} \\ \underline{0} & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & \underline{1} & & \end{array}$$

The corresponding β-set is $\{11, 10, 8, 5, 4, 2, 1, 0\}$ (representing the numbers of the zeros). This equals $\{8, 7, 5, 2, 1\}^{+3}$ and $\{8, 7, 5, 2, 1\}$ is the set of first column hook lengths of $P(\Lambda) = (4^2, 3, 1^2)$.

For completeness we note the following: If Λ is a partition sequence, we may define its *dual* Λ^0 as the partition sequence obtained by reading Λ from the right to the left with zeros and ones interchanged. We then have, of course:

Lemma (2.2). In the above notation

$$P(\Lambda)^0 = P(\Lambda^0).$$

Example.

$$\begin{aligned} \Lambda &= \underline{0}\,1\,0\,1\,1\,0\,1\,0\,0\,\underline{1}, \quad P(\Lambda) = (4, 4, 3, 1) \\ \Lambda^0 &= \underline{0}\,1\,1\,0\,1\,0\,0\,1\,0\,\underline{1}, \quad P(\Lambda^0) = (4, 3, 3, 2). \end{aligned}$$

Lemma (2.2) is in a certain sense generalized in the theory of cuts described below.

A *β-sequence* is an infinite sequence of zeros and ones, such that if we read the sequence from the left to the right all entries to the right of a certain point are ones. We number the entries in a β-sequence by $0, 1, 2, \ldots$ from the left to the right. Obviously β-sequences correspond bijectively to β-sets. Namely, if χ is a β-sequence we put

$$Q(\chi) = \{i \in N_0 \mid \text{the } i\text{th entry of } \chi \text{ is zero}\}$$

so that $Q(\chi)$ is a β-set. By adding an infinite number of zeros in front of a β-sequence, it is turned into a *corresponding* partition sequence.

Lemma (2.3). If χ is a β-sequence and Λ the corresponding partition sequence, then $Q(\chi)$ is a β-set for $P(\Lambda)$, i.e.

$$P^*(Q(\chi)) = P(\Lambda).$$

Proof. Obviously the numbering of χ described above gives a β-numbering of Λ, so the result follows.

Suppose that the partition λ' is obtained from another partition λ by removing a hook of length $h \in \mathcal{H}(\lambda)$. If X is a β-set for λ, then by Proposition (1.8) we obtain a β-set Y for λ' by replacing an element c ($\geq h$) in X by $c-h$, whereby $c-h \notin X$. Thus

$$Y = (X \cup \{c-h\})\backslash\{c\}, \quad |Y| = |X|.$$

This means that the β-sequence of Y is obtained from that of X by exchanging the zero in the c'th position with the one in the $(c-h)$'th position.

Therefore it is natural to define a *hook* in a partition sequence Λ as a pair of entries in Λ, a zero and a one, such that the one (called the *arm*) is to the left of the zero (called the *leg*). The hook is *removed* by exchanging the arm and the leg. The hook is said to be in the i-th *row* and the j-th *column* if its leg has the number i and its arm has the number j in the natural numbering of the zeros and the ones. By definition there are a_i ones to the left of the i-th zero, so there are a_i hooks in the i-th row. Thus there is a canonical bijection between the hooks of Λ and the hooks in the (Young diagram of the) partition $\lambda = P(\Lambda)$. Moreover the map P is compatible with the removal of hooks.

We may now give the following interpretation to the arm- and leglengths defined in section 1.

Lemma (2.4). The leglength $b_{ij}(\lambda)$ (armlength $a_{ij}(\lambda)$) of the (i,j)-hook in Λ or $\lambda = P(\Lambda)$ equals the number of zeros (ones) between the arm and the leg of the hook, excluding these.

Indeed the statement about $b_{ij}(\lambda)$ is a consequence of Remark (1.9) and the statement about $a_{ij}(\lambda)$ follows by duality.

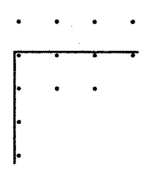
Example.

arm leg

$$\Lambda = \underline{0}\;1\;0\;0\;1\;1\;0\;1\;0\;0\;\underline{1}, \qquad \lambda = (4^2, 3, 1^2).$$

hook

The hook considered is the $(2,1)$-hook in λ as illustrated. (Boxes are indicated by nodes.)



The length is 7, the armlength and the leglength is 3. Removing the hook we get $\Lambda' = \underline{0}110110\underline{1}$ and $\lambda' = P(\Lambda') = (4,2)$.

Let us state explicitly that when the hook is removed, its leglength is the difference between the number of the leg in the natural numbering after and before the removal. (In the above example the leglength is $3 = 5-2$.)

A *cut* in a partition sequence Λ is a dividing line between two entries, such that Λ is divided into two disjoint parts Λ_1 and Λ_2, where Λ_1 (Λ_2) consists of all entries to the right (left) of the dividing line.

Example.

$$\Lambda: \quad \underbrace{\underline{0}\ 1\ 0\ 0\ 1\ 1}_{\Lambda_2} \overset{\text{cut}}{\mid} \underbrace{0\ 1\ 0\ 0\ \underline{1}}_{\Lambda_1},$$

The *coordinates* of the cut are (g, h), where g is the number of zeros in Λ_1 and h is the number of ones in Λ_2. The *position* of the cut is then defined as $h - g$.

Example. In the above example the coordinates are (3,3) and the position is $0 = 3 - 3$.

Suppose that a cut with the coordinates (g, h) is moved one step to the right, so that an entry is added to Λ_2 and the same entry is deleted in Λ_1. Then the new cut has the coordinates

$$\begin{array}{ll} (g, h+1) & \text{if the entry is one,} \\ (g-1, h) & \text{if the entry is zero.} \end{array}$$

Thus *in any case the position is increased by* 1. This shows:

Lemma (2.5). For each $i \in \mathbb{Z}$ there is exactly one cut in Λ with i as its position.

We fix a partition sequence Λ and let $P(\Lambda) = \lambda$ be as above. Moreover we let (g_i, h_i) be the coordinates of the i-th cut in Λ (the cut with position i), so that

(5) $$h_i - g_i = i \quad \text{for all} \quad i \in \mathbb{Z}.$$

Each cut in Λ determines two β-sequences, namely Λ_1 and Λ_2^0. Here Λ_2^0 is the dual of Λ_2, that is Λ_2 read from the right to the left with zeros and ones exchanged. If the cut is in the i-th position, $i \in \mathbb{Z}$, we write

(6) $$F_i(\Lambda) = F_i(\lambda) = (X_i \mid Y_i)$$

where

$$X_i = Q(\Lambda_1), \quad Y_i = Q(\Lambda_2^0)$$

and call $F_i(\Lambda) = F_i(\lambda)$ the i-th *Frobenius symbol* of Λ or λ. Then X_i and Y_i are β-sets and by definition we get

(7) $$|X_i| = g_i, \quad |Y_i| = h_i$$

so that $|Y_i| - |X_i| = i$ by (5). In $(X_i \mid Y_i)$ we simply write the elements of X_i and Y_i in decreasing order.

Example. Consider the cut with position 0 in the previous example. We have

$$X_0 = \{3, 2, 0\}, \quad Y_0 = \{3, 1, 0\},$$

so

$$F_0(\Lambda) = (3, 2, 0 \mid 3, 1, 0).$$

Further examples of Frobenius symbols for our given Λ are

$$\begin{aligned} F_{-1}(\Lambda) &= (4, 3, 1 \mid 3, 0), \\ F_{-5}(\Lambda) &= (8, 7, 5, 2, 1 \mid \emptyset), \\ F_1(\Lambda) &= (2, 1 \mid 5, 2, 1), \\ F_4(\Lambda) &= (\emptyset \mid 8, 5, 4, 2). \end{aligned}$$

(Note that 8,7,5,2,1 and 8,5,4,2 are the first column and first row hooklengths of λ.)

From the definition and Lemma (2.2) we get

Proposition (2.6). If $F_i(\Lambda) = (X \mid Y)$ then $F_{-i}(\Lambda^0) = (Y \mid X)$.

It is also easy to see that the following holds:

Proposition (2.7). If X and Y are β-sets, there exists a unique partition $\lambda = P(X \mid Y)$ having $(X \mid Y)$ as a Frobenius symbol. Then $(X \mid Y) = F_i(\lambda)$, where $i = |Y| - |X|$.

Proof. We simply write the dual β-sequence of Y in front of the β-sequence of X to get a partition sequence with a cut at the place where the β-sequences meet.

Example. $X = \{4, 2\}$, $Y = \{3, 2, 1\}$.

$$\begin{array}{lcccccccccccc} \beta\text{-sequence for X:} & 1 & 1 & 0 & 1 & 0 & \underline{1}. & & & & & & \\ \text{Dual } \beta\text{-sequence for } Y\text{:} & \underline{0} & 1 & 1 & 1 & 0. & & & & & & & \\ \text{Partition sequence:} & \underline{0} & 1 & 1 & 1 & 0 & \mid & 1 & 1 & 0 & 1 & 0 & \underline{1}. \\ \text{Partition } \lambda = (6, 5, 3). & & & & & & & & & & & & \end{array}$$

If again $F_i(\Lambda) = (X_i \mid Y_i)$, then X_i and Y_i are β-sets for certain partitions, say

$$\lambda_i^c := P^*(X_i), \quad \lambda_i^r := P^*(Y_i).$$

Proposition (2.8). In the above notation we have

(i) The Young diagram $Y(\lambda_i^c)$ is obtained by removing the first h_i columns from the Young diagram $Y(\lambda)$, i.e.

$$\lambda_i^c = (b_{h_i+1}, b_{h_i+2}, \dots, b_l)^0.$$

(ii) The Young diagram $Y(\lambda_i^r)^0$ is obtained by removing the first g_i rows from the Young diagram $Y(\lambda)$, i.e.

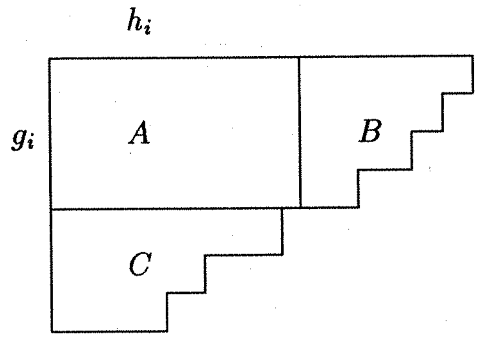
$$\lambda_i^r = (a_{g_i+1}, a_{g_i+2}, \dots, a_m)^0.$$

In particular $\lambda_i^c = 0$ if $h_i \geqq l$ and $\lambda_i^r = 0$ of $g_i \geqq m$.

Proof. The hooks of λ are recognized as pairs of zeros and ones in Λ as described above. But the partition sequence of λ_i^c is obtained from that of λ by changing the first h_i ones (in the natural numbering) into zeros. The result (i) follows and (ii) is proved similarly.

We now give a complete description of the hooks of $\Lambda(\lambda)$ in terms of $F_i(\Lambda) = (X_i \mid Y_i)$, where again $X_i = Q(\Lambda_i)$, $Y_i = Q(\Lambda_2^0)$.

Proposition (2.8) suggests that the Young diagram of λ may be decomposed into three parts as follows:



Part A is the intersection of the first g_i rows with the first h_i columns. The nodes in A correspond to those hooks of Λ whose leg is in Λ_1 (to the right of the cut), and whose arm is in Λ_2 (to the left of the cut). These hooks are called *mixed* (relative to $(X_i \mid Y_i)$). Part B is the Young diagram of λ_i^c and the nodes represent hooks, whose arm and leg are in Λ_1. Similarly part C is the Young diagram of $(\lambda_i^r)^0$ and the nodes represent hooks whose arm and leg are in Λ_2. The hooks in the parts B and C are called *unmixed.* The terms mixed and unmixed reappear in section 4, where the above concepts are applied.

To compute the mixed hooklengths write

$$\begin{aligned} X_i &= \{c_1^i, c_2^i, \ldots, c_{g_i}^i\}, \quad c_i^i > c_2^i > \ldots > c_{g_i}^i \geqq 0 \\ Y_i &= \{d_1^i, d_2^i, \ldots d_{h_i}^i\}, \quad d_1^i > d_2^i > \ldots > d_{h_i}^i \geqq 0. \end{aligned}$$

As before $a_{kk'}(\lambda), b_{kk'}(\lambda), h_{kk'}(\lambda)$ denotes the armlength, leglength and length of the (k, k')-hook in λ.

Lemma (2.9).

(i) For $1 \leq k \leq g_i : c_k^i = a_k - k - i = a_{kk}(\lambda) - i$.
(ii) For $1 \leq k' \leq h_i : d_{k'}^i = b_{k'} - k' + i = b_{k'k'}(\lambda) + i$.

Proof. Since X_i is a β-set for λ_i^c we have that for $1 \leq k \leq g_i$, $c_k^i - (g_i - k)$ has to equal the k-th part of the partition in λ_i^c, that is $a_k - h_i$ (by (2.8)). Thus using (5) we have

$$c_k^i = g_i - k + a_k - h_i = a_k - k - i.$$

Trivially $a_k - k = a_{kk'}(\lambda)$, so (i) is proved and (ii) is proved in a similar way.

(It should perhaps be noted that in the above notation $a_k - h_i \geqq 0$ for $k = 1, 2, \ldots, g_i$, since by the definition of the coordinate of a cut, the first h_i ones are to the left of the first g_i zeros. Thus the first g_i parts of λ are at least equal to h_i.)

In particular for $i = 0$ we have

Corollary (2.10). For $k = 1, 2, \ldots, h_0 = g_0$:

$$c_k^0 = a_k - k, \quad d_k^0 = b_k - k.$$

This corollary shows that the entries of $F_0(\lambda)$ are also the entries in Frobenius' "characteristic" for λ ([22], §4) (see also [31], p.3, or [56], p.49). So $F_0(\lambda)$ consists of the armlengths and leglengths of the diagonal hooks in λ. If $i > 0$, we shift the diagonal by i positions starting then at the $(1, i+1)$-node. If $i < 0$, we start at the $(-i+1, 1)$-node. Going diagonally we shall in any case hit a rim node in the position (g_i, h_i). Then the hook lengths $h_{kk'}(\lambda)$ in Part A of the Young diagram are exactly all the possible sums of an entry in X_i with an entry in Y_i plus 1. Indeed, adding the equations in (2.9) we get:

Corollary (2.11). For $1 \leqq k \leqq g_i$, $1 \leqq k' \leqq h_i$ we have

$$c_k^i + d_{k'}^i = h_{kk'}(\lambda) - 1.$$

(Note also that

(8) $$a_{kk'}(\lambda) = c_k^i + i, \quad b_{kk'}(\lambda) = d_{k'}^i - i,$$

which explains the shifting mentioned above.)

Collecting the information above we have:

Proposition (2.12). In the above notation:

(i) For $1 \leq k \leq g_i$ the hook lengths in the k-th row of λ are

$$\{c_k^i + d_{k'}^i + 1 \mid k' = 1, 2, \ldots, h_i\} \cup \{1, 2, \ldots, c_k^i\} \backslash \{c_k^i - c_l^i \mid l > k\}.$$

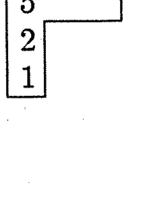
(ii) For all $j = 1, 2, \ldots, h_i$

$$\{h_{g_i+1,j}(\lambda), h_{g_i+2,j}(\lambda), \ldots, h_{b_i,j}(\lambda)\} = \{1, 2, \ldots, d_j^i\} \backslash \{d_j^i - d_{j'}^i \mid j' > j\}.$$

Proof. (i) describes the hook lengths in parts A and B using (2.11) and (2.8) (1), and (ii) describe the hook lengths in part C using (2.8) (2).

Example. Let Λ be as in (5),

$$\lambda = (4^2, 3, 1^2), \quad F_1(\Lambda) = (2, 1 \mid 5, 2, 1), \quad g_1 = 2, \; h_1 = 3.$$



The hook lengths in the first row are

$$\{2+5+1, 2+2+1, 2+1+1\} \cup \{1,2\} \backslash \{2-1\} = \{8,5,4,2\}.$$

We may of course also compute directly the first column hook lengths of λ, that is X_λ, from the Frobenius symbol:

Proposition (2.13). In the above notation

$$X_\lambda = \{c^i_j + d^i_1 + 1 \mid j = 1, 2, \dots g_i\} \cup \{1, 2, \dots, d^i_1\} \backslash \{d^i_1 - d^i_j \mid j = 2, \dots, h_i\}.$$

Note 1. Mixed hooks (relative to $(X_i \mid Y_i)$) correspond canonically to pairs of elements (c, d), $c \in X_i$, $d \in Y_i$. The corresponding hook length is $c + d + 1$, by (2.11). Removing this hook we get a partition having $(X_i \backslash \{c\} \mid Y_i \backslash \{d\})$ as Frobenius symbol!

Note 2. We may compute $|\lambda| = a_1 + \ldots + a_m$ from $(X_i \mid Y_i)$. Indeed

$$|\lambda| = \sum_k c^i_k + \sum_{k'} d^i_{k'} + \frac{1}{2}(g_i + h_i) - \frac{1}{2}(g_i - h_i)^2.$$

This follows easily, since part A contains $g_i \cdot h_i$ nodes, part B contains $\sum_k c^i_k - \binom{g_i}{2}$ nodes and part C contains $\sum_{k'} d^i_{k'} - \binom{h_i}{2}$ nodes.

For $i = 0$, the formula coincides with Frobenius' [22], formula (7) in §4.

Note 3. (2.12) allows us to generalize two further formulas of Frobenius (for the degrees of the irreducible characters in S_n). Let

$$\Delta(x_1, \dots, x_t) = \Pi_{1 \le i < j \le t}(x_j - x_i).$$

Let f_λ be the degree of the irreducible character of S_n corresponding to λ (see section 6). Then

$$f_\lambda = \frac{n! \Delta(c^i_1, c^i_2, \dots, d^i_{g_i}) \Delta(d^i_1, d^i_2, \dots, d^i_{h_i})}{c^i_1! c^i_2! \dots c^i_{g_i}! d^i_1! d^i_2! \dots d^i_{h_i}! \Pi_{k,k'}(c^i_k + d^i_{k'} + 1)}$$

This is a common generalization of [22], (6) in §3 and (9) in §4.

3 Cores and quotients for partitions

We describe for a given partition λ and a given positive integer p, not necessarily a prime, the p-core $\lambda_{(p)}$ and the p-quotient $\lambda^{(p)}$ of λ. Given $\lambda_{(p)}$ and $\lambda^{(p)}$ it is possible to recover λ. There is a certain analogy to the division process of an integer by p where $\lambda_{(p)}$ is the "remainder" and $\lambda^{(p)}$ the "multiple" of p occurring in the division. The p-core $\lambda_{(p)}$ has no hooks of length (divisible by) p and in the p-quotient all information about hooks of length divisible by p in λ may be read off.

The *p-core* $\lambda_{(p)}$ may be obtained from λ as follows: If λ has no hooks of length p, $\lambda = \lambda_{(p)}$. Otherwise remove a p-hook from λ to get λ'. If λ' has no hooks of length p, $\lambda_{(p)} = \lambda'$. Otherwise remove a p-hook from λ'. Continuing like this we eventually reach a partition with no p-hooks which is then the p-core $\lambda_{(p)}$. It is not obvious that the partition thus obtained by removing all p-hooks from λ is unique. This may be proved using a β-set for λ.

Let X be a β-set and $p \in \mathbb{N}$. For $0 \leq i \leq p-1$ define

$$X_i^{(p)} = \{a \in \mathbb{N}_0 \mid ap + i \in X\}$$

and

$$X_{(p)} = \cup_{i=0}^{p-1}\{ap + i \mid 0 \leq a \leq |X_i^{(p)}|\}.$$

We have as is easily calculated

Lemma (3.1). In the above notation

(i) For $i > 0$ $(X^{+1})_i^{(p)} = X_{i-1}^{(p)}$.

(ii) $(X^{+1})_0^{(p)} = (X_{p-1}^{(p)})^{+1}$.

(iii) $(X^{+1})_{(p)} = (X_{(p)})^{+1}$.

(iv) $P^*\left((X^{+1})_{(p)}\right) = P^*(X_{(p)})$.

(v) $P^*\left((X^{+1})_i^{(p)}\right) = P^*(X_{i-1}^{(p)})$, when the i's are computed modulo p.

The equation (iv) shows that if X is a β-set for λ, i.e. $P^*(X) = \lambda$, then the partition $\lambda_{(p)} = P^*(X_{(p)})$ does not depend on the choice of X. We call $\lambda_{(p)}$ the *p-core* of λ. However, the partitions $P^*(X_e^{(p)})$ are permuted cyclicly when X is replaced by X^{+1}, by (v). These partitions form the p-quotient of λ, but in order to avoid ambiguities we must fix the cardinality of X mod p.

Suppose that X is a β-set for λ satisfying $p \mid |X|$. Then the *p-quotient* $\lambda^{(p)}$ of λ is defined as

$$\lambda^{(p)} = (\lambda_0, \dots, \lambda_{p-1})$$

where $\lambda_i = P^*(X_i^{(p)})$.

To see that $\lambda_{(p)} = P^*(X_{(p)})$ is obtained from λ by removing p-hooks and that $\lambda_{(p)}$ has no p-hooks we place the elements of X on a p-abacus (as suggested by G. James). The abacus has p runners going from north to south numbered $0, 1, \dots, p-1$. On each runner there are positions numbered $0, 1, 2, \dots$. We visualize X on this abacus by placing a bead in the a-th position on the i-th runner if and only if $ap + i \in X$. Therefore, the sets $X_i^{(p)}$ list the positions of the beads on the i-th runner. On the $(p-)$-abacus of X the hooks of length p in λ are easily recognized. Indeed if $X = \{h_1, \dots, h_t\}$ there is a hook in the j-th row of λ of length p if and only if $h_j - p \geq 0$ and $h_j - p \notin X$, by (1.5). Thus there has to be a vacant position just above the bead representing h_j on the abacus. Moving the bead to the vacant positions gives the bead configuration for a β-set Y of the partition obtained from λ by removing the p-hook in row j, by (1.8).

Let us consider an example: Let $\lambda = (6, 5, 2, 1^3)$, $p = 3$. As β-set for λ we choose $X = \{11, 9, 5, 3, 2, 1\}$ of cardinality 6 (divisible by $p = 3$). The 3-abacus for X and the hook diagram for λ are:

0	①	②
③	4	⑤
6	7	8
⑨	10	⑪
......	......	......

11	7	5	4	3	1
9	5	3	2	1	
5	1				
3					
2					
1					

The beads 11, 9 and 3 have empty positions immediately above them, so λ has three 3-hooks. (The empty position 0 above 9 represents a hook of length $9 - 0 = 9$.) Movement from 11 to 8 yield the β-set for the partition partition $\lambda \backslash H_{15} = (4^2, 2, 1^3)$. Movement from 9 to 6 yields $\lambda \backslash H_{23} = (6, 2^2, 1^3)$ and movement from 3 to 0 yields $\lambda \backslash H_{41} = (6, 5, 2)$.

Generally, the removal of p-hooks in λ corresponds to moving beads upwards on a runner. The removal process stops when all beads on the abacus are in their highest possible position (i.e. with the lowest numbers). Then we have the bead configuration of the set $X_{(p)}$, so we have proved:

Proposition (3.2). If $P^*(X) = \lambda$, $p \in \mathbb{N}$, then the *p-core*

$$\lambda_{(p)} = P^*(X_{(p)})$$

is the unique partition without p-hooks which may be obtained from λ by removal of a series of p-hooks. In particular, if μ is a partition obtained by removing some p-hooks from λ, then $\lambda_{(p)} = \mu_{(p)}$.

We proceed to study the p-quotient of λ. When $p \mid |X|$, then we have defined

$$\lambda^{(p)} = (\lambda_0, \lambda_1, \dots, \lambda_{p-1}),$$

where $\lambda_i = P^*(X_i^{(p)})$. We note that if we consider a given runner i on the p-abacus as the unique runner in an 1-abacus, then the beads on that runner represent the β-set $X_i^{(p)}$ for

λ_i. Using again (1.8) on X and on the $X_i^{(p)}$'s we get easily the following: Define a hook in the p-quotient $\lambda^{(p)}$ as a hook in one of the partitions λ_i, and the removal of a hook in $\lambda^{(p)}$ analogously. Then

Theorem (3.3). There is a canonical bijection f between the (multi-)set of hooks of λ of length divisible by p and the (multi-)set of hooks of $\lambda^{(p)}$. Thereby a hook $H_{ij}(\lambda)$ of λ of length $p \cdot h$ is mapped unto a hook of length h. For the removal of the hooks we have

$$(\lambda \backslash H_{ij}(\lambda))^{(p)} = \lambda^{(p)} \backslash f\left(H_{ij}(\lambda)\right).$$

Moreover, $f(H_{ij}(\lambda))$ is a hook in λ_e, where $e \equiv a_i - i (\text{mod } p)$.

Proof. Suppose that $X = \{h_1, h_2, \ldots, h_{cp}\}$ is a β-set for λ with $p \mid |X|$. If $h_{ij} = hp$, then by (1.8) $h_i - hp \geq 0$ and $h_i - hp \notin X$. Now $a_i = h_i - (cp - i)$ so that $h_i \equiv a_i - i (\text{mod } p)$. If the bead representing h_i is paced on the e-th runner of the p-abacus then by definition $e \equiv h_i \equiv a_i - i$ (mod p). Writing $h_i = h'p + e$ we get from the assumption that $h' \in X_e^{(p)}$, $h' - h \geq 0$ and $h' - h \notin X_e^{(p)}$. Thus λ_e contains a hook of length h in the row of λ_e corresponding to the element h' in the β-set $\lambda_e^{(p)}$ for λ_e. This unique hook of $\lambda^{(p)}$ is defined to be $f(H_{ij}(\lambda))$. Then the result follows easily from (1.8), applying the definition of p-quotient to the β-sets X for λ and $Y = X \cup \{h_i - hp\} \backslash \{h_i\}$ for $\lambda \backslash H_{ij}(\lambda)$. Note that $|Y| = |X|$ has cardinality divisible by p.

Remark (3.4). The Young diagrams for the partitions λ_e in the p-quotient $\lambda^{(p)} = (\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$ of λ may be visualized as subdiagrams of nodes with hooklengths divisible by p in the Young diagram $Y(\lambda)$ of λ as follows: Let e be given, $0 \leq e \leq p-1$. Suppose that $a_{i_1} \geq a_{i_2} \geq \cdots \geq a_{i_r}$, $i_1 > i_2 > \cdots i_r$ are exactly the parts a_i of λ satisfying $a_i - i \equiv e (\text{mod } p)$. Suppose that the i_j-th row of λ contains $a_j^{(e)}$ nodes of length divisible by p. Then $\lambda_e = (a_1^{(e)}, \ldots, a_r^{(e)})$. This may be seen from (3.3) and the results of section 1.

Example.

$p = 3$

											$a_i - i$ (mod 3)
(15)	14	13	10	**(9)**	8	**(6)**	5	4	**(3)**	1	1
13	12	11	8	7	6	4	3	2	1		2
8	7	6	3	2	1						0
4	3	2									2
(3)	2	1									1

The circled nodes form $Y(3,1)$, where $\lambda_1 = (3,1)$. From row 2 and 4 we see that $\lambda_2 = (3,1)$ and from row 3 that $\lambda_0 = (2)$.

Proposition (3.5). Let λ be a partition with p-core $\lambda_{(p)}$ and p-quotient $\lambda^{(p)} = (\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$. Let λ^0 be the conjugate partition to λ. Then

$$(\lambda^0)_{(p)} = (\lambda_{(p)})^0$$

and

$$(\lambda^0)^{(p)} = (\lambda^0_{p-1}, \lambda^0_{p-2}, \dots, \lambda^0_0)$$

Proof. The statement about $(\lambda^0)_{(p)}$ is clear from the description of the p-core as the partition obtained by removing all p-hooks. The statement about $(\lambda^0)^{(p)}$ may be deduced most easily from (3.3) and (3.4) by noting the following: By definition $h_{ij}(\lambda) = a_i - i + b_j - j + 1$ where b_j is the j-th part of λ^0. If $h_{ij}(\lambda) = hp$ and if $a_i - i \equiv e (\text{mod } p)$, where $0 \leq e \leq p-1$, then $b_j - j \equiv (p-1) - e (\text{mod } p)$.

The next definition is very important for our work: Let λ be a partition. If $\lambda^{(p)} = (\lambda_0, \dots, \lambda_{p-1})$, then the integer

$$w_p(\lambda) = |\lambda_0| + |\lambda_1| + \dots + |\lambda_{p-1}|$$

is called the *p-weight* of λ.

Proposition (3.6). Let λ be a partition. We have

(i) The number of p-hooks to be removed to go from λ to $\lambda_{(p)}$ is $w_p(\lambda)$.

(ii) $|\lambda| = |\lambda_{(p)}| + pw_p(\lambda)$.

(iii) Exactly $w_p(\lambda)$ hooklengths in λ are divisible by p.

Proof. By (3.3) the removal of a p-hook in λ is equivalent to the removal of a 1-hook in the p-quotient $\lambda^{(p)}$ of λ. Since exactly $w_p(\lambda) = \sum_i |\lambda_i|$ 1-hooks may be removed from $\lambda^{(p)}$, (i) follows. (ii) is immediate from (i) and (iii) is a consequence of the bijection established in (3.3).

By definition a *p-core of n* is a partition of n without p-hooks. A *p-quotient of n* is a p-tuple $(\lambda_0, \lambda_1, \dots, \lambda_{p-1})$ of partitions satisfying $\sum |\lambda_i| = n$. The set of p-quotients of n is denoted $K(p,n)$. The cardinality of $K(p,n)$ is $k(p,n)$, see (3.11) and also (9.13) (iv).

Proposition (3.7). Given a p-core κ and a p-quotient $(\lambda_0, \lambda_1, \dots, \lambda_{p-1})$ there exists a unique partition λ such that $\lambda_{(p)} = \kappa$ and $\lambda^{(p)} = (\lambda_0, \lambda_1, \dots, \lambda_{p-1})$. Moreover, $|\lambda| = |\kappa| + pw$, where $w = |\lambda_0| + \dots + |\lambda_{p-1}|$).

Proof. Choose a β-set Y for κ satisfying that $p \mid |Y|$ and that for all i, $0 \leq i \leq p-1$ $Y_i^{(p)}$ contains at least $l(\lambda_i)$ elements. By (3.1) such a choice is possible. Choose β-sets $X_0, X_1, \dots, X_{p-1}$ for $\lambda_0, \lambda_1, \dots, \lambda_{p-1}$ respectively satisfying $|X_i| = |Y_i^{(p)}|$ for all i. This is possible by (1.3). Then by definition

$$X = \cup_{i=0}^{p-1} \{xp + i \mid x \in X_i\}$$

is a β-set for a partition λ with $\lambda_{(p)} = \kappa$ and $\lambda^{(p)} = (\lambda_0, \dots, \lambda_{p-1})$. Moreover, if μ is another partition with the same p-core and p-quotient as λ, choose a β-set X' for μ with $|X'| = |X|$. Then the definition of p-core and p-quotient above forces $X = X'$, i.e. $\lambda = \mu$. The fact that $|\lambda| = |\kappa| + pw$ follows from (3.6) (2).

Let κ be a p-core and $w \in \mathbb{N} \cup \{0\}$. We define $\mathcal{B}(w,\kappa) = \{\lambda \in \mathcal{P}(|\kappa| + wp) \mid \lambda_{(p)} = \kappa\}$, the set of partitions with p-weight w and p-core κ.

Corollary (3.8). Let κ and κ' be arbitrary p-cores, $w \in \mathbb{N} \cup \{0\}$. There is a bijection

$$\ominus^{w}_{\kappa,\kappa'} : \mathcal{B}(w,\kappa) \to \mathcal{B}(w,\kappa')$$

satisfying that $\ominus^{w}_{\kappa,\kappa'}(\lambda) = \lambda'$ if and only if λ and λ' have the same p-quotient.

Proof. Immediate from (3.7).

Remark (3.9). The previous Corollary (3.8) is of particular interest when one of the p-cores, say κ', is chosen as the partition (0) of 0. Then $\ominus^{w}_{\kappa} = \ominus^{w}_{\kappa(0)}$ is called a *reduction map.* The effect of $\ominus^{w}_{\kappa}$ is to "throw away the p-core". If one would like to replace the p-tuple of partitions in the p-quotient of a partition λ by a single partition which incorporates the complete p-hook structure of λ and has an empty p-core, this partition should be $\lambda' = \ominus^{w}_{\kappa}(\lambda)$, where $w = w_p(\lambda)$ and $\kappa = \lambda_{(p)}$. Note that by (3.6) (2) $|\lambda| = |\lambda_{(p)}| + |\lambda'|$.

Remark (3.10). By (3.9) if κ is a p-core and $w \geq 0$ then $|\mathcal{B}(w,\kappa)| = |\mathcal{B}(w,(0))|$ equals the number of p-quotients $(\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$ with of w. Thus $|\mathcal{B}(w,\kappa)|$ depend only on w and p. Thus cardinality play an important role in the following and is denoted $k(p,w)$. We have obviously

(3.11) $$k(p;w) = \textstyle\sum_{(w_0,w_1,\ldots,w_{p-1})} p(w_0)p(w_1)\cdots p(w_{p-1}),$$

where $(w_0, \ldots, w_{p-1})$ runs through all p-tuples of non-negative integers with $\sum_i w_i = w$. In particular $k(p,1) = p$, $k(1,w) = p(w)$.

Let us mention that for any p-core κ and any $w \geq 1$ there is an action of the wreath product $\mathbb{Z}_2$ wr S_p on $\mathcal{B}(w,\kappa)$. Indeed we let $\mathbb{Z}_2$ wr S_p act on the set $K(p,w)$ of p-quotients of w which then by (3.7) induces an action on $\mathcal{B}(w,\kappa)$. The base subgroup $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ acts on the quotients by conjugating the partitions in a quotient and S_p by permuting the partitions in a quotient. Since $\mathcal{P}(n)$, the set of all partitions of an integer n, is a union of sets $\mathcal{B}(w,\kappa)$ for suitable w's and κ's we get an action of $\mathbb{Z}_2$ wr S_p on $\mathcal{P}(n)$. To the knowledge of the author, this interesting action has never been studied seriously.

We finish by some miscellanous results and remarks, some of which are needed later.

First we describe a nice theory of relative p-signs for partitions having the same p-core. (See (11.6) later.)

Let X be a β-set. We number the elements of X with the numbers $1, 2, \ldots, |X|$ in two different ways:

In the *natural numbering* the elements of X are numbered according to their size in increasing order. (Thus c is numbered lower than d if and only if $c < d$.)

Next, let p be an arbitrary positive integer. We represent the elements of X as beads on the p-abacus as above. (So there is a bead in the a-th row of the i-th runner $(0 \leq i \leq p-1)$ if and only if $ap + i \in X$.) The j-th *layer*, $j \geq 1$, consists of those beads which have $(j-1)$ beads above them on their respective runners. In the *p-numbering* the element of X

represented by the j-th bead in the i-th runner is numbered before the element represented by the j_1-th bead on the i_1-th runner if and only if

$$j < j_1 \quad \text{or} \quad j = j_1 \quad \text{and} \quad i < i_1$$

(i.e. according to the layers!).

When we compare the natural numbering of X with the p-numbering of X we get a permutation $\pi_p(X)$ of $\{1, 2, \dots, |X|\}$, whose *sign* is denoted by $\delta_p(X)$. It is easily verified that $\delta_p(X) = \delta_p(X^{+1})$ and so if X is any β-set for the partition λ we may define

$$\delta_p(\lambda) = \delta_p(X),$$

the *p-sign of* λ.

Example. Let $p = 3$ and consider the β-set $X = \{9, 8, 7, 4, 1, 0\}$ for the partition $\lambda = (4, 4, 4, 2)$. The 3-abacus of X is

$$\begin{array}{ccc} \textcircled{0} & \textcircled{1} & 2 \\ 3 & \textcircled{4} & 5 \\ 6 & \textcircled{7} & \textcircled{8} \\ \textcircled{9} & 10 & 11 \end{array}$$

The first layer is 0,1,8, the second layer is 9,4, and the third layer is 7. Thus

Natural numbering:
$$\begin{array}{ccc} \textcircled{0}_1 & \textcircled{1}_2 & 2 \\ 3 & \textcircled{4}_3 & 5 \\ 6 & \textcircled{7}_4 & \textcircled{8}_5 \\ \textcircled{9}_6 & 10 & 11 \end{array}$$

3-numbering:
$$\begin{array}{ccc} \textcircled{0}_1 & \textcircled{1}_2 & 2 \\ 3 & \textcircled{4}_5 & 5 \\ 6 & \textcircled{7}_6 & \textcircled{8}_3 \\ \textcircled{9}_4 & 10 & 11 \end{array}$$

and so

$$\pi_3(X) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 3 & 4 \end{pmatrix} = (35)(46)$$

and

$$\delta_3(X) = \delta_3(\lambda) = 1.$$

Suppose that the partition μ is obtained from λ by removing a sequence of p-hooks. Then we define the *relative sign* $\delta_p(\lambda, \mu)$ by

$$\delta_p(\lambda, \mu) = \delta_p(\lambda)\delta_p(\mu).$$

Trivially, the relative sign is transitive. If μ is obtained from λ by removing a sequence of p-hooks, and ζ is obtained from μ by removing a sequence of p-hooks, then $\delta_p(\lambda,\zeta) = \delta_p(\lambda,\mu)\delta_p(\mu,\zeta)$. Therefore we are led to consider the minimal case where μ is obtained from λ by removing a single p-hook $H_{ij}(\lambda)$ say. Then a β-set Y for μ is obtained from X by replacing h_i by $h_i - p$ and by (1.9) the leglength of the hook is $b_{ij}(\lambda) = |\{d \in X \mid h_i - p < d < h_i\}|$.

Since the number of beads above the one representing h_i on the relevant runner on the p-abacus for X equals the same number for $h_i - p$ on the p-abacus for Y, the p-number of $h_i \in X$ equals the p-number of $h_i - p \in Y$. But obviously the natural numbering is changed by a cycle of length $b_{ij}(\lambda) + 1$. Thus

Lemma (3.12). If μ is obtained by removing the single p-hook $H_{ij}(\lambda)$ from λ, then

$$\delta_p(\lambda,\mu) = (-1)^{b_{ij}(\lambda)}$$

where $b_{ij}(\lambda)$ is the leglength of the hook.

From this and the transitivity of the relative sign we get:

Proposition (3.13). If μ is obtained from λ by removing a sequence of v p-hooks having leglength $b_1, b_2, \ldots, b_v$, then

$$\delta_p(\lambda,\mu) = (-1)^b$$

where $b = \sum_i b_i$. In particular the residue of $b (\mathrm{mod}\ 2)$ does not depend on the choice of p-hooks being removed in going from λ to μ.

This result can be applied in particular to the case where $\mu = \lambda_{(p)}$, the p-core of λ. For a β-set for $\lambda_{(p)}$ the natural numbering and the p-numbering coincide since the beads on the p-abacus are all in the highest possible position. Thus $\delta_p(\lambda_{(p)}) = 1$ and we have

Corollary (3.14). For any partition

$$\delta_p(\lambda) = \delta_p\left(\lambda, \lambda_{(p)}\right).$$

Note. This shows that $\delta_p(\lambda)$ is the sign σ of Robinson [56] or [25], p.82.

Remark (3.15) (On the p-core/defect 0 problem). Let $c(p,n)$ be the number of p-cores of n. We are concerned with the following question:

For which p and n is $c(p,n) \neq 0$?

This apparently purely combinatorial question is of importance in the representation theory of symmetric (and other) groups. As we shall see in section 6, when p is a prime, then $c(p,n)$ equals the number of p- blocks of defect 0 in S_n whereas for general p $c(p,n)$ counts unipotent characters of general linear groups $GL(n,q)$ which are of defect 0 for suitable primes *not* dividing q (the "non-defining characteristic"), see (8.8).

Trivially all partitions of n are p-cores when $n < p$, so in that case $c(p,n) \neq 0$. Moreover, $c(1,n) = 0$ when $n \neq 0$. It has been known for a long time that

$$c(2,n) = \begin{cases} 1 & \text{if} \quad n = k(k+1)/2 \quad \text{for some} \quad k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The 2-core of $k(k+1)/2$ is the partition $(k, k-1, \ldots, 1)$.

It is also known that $c(3,n) = 0$ for infinitely many n. For example $c(3,7) = 0$.

However, looking at tables of $c(p,n)$ for $p \geq 4$ it seems reasonable to conjecture.

Conjecture (3.16). For all n and all $p \geq 4$

$$c(p,n) \neq 0.$$

The following is known to the author about (3.16):

(i) Many authors (Klyachko, Atkin, Andrews, ...) have given an explicit formula for $c(5,n)$ implying in particular $c(5,n) \neq 0$ for all n ([27]).

(ii) Erdmann and Michler ([17]) have explicitly exhibited how to construct a 5-core of n for any given n.

(iii) From (1.6) we get: If $p \mid p'$ then a p-core is also a p'-core. Thus: If $p \mid p'$ then $c(p,n) \leq c(p',n)$. In particular, if $c(p,n) \neq 0$ then $c(p',n) \neq 0$.

(iv) K. Ono has shown: $c(p,n) \neq 0$ if $p \geq 4$ is *even* (see [53]).

(v) Several authors (Klyachko, Atkin, Ono, ...) have noticed that for $p \geq 4$ there are at most finitely many n, s.t. $c(p,n) \neq 0$. However, it does not seem to be possible for a given p to give an explicit bound m, s.t. $c(p,n) \neq 0$ for $n \geq m$.

(vi) If p is divisible by 7, 9, or 11 then $c(p,n) \neq 0$ for all n (Atkin, Ono, ...). This list of odd numbers/primes may probably be extended by a case-by-case analysis.

Most of the non-trivial results above including (iv) – (vi) have been proved using the theory of modular forms. The point is that the generating functions for $c(p,n)$ (see (9.17)), given p, may be expressed using a quotient of Dedekind's η-function which is a modular form of weight $\frac{p-1}{2}$ on $\Gamma_0(p)$ (see [27], [53],... for further details). An alternative approach which is quite interesting but does not appear to be easy to use for obtaining results is this:

Let X be a minimal β-set for a p-core. Then $0 \notin X$ and the 0-th runner on the p-abacus for X must be empty and $X = X_{(p)}$. This means that X is characterized by a $(p-1)$-tuple $(x_1, x_2, \ldots x_{p-1})$ of non-negative integers where $|X_i^{(p)}| = x_i$ for $1 \leq i \leq p-1$ ($|X_0^{(p)}| = 0$).

If X is characterized by $(x_1, \dots x_{p-1})$ is a β-set for the p-core κ, then we get by an easy calculation that

$$|\kappa| = \sum_{i=1}^{p-1} \left[p \binom{x_i}{2} + i x_i \right] - \binom{\sum_{i=1}^{p-1} x_i}{2}.$$

The right hand side may be seen as a polynomial of degree 2 in the x_i's. Then Conjecture (3.15) is equivalent to the question whether any positive integer is represented by this polynomial for non-negative integral x_i's. A variation on this is presented in [23]. This finishes our remarks on (3.16).

It is quite easy to prove some surprising results about 2-cores of classes of partitions. As an example, let us look at the following result:

Proposition (3.17). Let $p > 0$ be *odd*. Let κ be any p-core. Then the p partitions in the set $\mathcal{B}(1, \kappa)$ have at most two different 2-*cores*.

This result is proved using β-sets. We have seen that the 2-cores are exactly the following partitions κ_k where

$$\kappa_k = (k, k-1, \dots, 1)$$

$k \geq 0$. From (3.2) and its proof we get easily the following:

Lemma (3.18). Let X be a β-set for the partitions λ. Suppose that X contains r odd integers and s even integers. Then in the above notation $\lambda_{(2)} = \kappa_k$, where

$$k = \begin{cases} s - r & \text{if } s \geq r \\ r - s - 1 & \text{if } s < r \end{cases}$$

Proof of (3.17). Let Y be a β-set for κ with the property that all runners in the p-abacus for Y are non-empty ($Y_e^{(p)} \neq \emptyset$ for all e). By (1.8) we obtain β-sets for the p partitions in $\mathcal{B}(1, \kappa)$ by replacing elements $h_i \in Y$ by $h_i + p$. (The h_i's are represented as those beads on the p-abacus having an empty position immediately below them.) Obviously h_i and $h_i + p$ have different parity since p is odd. If Y has r odd integers and s even integers then the β-sets for the partitions in $\mathcal{B}(1, \kappa)$ has $r - 1$ odd integers and $s + 1$ even integers (if h_i is odd) or $r + 1$ odd integers and $s - 1$ even integers (if h_i is even). The result follows from (3.17).

Remark (3.19). In (3.17) κ does not really need to be a p-core: If κ is any partition, p odd and if $\mathcal{B}$ is the set of partitions obtained by adding an arbitrary p-hook to κ, then the partitions in $\mathcal{B}$ have two different 2-cores at most. The same is true for the set $\mathcal{B}'$ of partitions obtained by *removing* a p-hook from κ.

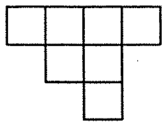
4 On bar partitions

In this section we describe a non-trivial analogue of the results of sections 1 and 3 for partitions into distinct parts. This is needed for the study of spin characters of the covering groups of S_n.

If λ is a partition of n with all parts different we call λ a *bar partition.* (Such a partition is usually also called 2-regular.) We assume that

$$(1) \qquad \lambda = (a_1, \dots, a_m), \quad a_1 > a_2 > \cdots > a_m > 0$$

is a bar partition of n. The set of bar partitions of n is denoted $\bar{\mathcal{P}}(n)$ and $|\bar{\mathcal{P}}(n)| = q(n)$. Let $S(\lambda)$ be the shifted Young diagram of λ. It is obtained from the usual Young diagram of λ by shifting the i-th row $(i-1)$-positions to the right. For example, if $\lambda = (4, 2, 1)$ then

$S(\lambda)$: 

The j-th node in the i-th row will be called the (i,j)-node. To each node in $S(\lambda)$, we associate an integer *bar length* as follows: The bar lengths in the i-th row are obtained by writing the elements of the following set in decreasing order:

$$(2) \qquad \bar{\mathcal{H}}_i(\lambda) = \{1, 2, \dots, a_i\} \cup \{a_i + a_j \mid j > i\} \backslash \{a_i - a_j \mid j > i\}.$$

(So in $\{1, 2, \dots, a_i\}$ one replaces $a_i - a_j$ by $a_i + a_j$). Thus the first $(m-i)$-bar lengths are $a_i + a_{i+1}, \dots, a_i + a_m$ (in this order), and then the remaining $a_i - (m-i)$-bar lengths are the *hook lengths* in the i-th row of the partition $\lambda^* = P^*(\lambda)$ (considering λ as a β-set!). As was the case with hooks we put $\bar{\mathcal{H}}(\lambda) = \cup_i \bar{\mathcal{H}}_i(\lambda)$, a multiset.

In the above example the bar lengths are

$$\begin{matrix} 6 & 5 & 4 & 1 \\ & 3 & 2 & \\ & & 1 & \end{matrix}$$

(Note that $\begin{matrix} 4 & 1 \\ 2 & \\ 1 & \end{matrix}$ is the hook diagram for $(2, 1^2) = P^*(\{4, 2, 1\})$.)

We denote the (i,j)-bar length by $\bar{h}_{ij}(\lambda)$. To each node (i,j) in $S(\lambda)$ we associate a *bar*, i.e. a subdiagram of $S(\lambda)$ consisting of $\bar{h}_{ij}(\lambda)$ nodes. If $i + j > m$ the (i,j)-bar consists of the last $\bar{h}_{ij}(\lambda)$ nodes in the i-th row of $S(\lambda)$. Such a bar is called *unmixed.* If $i + j \leqq m$, the (i,j)-bar consists of all the nodes in the i-th and all the nodes in the $(i+j)$-th row of $S(\lambda)$ (a *mixed* bar). If we remove the nodes of the (i,j)-bar from $S(\lambda)$ and rearrange the rows of the diagram obtained according to size we obtain a new shifted diagram $S(\mu)$ and say that μ is obtained from λ by *removing the* (i,j)*-bar.* Thus $\mu \in \bar{\mathcal{P}}(n - \bar{h}_{ij}(\lambda))$. The parts of μ are

$$a_1, a_2, \dots, a_{i-1}, a_i - \bar{h}_{ij}(\lambda), a_{i+1}, \dots, a_m \quad \text{if} \quad i + j > m$$

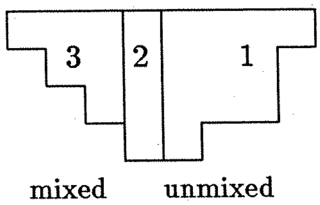
and

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{i+j-1}, a_{i+j+1}, \dots, a_m \quad \text{if} \quad i + j \leqq m.$$

If the (i,j)-bar is removed from λ to get μ we write $\mu = \lambda \backslash \bar{H}_{ij}(\lambda)$. A bar of length l is called an *l-bar*. The (i,j)-bar is said to be of

$$\text{(3)} \qquad \left\{ \begin{array}{lll} \textit{Type } 1, & \text{if} & i+j \geqq m+2 \\ \textit{Type } 2, & \text{if} & i+j = m+1 \\ \textit{Type } 3, & \text{if} & i+j \leqq m \end{array} \right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{ unmixed bar} \\ \quad \text{mixed bar} \end{array}$$

This is illustrated in the following shifted diagram:



We define the *leglength* $\bar{b}_{ij}(\lambda)$ of the (i,j)-bar of λ as follows:

$$\bar{b}_{ij}(\lambda) = \left\{ \begin{array}{ll} |\{k \mid a_i > a_k > a_i - \bar{h}_{ij}(\lambda)\}| & \text{for bars of type 1 and 2} \\ a_{i+j} + |\{k \mid a_i > a_k > a_{i+j}\}| & \text{for bars of type 3.} \end{array} \right.$$

We investigate all bars in λ whose lengths are divisible by a fixed positive integer p. As was done in section 3 (for β-sets of partitions) we represent the *parts* of λ as beads on a *p-abacus*. This abacus has runners and positions on the runners as described in section 3. For $0 \leq i \leq p-1$ we define

$$X_i^\lambda = \{a \in \mathbb{N}_0 \mid \text{There exists a } k, 1 \leq k \leq m, \text{ such that } a_k = ap + i\},$$

so that the X_i^λ are β-sets. The bead configuration of λ on the p-abacus then includes a bead in the j-th position of the i-th runner if and only if $j \in X_i^\lambda$.

(The material on the following pages has been taken from [48].)

We define a p-bar core $\lambda_{(\bar{p})}$ (abbreviated *$\bar{p}$-core*) and a p-bar quotient $\lambda^{(\bar{p})}$ (*$\bar{p}$-quotient*) for λ having properties analogous to the p-core and p-quotient for arbitrary partitions described in section 3. Especially we want the structure of $\lambda^{(\bar{p})}$ to describe the bars of length divisible by p. In general this is only possible for p *odd.* Consider the following:

Example. $\lambda = (3,1)$. The bar diagram is $\begin{matrix} 4 & 3 & 1 \\ & 1 & \end{matrix}$. So λ contains a 4-bar, but no 2-bar.

This shows that for $p = 2$ there cannot be a $\bar{p}$-quotient having the property of Theorem (4.3) below. The same difficulty arises when p is even. (Then $(3p/2, p/2)$ has a $2p$-bar but no p-bar.) This also shows that both statements of the Corollary in [38], p.31, are false for $p(= q)$ even. Another difficulty arises also when p is even. Part of it is reflected in [38], Theorem 2(2). In [38], $\bar{p}$-quotients are also defined for p even but they are not suitable for our purposes.

However, a careful examination of the arguments below shows that the difficulties will not arise for λ when $X_{p/2}^\lambda = \emptyset$.

We assume from now on that p is odd and put $t = (p-1)/2$. Then we define the $\bar{p}$-quotient

$$\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \dots, \lambda_t),$$

where λ_0 is the bar partition whose parts are the elements in X_0^λ and where for $1 \leqq j \leqq t$

$$\lambda_j = P\left(X_j^\lambda \mid X_{p-j}^\lambda\right) \tag{4}$$

in the notation of (2.7). Thus $\lambda_1, \dots, \lambda_t$ are partitions. The runner 0 determines λ_0 and for $1 \leqq j \leqq t$ the *conjugate runners* j and $p-j$ determines λ_j.

The removal of a p-bar is registered on the abacus as follows (see (3)):

(5)
- *Type* 1 : Move a bead one position up to the same runner.
- *Type* 2 : Remove the bead in the first position on the 0-th runner.
- *Type* 3 : Remove the two beads in the 0-th position on the j-th and the $(p-j)$-th runner for some j, $1 \leqq j \leqq t$.

When we remove recursively all p-bars from λ we see (using (5) repeatedly) that the p-abacus configuration is changed until the following configuration is obtained:

(i) There are no beads on the 0-th runner.

(ii) For $1 \leqq j \leqq p-1$ the j-th runner contains $l_j = \text{Max}\,(|X_j^\lambda| - |X_{p-j}^\lambda|, 0)$ beads in the l_j first positions.

Thus:

Proposition (4.1). The $\bar{p}$-core $\lambda_{(\bar{p})}$ of λ obtained by removing all p-bars from λ is uniquely determined by λ and p.

Putting $f_j = |X_j^\lambda| - |X_{p-j}^\lambda|$ for $1 \leqq j \leqq t$, we see that the t-tuple $(f_1, \dots, f_t)$ of integers determines $\lambda_{(\bar{p})}$ completely. We call it the *characteristic* of $\lambda_{(\bar{p})}$.

Example. $\lambda = (17, 14, 13, 11, 9, 5, 2)$, $p = 5$:

5-abacus:				
0	1	(2)	3	4
(5)	6	7	8	(9)
10	(11)	12	(13)	(14)
15	16	(17)	18	19

$X_0^\lambda = \{1\}$, $X_1^\lambda = \{2\}$, $X_2^\lambda = \{3, 0\}$, $X_3^\lambda = \{2\}$, $X_4^\lambda = \{2, 1\}$. Thus $\lambda_0 = (1)$, $\lambda_1 = P(2 \mid 2, 1) = (4, 2)$, $\lambda_2 = P(3, 0 \mid 2) = (3, 1^3)$.

Partition sequence for λ_1:	$\underline{0}\,1\,1\,0 \mid 1\,1\,0\,\underline{1}$.
Partition sequence for λ_2:	$\underline{0}\,1\,0\,0 \mid 0\,1\,1\,0\,\underline{1}$.
Characteristic: (-1,1)	$\lambda_{(\bar{5})} = (4,2)$.

Suppose that $(\lambda_0, \lambda_1, \dots, \lambda_t)$ is given where λ_0 is a bar partition and $\lambda_1, \dots, \lambda_t$ are partitions, and suppose that $(f_1, \dots, f_t) \in Z^t$. Let X_0 be the set of parts of λ_0 and for $1 \leqq i \leqq t$ let

$$F_{f_i}(\lambda_i) = (X_i \mid X_{p-i}).$$

Then the bar partition λ having $X_i^\lambda = X_i$ for $i = 0, 1, \dots, p-1$ has $(\lambda_0, \lambda_1, \dots, \lambda_t)$ as a $\bar{p}$-quotient and its $\bar{p}$-core has $(f_1, \dots, f_t)$ as characteristic. (As in section 2 F denotes a Frobenius symbol.) We have proved:

Proposition (4.2). A bar partition determines and is uniquely determined by its $\bar{p}$-core and its $\bar{p}$-quotient.

We may generalize (5) as follows:

The removal of an lp-bar, $l \geqq 1$ is registered on the p-abacus as follows:

(6)

Type 1: Move a bead l positions up to the same runner.

Type 2: Remove the bead in the l-th position on the 0-th runner.

Type 3: (i) Remove the two beads in the 0-th runner in the l_1-th and l_2-th position, $1 \leqq l_1 < l_2$, $l_1 + l_2 = l$.

(ii) Remove the bead in the l_1-th position and the j-th runner and the bead in the l_2-th position and the $(p-j)$-th runner, $1 \leqq j \leqq t$, $l_1 + l_2 = l - 1$.

Defining an *l-bar* in $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \dots, \lambda_t)$ to be either an l-bar in λ_0 or an l-hook in one of the partitions $\lambda_1, \dots, \lambda_t$ and the *removal* of an l-bar in $\lambda^{(\bar{p})}$ correspondingly, we have the following important result:

Theorem (4.3). There exists a canonical bijection g between the set of bars of λ of length divisible by p and the set of bars in $\lambda^{(\bar{p})}$. Thereby an lp-bar is mapped on an l-bar. Moreover, for the removal of corresponding bars we have

$$(\lambda \backslash \bar{H})^{(\bar{p})} = \lambda^{(\bar{p})} \backslash g(\bar{H}).$$

Proof. Suppose that $\bar{H}$ is an lp-bar, $l \geqq 1$. We adapt the notation of (6).

Type 1. The removal of $\bar{H}$ is registered by moving a bead l positions up on (say) the j-th runner. In $\lambda^{(\bar{p})}$ this corresponds
if $j = 0$ to the removal of an l-bar of type 1 in λ_0,
if $1 \leqq j \leqq t$ to the removal of an l-hook in part B of the Young diagram of λ_j,
if $t+1 \leqq j \leqq p-1$ to the removal of an l-hook in the part C of the Young diagram of λ_{p-j}.
(The letters B and C refer to the decomposition of a Young diagram described in section 2.)

Type 2. The removal of $\bar{H}$ is registered by removing a bead from the 0-th runner. Correspondingly the l-bar of type 2 is removed in λ_0.

Type 3. (i) The removal of $\bar{H}$ is registered by removing beads representing l_1p and l_2p. Correspondingly a mixed l-bar (of type 3) consisting of the parts l_1, l_2 is removed from λ_0. (ii) The removal of $\bar{H}$ is registered by removing beads on the j-th and $p-j$-th runner $(1 \leqq j \leqq t)$. Correspondingly a mixed hook relative to $(X_i^\lambda \mid X_{p-j}^\lambda)$ is removed from λ_j (in part A of the Young diagram).

The above describes how the map g has to be defined in order to be compatible with the removal of bars and shows its existence. Thus (4.3) is proved.

If $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$ we define the *$\bar{p}$-weight* of λ as

$$w_{\bar{p}}(\lambda) = |\lambda_0| + |\lambda_1| + \cdots + |\lambda_t|.$$

Repeated use of (4.3) (with $l = 1$) shows (cfr. also (3.6)):

Corollary (4.4). Notation as above.

(i) The number of p-bars to be removed to go from λ to $\lambda_{(\bar{p})}$ is $w_{\bar{p}}(\lambda)$.

(ii) $|\lambda| = |\lambda_{(\bar{p})}| + pw_{\bar{p}}(\lambda)$.

(iii) Exactly $w_{\bar{p}}(\lambda)$ bar lengths in λ are divisible by p.

Corollary (4.5). If λ contains an lp-bar, $l \geq 2$, p odd, then λ contains also a p-bar.

As we have seen, $w_{\bar{p}}(\lambda)$ p-bars have to be removed going from λ to $\lambda_{(\bar{p})}$. One may ask how many of these p-bars are of type 1,2 and 3 respectively. This number will depend on $\lambda_{(\bar{p})}$ and $\lambda^{(\bar{p})}$ as follows (let as usual $l(\lambda)$ denote the number of parts in the bar partition λ, i.e. the length of λ):

Corollary (4.6). Let $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$.

(i) The number of p-bars of type 2 being removed going from λ to $\lambda_{(\bar{p})}$ is $l(\lambda_0)$.
(ii) The number of p-bars of type 3 being removed going from λ to $\lambda_{(\bar{p})}$ is $\frac{1}{2}(l(\lambda) - l(\lambda_{(\bar{p})}) - l(\lambda_0))$.

Proof. (i) is an easy consequence of the proof of (4.3).

(ii) By the removal of a bar of type 2, the number of parts in a bar partition is reduced by 1, and by the removal of a bar of type 3 the number of parts is reduced by 2. So if a is the number of p-bars of type 3 being removed going from λ to $\lambda_{(\bar{p})}$, then by (i)

$$l(\lambda) - l(\lambda_{(\bar{p})}) = l(\lambda_0) + 2a.$$

The result follows.

Let κ be a $\bar{p}$-core and $w \in \mathbb{N} \cup \{0\}$. We define

$$\bar{\mathcal{B}}(w, \kappa) = \{\lambda \in \bar{\mathcal{P}}(|\kappa| + wp) \mid \lambda_{(\bar{p})} = \kappa\}$$

the set of bar partitions of $\bar{p}$-weight w with $\bar{p}$-core κ.

Corollary (4.7). Let κ and κ' be arbitrary $\bar{p}$-cores, $w \in \mathbb{N} \cup \{0\}$. There is a bijection

$$\bar{\mathcal{O}}^w_{\kappa\kappa'} \; : \; \bar{\mathcal{B}}(w,\kappa) \to \bar{\mathcal{B}}(w,\kappa')$$

satisfying that $\bar{\Theta}^w_{\kappa\kappa'}(\lambda) = \lambda'$ if and only if λ and λ' have the same $\bar{p}$-quotient.

Remark (4.8). By (4.7) if κ is a $\bar{p}$-core and $w \geq 0$ then $|\bar{\mathcal{B}}(\kappa, w)|$ equals the number of $\bar{p}$-quotients $(\lambda_0, \lambda_1, \ldots, \lambda_t)$, with $\sum_i |\lambda_i| = w$. Here $t = (p-1)/2$, λ_0 is a bar partition and $\lambda_1, \ldots, \lambda_t$ are arbitrary partitions. Thus (see also (3.10))

$$\begin{aligned} |\bar{\mathcal{B}}(\kappa, w)| &= k(\bar{p}, w), \quad \text{where} \\ q(\bar{p}, w) &= \sum_{i=0}^{w} q(i) k(t, w-i) \end{aligned}$$

It is possible to prove analogous (considerably more complicated) of the results on relative p-signs in the previous section (see (3.12) – (3.14)) in the case where λ is a bar partition and where bars replace hooks. Again it is reasonable to restrict ourselves to the case where p is odd. The relative $\bar{p}$-sign is of importance when you consider iterated versions of the spin analogue of the Murnaphan-Nakayama formula (see [37] from which the following is taken) and may also be of relevance in another connection (see (13.15)).

A γ-set $\mathfrak{X}$ is a finite subset of $\mathbb{Z}$ with the following property: If $i \in \mathfrak{X}$ and $i < 0$ then $i+1 \in \mathfrak{X}$.

The γ-set $\mathfrak{X}$ consists of two disjoint subsets

$$\begin{aligned} \mathfrak{X}^- &= \{i \in \mathfrak{X} \mid i \leqq 0\} \\ \mathfrak{X}^+ &= \{i \in \mathfrak{X} \mid i > 0\}. \end{aligned}$$

If the elements of $\mathfrak{X}^+$ form the parts of the bar partition λ we call $\mathfrak{X}$ a *γ-set for* λ and write $\bar{P}^*(\mathfrak{X}) = \lambda$. Thus a bar partition λ has infinitely many γ-sets. These are distinguished simply by $|\mathfrak{X}^-|$ (since $|\mathfrak{X}^-|$ of course determines $\mathfrak{X}^- = \{0, -1, \ldots, -(|\mathfrak{X}^-|-1)\}$).

Example. $\mathfrak{X} = \{-2, -1, 0, 1, 4, 5\}$ is a γ-set for $\bar{P}^*(\mathfrak{X}) = \lambda = (4, 5, 1) \in \bar{\mathcal{P}}(10)$.

Our idea is to describe the removal of p-bars in terms of γ-sets which have the same cardinality for these partitions. This will make it possible to define the p-bar sign ($\bar{p}$-sign) of λ as the sign of a permutation.

Suppose then that $\bar{P}^*(\mathfrak{X}) = \lambda$, $\bar{P}^*(\mathfrak{Y}) = \mu$, that $|\mathfrak{X}| = |\mathfrak{Y}|$, and that μ is obtained from λ by removing a p-bar. Then

$$\begin{array}{llll} |\mathfrak{Y}^-| = |\mathfrak{X}^-|, & (\mathfrak{Y}^+ \backslash \{a\}) \cup \{a-p\}, \quad a > p & & \text{if the bar is of } \textit{type } 1, \\ |\mathfrak{Y}^-| = |\mathfrak{X}^-| + 1, & \mathfrak{Y}^+ = \mathfrak{X}^+ \backslash \{p\}, & & \text{if the bar is of } \textit{type } 2, \\ |\mathfrak{Y}^-| = |\mathfrak{X}^-| + 2, & \mathfrak{Y}^+ = \mathfrak{X}^+ \backslash \{a, a'\}, & (a + a' = p) & \text{if the bar is of } \textit{type } 3. \end{array}$$

As was the case for β-sets in section 3, we number the elements of a γ-set $\mathfrak{X}$ by $1, 2, \dots, |\mathfrak{X}|$ in two ways:

In the *natural numbering* the elements of $\mathfrak{X}$ are numbered according to their increasing order. For an odd number p we define the *$\bar{p}$-numbering* of $\mathfrak{X}$ as follows: The elements of $\mathfrak{X}^+$ are placed on the p-abacus as usual. Two elements of $\mathfrak{X}^+$ are called *conjugate* if

(i) their beads are situated on two conjugate runners,

(ii) they are in the same layer, i.e. the number of beads above these beads on their respective runners is the same.

An element in $\mathfrak{X}^+$ having no conjugate element will be called *isolated.* We then decompose $\mathfrak{X}^+$ as

$$\begin{aligned} \mathfrak{X}_0^+ &= \{a \in \mathfrak{X}^+ \mid p \text{ divides } a\} \\ \mathfrak{X}_c^+ &= \{a \in \mathfrak{X}^+ \mid a \text{ has a conjugate element in } \mathfrak{X}^+\} \\ \mathfrak{X}_i^+ &= \{a \in \mathfrak{X}^+ \mid p \nmid a \text{ and } a \text{ is isolated}\}. \end{aligned}$$

We first number the elements of $\mathfrak{X}^-$, then those in $\mathfrak{X}_0^+$, then $\mathfrak{X}_c^+$, and finally $\mathfrak{X}_i^+$. The elements of $\mathfrak{X}^-$ are given the same numbers as in the natural numbering. Then the elements of $\mathfrak{X}_0^+$ are numbered in their increasing order. The elements of $\mathfrak{X}_c^+$ are numbered as follows:

(i) Suppose that a and a' are conjugate elements. Then they will be given two consecutive numbers k and $k+1$ with the condition that a will get the lowest number k if and only if the corresponding bead is on a runner with an even number.

(ii) If (a, a') and (b, b') are pairs of conjugate elements, then a and a' are numbered before b and b', if and only if $\min\{a, a'\} < \min\{b, b'\}$.

Finally, the elements of $\mathfrak{X}_i^+$ are numbered according to layers as follows: An element represented by the j-th *isolated* bead on the i-th runner is numbered before the element represented by the j_1-th isolated bead on the i_1-th runner if and only if $j < j_1$ or $j = j_1$ and $i < i_1$.

When we compare the natural numbering of $\mathfrak{X}$ with the $\bar{p}$- numbering of $\mathfrak{X}$ we get a permutation $\pi_{\bar{p}}(\mathfrak{X})$, whose sign is denoted by $\delta_{\bar{p}}(\mathfrak{X})$. Since the numbering of the negative elements of $\mathfrak{X}$ is the same in the natural and in the $\bar{p}$-numbering we have that $\delta_{\bar{p}}(\mathfrak{X}) = \delta_{\bar{p}}(\mathfrak{Y})$ if $\mathfrak{X}$ and $\mathfrak{Y}$ are γ-sets with $\mathfrak{X}^+ = \mathfrak{Y}^+$. Therefore we may define the *$\bar{p}$-sign of* λ by

$$\delta_{\bar{p}}(\lambda) := \delta_{\bar{p}}(\mathfrak{X})$$

if $\mathfrak{X}$ is a γ-set with $\bar{P}^*(\mathfrak{X}) = \lambda$.

Since the above rules and definitions are quite complicated, we illustrate them with an example.

Example. $\mathfrak{X} = \{-1, 0, 1, 6, 8, 9, 10, 13, 14, 15, 16, 17, 21\}$.
$\bar{P}^*(\mathfrak{X}) = \lambda = (21, 17, 16, 15, 14, 13, 10, 9, 8, 6, 1)$. Let $p = 5$.

$\mathfrak{X}^- = \{-1, 0\}$.

$\mathfrak{X}_0^+ = \{10, 15\}$.

$\mathfrak{X}_c^+ = \{1, 6, 8, 9, 14, 17\}$.

Conjugate pairs:

(1,9), (6,14), (8,17).

$\mathfrak{X}_i^+ = \{13, 16, 21\}$.

5-abacus for $\mathfrak{X}^+$

0	$\textcircled{1}$	2	3	4
5	$\textcircled{6}$	7	$\textcircled{8}$	$\textcircled{9}$
$\textcircled{10}$	11	12	$\textcircled{13}$	$\textcircled{14}$
$\textcircled{15}$	$\textcircled{16}$	$\textcircled{17}$	18	19
20	$\textcircled{21}$	22	23	24

The elements of $\mathfrak{X}^-$ are numbered 1, 2 in both numberings. After that we have

Natural numbering

0	$\textcircled{1}_3$	2	3	4
5	$\textcircled{6}_4$	7	$\textcircled{8}_5$	$\textcircled{9}_6$
$\textcircled{10}_7$	11	12	$\textcircled{13}_8$	$\textcircled{14}_9$
$\textcircled{15}_{10}$	$\textcircled{16}_{11}$	$\textcircled{17}_{12}$	18	19
20	$\textcircled{21}_{13}$	22	23	24

$\bar{5}$-numbering

0	$\textcircled{1}_6$	2	3	4
5	$\textcircled{6}_8$	7	$\textcircled{8}_{10}$	$\textcircled{9}_5$
$\textcircled{10}_3$	11	12	$\textcircled{13}_{12}$	$\textcircled{14}_7$
$\textcircled{15}_4$	$\textcircled{16}_{11}$	$\textcircled{17}_9$	18	19
20	$\textcircled{21}_{13}$	22	23	24

(In the $\bar{5}$-numbering 9 is numbered before 1 because 9 is on the 4-th (even) runner and 17 is before 8, since 17 is on the second runner. The layers of $\mathfrak{X}_i^+$ are $\{13, 16\}$, $\{21\}$. Then 16 is before 13, because 16 is on the first runner and 13 on the third.) We have

$$\begin{aligned}\pi_{\bar{5}}(\mathfrak{X}) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 2 & 6 & 8 & 10 & 5 & 3 & 12 & 7 & 4 & 11 & 9 & 13 \end{pmatrix} \\ &= (3, 6, 5, 10, 4, 8, 12, 9, 7)\end{aligned}$$

and so,

$$\delta_{\bar{5}}(\mathfrak{X}) = \delta_{\bar{5}}(\lambda) = (-1)^8 = 1.$$

If μ is obtained from λ by removing a sequence of p-bars we define the *relative sign* $\delta_{\bar{p}}(\lambda, \mu)$ as

$$\delta_{\bar{p}}(\lambda, \mu) = \delta_{\bar{p}}(\lambda)\delta_{\bar{p}}(\mu).$$

We trivially have transitivity. The following may be proved by considering bars of type 1, 2 and 3 separately. We omit the details:

Lemma (4.9). If μ is obtained from λ by removing a single p-bar $\bar{H}$, then

$$\delta_{\bar{p}}(\lambda, \mu) = (-1)^{\bar{b}(\bar{H})}$$

where $\bar{b}(\bar{H})$ is the leglength of the p-bar $\bar{H}$.

Analogous to section 3 we get:

Proposition (4.10). If μ is obtained from λ by removing a sequence of v p-bars with leglengths $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_v$ then

$$\delta_{\bar{p}}(\lambda, \mu) = (-1)^{\sum_i \bar{b}_i}.$$

In particular, the residue of $\sum \bar{b}_i (\mathrm{mod}\ 2)$ does not depend on the choice of p-bars being removed in going from λ to μ.

If λ is a $\bar{p}$-core and $\mathfrak{X}$ is a γ-set for λ, then $\mathfrak{X}^+ = \mathfrak{X}_i^+$ and so the natural numbering of $\mathfrak{X}$ coincides with the $\bar{p}$-numbering. Thus $\delta_{\bar{p}}(\lambda) = 1$.

Corollary (4.11). If λ is a bar partition, then

$$\delta_{\bar{p}}(\lambda) = \delta_{\bar{p}}(\lambda, \lambda_{(\bar{p})}).$$

Remark (4.12) (The $\bar{p}$-core problem). Let $p \in \mathbb{N}$ be odd. A $\bar{p}$-core κ is called a $\bar{p}$-core of n if $|\kappa| = n$. Let $c(\bar{p}, n)$ be the number of $\bar{p}$-cores of n. Above we have developed a theory of $\bar{p}$-cores and $\bar{p}$-quotients in complete analogy with the theory of p-cores and p-quotients in section 3. Therefore it seems reasonable to ask (see (3.15)):

For which p and n, p odd, is $c(\bar{p}, n) \neq 0$?

As we shall see, $c(\bar{p}, n)$ equals the number of associate classes of spin characters of p-defect 0. Numerical evidence seems to suggest:

Conjecture (4.13). For all n and all odd $p \geq$ ~~5~~ 7

$$c(\bar{p}, n) \neq 0.$$

Contrary to (3.15) not much is known about this question. The main reason is that methods involving modular forms cannot be applied here. This has to do with the form of the generating function for $c(\bar{p}, n)$ (see section 9). However, recently Erdmann and Michler [18] verified (4.13) for $p =$ [illegible] 7 using another method.

5 On Lusztig's symbols

The unipotent characters of the finite classical groups are indexed by combinatorial objects which G. Lusztig called "symbols". They may be seen as generalized partitions/β-sets and the ideas of sections 1 and 3 may be applied to study the unipotent character degrees as we shall see in section 8.

In section 2 we described "Frobenius symbols" for partitions as an ordered pair of β-sets. Lusztig's symbols are infinite classes of such pairs of β-sets. The theory developed in section 2 does not really have any connection with the following. The material in this section has been taken from [46].

If X is a β-set for the partition λ we define the *rank* of X by

$$\operatorname{rk} X = |\mathcal{P}^*(X)| = |\lambda|.$$

We fix $e \in \mathbb{N}$. Suppose that X is a β-set with $X = X_{(e)}$ (see section 3), i.e. a β-set for an e-core. Suppose that $\lambda_0, \lambda_1, \ldots, \lambda_{e-1}$ are partitions such that there exist for each i a β-set Y_i for λ_i with $|Y_i| = |X_i^{(e)}|$. Then we define

(1) $$X * (\lambda_0, \lambda_1, \ldots, \lambda_{e-1}) = \bigcup_{i=0}^{e-1} \{xe + i \mid x \in Y_i\}$$

The proof of (3.7) shows that $P^*(X * (\lambda_0, \ldots, \lambda_{e-1})) = \lambda$ is a partition with $\lambda_{(e)} = P^*(X)$, $\lambda^{(e)} = (\lambda_0, \ldots, \lambda_{e-1})$.

A *symbol* is an unordered pair $[X, Y]$ of finite subsets of $\mathbb{N}_0$ (β-sets). One identifies the symbols $[X, Y]$ and $[X^{+t}, Y^{+t}]$ for all $t \geq 0$. Define then

(2) $$\pi[X, Y] := [P^*(X), P^*(Y); |X| - |Y|],$$

so $\pi[X, Y]$ is a triple $[\lambda, \mu; z]$ where λ and μ are partitions and $z \in \mathbb{Z}$. The map π is compatible with the identification of symbols. One has to identify $[\lambda, \mu; z]$ with $[\mu, \lambda; -z]$ since a symbol is unordered. Then π is a bijection between the set of symbols and the set of such triples. Lusztig calls $||X| - |Y||$ the defect of the symbol $[X, Y]$. However, to avoid confusion with a concept in modular representation theory we call

$$d[X, Y] = ||X| - |Y||$$

the *difference* of the symbol. We call $P^*(X)$ and $P^*(Y)$ the *associated partitions* of the symbol $[X, Y]$. There are always infinitely many symbols having the same pair of associated partitions. These are distinguished by their differences.

Throughout this section we assume $X = \{a_1, a_2, \ldots, a_k\}$, $Y = \{b_1, b_2, \ldots, b_l\}$, where $a_1 > a_2 > \cdots > a_k \geq 0$, $b_1 > b_2 > \cdots > b_l \geq 0$. Then the *rank* of $[X, Y]$ is defined by

(3) $$\operatorname{rk}[X, Y] = \sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j - \left[\left(\frac{k+l-1}{2}\right)^2\right],$$

where $[c]$ denoted the integral part of c. Using that

$$\binom{k}{2} + \binom{l}{2} = \left[\left(\frac{k+l-1}{2}\right)^2\right] + \left[\left(\frac{k-l}{2}\right)^2\right] \quad \text{for} \quad k, l \in \mathbb{N}_0$$

we find

$$\text{(4)} \qquad \operatorname{rk}[X,Y] = \operatorname{rk} X + \operatorname{rk} Y + \left[\left(\frac{d[X,Y]}{2}\right)^2\right].$$

In particular $\operatorname{rk}[X,Y] \geq \operatorname{rk} X + \operatorname{rk} Y$ with equality if and only if $d[X,Y] \leq 1$.

We now consider hooks and cohooks of a symbol $[X,Y]$. Put

$$\mathcal{H}^+[X,Y] := \mathcal{H}(X) \dot{\bigcup} \mathcal{H}(Y).$$

Then $\mathcal{H}^+[X,Y]$ is the multiset of *hooklengths* of $[X,Y]$. Thus a hooklength of $[X,Y]$ is simply a hooklength of one of the associated partitions (by Proposition (1.4)). If $e \in \mathcal{H}^+[X,Y]$, say $e \in \mathcal{H}_i(X)$ we remove the corresponding hook of $[X,Y]$ by replacing $[X,Y]$ by $[\{a_1,\ldots,a_{i-1},a_i-e,a_{i+1},\ldots,a_k\},Y]$. We proceed analogously if $e \in \mathcal{H}_j(Y)$. By Proposition (1.8) removing an e-hook of $[X,Y]$ means simply removing an e-hook in one of the associated partitions.

For $1 \leq i \leq k$ and $1 \leq j \leq l$ we define

$$\begin{aligned}
&\text{(5)} & \mathcal{H}_i(X,Y) &:= \{1,2,\ldots,a_i\}\backslash\{a_i - b_j \mid b_j < a_i\} \\
&\text{(6)} & \mathcal{H}_j(Y,X) &:= \{1,2,\ldots,b_j\}\backslash\{b - a_i \mid a_i < b_j\} \\
&\text{(7)} & \mathcal{H}(X,Y) &:= \dot{\bigcup_i} \mathcal{H}_i(X,Y) \\
&\text{(8)} & \mathcal{H}(Y,X) &:= \dot{\bigcup_i} \mathcal{H}_j(Y,X)
\end{aligned}$$

and

$$\mathcal{H}^-[X,Y] := \mathcal{H}(X,Y) \dot{\bigcup} \mathcal{H}(Y,X).$$

Then $\mathcal{H}^-[X,Y]$ is the multiset of *cohooklengths* of $[X,Y]$. If $e \in \mathcal{H}^-[X,Y]$, say $e \in \mathcal{H}_i(X,Y)$ we remove the corresponding cohook of $[X,Y]$ by replacing $[X,Y]$ by $[X\backslash\{a_i\}, Y \cup \{a_i - e\}]$.

When we remove and e-hook from $[X,Y]$ the rank is reduced by e and the difference is unchanged. When we remove an e-cohook the rank is reduced by e and the difference is changed modulo 2.

As in section 3 it is convenient to visualize a symbol on two runners of an abacus, the runners going from north to south. On one runner the positions of the beads indicate the elements of X (a bead in the i-th position if $i \in X$) and similarly the beads in the second runner indicate Y. To remove an e-hook (e-cohook) means on the abacus that you move a bead e positions up to an empty space on the same runner (the opposite runner).

The hooklengths of $\mathcal{H}^+[X,Y]$ may be arranged in the hookdiagrams of the associated partitions. In a similar way the cohooklengths of $\mathcal{H}(X,Y)$ and $\mathcal{H}(Y,X)$ may be arranged in two "cohookdiagrams" as illustrated by the following example:

Example. Let $X = \{9,7,4,2\}$ and $Y = \{3,1,0\}$ so $\pi[X,Y] = [(6,5,3,2),(1);1]$.

$$\text{Hookdiagrams:} \qquad \mathcal{H}(X) : \begin{array}{llllll} 9 & 8 & 6 & 4 & 3 & 1 \\ 7 & 6 & 4 & 2 & 1 & \\ 4 & 3 & 1 & & & \\ 2 & 1 & & & & \end{array} \qquad \mathcal{H}(Y) : \; 1$$

$$\text{Cohookdiagrams:} \qquad \mathcal{H}(X,Y) : \begin{array}{llllll} 7 & 5 & 4 & 3 & 2 & 1 \\ 5 & 3 & 2 & 1 & & \\ 2 & & & & & \end{array} \qquad \mathcal{H}(Y,X) : \begin{array}{ll} 3 & 2 \\ 1 & \end{array}$$

Here the first row of $\mathcal{H}(X,Y)$ is $\{1,2,\dots,9\}\backslash\{9-3, 9-1, 9-0\} = \{7,5,4,3,2,1\}$ and the first row of $\mathcal{H}(Y,X)$ is $\{1,2,3\}\backslash\{3-2\} = \{3,2\}$.

Let us list the formulas for the numbers of hooks and cohooks of a symbol. If $h^+[X,Y] := |\mathcal{H}^+[X,Y]|, h^-[X,Y] := |\mathcal{H}^-[X,Y]|$ then we have

$$(9) \qquad h^+[X,Y] = \operatorname{rk} X + \operatorname{rk} Y = \sum_i a_i + \sum_j b_j - \binom{k}{2} - \binom{l}{2}$$

$$(10) \qquad h^-[X,Y] = \sum_i a_i + \sum_j b_j - kl + |X \cap Y|.$$

The formula (9) is obvious by the definition. The set of cohooklengths is obtained by subtracting the $kl - |X \cap Y|$ elements $\{|a_i - b_j| \mid a_i \neq b_j\}$ from $\dot{\bigcup}_i\{1,\dots,a_i\} \dot{\cup} \dot{\bigcup}_k\{1,2,\dots,b_j\}$. This proves (10). Whereas by (4) $h^+[X,Y] \leq \operatorname{rk}[X,Y]$ it is not true that in general $h^-[X,Y] \leq \operatorname{rk}[X,Y]$.

The following result generalizes a result of Nakayama on hooklengths, see (1.6). The proof is similar.

Lemma (5.1). Let $\epsilon = \pm$ be a sign, $t, e \in \mathbb{N}$. Suppose that $te \in \mathcal{H}^{\epsilon^t}[X,Y]$. Then $e \in \mathcal{H}^{\epsilon}[X,Y]$.

There are examples to show that generally, the elements of $\mathcal{H}^-[X,Y]$ cannot be arranged in a union of hookdiagrams (i.e., there do not exist sets $X_1, \dots, X_t$ with $\mathcal{H}^-[X,Y] = \mathcal{H}(X_1) \dot{\cup} \cdots \dot{\cup} \mathcal{H}(X_t)$). It is not known whether the same is true for $\mathcal{H}^+[X,Y] \dot{\cup} \mathcal{H}^-[X,Y]$. This question appears not to be easy to answer. But there exists a partition whose hook structure determines $\mathcal{H}^+[X,Y]$ and $\mathcal{H}^-[X,Y]$ in a unique way.

Consider the set

$$Z := (2X) \cup (2Y+1) = \{2a_i \mid 1 \leq i \leq k\} \cup \{2b_j + 1 \mid 1 \leq j \leq l\}.$$

Then $P^*(Z)$ is called a *parity partition* of $[X,Y]$. Note that if X and Y are replaced by X^{+t}, X^{+t} then Z is replaced by Z^{+2t} so that the partition is unchanged. If X and Y are interchanged we get the other parity partition $P^*(Z')$ where $Z' = (2Y) \cup (2X+1)$. By definition the 2-quotient of $P^*(Z')$ is $(P^*(X), P^*(Y))$ if $k+l$ is even and $(P^*(Y), P^*(X))$ if $k+l$ is odd (see section 3). The 2-core of $P^*(Z)$ is the partition $(z-1, z-2, \dots, 1)$ of

$z = k - l \geq 0$ and $(-z, -z-1, \ldots, 1)$ if $z = k - l < 0$. This follows from (3.18). Then it is easily seen that

$$|P^*(Z)| - |P^*(Z')| = l - k,$$

so

$$d[X,Y] = ||P^*(Z)| - |P^*(Z')||.$$

The above remarks show that we may assume without loss of generality that $0 \notin X$, i.e. $0 \notin Z$. Consider then the hooks of $\rho := P^*(Z)$. By Thereom (3.3) the hooks of even length in ρ are in bijective correspondence with $\mathcal{H}(X) \cup \mathcal{H}(Y) = \mathcal{H}^+[X,Y]$. Thereby a hook of length $2e$ in ρ corresponds to a hook of length e in $\mathcal{H}^+[X,Y]$.

Similarly, with the exception of $|Y| - |X \cap Y|$ hooks of length 1 (which occur in rows of ρ whose largest hooklength is odd), the hooks of odd length in ρ are in bijective correspondence with the cohooks of $[X,Y]$. Indeed, let $Z = \{c_1, c_2 \ldots, c_{k+l}\}$, where $c_1 > c_2 > \cdots > c_{k+l} > 0$. Suppose that $h \in \mathcal{H}_j(Z)$ is odd. If $c_j = 2a_i$ is even then $(h+1)/2 \in \mathcal{H}_i(Y,X)$ and if $c_j = 2b_i + 1$ is odd then $(h-1)/2 \in \mathcal{H}_i(Y,X)$. This is not difficult to see from the definition. If $a \in X \cap Y$ then for some j $c_j = 2a+1$, $c_{j+1} = 2a$, so $1 \notin \mathcal{H}_j(Z)$, whereas if $a \in Y$, $a \notin X$ then $1 \in \mathcal{H}_j(Z)$ if $c_j = 2a+1$.

Let us briefly discuss the analogues of cores and quotients. Again we want to register in a natural way all hook- (and cohook-) lengths divisible by a given $e \in \mathbb{N}$.

The *e-core* $[X,Y]^+_{(e)}$ of a symbol $[X,Y]$ is obtained by removing recursively all hooks of length e from it. As in the case of partitions, the final result is independent of the order in which these hooks are removed. (The argument is the same.) Using the results of section 3 we see in fact that if $\pi[X,Y] = [\lambda, \rho; z]$ then $\pi[X,Y]^+_{(e)} = [\lambda_{(e)}, \mu_{(e)}; z]$. The *e-weight* is defined by

$$w_e^+[X,Y] = w_e(\lambda) + w_e(\mu).$$

By replacing X, Y by X^{+t}, Y^{+t} we may assume that $k + l \equiv 0$ or $1 (\mathrm{mod}\ 2e)$. Then for $0 \leq i \leq (e-1)$ we define $X_i^{(e)}$ and $Y_i^{(e)}$ as in section 3 and put

$$\lambda_i := P^*(X^{(e)})_i, \mu_i := P^*(Y_i^{(e)}) 0 \leq i \leq e-1.$$

Then

(11) $$[X,Y]_+^{(e)} := [\lambda_0, \lambda_1, \ldots, \lambda_{e-1}; \mu_0, \mu_1, \ldots, \mu_{e-1}]$$

is the *e-quotient* of $[X,Y]$. Note that if we replace X by X^{+te} and Y by Y^{+te} the condition $k + l \equiv 0$ or $1 (\mathrm{mod}\ 2e)$ is still fullfilled and the partitions in (7) are exactly the same. If X and Y are interchanged we get by the same construction $[\mu_0, \mu_1, \ldots, \mu_{e-1}; \lambda_0, \lambda_1, \ldots, \lambda_{e-1}]$ so one has to identify $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$ with $[\mu_0, \ldots, \mu_{e-1}; \lambda_0, \lambda, \ldots, \lambda_{e-1}]$. So the e-quotient is an unordered pair of e-tuples of partitions satisfying

(12) $$\sum_i |\lambda_i| + \sum_j |\mu_j| = w_e^+[X,Y].$$

Just in the case of partitions ((3.3)) we see the following

Proposition (5.2). There is a bijection between the hooks of $[X,Y]$ of a length divisible by e and $\mathcal{H}_+^{(e)}[X,Y] := \dot{\bigcup}_i \mathcal{H}(\lambda_i) \dot{\cup} \dot{\bigcup}_j \mathcal{H}(\mu_j)$, where $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-q}]$ is the e-quotient of $[X,Y]$. Thereby a hook of length te is mapped onto a hook of length t.

Obviously a symbol determined its e-core and e-quotient in a unique way, but contrary to the case of partitions there may be two different symbols having the same e-core and e-quotient. Indeed, suppose that $[X', Y']$ is an e-core (i.e., a symbol without an e-hook) and $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$ an unordered pair of e-tuples of partitions satisfying $\sum_i |\lambda_i| + \sum_j |\mu_j| = w$. Let $|X'| = k'$, $|Y'| = l'$. By replacing X'^{+t}, Y'^{+t} by X'^{+t}, Y'^{+t} for suitable t we may assume $k' + l' \equiv 0$ or $1 (\mathrm{mod}\ 2e)$ and that for all $i, 0 \leq i \leq e-1$, $\left|X'^{(e)}{}_i\right| \geq w$ and $\left|Y'^{(e)}{}_i\right| \geq w$. Then the symbols $[X' * (\lambda_0, \lambda_1, \ldots, \lambda_{e-1}),\ Y' * (\mu_0, \mu_1, \ldots, \mu_{e-1})]$ and $[X' * (\lambda_0, \ldots, \lambda_{e-1}),\ Y' * (\mu_0, \ldots, \mu_{e-1})]$ (see (1) for the definition of the operation $*$) have the same e-core $[X', Y']$ and e-quotient $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$. These symbols may be equal, but generally they are not. We have in fact, as is easily seen,

Proposition (5.3). In the above notation

$$[X' * (\lambda_0, \ldots, \lambda_{e-1}), Y' * (\mu_0, \ldots, \mu_{e-1})] = [X' * (\mu_0, \ldots, \mu_{e-1}), Y' * (\lambda_0, \ldots, \lambda_{e-1})]$$

if and only if $X' = Y'$ or $(\lambda_0, \ldots, \lambda_{e-1}) = (\mu_0, \ldots, \mu_{e-1})$.

Let us now consider the cohooks. We want the e-coquotient to register those hooklengths which are an even multiple of e and those e-cohooklengths which are an odd multiple of e. These hooklengths and cohooklengths may be seen more conveniently in the e-quotient of the "e-twisting" of the symbol. Assume again that $k + l \equiv 0$ or $1 (\mathrm{mod}\ 2e)$. Put

$$\begin{aligned}\tilde{X} &:= \left\{a \in X \;\middle|\; \left[\frac{a}{e}\right] \text{ is even}\right\} \bigcup \left\{b \in Y \;\middle|\; \left[\frac{b}{e}\right] \text{ odd}\right\} \\ \tilde{Y} &:= \left\{a \in X \;\middle|\; \left[\frac{a}{e}\right] \text{ is odd}\right\} \bigcup \left\{b \in Y \;\middle|\; \left[\frac{b}{e}\right] \text{ even}\right\}\end{aligned}$$

Then we call $[\tilde{X}, \tilde{Y}]$ the *e-twisting* of $[X,Y]$. Note that if X, Y are replaced by X^{+te}, Y^{+te} or interchanged, the symbol $[\tilde{X}, \tilde{Y}]$ is unchanged. So the e-twisting is an involutory bijection on the set of symbols. By e-twisting the difference of the symbol is changed modulo 2. (Here, as also in the case of the definition of e-quotients the condition $k+l \equiv 0$ or $1 (\mathrm{mod}\ 2e)$ is only to assure that that we do is well defined.) Using the visualization of a symbol on two runners on an abacus described above we see that the following holds:

Proposition (5.4). Let $[\tilde{X}, \tilde{Y}]$ be the e-twisting of $[X,Y]$.

(i) If t is odd there is a bijection between the cohooks of lengths te in $[X,Y]$ and the hooks of lengths te in $[\tilde{X}, \tilde{Y}]$.

(ii) If t is even there is a bijection between the hooks of lengths te in $[X,Y]$ and the hooks of length te in $[\tilde{X}, \tilde{Y}]$.

The *e-cocore* $[X,Y]^{-}_{(e)}$ is obtained from $[X,Y]$ by removing recursively all e-cohooks from $[X,Y]$. Again the final result does not depend on the order in which the cohooks are removed. By the above remarks it is quite obvious that the e-cocore is also the e-twisting of the e-core of $[\tilde{X},\tilde{Y}]$ where $[\tilde{X},\tilde{Y}]$ is the e-twisting of $[X,Y]$. Let

$$w_e^{-}[X,Y] = \left(\mathrm{rk}\,[X,Y] - \mathrm{rk}\,[X,Y]^{-}_{(e)}\right)/e,$$

the *e-coweight.* Define the *e-coquotient* $[X,Y]^{(e)}_{-}$ as $[\tilde{X},\tilde{Y}]^{(e)}_{+}$, i.e., the e-quotient of the e-twisting. From (5.2) and (5.4) we get

Proposition (5.5). Let $[X,Y]^{(e)}_{-} = [\lambda_0,\lambda_1,\dots,\lambda_{e-1};\mu_0,\mu_1,\dots,\mu_{e-1}]$.

(i) If t is odd there is a bijection between the cohooks of length te in $[X,Y]$ and the hooks of length t in $\dot{\bigcup}_i \mathcal{H}(\lambda_i) \,\dot{\cup}\, \dot{\bigcup}_j \mathcal{H}(\mu_j) := \mathcal{H}^{(e)}[X,Y]$.

(ii) If t is even there is a bijection between the hooks of length te in $[X,Y]$ and the hooks of length t in $\mathcal{H}^{(e)}_{-}[X,Y]$.

Similar to the case of e-cores and e-quotients a symbol determines uniquely its e-cocore and e-coquotient. There may be one or two symbols sharing the same e-cocore $[X',Y']$ and e-coquotient $[\lambda_0,\dots,\lambda_{e-1};\mu_0,\dots,\mu_{e-1}]$. As in (5.3) the former case occurs if and only if $X'=Y'$ or $(\lambda_0,\dots,\lambda_{e-1}) = (\mu_0,\dots,\mu_{e-1})$.

Finally we want to count the number of symbols of a given e-weight having a given e-core. This gives then also a count of the number of symbols having a given e-coweight and a given e-cocore.

As in section 3, $k(s,t)$ is the number of s-tuples of partitions $(\lambda_1,\lambda_2,\dots,\lambda_s)$ satisfying $\sum_i |\lambda_i| = t$. The symbol $[X,Y]$ is called *special* if $X=Y$. Using methods similar to those of section 3 we can prove:

Proposition (5.6). The number of symbols of e-weight w having given non-special e-core is $k(2e,w)$.

Proposition (5.7). Let $[X,X]$ be a special e-core.

(i) If w is odd then the number of symbols having $[X,X]$ as e-core and e-weight w is $\frac{1}{2}k(2e,w)$. These symbols are all nonspecial.

(ii) If $w=2w_1$ is even then the number of special symbols having $[X,X]$ as e-core and e-weight w is $k(e,w_1)$ and the number of nonspecial symbols having $[X,X]$ as e-core and e-weight w is $\frac{1}{2}(k(2e,w)-k(e,w_1))$.

II. Character degrees

In this chapter we study the degrees of irreducible characters of the covering groups of S_n and of the unipotent characters of certain linear groups. Basically we want for a given prime p to describe the power of p dividing the degree of a given character and to compute the socalled McKay numbers for some of these groups. Generally, if p is a prime, $a \geq 0$ and G a finite group then $\mathrm{Irr}(G)$ is the set of irreducible characters of G and

$$m_a(p,G) = |\{\chi \in \mathrm{Irr}(G) \mid p^a \text{ is the exact power of } p \text{ dividing } \chi(1)\}|$$

the *a-th McKay number* of G (w.r.t. p). We will be interested in the *height* and *defect* of irreducible characters, defined by $h_p(\chi) = \nu_p(\chi(1))$, $d_p(\chi) = \nu_p |G| - h_p(\chi)$.

6 Character degrees in S_n and A_n

As is wellknown [25], [56] the irreducible characters (representations) of the symmetric group S_n are parametrized by the partitions λ of n. The character corresponding to $\lambda \in \mathcal{P}(n)$ is denoted by $[\lambda]$ and its degree (dimension) by $f_\lambda = [\lambda](1)$.

It follows already from Frobenius' work that if $X = \{h_1, h_2, \dots, h_t\}$ is a β-set for λ, then

(6.1) $\qquad f_\lambda = \frac{n!\Delta(h_1,\dots,h_t)}{\Pi_{i\geq 1} h_i!} = \frac{n!\Pi_{1\leq i<j\leq t}(h_i-h_j)}{\Pi h_i!}$

(see the end of section 2 above) and then the definition of $\mathcal{H}_i(X)$ in section 1 shows

$$f_\lambda = \frac{n!}{\Pi_{h\in\mathcal{H}(X)} h}$$

and from (1.4) we then get the famous "Hook formula"

(6.2) $\qquad f_\lambda = \frac{n!}{\Pi_{h\in\mathcal{H}(\lambda)} h}$

When m is an integer, p a prime, then $\nu_p(m)$ is the largest integer $a \geq 0$, s.t. $p^a \mid m$. We put

$$\begin{aligned} h_p(\lambda) &= \nu_p(f_\lambda), \quad \text{the } \textit{height of } [\lambda] \textit{ in } S_n \\ d_p(\lambda) &= \nu_p\left(\Pi_{h\in\mathcal{H}(\lambda)} h\right) = \nu_p(n!) - h_p(\lambda), \quad \textit{the defect of } [\lambda] \text{in } S_n. \end{aligned}$$

The relevant facts about cores and quotients needed now were proved in section 3.

For a given $\lambda \in \mathcal{P}(n)$, $p \in \mathbb{N}$, we define its *p-core tower* as follows: It has rows numbered by $i = 0, 1, \dots$. Its i-th row contains p^i p-cores. The 0-th row is $\lambda_{(p)}$, the p-core of λ. The first row is $\lambda_{0(p)}, \dots, \lambda_{p-1(p)}$, where $\lambda^{(p)} = (\lambda_0, \dots, \lambda_{p-1})$. Let $\lambda_i^{(p)} = (\lambda_{i0}, \dots, \lambda_{ip-1})$. Then the second row is the p-cores of $\lambda_{00}, \dots, \lambda_{0p-1}, \lambda_{10}, \dots, \lambda_{p-1p-1}$, in that order. Continuing this process of taking cores of quotients gives us the p-core tower of λ. A partition is described uniquely by its p-core tower, which contains only finitely many nonzero partitions. This follows by repeated use of the fact that by (3.7) a partition is uniquely determined by its p-core and p-quotient.

Example. $\lambda = (5, 4, 2^2, 1^2) \in \mathcal{P}(15)$, $p = 2$. The hooklengths of λ are

12	7	4	3	1
8	5	2	1	
5	2			
4	1			
2				
1				

We get $\lambda_{(2)} = (1)$, $\lambda^{(2)} = ((2^2, 1^2), (1))$. The 2-core tower of λ looks like this

$$\begin{array}{ccccccccccccccc} & & & & & & & (1) & & & & & & & \\ & & & (0) & & & & & & & & (1) & & & \\ & (0) & & & & (1) & & & & (0) & & & & (0) & \\ (1) & & (0) & & (0) & & (0) & & (0) & & (0) & & (0) & & (0). \end{array}$$

Let for a given partition λ and for $i \geqq 0$, $\beta_i(p, \lambda)$ be the sum of the cardinalities of the partitions in the i-th row of the p-core tower of λ. Thus $\beta_0(p, \lambda) = |\lambda_{(p)}|$. It follows from the construction of the p-core that the following holds (see (3.6) (ii)):

Lemma (6.3). $n = |\lambda| = \sum_i \beta_i(p, \lambda) p^i$, where the summation is on $i \geqq 0$.

If $a_0 + a_1 p + \ldots + a_k p^k$ is the p-adic decomposition of n and $\lambda \in \mathrm{Par}(n)$, define the *p--deviation of* λ as

$$e_p(\lambda) = \left(\sum_i \beta_i(p, \lambda) - \sum_j a_j \right) / (p - 1) \quad (i \geqq 0,\ 0 \leqq j \leqq k).$$

Since $p^i \equiv 1 (\mathrm{mod}\ p - 1)$ for all $i \geqq 0$, we get

$$n \equiv \sum_i \beta_i(p, \lambda) \equiv \sum_j a_j \,(\mathrm{mod}\ (p-1)),$$

so $e_p(\lambda)$ is a nonnegative integer.

Proposition (6.4). If $\lambda \in \mathcal{P}(n)$, and p is a prime then $h_p(\lambda) = e_p(\lambda)$.

Proof. It is wellknown that

$$(1) \qquad \nu_p(n!) = \left(n - \sum_{i \geq 0} a_i \right) / (p - 1).$$

Moreover, if $\lambda^{(p)} = (\lambda_0, \lambda_1, \ldots, \lambda_{p-1})$ then by (3.3)

$$(2) \qquad d_p(\lambda) = w_p(\lambda) + \sum_{i=0}^{p-1} d_p(\lambda_i)$$

Now, as is seen by (6.3)

(3) $$w_p(\lambda) = \sum_{i\geq 1} \beta_i(p,\lambda)p^{i-1}.$$

By iteration of (2) we get then that

(4) $$\begin{aligned} d_p(\lambda) &= \sum_{i\geq 1} \beta_i(p,\lambda)(1+p+\ldots+p^{i-1}) \\ &= \left(\sum_{i\geq 0} \beta_i(p,\lambda)(p^i-1)\right)/(p-1) \\ &= \left(n - \sum_{i\geq 0} \beta_i(p,\lambda)\right)/(p-1) \end{aligned}$$

using (6.3) for the last equality. We then subtract (4) from (1) to get the result.

Remark (6.5). The above proof shows that $d_p(\lambda) = 0 \Leftrightarrow w_p(\lambda) = 0$. Thus the characters of p- defect 0 of S_n are labelled by the p-cores of n.

The above result gives a purely combinatorial interpretation of the height of $[\lambda]$ in S_n $h_p(\lambda)$, i.e. the exponent of p dividing $[\lambda](1) = f_\lambda$. We proceed to compute the McKay numbers:

Let again $p \in \mathbb{N}$ be arbitrary. An *p-expansion* of n is a sequence $(\alpha_0, \alpha_1, \ldots)$ of nonnegative integers such that $n = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots$. The set of these expansions is denoted $E(p,n)$. If $\alpha = (\alpha_0, \alpha_1, \ldots) \in E(p,n)$, let the *deviation* of α be

$$e(p,\alpha) = \left(\sum_i \alpha_i - \sum_j a_j\right)/(p-1) \quad (i \geqq 0,\ 0 \leqq j \leqq k)$$

where the a_j's are as above. $E_a(p,n)$ denotes the set of p-expansions of n of deviation a. Let us note that $(a_0, a_1, \ldots, a_k, 0, 0, \ldots)$ is the unique element in $E_0(p,n)$.

As in section 3, $c(p,n)$ is the number of r-cores of n. Moveover, $C_p(r,n)$ denotes the set of r-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_r)$ of p-cores s.t. $\sum |\kappa_i| = n$ and $c_p(r,n) = |C_p(r,n)|$. Thus

(5) $$c_p(r,n) = \sum_{(w_1,\ldots,w_r)} c(p,w_1)c(p,w_2)\cdots c(p,w_s)$$

where $(w_1, \ldots, w_r)$ runs through all r-tuples of non-negative integers with $\sum w_i = n$. (Compare with (3.11).)

Let us note that $c_p(r,n) = k(r,n)$ if $n < p$, since any partition of $w < p$ is a p-core.

Proposition (6.6). Let p be a prime, $a \geq 0$. Then

$$m_a(p,S_n) = \sum_\alpha c_p(1,\alpha_0)c_p(p,\alpha_1)c_p(p^2,\alpha_2)\ldots$$

where the summation is over all $\alpha = (\alpha_0, \alpha_1, \ldots) \in E_a(p,n)$. In particular

$$m_0(p,S_n) = c_p(1,a_0)c_p(p,a_1)c_p(p^2,a_2)\cdots.$$

Proof. By (6.4) if $\lambda \in \mathcal{P}(n)$, then $h_p(\lambda) = a$ if and only if $(\beta_0(p, \lambda), \beta_1(p, \lambda), \dots) \in E_a(p, n)$. We then count the partitions λ with $h_p(\lambda) = a$ by counting for each $\alpha = (\alpha_0, \alpha_1, \dots) \in E_a(p, n)$ the possibilities for the p-core towers s.t. the sum of the cardinalities of the partitions in the i-th row is α_i, $i \geq 0$.

To compute the McKay numbers for A_n we need to discuss associate classes of characters.

Remark (6.7). Suppose that the finite group G has a subgroup H of index 2, $|G : H| = 2$. Let σ be the non-trivial linear character of G with kernel H. If χ is an irreducible character of G, then so is $\chi' = \chi \cdot \sigma$. Either $\chi = \chi'$ and then χ is called *self-associate* (s.a) or $\chi \neq \chi'$ and then χ and χ' are called *associated*, and χ, χ' a pair of *non-selfassociate* (n.s.a.) characters. By Clifford's theorem the restriction of s.a. characters to H is a sum of two irreducible characters of H (which are conjugate in G) whereas two associated characters have the same irreducible restriction to H. This remark applies also to modular irreducible characters in characteristic > 2.

We apply (6.7) to $G = S_n$, $H = A_n$ and later to the covering group of S_n.

In the case of $G = S_n$, $H = A_n$, the linear character σ in (6.7) is of course the sign character $\sigma = [1^n]$. The trivial character of S_n is $[n]$. It is not difficult to prove that for $\lambda \in \mathcal{P}(n)$ we have $[\lambda]' = [\lambda^0]$ where λ^0 is the partition conjugate to λ.

As in section 3 $K(r, w)$ is the set of r-quotients of w, i.e. the set of sequences $(\lambda_0, \lambda_1, \dots, \lambda_{r-1})$ of partitions with $\sum |\lambda_i| = w$ and $C_p(r, w)$ is the corresponding set where the λ_i's are all p-cores. We have a "conjugation map" on $K(r, w)$ and $C_p(r, w)$ by mapping $\underline{\lambda} = (\lambda_0, \dots, \lambda_{r-1})$ onto $\underline{\lambda}^0 = (\lambda^0_{r-1}, \lambda^0_{r-2}, \dots, \lambda^0_0)$ and we call $\underline{\lambda}$ self-conjugate if $\underline{\lambda} = \underline{\lambda}^0$. Obviously $\underline{\lambda}$ is self-conjugate if and only if $\lambda^0_i = \lambda_{r-1-i}$ for all i. In particular if $r = 2t + 1$ is odd then $\lambda_t = \lambda^0_t$. Let

$$k^0(r, w) = \left|K^0(r, w)\right|, \quad c^0_p(r, w) = \left|C^0_p(r, w)\right|$$

be the cardinalities of the subsets of self-conjugate elements in $K(r, w)$, $C_p(r, w)$. In particular $k^0(1, w) = p^0(w)$ ($c^0_p(1, w) = c^0(p, w)$) is the number of self-conjugate partitions (p-cores) of w. We obviously have

Lemma (6.8). Notation as above.

(i) If $r = 2t$ is even, then

$$\begin{aligned} k^0(r, w) &= k(t, w), \\ c^0_p(r, w) &= c_p(t, w). \end{aligned}$$

(ii) If $r = 2t + 1$ is odd then

$$\begin{aligned} k^0(r, w) &= \sum_{i=0}^{w} k(t, i) p^0(w - i) \\ c^0_p(r, w) &= \sum_{i=0}^{w} c_p(t, i) c^0(p, w - i) \end{aligned}$$

Repeated use of (3.5) shows that given λ and p then $\lambda = \lambda^0$ if and only if each row in the p-core tower of λ is self-conjugate. Using the above observations and notation we may compute the McKay numbers of A_n:

Proposition (6.9). If p is an *odd* prime, $a \geq 0$. Then

$$m_a(p, A_n) = 2m_a^0(p, S_n) + \frac{1}{2}\left(m_a(p, S_n) - m_a^0(p, S_n)\right)$$

where $m_a(p, S_n)$ is given in (6.6) and $m_a^0(p, S_n)$ is defined analogously with each $c_p(p^i, \alpha)$ replaced by $c_p^0(p^i, \alpha)$.

We omit the proof which is straightforward.

For $p = 2$ a similar formula may be obtained but there is a complication since the power of 2 occurring in $[\lambda](1)$ and in the degrees of the constituents of $[\lambda]_{|A_n}$ differ by 1 if $\lambda = \lambda^0$.

7 Character degrees in $\hat{S}_n$

Schur [57] proved in 1911 that the symmetric groups S_n have covering groups $\hat{S}_n$ of order $2|S_n| = 2n!$. Thus there is a non-split exact sequence

$$1 \to \langle z \rangle \to \hat{S}_n \to S_n \to 1$$

where $\langle z \rangle$ is a central subgroup of order z in $\hat{S}_n$.

The irreducible characters of $\hat{S}_n$ which have $\langle z \rangle$ in their kernel will be referred to as *ordinary characters* of $\hat{S}_n$. They are of course exactly the irreducible characters of $S_n \cong \hat{S}_n/\langle z \rangle$ and were studied in the previous section. The other irreducible characters of $\hat{S}_n$ are referred to as *spin characters.*

The results in this section were taken from [48].

The associate classes of spin characters of $\hat{S}_n$ (cfr. (6.7)) are labelled canonically by the bar partitions of n. The set $\bar{\mathcal{P}}(n)$ of bar partitions (see section 4) is divided into

$$\begin{aligned}
\bar{\mathcal{P}}^+(n) &= \{\lambda \in \bar{\mathcal{P}}(n) \mid n - l(\lambda) \text{ even}\} \\
\bar{\mathcal{P}}^-(n) &= \{\lambda \in \bar{\mathcal{P}}(n) \mid n - l(\lambda) \text{ odd}\}
\end{aligned}$$

where $l(\lambda)$ is the length of λ. Put $q^+(n) = |\bar{\mathcal{P}}^+(n)|$, $q^-(n) = \left|\bar{\mathcal{P}}^-(n)\right|$. Then each $\lambda \in \bar{\mathcal{P}}^-(n)$ labels a pair of non self-associate spin characters $\langle \lambda \rangle$ and $\langle \lambda \rangle'$ and each $\lambda \in \bar{\mathcal{P}}^+(n)$ labels a self-associate spin character $\langle \lambda \rangle$ of $\hat{S}_n$. We write $\sigma(\lambda) = \sigma$ if $\lambda \in \bar{\mathcal{P}}^\sigma(n)$ and call $\sigma(\lambda) = (-1)^{n-l(\lambda)}$, the *sign* of λ.

When $\lambda \in \bar{\mathcal{P}}(n)$ then $\bar{f}_\lambda$ denotes the degree of the spin character(s) labelled by λ. Schur proved that if $\lambda = (a_1, a_2, \ldots, a_m) \in \bar{\mathcal{P}}(n)$, then

(7.1) $$\bar{f}_\lambda = 2^{[n-m/2]}\frac{n!}{\prod_i a_i!}\prod_{i<j}\frac{a_i - a_j}{a_i + a_j}$$

(see [58], §41). Then the definition of $\bar{\mathcal{H}}(\lambda)$ in section 4 shows that

$$(7.2) \qquad \bar{f}_\lambda = 2^{[n-m/2]} \frac{n!}{\prod_{h \in \bar{\mathcal{H}}(\lambda)} h}$$

(Here $[\cdot]$ signifies "integral part of".) Apart from the power of two which is a special feature when dealing with spin characters the formula (7.2) has a striking similarity to (6.2). For spin characters "hooks" are replaced by "bars". We put when p is an odd prime

$$(1) \qquad \begin{aligned} \bar{h}_p(\lambda) &= \nu_p(\bar{f}_\lambda), \\ \bar{d}_p(\lambda) &= \nu_p(n!) - \bar{h}_p(\lambda) \\ &= \nu_p\left(\prod_{h \in \bar{\mathcal{H}}(\mathcal{H})}\right). \end{aligned}$$

Consider $\lambda \in \mathcal{P}(n)$. Let $\lambda_{(\bar{p})}$ and $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \dots, \lambda_t)$, $t = p - 1/2$, be the $\bar{p}$-core and $\bar{p}$-quotient of λ. Let $w = w_{\bar{p}}(\lambda)$ be the $\bar{p}$-weight of λ (see section 4). By (4.4) (ii)

$$(2) \qquad n = |\lambda_{(\bar{p})}| + wp$$

Moreover, by (4.3)

$$(3) \qquad \bar{d}_p(\lambda) = w_{\bar{p}}(\lambda) + \bar{d}_p(\lambda_0) + \sum_i d_p(\lambda_i)$$

Now we may of course also apply (3) to the bar partition λ_0 and formula (3) of section 6 to the partitions λ_i, $1 \le i \le t$.

In analogy with the previous section we define the $\bar{p}$-*core tower* of $\lambda \in \bar{\mathcal{P}}(n)$ as follows: It has rows numbered $0, 1, 2, \dots$, where the i-th row contains one $\bar{p}$-core and $(p^i - 1)/2$ p-cores. The zeroth row consists of $\lambda_{(\bar{p})}$. If $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \dots, \lambda_t)$, the first row consists of $\lambda_{0(\bar{p})}, \lambda_{1(\bar{p})}, \dots, \lambda_{t(\bar{p})}$. Now the bar partition λ_0 has a $\bar{p}$-quotient and the partitions $\lambda_1, \dots, \lambda_t$ have p-quotients. Then the second row consists of the $\bar{p}$-core of the first partition in $\lambda_0^{(\bar{p})}$ and the p-cores of the remaining partitions in $\lambda_0^{(\bar{p})}$, followed by the p-cores of the partitions in $\lambda_1^{(p)}, \dots, \lambda_t^{(p)}$. Continuing this way we get the $\bar{p}$-core tower.

Example. Consider again the example following (4.1). We have

$$\lambda_{(\bar{5})} = (4,2), \quad \lambda^{(\bar{5})} = \left((1), (4,2), (3, 1^3)\right).$$

The $\bar{5}$-core tower is the following:

					(4,2)							
	(1)				(1)					(3, 1^3)		
(0)	(0)	(0)	(0)	(0)	(0)	(1)	(0)	(0)	(0)	(0)	(0)	(0)

The remaining entries are (0).

Using (3.7) and (4.2) we see that we may always recover a bar partition from its $\bar{p}$-core tower. Let $\beta_i(\bar{p}, \lambda)$ be the sum of the cardinalities of the partitions in the i-th row of the $\bar{p}$-core tower of λ. Then we get from the definition

$$(4) \qquad |\lambda| = \sum_{i \geq 0} \beta_i(\bar{p}, \lambda) p^i.$$

If $a_i + a_1 p + \ldots + a_k p^k$ is the p-adic decomposition of n and $\lambda \in \bar{\mathcal{P}}(n)$ let

$$\bar{e}_p(\lambda) = \left(\sum_i \beta_i(\bar{p}_1, \lambda) - \sum_j a_j \right) / (p-1).$$

Then a proof completely analogous to that of (6.4) shows:

Proposition (7.3). If $\lambda \in \bar{\mathcal{P}}(n)$ and p is an odd prime then $\bar{h}_p(\lambda) = \bar{e}_p(\lambda)$.

Let us define $\bar{m}_a(p, \hat{S}_n) = m_a(p, \hat{S}_n) - m_a(p, S_n)$ which is then just the number of spin characters $\langle \lambda \rangle$ of S_n with $\bar{h}_p(\lambda) = a$ and correspondingly $\bar{m}_a(p, \hat{A}_n)$.

As shown by (7.4) below, contrary to the case of S_n there is a nice duality between the formulas for the McKay numbers of spin characters in $\hat{S}_n$ and $\hat{A}_n$. This duality also occurs for the p-blocks, p odd, as we shall see. This has of course to do with (6.7) and the fact that the partitions in $\bar{\mathcal{P}}(n)$ label associated classes of characters.

Let ϵ be a sign and $\lambda \in \bar{\mathcal{P}}(n)$. As above we write $\sigma(\lambda) = \sigma$ if $\lambda \in \bar{\mathcal{P}}^\sigma(n)$. Moreover for $r \geq 0$, $w \geq 0$,

$$\bar{c}_p^\sigma(2r+1, w) = \left| \left\{ (\lambda_0, \lambda_1, \ldots, \lambda_r) \;\middle|\; \begin{array}{l} \lambda_0 \ \bar{p}\text{-core}, \lambda_1, \ldots, \lambda_r \ p\text{-cores} \\ \sum |\lambda_i| = w, (-1)^{w - l(\lambda_0)} = \sigma \end{array} \right\} \right|.$$

Obviously $c_p^{-\epsilon}(2r+1, w)$ is expressible in terms of the integers $c_p(r, w)$ of section 5 and the numbers $c^{\epsilon'}(\bar{p}, w)$ of $\bar{p}$-cores of w with sign ϵ'. (Their generating functions are computed in section 9, see also (13.9).)

If $\lambda \in \bar{\mathcal{P}}(n)$, then $l(\lambda) \equiv l(\lambda_{(\bar{p})}) + l(\lambda_0)$ if $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$. This follows from (4.6) (ii) and the definition of $\bar{p}$- core and $\bar{p}$-quotient. Using this and (4) above we get $\sigma(\lambda) = \sigma(\lambda_{(\bar{p})}) \cdot (-1)^{w - l(\lambda_0)}$, where $w = w_{\bar{p}}(\lambda)$, since p is odd. We may apply this also to the partition λ_0. Iterating this we see that the sign $\sigma(\lambda)$ of $\lambda \in \bar{\mathcal{P}}(n)$ is the product of the signs of the rows in the $\bar{p}$-core tower of λ where generally

$$\sigma(\lambda_0, \lambda_1, \ldots, \lambda_r) = (-1)^{\sum_i |\lambda_i| - l(\lambda_0)}.$$

(In particular, $\sigma(\lambda) = \sigma(\lambda^{(\bar{p})}) \sigma(\lambda_{(\bar{p})})$.)

Finally, if σ is a sign, $a \geq 0$, put

$$\bar{m}_a^\sigma(p, n) = \sum_{\substack{\alpha = (\alpha_0, \alpha_1, \ldots) \\ \sigma = (\sigma_0, \sigma_1, \ldots)}} \Pi_i \bar{c}_p^{\sigma_i}(\bar{p}^i, \alpha_i)$$

where α runs through $E_a(p,n)$ (as in section 6) and σ runs through all sequences of signs satisfying $\Pi\sigma_i = \sigma$.

From the above we get then

Proposition (7.4). In the notation above

$$\begin{aligned} \bar{m}_a(p, \hat{S}_n) &= \bar{m}_a^+(p,n) + 2\bar{m}_{\bar{a}}(p,n) \\ \bar{m}_a(p, \hat{A}_n) &= \bar{m}_{\bar{a}}(p,n) + 2\bar{m}_a^+(p,n) \end{aligned}$$

From (6.6), (6.9) and (7.4) we may then compute the McKay numbers of $\hat{S}_n$ and $\hat{A}_n$ for p odd.

For $p = 2$ the situation is quite different. Again let

$$\begin{aligned} \bar{h}_2(\lambda) &= \nu_2(\bar{f}_\lambda), \\ \bar{d}_2(\lambda) &= \nu_2(n!) - \bar{h}_2(\lambda). \end{aligned}$$

Note that here $\bar{d}_2(\lambda)$ is not the "defect" of the character $\langle\lambda\rangle$ (see the beginning of chapter II or section 10). To get the defect you have to add 1, since $\nu_2(|\hat{S}_n|) = 1 + \nu(n!)$. We define

$$\begin{aligned} t_2(\lambda) &= \nu_2\left(\Pi_{h\in\bar{\mathcal{H}}(\lambda)} h\right) \\ a(\lambda) &= [n - l(\lambda)/2] \end{aligned}$$

so that

$$\bar{d}_2(\lambda) = t_2(\lambda) - a(\lambda).$$

If $\lambda = (a_1, a_2, \ldots, a_m) \in \bar{\mathcal{P}}(n)$, $m = l(\lambda)$, let $\lambda_o(\lambda_e)$ be the partition consisting of all the odd (even) parts of λ.

Example. $\lambda = (8,5,2,1)$, $\lambda_o = (5,1)$, $\lambda_e = (8,2)$.

Lemma (7.5). In the above notation

$$t_2(\lambda) = t_2(\lambda_o) + t_2(\lambda_e).$$

Proof. The power of 2 dividing the product of the integers in the set

$$(\{1,2,\ldots,a_i\} \cup \{a_i + a_j \mid j > i\}) \setminus \{a_i - a_j \mid j > i\}$$

(which are the bar lengths in the i-th row of λ) equals the power of 2 dividing the product of the inters in

$$(\{1,2,\ldots,a_i\} \cup \{a_i + a_j \mid j > i,\ a_i \equiv a_j (\text{mod } 2)\}) \setminus \{a_i - a_j \mid j > i,\ a_i \equiv a_j (\text{mod } 2)\}$$

because when $a_i \not\equiv a_j(\text{mod } 2)$, then $\nu_2(a_i + a_j) = \nu_2(a_i - a_j) = 0$. From this (7.5) follows easily.

Lemma (7.6). If all the parts of λ are even, let χ be the bar partition obtained by dividing all the parts of λ by 2. In that case

$$t_2(\lambda) = |\lambda'| + t_2(\lambda').$$

Proof. Let $\lambda = (a_1, \dots, a_m)$, $a_i = 2\alpha_i$. Then the even bar lengths in the i-th row of λ are in the set

$$(\{2, 4, \dots, 2\alpha_i\} \cup \{2\alpha_i + 2\alpha_j \mid j > i\}) \backslash \{2\alpha_i - 2\alpha_j \mid j > i\}.$$

Dividing all integers in this set (which is of cardinality α_i) by 2 we get the bar lengths in the th row of λ'.

This is all we need to derive a formula for $t_2(\lambda)$ in terms of partitions with all parts odd. We define for $i \geqq 0$ inductively bar partitions $\lambda^{(i)}$, $\lambda_o^{(i)}, \lambda_e^{(i)}$ as follows:

(i) $\lambda^{(0)} = \lambda$,

(ii) $\lambda_o^{(i)}(\lambda_e^{(i)})$ is the partition consisting of the odd (even) parts of $\lambda^{(i)}$,

(iii) $\lambda^{(i+1)}$ is obtained by dividing all parts of $\lambda_e^{(i)}$ by 2.

(Of course these partitions are not relevant for the cores or quotients of λ.)

The above lemmas imply

$$\begin{aligned} \left|\lambda^{(i)}\right| &= \left|\lambda_o^{(i)}\right| + 2\left|\lambda^{(i+1)}\right| \quad \text{for} \quad i \geqq 0 \\ t_2(\lambda^{(i)}) &= t_2(\lambda_o^{(i)}) + \left|\lambda^{(i+1)}\right| + t_2(\lambda^{(i+1)}) \quad \text{for} \quad i \geqq 0. \end{aligned}$$

Using these equations we get by an easy calculation

$$\begin{aligned} n = |\lambda| &= \sum_{i \geq 0} \left|\lambda_o^{(i)}\right| 2^i \\ t_2(\lambda) &= \sum_{i \geq 0} t_2\left(\lambda_o^{(i)}\right) + \sum_{i \geq 1} (2^i - 1)\left|\lambda_o^{(i)}\right| \\ &= \sum_{i \geq 0} t_2\left(\lambda_o^{(i)}\right) + \left(n - \sum_{i \geq 0}\left|\lambda_o^{(i)}\right|\right). \end{aligned}$$

To simplify notation, we put $\lambda^i = \lambda_o^{(i)}$, $i \geqq 0$ and so each λ^i has only odd parts. In fact, t is a part of λ^i if and only if t is odd and $2^i t$ is a part of λ. We have proved:

Proposition (7.7). If $\lambda \in \bar{\mathcal{P}}(n)$ write

$$\lambda = \lambda^0 + 2\lambda^1 + 4\lambda^2 + \dots$$

where each λ^i is a bar partition with all parts odd. Then

$$n = \sum_{i\geq 0} |\lambda^i| 2^i \tag{5}$$

$$t_2(\lambda) = \sum_{i\geq 0} t_2(\lambda^i) + \left(n - \sum_{i\geq 0} |\lambda^i|\right). \tag{6}$$

Next we make a similar decomposition for $a(\lambda) = [(n - l(\lambda))/2]$.

Lemma (7.8). In the notation of (7.7)

$$a(\lambda) = \sum_{i\geq 0} a(\lambda^i) + \left[\frac{1}{2}\left(n - \sum_{i\geq 0} |\lambda^i|\right)\right]. \tag{7}$$

Proof. Trivially $l(\lambda) = \sum_{i\geq 0} l(\lambda^i)$. Moreover $|\lambda^i| - l(\lambda^i)$ is even for all i, since λ^i has only odd parts, so $a(\lambda^i) = (|\lambda^i| - l(\lambda^i))/2$. Using this and (5), (7) follows by an easy calculation.

Proposition (7.9). In the notation of (7.7)

$$\begin{aligned} \bar{d}_2(\lambda) &= \sum_{i\geq 0} \bar{d}_2(\lambda^i) + \left[\frac{1}{2}\left(n + 1 - \sum_{i\geq 0} |\lambda^i|\right)\right] \\ &= \sum_{i\geq 0} \bar{d}_2(\lambda^i) + \left[\frac{1}{2}\left(1 + \sum_{i\geq 1} |\lambda^i|(2^i - 1)\right)\right] \end{aligned} \tag{8}$$

Proof. Use (7.7), (7.8) and the fact that for any integer k we have $k = [k + 1/2] + [k/2]$.

Corollary (7.10). For all $\lambda \in \bar{\mathcal{P}}(n)$, $\bar{d}_2(\lambda) \geq 0$.

Proof. By (8) it suffices to show $\bar{d}_2(\lambda) \geqq 0$, when λ has all parts odd. In that case, $a(\lambda)$ is just the number of even bar lengths in λ, as is easily seen. (There are $(a_i - 1)/2$ even bar lengths in the i-th row of λ.)

(Of course this result may also be deduced from the fact $\hat{S}_n$ cannot have 2-blocks of defect 0.)

We have reduced our problem to the case where all parts of λ are odd (i.e. $\lambda = \lambda^0$). As we just mentioned, $a(\lambda)$ is then the number of even bar lengths in λ. Therefore, to compute $t_2(\lambda)$ or $\bar{d}_2(\lambda)$ we need only consider bars of a length *divisible by* 4. Let (as in section 4)

$$\begin{aligned} X_1^\lambda &= \{a \in \mathbb{N}_0 \mid a_j = 2a + 1 \quad \text{for some} \quad j \in \{1, 2, \ldots, m\}\} \\ X_3^\lambda &= \{a \in \mathbb{N}_0 \mid a_j = 4a + 3 \quad \text{for some} \quad j \in \{1, 2, \ldots, m\}\} \end{aligned}$$

(so $|X_1^\lambda| + |X_3^\lambda| = m$) and put

$$\mu(\lambda) := P(X_1^\lambda \mid X_3^\lambda)$$

in the notation of section 2. So $\mu(\lambda)$ is a partition. An argument analogous to the one used in the proof of (4.3) (the bars of type 2 do not occur) shows that we have:

Proposition (7.11). Let $\lambda \in \bar{\mathcal{P}}(n)$ have all parts odd. For $l \geqq 1$ there is a canonical bijection g between the set of $4l$-bars in λ and the set of l-hooks in $\mu(\lambda)$. If $\bar{H}$ is a $4l$-bar in λ we have

$$\mu(\lambda \backslash \bar{H}) = \mu(\lambda) \backslash g(\bar{H}).$$

Note. The reason that the difficulties mentioned in section 4 do not occur here is really that λ has no parts which are congruent 2 modulo 4.

If μ is an ordinary partition, let $d_2(\mu) = \nu_2(h(\mu))$, where $h(\mu)$ is the product of the hook lengths of μ. Then $d_2(\mu)$ has been computed in section 6. We have

Proposition (7.12). Let $\lambda \in \bar{\mathcal{P}}(n)$ have all parts odd. Then

$$t_2^*(\lambda) = |\mu(\lambda)| + d_2(\mu(\lambda))$$

Proof. If $\mu = \mu(\lambda)$, then $a(\lambda)$ bar lengths of λ are divisible by 2 and $|\mu|$ bar lengths are divisible by 4 (using (7.11)). The bars of length divisible by at least 8 are registered as hook of even length in μ (by (7.11)). Thus

$$t_2(\lambda) = a(\lambda) + |\mu| + d_2(\mu)$$

as desired.

This finishes our study of the power of two dividing spin character degrees. We do not compute the McKay numbers in this case.

8 Character degrees in linear groups

In this section we are concerned with character degrees of finite linear groups, more specifically the general linear, unitary, symplectic and orthogonal groups. For the last two classes of groups we consider only unipotent characters.

8.1 The case of $GL(n,q)$ and $U(n,q)$

We let $G = G(n,q)$ be $GL(n,q)$ or $U(n,q)$ where we for U assume that $q = q_0^2$. The irreducible characters of G are indexed by G-conjugacy classes of pairs (t, ψ) where t is a semisimple element of G and ψ is a unipotent character of $C_G(t)$.

Let $F = GF(q)$ be the field with q elements. Let $F[X]$ be the polynomial ring in the indeterminant X over F, and $\mathcal{F}_0$ the subset of monic irreducible polynomials different from X. In the case q is a square, say $q = q_0^2$, let $\sim$ be the permutation of $\mathcal{F}_0$ of order 2 defined by mapping

$$\Delta(X) = X^m + \alpha_{m-1}X^{m-1} + \ldots + \alpha_1 X + \alpha_0$$

onto $\tilde{\Delta}(X) = (\alpha_0^{-1})^J X^m \Delta^S J(X^{-1})$. (Here J is the unique automorphism of order 2 of F.) In particular α is a root or $\tilde{\Delta}(X)$ if and only if α^{q_0} is a root of $\tilde{\Delta}(X)$. Thus $\Delta = \tilde{\Delta}$ if and only if Δ has odd degree d and the roots of Δ have order dividing $q_0^d + 1$. We let

$$\begin{aligned} \mathcal{F}_1 &= \{\Delta \mid \Delta \in \mathcal{F}_0, \Delta = \tilde{\Delta}\} \\ \mathcal{F}_2 &= \{\Delta\tilde{\Delta} \mid \Delta \in \mathcal{F}_0, \Delta \neq \tilde{\Delta}\}. \end{aligned}$$

The polynomials in $\mathcal{F}_1$ and $\mathcal{F}_2$ have odd and even degrees respectively. The degree of a polynomial Γ will be denoted by d_Γ. In addition, we define a *reduced degree* δ_Γ for polynomials in $\mathcal{F}_0 \cup \mathcal{F}_2$ by

$$\delta_\Gamma = \begin{cases} d_\Gamma & \text{if } \Gamma \in \mathcal{F}_0 \\ \frac{1}{2} d_\Gamma & \text{if } \Gamma \in \mathcal{F}_2. \end{cases}$$

Following [20] we let $\mathcal{F} = \mathcal{F}_0$ if G is general linear and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ if G is unitary.

If the semisimple element $t \in G$ has primary decomposition $t = \Pi_{\Gamma \in \mathcal{F}} t_\Gamma$, then

$$C_G(t) = \prod_{\Gamma \in \mathcal{F}} G_\Gamma(m_\Gamma(t), q^{\delta_\Gamma})$$

where

$$G_\Gamma = \begin{cases} U & \text{if } G \text{ is unitary and } \Gamma \in \mathcal{F}_1 \\ GL & \text{otherwise} \end{cases}$$

and $m_\Gamma(t)$ is the multiplicity. We know from the decomposition of the underlying vector space that

(1) $$n = \sum_\Gamma m_\Gamma(t) d_\Gamma.$$

Correspondingly a unipotent character ψ of $C_G(t)$ is decomposed $\psi = \Pi_\Gamma \psi_\Gamma$ where ψ_Γ is a unipotent character of $C_G(t)_\Gamma = G_\Gamma(m_\Gamma(t), q^{\delta_\Gamma})$. Thus each ψ_Γ corresponds canonically to a partition ρ_Γ of $m_\Gamma(t)$. Since only the partitions play a role for us, we will denote the labels of irreducible characters by (t, ρ) where t is as above and $\rho = \Pi_\Gamma \rho_\Gamma$. Alternatively we may see the irreducible characters χ of G labelled by maps $\Theta : \mathcal{F} \to P$ satisfying

(2) $$\sum_{\Gamma \in F} |\Theta(\Gamma)| \, d_\Gamma = n$$

where $\mathcal{P} = \bigcup_{n \geq 0} \mathcal{P}(n)$ is the set of all partitions.

The characters where $m_{X-1}(t) = n$ (or $\Theta(X-1) \in \mathcal{P}(n)$) are then the *unipotent characters* of G.

For $\lambda \in \mathcal{P}(n)$, $\lambda = (a_1, a_2, \dots, a_m)$, we define a $n(\lambda) = \sum_{i=1}^n (i-1)a_i$. If $\lambda^0 = (b_1, b_2, \dots, b_l)$ then it is not difficult to see that

(3) $$n(\lambda) = \sum_{i=1}^{l} \binom{b_i}{2}.$$

We define also $n'(\lambda) = n(\lambda^0) = \sum_{i=1}^{m} \binom{a_i}{2}$. If λ is written exponentially $\lambda = (j^{m_j})$ (i.e. m_j parts of λ equals j), then obviously

(4) $$n'(\lambda) = \sum_{j \geq 1} \binom{j}{2} m_j$$

Given λ as above we also define a polynomial

$$f_\lambda(X) = \frac{X^{n(\lambda)} \prod_{i=1}^{n} (X^i - 1)}{\prod_{h \in H(\lambda)} (X^h - 1)}.$$

The unipotent character of G corresponding to λ then has degree

(5) $$\tilde{f}_\lambda(q) = \begin{cases} f_\lambda(q) & \text{if } G = GL \\ f_\lambda(-q) & \text{if } G = U. \end{cases}$$

Generally, the character $\chi_{(t,\rho)}$ of G labelled by (t, ρ) has degree

(6) $$\chi_{(t,\rho)}(1) = |G : C_G(t)|_{q'} \prod_{\Gamma \in \mathcal{F}} \tilde{f}_{\rho_\Gamma}(q^{\delta_\Gamma})$$

where $\ldots_{q'}$ means the part of a polynomial in q which is prime to q. (Thus $(q^2 - q)_{q'} = (q-1)$.) Here $\tilde{f}_{\rho_\Gamma}(q^{\delta_\Gamma})$ denotes the degree of a unipotent character of G_Γ.

Let us first look at the case of the *defining characteristic*, i.e. p is the prime dividing q. Since the degrees of characters are polynomials in q it is reasonable to consider $m_q(a, G)$ as the number of irreducible charcters of G whose degree is divisible by q^a exactly. Thus if $q = p^f$, then

$$m_p(a', G) = \begin{cases} 0 & \text{if } f \nmid a' \\ m_q(a, G) & \text{if } a' = fa. \end{cases}$$

By (6) the power of q dividing $\chi_{t,\rho}$ is

(7) $$h_q(t, \rho) = \sum_{\Gamma \in F} \delta_\Gamma n(\rho_\Gamma).$$

For each $\Gamma \in F$ write $\rho_\Gamma^0 = (j^{m_\Gamma^j})$ exponentially and put $\mu(t, \rho) = (r^{m_j})$, $\mu'(t, \rho) = (r^{m'_j})$ where $m_j = \sum_\Gamma d_\Gamma m_\Gamma^j$, $m'_j = \sum_\Gamma \delta_\Gamma m_\Gamma^j$.

Thus $|\mu(t, \rho)| = n$ and $n(\mu'(t, \rho)) = h_q(t, \rho)$. Notice that if $G = GL$, then $\mu(t, \rho) = \mu'(t, \rho)$ for all t, ρ.

Lemma (8.1). Let $\mu = (j^{m_j}) \in \mathcal{P}(n)$ be given. The number of pairs (t, ρ) as above with $\mu(t, \rho) = \mu$ equals the number of semisimple conjugacy classes in the group

$$\Pi_{j \geq 1} G(m_j, q).$$

This number equals $q^{l(\mu) - l'(\mu)}(q-1)^{l'(\mu)}$ for $G = GL$ and $q_0^{l(\mu) - l'(\mu)}(q_0 + 1)^{l'(\mu)}$ for $G = U$ where $l'(\mu)$ is the number of *different* parts in μ.

Proof. Given μ, we have for each j to count all decompositions on the form

$$m_j = \sum_{\Gamma} d_\Gamma m_\Gamma$$

where the m_Γ are non-negative integers. Trivially each such decomposition corresponds to the semisimple conjugacy class of elements in $G(m_j, q)$ whose characteristic polynomial is $\Pi_\Gamma \Gamma^{m_\Gamma}$. If m is arbitrary, the number of semisimple conjugacy classes in $GL(m,q)$ equals the number of monic polynomials of degree m over $GF(q)$ with non-zero constant term, i.e. $(q-1)q^{m-1}$. (Just count the possibilities for the coefficients in the monic polynomial.) The mapping $\tilde{\ }$ on $\mathcal{F}_0$ may of course be extended to a mapping on the set of all monic polynomials with non-zero contant term. Then it is easy to see that the number of monic polynomials of degree m with non-zero constant term fixed by $\sim$ is $(q_0+1)q_0^{m-1}$. Therefore this is the number of semisimple conjugacy classes in $U(m,q)$. From this the result follows by a simple calculation. (This argument has been taken from [51].)

Corollary (8.2). Let $G = GL(n,q)$, $a \geq 0$. Then

$$m_q(a,G) = \sum_{\{\mu \in \mathcal{P}(n) \mid n'(\mu)=a\}} q^{l(\mu)-l'(\mu)}(q-1)d^{l'(\mu)}.$$

Proof. Follows immediately from (8.1) and the remarks preceding it.

Remark (8.3). If $G = U(n,q)$, $a \geq 0$ the obvious analogue of (8.2) (replacing $q-1$ by q_0+1 and q by q_0) does *not* hold. The reason is that $\mu(t,\rho)$ may be different from $\mu'(t,\rho)$ and it is $\mu'(t,\rho)$ which determines the relevant a. There does not appear to exist an elegant closed formula like (8.2) in the unitary case. However using the above it is not difficult for any given a and n to compute explicitely $m_a(q, U(n,q))$.

We now consider the case of the *non-defining characteristic* and a prime r, such that $r \nmid q$. We will assume that $r \neq 2$ and that $r \mid |G|$.

Let $e = \min\{i \in \mathbb{N} \mid r \mid (q^i - 1)\}$, i.e. the multiplicative order of q modulo r. Let $c = \nu_r(q^e - 1)$. Generalizing this let for $\Gamma \in F$

$$\epsilon_\Gamma = \left\{ \begin{array}{ll} (-1) & \text{if } G \text{ is unitary and } \Gamma \in \mathcal{F}_1 \\ 1 & \text{otherwise} \end{array} \right\}$$

and

$$e_\Gamma = \min\left\{i \in \mathbb{N} \mid r \mid \left((\epsilon_\Gamma q)^{\delta_\Gamma i} - 1\right)\right\}.$$

Thus e_Γ is the minimal integer such that $r \mid \left|G_\Gamma(e_\Gamma, q^{\delta_\Gamma})\right|$. Of particular importance is $e' = e_{X-1}$. An easy calculation shows that the following holds ([20], [34]):

Lemma (8.4). (i) If G is linear then $e = e'$. If G is unitary then

$$e' = \left\{ \begin{array}{lll} e & \text{if } r & \mid (q_0^e - (-1)^e) \\ 2e & \text{if } r & \mid (q_0^e + (-1)^e). \end{array} \right.$$

(ii) In any case we have for all $\Gamma \in \mathcal{F}$ that

$$e_\Gamma = e'/(e, \delta_\Gamma).$$

Lemma (8.5). Let $\Gamma \in \mathcal{F}$, $h \in \mathbb{N}$. We have

$$r \;\Big|\; \left((\epsilon_\Gamma q)^{\delta_\Gamma h} - 1\right) \Leftrightarrow e_\Gamma \mid h.$$

If $e_\Gamma \mid h$ then $\nu_r((\epsilon_\Gamma q)^{\delta_\Gamma h} - 1) = c + \nu_r(\delta_\Gamma h)$.

Using the degree formula (6) above it is easy to describe the ($r-$) defect of a character $\chi_{(t,\rho)}$ of G. Indeed

$$d_r\left(\chi_{(t,\rho)}\right) = \sum_{\Gamma \in \mathcal{F}} \sum_{h \in \mathcal{H}(\rho_\Gamma)} \nu_r\left((\epsilon_\Gamma q)^{\delta_\Gamma h} - 1\right). \tag{8}$$

Let us abbreviate for a partition λ

$$d_r^\Gamma(\lambda) = \sum_{h \in \mathcal{H}(\lambda)} \nu_r\left((\eta_\Gamma q)^{\delta_\Gamma h} - 1\right) \tag{9}$$

so that

$$d_r\left(\chi_{(t,\rho)}\right) = \sum_\Gamma d_r^\Gamma(\rho_\Gamma) \tag{10}$$

For a given $\lambda \in \mathcal{P}(n)$ and integers e and r (which are assumed to be relative prime) we may generally define the (e,r)-*core tower* of λ as follows: It has rows numbered $0, 1, 2, \ldots$. The $0'$th row is $\lambda_{(e)}$, the e-core of λ. If $\lambda^{(e)} = (\lambda_0, \ldots, \lambda_{e-1})$ then the first row is $\lambda_{0(p)}, \ldots, \lambda_{e-1(p)}$. The next row contains the r-cores of the partitions in $\lambda_0^{(r)}, \lambda_1^{(r)}, \ldots$ etc. Thus for $i \geq 1$ the ith row consists of the $(i-1)$th row of the r-core tower of λ_0, followed by the $(i-1)$th row of the r-core tower of λ_1, etc. Of course the (e,r)-core tower is a generalization of the p-core tower introduced in section 6. (Indeed the $(1,r)$-core tower of λ is just the r-core tower of λ with an added initial row containing the partition (0).)

We let $\beta_i(e,r,\lambda)$ denote the sum of the cardinalities of the partitions in the ith row of the (e,r)-core tower of λ. By repeated use of (3.7) we see that λ is uniquely determined by its (e,r)-core tower.

Lemma (8.6). Let $\lambda \in \mathcal{P}(n)$. Let e, r be given. We have the following:

(i) $n = |\lambda| = \beta_0(e,r,\lambda) + \sum_{i \geq 1} \beta_i(e,r,\lambda) e r^{i-1}$

(ii) $w_e(\lambda) = \sum_{i \geq 1} \beta(e,r,\lambda) r^{i-1}$

Proof. By definition $\beta_0(e,r,\lambda) = \left|\lambda_{(e)}\right|$. Let $\lambda^{(e)} = (\lambda_0, \lambda_1, \ldots, \lambda_{e-1})$. Then $w_e(\lambda) = \sum_{j=0}^{e-1} |\lambda_j|$ by definition, and applying (6.3) (with $p = r$) to each of the partitions λ_j and adding the results yields (ii). Then (i) follows by (3.6).

Consider again the integer $d_r^\Gamma(\lambda)$ defined by (9). We have

Proposition (8.7). Let $\Gamma \in \mathcal{F}$, λ a partition. Then

$$d_r^\Gamma(\lambda) = \sum_{i\geq 0} \beta_{i+1}(e_\Gamma, r, \lambda)\left[(c+\nu_r(d_\Gamma))\, r^i + \left((r^i-1)/(r-1)\right)\right].$$

Proof. By (3.6) exactly $w_{e_\Gamma}(\lambda)$ hooks of λ have length divisible by λ. Therefore, by (8.5) we get (using also (3.3))

$$(*) \qquad d_r^\Gamma(\lambda) = w_{e_\Gamma}(\lambda)[c+\nu_r(d_\Gamma)] + \sum_{j=0}^{e-1} d_r(\lambda_j)$$

where $\lambda^{(e_\Gamma)} = (\lambda_0, \lambda_1, \ldots, \lambda_{e_\Gamma - 1})$. The integers $d_r(\lambda_j)$ may be computed using formula (4) of section 6. We get

$$\begin{aligned}(**) \qquad \sum_j d_r(\lambda_j) &= \sum_j \left(|\lambda_j| - \sum_{i\geq 0}\beta_i(r,\lambda_j)\right)/(r-1)\\ &= \left(w_{e_\Gamma}(\lambda) - \sum_{j=0}^{e-1}\sum_{i\geq 0}\beta_i(r,\lambda_j)\right)/(r-1)\\ &= \left(w_{e_\Gamma}(\lambda) - \sum_{i\geq 0}\beta_{i+1}(e_\Gamma,r,\lambda)\right)/(r-1)\end{aligned}$$

since by definition $\sum_j \beta_i(r,\lambda_j) = \beta_{i+1}(e_\Gamma, r, \lambda)$. By (8.6) (ii)

$$(***) \qquad w_{e_\Gamma}(\lambda) = \sum_{i\geq 0}\beta_{i+1}(e_\Gamma, r, \lambda) r^i.$$

Now (8.7) follows by a straightforward calculation from $(*)$, $(**)$ and $(***)$.

From (8.7) and formula (8) above we may compute then the defect (and therefore also the height) of any irreducible character in G. Let us note the following:

Corollary (8.8). Let $\chi = \chi_{(t,\rho)}$ be an irreducible character of G. Then χ has r-defect 0 if and only if ρ_Γ is an e_Γ-core for all $\Gamma \in \mathcal{F}$ (see [20], [10] for the general block theory of these groups).

The above illustrates clearly how the general theory of cores and quotients of partitions may be used in the study of character degrees in general linear and unitary groups. Next we explain briefly how the analogous theory for Lusztig's symbols (section 5) may be applied for the classical groups.

8.2 The case of classical groups

As was shown in [30] the unipotent characters of finite classical groups (symplectic and orthogonal) are labelled by symbols as considered in section 5.

Consider a symbol $[X,Y]$ where $X = \{a_1, a_2, \dots, a_k\}$ and $Y = \{b_1, b_2, \dots, b_l\}$ are β-sets and $a_1 > a_2 > \dots > a_k$, $b_1 > b_2 > \dots > b_l$. We need some notation in addition to the notation used in section 5.

We write the elements of $X\dot{\cup}Y$ in decreasing order, $X\dot{\cup}Y = \{c_1, c_2, \dots, c_{k+l}\}$, $c_1 \geqq c_2 \geqq \dots \geqq c_{k+l} \geqq 0$. Let

$$
\begin{aligned}
(11) \qquad n^*(X) &= \sum_{i=1}^{k}(i-1)a_i, \qquad n(X) := n^*(X) - \sum_{i=1}^{k-1}\binom{i}{2} \\
(12) \qquad n^*(X,Y) &= \sum_{i=1}^{k+l}(i-1)c_i, \qquad n(X,Y) = n^*(X,Y) - \sum_{i=1}^{k+l-2}\left[\binom{i}{2}^2\right].
\end{aligned}
$$

Using the identity $\binom{n}{2} = [(n/2)^2] + [((n-1)/2)^2]$, $n \in \mathbb{N}$, we have

$$
(13) \qquad n(X,Y) = n^*(X,Y) - \sum_{i\geq 1}\binom{k+l-2i}{2}.
$$

Finally let

$$
(14) \qquad c(X,Y) = \begin{cases} \left[\frac{k+l-1}{2}\right] - |X \cap Y| & \text{if } X \neq Y \\ 0 & \text{if } X = Y \end{cases}
$$

Let us note that for $t \in \mathbb{N}_0$ we have $n(X^{+t}) = n(X)$, $n(X^{+t}, Y^{+t}) = n(X,Y)$, $c(X^{+t}, Y^{+t}) = c(X,Y)$, so $n(X,Y)$ and $c(X,Y)$ are invariants for the symbol. Moreover, if $P^*(X) = \lambda$ then $n(X)$ is the integer $n(\lambda)$ considered in the first part of this section.

Example. $X = \{9,7,4,1\}$, $Y = \{3,1,0\}$, $X\dot{\cup}Y = \{9,7,4,3,2,1,1,0\}$, $n(X) = 18-4 = 14$, $n(X,Y) = 33 - 13 = 20$, $c(X,Y) = 3 - 1 = 2$.

Let q be a power of the prime p and put (using section 5)

$$
\begin{aligned}
H(X,q) &:= \prod_{h\in\mathcal{H}(X)} (q^h - 1) \\
H(X,Y,q) &= \prod_{h\in\mathcal{H}^+[X,Y]} (q^h - 1) \prod_{h\in\mathcal{H}^-[X,Y]} (q^h + 1).
\end{aligned}
$$

If G is a general linear group over $GF(q)$, then the unipotent characters of G are indexed by partitions as we have seen. If $P^*(X) = \lambda$, then the degree of the unipotent character indexed by λ is

$$
q^{n(X)} \prod_{i=1}^{\operatorname{rk} X} (q^i - 1)/H(X,q) = q^{n(X)} \left|G\right|_{p'} / H(X,q)
$$

(see the first part of this section).

So the denominator has a factor for each hooklength of X (or λ). The same is true in case of the classical groups. We have

Proposition (8.9). Let G be a classical group over $GF(q)$ and $\chi_{[X,Y]}$ the unipotent character of G indexed by the symbol $[X,Y]$ (So rk $[X,Y]$ is also the rank of G). Then

$$\chi_{[X,Y]}(1) = q^{n(X,Y)}2^{-c(X,Y)}\left|G^{ad}\right|_{p'}/H(X,Y,q). \tag{15}$$

Proof. We assume $p \neq 2$. The proof for $p = 2$ is essentially the same. Put

$$\begin{aligned}
\Delta(X,q) &= \prod_{i<j}(q^{a_i} - q^{a_j}),\\
\Delta^*(X,q) &= \prod_{i<j}(q^{a_i-a_j} - 1),\\
\Delta(X,Y,q) &= \prod_{i,j}(q^{a_i} + q^{b_j}),\\
\Delta^*(X,Y,q) &= \prod_{\{(i,j)|a_i>b_j\}}(q^{a_i-b_j}+1)\prod_{\{(i,j)|a_i<b_j\}}(q^{b_j-a_i}+1)\\
\Theta(X,q) &= \prod_{i=1}^{k}\prod_{j=1}^{a_i}(q^j - 1)\\
\tilde{\Theta}(X,q) &= \prod_{i=1}^{k}\prod_{j=1}^{a_i}(q^j + 1)\\
c &= \left[\frac{k+l-1}{2}\right] \quad \text{if } X \neq Y, \quad c = k = l \text{ if } X = Y.
\end{aligned}$$

By section 8 in [30]

$$\chi_{[X,Y]}(1) = \left|G^{ad}\right|_{p'} D_{[X,Y]}(q),$$

where

$$D_{[X,Y]}(q) = \Delta(X,q)\Delta(Y,q)\Delta(X,Y,q)/\Theta(X,q^2)\Theta(Y,q^2)2^c q^d$$

with

$$d = \sum_{i\geq 1}\binom{k+l-2i}{2}.$$

First, we claim that $q^{n^*(X,Y)}$ is the exact power of q dividing $\Delta(X,q)\Delta(Y,q)\Delta(X,Y,q)$. Indeed, let $X\dot{\cup}Y = \{c_1, c_2, \ldots, c_{k+l}\}$ be as above, so $n^*(X,Y) = \sum_{1\leq i<j\leq k+l} c_j$, as is easily seen. Let $1 \leq i < j \leq k+l$. If $c_i, c_j \in X$ then $q^{c_j} \mid q^{c_t} - q^{c_j}$, a factor of $\Delta(X,q)$. We argue similarly, if $c_i, c_j \in Y$. In the remaining case $q^{c_j} \mid (q^{c_i} + q^{c_j})$, a factor of $\Delta(X,Y,q)$. Since we in each case consider the exact power of q dividing the factor, our claim is proved. Now it follows from (13) that $q^{n(X,Y)}$ is the exact power of q dividing $D_{[X,Y]}(q)$ and $\chi_{[X,Y]}(1)$. By (14) $c(X,Y) = c - |X \cap Y|$. That the power 2 occuring in (15) is correct is seen as follows: For each $a_i = b_j \in X \cap Y$ we get a factor 2 from the factor $q^{a_i} + q^{b_j} = 2q^{a_i}$ of $\Delta(X,Y,q)$.

By easy calculation we see that it remains to show that

$$\Theta(X,q^2)\Theta(Y,q^2)/\Delta^*(X,q)\Delta^*(Y,q)\Delta^*(X,Y,q) = H(X,Y,q). \tag{$*$}$$

Note that $\Theta(X, q^2) = \Theta(X, q)\tilde{\Theta}(X, q)$.

For a fixed $1 \leq i \leq k$ we have (by the definition of $\mathcal{H}_i(X)$ in section 1)

$$\prod_{h \in \mathcal{H}_i(X)} (q^h - 1) = \prod_{j=1}^{a_t} (q^j - 1) / \prod_{\{i' \mid i' > i\}} (q^{a_t - a_i} - 1)$$

so

$$\begin{aligned} H(X, q) &= \Theta(X, q)/\Delta^*(X, q), \\ H(Y, q) &= \Theta(Y, q)/\Delta^*(Y, q). \end{aligned}$$

A similar argument shows that

$$\prod_{h \in \mathcal{H}^-[X,Y]} (q^h + 1) = \tilde{\Theta}(X, q)\tilde{\Theta}(Y, q)/\Delta^*(X, Y, q).$$

Then $(*)$ follows and (15) is proved.

We now briefly consider the question which unipotent characters are of r-defect 0, where r is a prime, $r \neq 2$, $r \neq p$. This has also been done in [21]. Let e be multiplicative order of $q^2 \mod r$, so $r \mid (q^{2e} - 1)$. We call r *linear* if $r \mid q^e - 1$ and r *unitary* if $r \mid q^e + 1$ (following [21]).

Lemma (8.10). We have

$$r \nmid H(X, Y, q) \Leftrightarrow \begin{cases} e \notin \mathcal{H}^+[X, Y] & \text{if } r \text{ is linear} \\ e \notin \mathcal{H}^-[X, Y] & \text{if } r \text{ is unitary} \end{cases}$$

Proof. If r is linear, then $r \nmid (q^h + 1)$ for all h and $r \mid (q^h - 1)$ if and only if $e \mid h$. Thus $r \mid H(X, Y, q)$ if and only if $te \in \mathcal{H}^+[X, Y]$ for some $t \in \mathbb{N}$. By (5.1) this is equivalent to $e \in \mathcal{H}^+[X, Y]$. If r is unitary, then $r \mid (q^h + 1)$ if and only if $e \mid h$ and (h/e) is odd and $r \mid (q^h - 1)$ if and only if $e \mid h$ and (h/e) is even. Again (5.1) shows $r \mid H(X, Y, q)$ if and only if $e \in \mathcal{H}[X, Y]$.

Corollary (8.11). Let r and e be as above. If the unipotent character χ of a classical group is indexed by $[X, Y]$, then χ has r-defect 0 if and only if $e \notin \mathcal{H}^+[X, Y]$ if r is linear and $e \notin \mathcal{H}^-[X, Y]$ if r is unitary.

Proof. Follows immediately from (8.9) and (8.10).

The quotients and co-quotients of symbols may be used to give an explicit description of the power of r dividing $H(X, Y, q)$ for a given symbol $[X, Y]$. The method is the same as is applied in the proof of (8.7). We omit further details.

9 Some generating functions

For the computation of some block invariants which will be the topic of the next chapter it is useful to know some relations between the generating functions of the partition function and other functions. Generally, if $f : \mathbb{N} \cup \{0\} \to \mathbb{C}$ is a function then the formal power series

$$F(x) = \sum_{i=0}^{\infty} f(i)x^i \in \mathbb{C}[[x]]$$

is called the *generating function* for f. The idea is to try to find alternative simpler expressions for F. For example if f is the constant function $f(i) = 1$, then the generating function

$$F(x) = \sum x^i$$

equals $(1-x)^{-1}$, the inverse of the element $1-x$ in the ring $\mathbb{C}[[x]]$. Under certain circumstances it is possible to define the infinite sum or the infinite product of elements from $\mathbb{C}[[x]]$. If $f_1(x), \ldots, f_n(x), \ldots$ is an infinite sequence of elements in $\mathbb{C}[[x]]$, then $\sum_i f_i(x)$ makes sense as a well defined element of $\mathbb{C}[[x]]$ if for any given $i \geq 0$ only finitely many j_j's have the coefficients to x^i different from 0. Similarly $\Pi_i f_i(x)$ makes sense if all f_i have constant term equal to one and if for any given $i \neq 0$ only finitely many f_j's have nonvanishing coefficient to x^i. Only such infinite sums and products occur below. (The following has been taken from [49].)

Let $n \in \mathbb{N}_0$, σ a sign. We consider the following numbers where p is an *odd* prime:

- $p(n)$, the number of partitions of n (section 1)
- $q(n)$, the number of bar partitions of n (section 4)
- $q^\sigma(n)$, the number of bar partitions of n with sign σ (section 7)
- $\tilde{q}^\sigma(n) = q^{(-1)^n\sigma}(n)$
- $q(\bar{p}, n)$, the number of $\bar{p}$-quotients of weight n (section 4)
- $q^\sigma(\bar{p}, n)$, the number of $\bar{p}$-quotients of weight n with sign σ (section 7)
- $\tilde{q}^\sigma(\bar{p}, n) = q^{(-1)^n\sigma}(\bar{p}, n)$
- $c(\bar{p}, n)$, the number of $\bar{p}$-cores of n (section 4)
- $c^\sigma(\bar{p}, n)$, the number of $\bar{p}$-cores of n with sign σ (section 7)
- $\tilde{c}^\sigma(\bar{p}, n) = c^{(-1)^n\sigma}(\bar{p}, n)$

The corresponding generating functions will be denoted (in the same order)

$$P(x), Q(x), Q^\sigma(x), \tilde{Q}^\sigma(x), Q_{\bar{p})}(x), Q^\sigma_{\bar{p}}(x), \tilde{Q}^\sigma_p(x), F_{\bar{p}}(x), F^\sigma_{\bar{p}}(x), \tilde{F}^\sigma_{\bar{p}}(x).$$

It is wellknown that

Proposition (9.1).

$$P(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}.$$

Proof. If we want to compute the coefficient to x^n in

$$\prod_{i=1}^{\infty} \frac{1}{(1-x^i)} = (x^{0\cdot 1} + x^{1\cdot 1} + x^{2\cdot 1} + \ldots)(x^{0\cdot 2} + x^{2\cdot 1} + x^{2\cdot 2} + \ldots)(x^{0\cdot 3} + \ldots)\ldots$$

we get a contribution 1 to x^n whenever n is written as $n = (i_1 \cdot 1 + i_2 \cdot 2 + \ldots)$. Each expression $n = i_1 \cdot 1 + i_2 \cdot 2 + \ldots$ corresponds to a partition of n having i_1 parts equal to 1, i_2 parts equal to 2 etc.

The above argument is easily modified to show that

Proposition (9.2).

$$\begin{aligned} Q(x) &= \prod_{i=1}^{\infty}(1+x^i) \\ &= P(x)/P(x^2) \\ &= \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}. \end{aligned}$$

Proof. The first equality is clear since we are counting partitions without repetitions. Moreover, by (9.1)

$$P(x)/P(x^2) = \frac{\prod_{i=1}^{\infty}(1-x^{2i})}{\prod_{i=1}^{\infty}(1-x^i)}.$$

If we in this expression divide each $(1-x^i)$ into $(1-x^{2i})$ we get

$$P(x)/P(x^2) = \prod_{i=1}^{\infty}(1+x^i).$$

If instead we cancel all factors $(1-x^{2i})$ we get

$$P(x)/P(x^2) = \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}.$$

Proposition (9.3). We have

(i) $Q^+(x) + Q^-(x) = Q(x)$

(ii) $\tilde{Q}^+(x) + \tilde{Q}^-(x) = Q(x)$

(iii) $\tilde{Q}^+(x) - \tilde{Q}^-(x) = P(x)^{-1}$

(iv) $Q^+(x) - Q^-(x) = \tilde{Q}^+(-x) - \tilde{Q}^-(-x) = P(-x)^{-1}$.

Proof. (i) and (ii) are trivial. To prove (iii), we notice that $\tilde{q}^+(n) - \tilde{q}^-(n)$ is the coefficient to x^n in the product $(1-x)(1-x^2)\ldots(1-x^n)$: Writing n as a sum of different parts we get a contribution 1 (resp. -1) to this coefficient when the number of parts is even (resp. odd). Thus

$$\tilde{Q}^+(x) - \tilde{Q}^-(x) = \prod_{i\geq 1}^{\infty}(1 - x^i),$$

so (iii) follows from (9.1). Comparing coefficients it is easy to see that $Q^+(x) - Q^-(x) = \tilde{Q}^+(-x) - \tilde{Q}^-(-x)$, so (iv) follows from (iii).

Let us consider $P(-x)$, which is involved in several formulas in this section. Using (9.1) above we get easily

$$\begin{aligned} P(-x) &= P(x^2)^3/P(x)P(x^4) \\ P(-x)^{-1} &= P(x)P(x^4)/P(x^2)^3 = Q(x)/Q(x^2)P(x^2). \end{aligned}$$

Then an easy calculation shows

Proposition (9.4). We have

(i) $Q^\sigma(x) = \frac{1}{2}\frac{P(x)}{P(x^2)}\left(1 + \sigma\frac{P(x^4)}{P(x^2)^3}\right)$

(ii) $\tilde{Q}^\sigma(x) = \frac{1}{2}\frac{P(x)^2 + \sigma P(x^2)}{P(x)P(x^2)}$.

Thus using the results of section 7 we get

Proposition (9.5). The generating function for the number of spin characters in $\hat{S}_n$ is

$$\hat{P}(x) = \frac{P(x)}{P(x^2)}\left(\frac{3}{2} - \frac{1}{2}\frac{P(x^4)}{P(x^2)^2}\right).$$

Note. Of course $P(x)$ is the generating function for the number of irreducible characters in S_n.

Next we consider $\bar{p}$-quotients.

Proposition (9.6). We have (with $t = (p-1)/2$)

(i) $Q_{\bar{p}}(x) = Q(x)P(x)^t = P(x)^{t+1}/P(x^2)$

(ii) $Q^+_{\bar{p}}(x) + Q^-_{\bar{p}}(x) = Q_{\bar{p}}(x)$

(iii) $\tilde{Q}^+_{\bar{p}}(x) + \tilde{Q}^-_{\bar{p}}(x) = Q_{\bar{p}}(x)$

(iv) $\tilde{Q}^\sigma_{\bar{p}}(x) = \tilde{Q}^\sigma(x)P(x)^t$

(v) $\tilde{Q}^+_{\bar{p}}(x) - \tilde{Q}^-_{\bar{p}}(x) = P(x)^{t-1}$

(vi) $Q^+_{\bar{p}}(x) - Q^-_{\bar{p}}(x) = P(-x)^{t-1}$

Proof. (i) follows from the definition and (ii) and (iii) are tivial. (iv): Suppose that $(\lambda_0, \lambda_1, \ldots, \lambda_t)$ is a $\bar{p}$-quotient of weight n and sign $(-1)^n\sigma$. Then $(-1)^{l(\lambda_0)} = \sigma$. If $|\lambda_0| = n_0$ then $\sigma(\lambda_0) = (-1)^{n_0}\sigma$ and there are $\tilde{q}^\sigma(n_0)$ choices for λ_0 and then the remaining partitions $\lambda_1, \ldots, \lambda_t$ may be chosen arbitrarily subject to $|\lambda_1| + \cdots + |\lambda_t| = n - n_0$. The relation (v) follows from (iv) and (9.3) (iii) and (vi) is then true since $Q_{\bar{p}}^+(x) - Q_{\bar{p}}^-(x) = \tilde{Q}_{\bar{p}}^+(-x) - \tilde{Q}_{\bar{p}}^-(-x)$.

Corollary (9.7). We have

(i) $\tilde{Q}_{\bar{p}}^\sigma(x) = \frac{1}{2}P(x)^{t-1}\left(\frac{P(x)^2}{P(x^2)} + \sigma\right)$

(ii) $Q_{\bar{p}}^\sigma(x) = \frac{1}{2}\left(\frac{P(x)^{t+1}}{P(x^2)} + \sigma\frac{P(x^2)^{3t-3}}{P(x)^{t-1}P(x^4)^{t-1}}\right)$.

Proof. A straightforward calculation using (9.6).

Now we consider the $\bar{p}$-cores and defect 0 spin characters. If λ is a bar partition and $\sigma(\lambda) = 1$ then $\sigma(\lambda^{(\bar{p})}) = \sigma(\lambda_{(\bar{p})})$, by section 7. Since a bar partition is uniquely determined by its $\bar{p}$-core and its $\bar{p}$-quotient and since

$$|\lambda| = |\lambda_{(\bar{p})}| + p\sum_i |\lambda_i| \quad \text{if} \quad \lambda^{(\bar{p})} = (\lambda_0, \lambda_1, \ldots, \lambda_t)$$

(by (4.4) (ii)), we see

Proposition (9.8). We have

(i) $Q^+(x) = Q_{\bar{p}}^+(x^p)F_{\bar{p}}^+(x) + Q_{\bar{p}}^-(x^p)F_{\bar{p}}^-(x)$

(ii) $Q^-(x) = Q_{\bar{p}}^+(x^p)F_{\bar{p}}^-(x) + Q_{\bar{p}}^-(x^p)F_{\bar{p}}^+(x)$

(iii) $\tilde{Q}^+(x) = \tilde{Q}_{\bar{p}}^+(x^p)\tilde{F}_{\bar{p}}^+(x) + \tilde{Q}_{\bar{p}}^-(x^p)\tilde{F}_{\bar{p}}^-(x)$

(iv) $\tilde{Q}^-(x) = \tilde{Q}_{\bar{p}}^+(x^p)\tilde{F}_{\bar{p}}^-(x) + \tilde{Q}_{\bar{p}}^-(x^p)\tilde{F}_{\bar{p}}^+(x)$

We want to determine the number of spin characters of p-defect 0 in $\hat{S}_n$. The labels of such characters have to be $\bar{p}$-cores, e.g. by formula (3) in section 7. On the other hand $\bar{p}$-cores label defect 0 spin characters by (7.2). Thus the desired number of defect 0 spin characters is $c^+(\bar{p}, n) + 2c^-(\bar{p}, n)$. The corresponding generating function is $F_p^+(x) + 2F_p^-(x)$. Similarly the number of defect 0 characters in S_n equals $c(p, n)$, the number of p-cores of n whose generating function $F_p(x)$ is computed later in (9.14).

Proposition (9.9). We have

(i) $F_{\bar{p}}^+(x) + F_{\bar{p}}^-(x) = F_{\bar{p}}(x)$

(ii) $\tilde{F}_{\bar{p}}^+(x) + \tilde{F}_{\bar{p}}^-(x) = F_{\bar{p}}(x)$

(iii) $F_{\bar{p}}(x) = P(x)P(x^{2p})/P(x^2)P(x^p)^{t+1}$

(iv) $\tilde{F}^{\sigma}_{\bar{p}}(x) = \frac{1}{2}\left(F_{\bar{p}}(x) + \sigma P(x)^{-1}P(x^p)^{1-t}\right)$

(v) $F^{\sigma}_{\bar{p}}(x) = \frac{1}{2}\left(F_{\bar{p}}(x) + \sigma P(-x)^{-1}P(-x^p)^{1-t}\right)$

Proof. (i) and (ii) are trivial, so if we add (9.8) (i) and (ii) we get $Q(x) = Q_{\bar{p}}(x^p)F_{\bar{p}}(x)$ and using (9.2) and (9.6) (i) we then get (iii). If we substract (9.8) (iv) from (9.8) (iii) we get, using (9.3) (iii) and (9.6) (v),

$$P(x)^{-1} = P(x^p)^{t-1}\left(\tilde{F}^{+}_{\bar{p}}(x) - \tilde{F}^{-}_{\bar{p}}(x)\right).$$

Then

$$\tilde{F}^{+}_{\bar{p}}(x) - \tilde{F}^{-}_{\bar{p}}(x) = P(x)^{-1}P(x^p)^{1-t}$$

and substituting $x \to -x$ we get as before

$$F^{+}_{\bar{p}}(x) - F^{-}_{\bar{p}}(x) = P(-x)^{-1}P(-x^p)^{1-t}.$$

Combining the last formulas we get easily (iv) and (v).

Proposition (9.10). Let p be an odd prime. The generating function for the number of spin characters of p-defect 0 in $\hat{S}_n$ is

$$\hat{F}_{\bar{p}}(x) = F^{+}_{\bar{p}}(x) + 2F^{-}_{\bar{p}}(x).$$

Note. The results of section 7 or elementary block theroy shows that there are no spin characters in $\hat{S}_n$ or $\hat{A}_n$ of 2-defect 0.

The remarks about "duality" between $\hat{S}_n$ and $\hat{A}_n$ make the following "dual" results of (9.5) and (9.10) obvious.

Proposition (9.11). The generating function for the number of spin characters in $\hat{A}_n$ is

$$\hat{\hat{P}}(x) = \frac{P(x)}{P(x^2)}\left(\frac{3}{2} + \frac{1}{2}\frac{P(x^4)}{P(x^2)^2}\right).$$

(Indeed $\hat{\hat{P}}(x) = Q^{-}(x) + 2Q^{+}(x)$.)

Proposition (9.12). Let p be an odd prime. The generating function for the number of spin characters of p-defect 0 in $\hat{A}_n$ is

$$\hat{\hat{F}}_{\bar{p}}(x) = 2F^{+}_{\bar{p}}(x) + F^{-}_{\bar{p}}(x).$$

The similar calculations for S_n and A_n are somewhat simpler. We remind the reader that a partition λ is called self-conjugate if $\lambda = \lambda^0$ and by (3.5) λ is self-conjugate for a given $e \in \mathbb{N}$ if and only if both its e-core and e-quotient are self-conjugate (see the definition following (6.7)).

Let $n \in \mathbb{N}_0$, $e \in \mathbb{N}$. We consider the following integers:

- $p^s(n)$, the number of self-conjugate partitions of n
- $p^n(n)$, the number of non-self-conjugate partitions of n
- $k(e,n)$, the number of e-quotients of weight n (section 3)
- $k^s(e,n)$, the number of self-conjugate e-quotients of weight n (section 6)
- $k^n(e,n)$, the number of non-self-conjugate e-quotients of weight n
- $c(e,n)$, the number of e-cores of n (section 3)
- $c^s(e,n)$, the number of self-conjugate e-cores of n
- $c^n(e,n)$, the number of non-self-conjugate e-cores of n

The corresponding generating functions will be denoted (in the same order) $P^s(x)$, $P^n(x)$, $P_e(x)$, $P_e^s(x)$, $P_e^n(x)$, $F_e(x)$, $F_e^s(x)$, $F_e^n(x)$.

Proposition (9.13). We have

(i) $P^n(x) + P^s(x) = P(x)$

(ii) $P^s(x) = P(x)P(x^4)/P(x^2)^2$

(iii) $P^n(x) = P(x)(1 - P(x^4)/P(x^2)^2)$

(iv) $P_e(x) = P(x)^e$

(v) $P_e^n(x) + P_e^s(x) = P_e(x)$

(vi) $P_e^s(x) = \begin{cases} P(x^2)^f P^s(x) & \text{if } e = 2f+1 \text{ is odd} \\ P(x^2)^f & \text{if } e = 2f \text{ is even.} \end{cases}$

Proof. (i) and (v) are trivial. The diagonal hooklengths in a symmetric partition form the parts of a partition of n with all parts different and odd. Thus (arguing as in the beginning of this section)

$$\begin{aligned} P^s(x) &= (1+x)(1+x^3)(1+x^5)\dots(1+x^{2k+1})\dots \\ &= P(x)P(x^4)/P(x^2)^2 \end{aligned}$$

using (9.2). (iii) follows from (i) and (ii). Clearly $P(x)^e$ is the generating function for the number of e-tuples of partitions so that (iv) is trivial.

(vi): Let $(\lambda_1, \lambda_2, \dots, \lambda_e)$ be a self-conjugate e-quotient of weight n. If $e = 2f$ then $\lambda_1^0 = \lambda_e$, $\lambda_2^0 = \lambda_{e-1}, \dots, \lambda_f^0 = \lambda_{f+1}$. Thus the f-tuple $(\lambda_1, \dots, \lambda_f)$ determines the e-quotient completely. Since $|\lambda_1| + \dots + |\lambda_f| = |\lambda_{f+1}| + \dots + |\lambda_e| = n/2$, our claim follows. If $e = 2f+1$, λ_{f+1} has to be symmetric. Otherwise the argument is analogous.

Proposition (9.14). We have

(i) $F_e^n(x) + F_e^s(x) = F_e(x)$

(ii) $F_e(x) = P(x)/P(x^e)^e$

(iii) $F_e^s(x) = P^s(x)/P_e^s(x^e)$.

Proof. (i) is trivial. (ii) is a consequence of (3.7), (3.8) and (3.11): The number of partitions of n with a given e-core κ and of e-weight w is $k(e,w)$. The number of possibilities for κ is $c(e, n-pw)$. Thus

$$P(n) = \sum_{w \geq 0} k(e,w)c(e,n-pw)$$

which transforms into $P(x) = P(x^e)^e F_e(x)$ for the generating functions. This is (ii), and (iii) is proved in an analogous way.

Using the results of section 6 we deduce:

Proposition (9.15). The generating function for the number of irreducible characters in A_n is

$$\tilde{P}(x) = \frac{1}{2}P(x)\left(1 + 3P(x^4)/P(x^2)^2\right).$$

(Indeed, $\tilde{P}(x) = 2P(x) + \frac{1}{2}P^n(x)$.)

Proposition (9.16). Let p be an odd prime, $t = (p-1)/2$. The generating function for the number of irreducible characters in A_n of p-defect 0 is

$$\tilde{F}_p(x) = \frac{1}{2}\frac{P(x)P(x^4)}{P(x^2)^2}\left(1 + \frac{3}{P(x^{2p})^{t-2}P(x^p)P(x^{4p})}\right).$$

(Indeed $\tilde{F}_p(x) = 2F_p^s(x) + \frac{1}{2}F_p^n(x)$.)

Remark (9.17). We have of course that for any prime p $F_p(x)$ is the generating function for the number of blocks of defect 0 in S_n, i.e.

$$F_p(x) = P(x)/P(x^p)^p.$$

This is the generating function mentioned in (3.16).

III. Characters in Blocks

In this chapter we study the distribution of irreducible characters into p-blocks for some of the classes of groups described in chapter II. Basically the "cores" of the combinatorial objects determine the block containing the corresponding character whereas the "quotients" yield information about the height of the corresponding character in the block containing it. We will describe how to compute the block invariants and how to verify several general conjectures for blocks of the groups in question. We start by giving an overview of some of the interesting conjectures on block invariants. We assume knowledge of some basic facts from block theory ([19], [41]).

10 Some conjectures involving block invariants

When G is a finite group, $\mathrm{Irr}(G)$ denotes the set of ordinary irreducible characters in G and $\mathrm{IBr}_p(G)$ the set of irreducible Brauer characters of G in characteristic p, $p > 0$. When p is given we often write $\mathrm{IBr}(G)$ for $\mathrm{IBr}_p(G)$.

For $\chi \in \mathrm{Irr}(G)$, p a prime, $h_p(\chi) = \nu_p(\chi(1))$ is the *p-height* of χ (in G) and $d_p(\chi) = \nu_p|G| - \nu_p(\chi(1)) = \nu_p|G| - h_p(\chi)$ is the *p-defect* of χ (in G). These numbers were computed in many cases in chapter II.

It is a basic fact that given p, the sets $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$ are distributed into disjoint sets called the p-blocks of G (see [19], [41], ...). Let $\mathrm{Bl}_p(G)$ be the set of p-blocks of G .

For $B \in \mathrm{Bl}_p(G)$ we let $\mathrm{Irr}(B)$ and $\mathrm{IBr}_p(B)$ denote the set of ordinary (modular) irreducible characters in B and put

$$d(B) = \max\{d_p(\chi) \mid \chi \in \mathrm{Irr}(B)\},$$

the *defect of* B. If $\chi \in \mathrm{Irr}(B)$, we put

$$h_B(\chi) = d(B) - d_p(\chi),$$

the *height of* χ *in* B which is a non-negative integer. We have

$$h_p(\chi) - h_B(\chi) = \nu_p|G| - d(B).$$

The following *block invariants* will be studied for $B \in \mathrm{Bl}_p(G)$.

$k(B)$: the number of ordinary irreducible characters in B ($= |\mathrm{Irr}(B)|$)
$k_i(B)$: the number of ordinary irreducible characters of height i in B
$l(B)$: the number of modular irreducible characters in B ($= |\mathrm{IBr}(B)|$)

We also need

$k(G)$: the number of irreducible characters of G ($= |\mathrm{Irr}(G)|$)
$l(G)$: the number of modular irreducible characters in G ($= |\mathrm{IBr}(G)|$)

We have the trivial relations

$$k(G) = \sum_{B\in \mathrm{Bl}_p(G)} k(B); \quad l(G) = \sum_{B\in \mathrm{Bl}_p(G)} l(B)$$
$$k(B) = \sum_{i\geq 0} k_i(B) \quad \text{for} \quad B \in \mathrm{Bl}_p(G).$$

To $B \in \mathrm{Bl}_p(G)$ there is also associated a *defect group* $\Delta(B)$, determined to G-conjugacy, such that $|\Delta(B)| = p^{d(B)}$. The defect group has many important properties, as is well-known.

If H is a subgroup of G and b is a block of H it is under certain circumstances possible to define the "induced block" b^G which is a block of G. We put

$$\mathrm{Bl}(H, B) = \{b \in \mathrm{Bl}_p(H) \mid b^G \text{ is defined and } b^G = B\}.$$

For instance, if H is the centralizer or the normalizer of a p-subgroup in G then b^G is defined for all $b \in \mathrm{Bl}_p(H)$.

In the particular case where $B \in \mathrm{Bl}_p(B)$ and $H = N_G(\Delta(B))$ there is a unique block $b \in \mathrm{Bl}(H)$ with $b^G = B$. This block b is called the *Brauer correspondent* of B.

Here are some conjectures: B is a p-block of G.

Conjecture (10.1). $k(B) \leq |\Delta(B)|$.

Conjecture (10.2). $k(B) = k_0(B) \Longleftrightarrow \Delta(B)$ is abelian.

Conjecture (10.3). $k_0(B) \leq |\Delta(B) : \Delta(B)'|$ ($\Delta(B)'$ is the commutator subgroup of $\Delta(B)$).

Conjecture (10.4). $k_0(B) = k_0(b)$ where b is the Brauer correspondent of B.

The conjectures (10.1) and (10.2) are based on observations by R. Brauer, (10.3) is due to the author and (10.4) is known as the *Alperin-McKay conjecture.*

To state more conjectures we need some additional notation. It is wellknown that if $B \in \mathrm{Bl}_p(G)$, then $d(B) = 0$ if and only if $k(B) = 1$. The set of p-blocks of defect 0 in G is denoted $\mathrm{Bl}_p^0(G)$. Note that the irreducible characters $\chi \in \mathrm{Irr}(G)$ with $d_p(\chi) = 0$ (i.e. of defect 0) are exactly the characters in the p-blocks of defect 0. A *(p)-modular weight* in G is a pair (P, β) where P is a p-subgroup of G and $\beta \in \mathrm{Bl}_p^0(N_G(P)/P)$. Each modular weight in G is associated to a unique block B of G. Indeed if (P, β) is as above, then the unique character ψ of β may be considered as a character of $H = N_G(P)$ and as such it is contained in a block b of H. We let $B = b^G$ and call (P, β) a *modular B-weight* or simply a *B-weight. Alperin's weight conjecture* [1] is the following:

Conjecture (10.5). Let $B \in \mathrm{Bl}_p(G)$. Then $l(B)$ equals the number of G-conjugacy classes of B-weights.

If Q is a normal p-subgroup of G, then $Q \leq \Delta(B)$ for all $B \in \mathrm{Bl}_p(G)$. In particular, if (P, β) is a modular weight, then $O_p(N_G(P)/P) = 1$, i.e. $P = O_p(N_G(P))$. A p-subgroup P of G satisfying $P = O_p(N_G(P))$ is called a *radical subgroup* of G.

Conjecture (10.5) has inspired a lot of subsequent work in form of reformulations and generalizations. We proceed to give a short survey of some of these. In [28] chains of p-subgroups were introduced into the subject. Consider a chain C of subgroups

$$C : Q_0 = \{1\} \subset \ldots \subset Q_n,$$

where $Q_i \neq Q_{i+1}$ for all i. Write

$C \in \mathcal{P}$ if all Q_i's are p-subgroups in G
$C \in \mathcal{E}$ if all Q_i's are elementary abelian p-subgroups in G
$C \in \mathcal{N}$ if all Q_i's are p-subgroups in G and $Q_i \lhd Q_n$ for all i
$C \in \mathcal{U}$ if all Q_i's are radical p-subgroups in G

The normalizer $N_G(C)$ is defined as

$$N_G(C) = \bigcap_{i=0}^{n} N_G(Q_i).$$

If C is as above we let C_i be the chain $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_i$ for $0 \leq i \leq n$. We write $C \in \mathcal{R}$ if $Q_i = O_p(N_G(C_i))$ for $0 \leq i \leq n$. Moreover, if C is as above we let $|C| = n$.

We have defined five classes of chains of subgroups in G. Obviously G acts on these classes by conjugation. If f is a function from the set of subgroups of G into $\mathbb{Z}$, which is invariant under G-conjugation, and if $\mathcal{A}$ is any of the above classes, put

$$S(f, \mathcal{A}) = \sum_{C \in \mathcal{A}/G} (-1)^{|C|} f(N_G(C)).$$

It has been shown (see [28], [14]): Let $\mathcal{A}, \mathcal{A}'$ be any of the classes above and f as above. Then $S(f, \mathcal{A}) = S(f, \mathcal{A}')$ except possibly when $O_p(G) \neq 1$ and $\mathcal{A}$ or $\mathcal{A}'$ equals $\mathcal{R}$. If $O_p(G) \neq 1$ then $S(f, \mathcal{R}) = 0$, since $\mathcal{R} = \emptyset$.

If $C \in \mathcal{A}$, where $\mathcal{A}$ is any of the above classes and if $b \in \mathrm{Bl}(N_G(C))$, then it can be shown that b^G is defined. For $B \in \mathrm{Bl}_p(G)$ put

$$l_B(N_G(C)) = \sum_{b \in \mathrm{Bl}(N_G(C), B)} l(b)$$

and for $d \geq 0$

$$k_B^d(N_G(C)) = \sum_{b \in \mathrm{Bl}(N_G(C), B)} k^d(b)$$

where $k^d(b) = |\{\psi \in \mathrm{Irr}(b) \mid d_p(\psi) = d\}|$.

In [28] it was shown that (10.5) is equivalent to

Conjecture (10.6). Let $B \in \mathrm{Bl}_p(G)$ and $d(B) > 0$. Then $S(l_B, \mathcal{N}) = 0$

and in [14] it was conjectured that

Conjecture (10.7). Let $B \in \mathrm{Bl}_p(G)$, $d(B) > 0$, $O_p(G) = 1$, $d \geq 0$. Then $S(k_B^d, \mathcal{R}) = 0$.

As may be seen from the above remarks we may in both conjectures choose to replace $\mathcal{N}$ and $\mathcal{R}$ by any $\mathcal{A}$ as before and get equivalent conjectures.

Several of the above conjectures may be seen as an attempt to determine the block invariants of a block $B \in \mathrm{Bl}_p(G)$ "locally", i.e. in terms of invariants of proper subgroups. Since this is the case one may hope to prove the conjectures by induction, e.g. by reducing them to (almost) simple groups and then use the classification of finite simple groups. However this seems to be extremely difficult. E. Dade [14] announces (10.7) to be the first in a series of increasingly delicate conjectures of which the last one will have the property that it holds for all finite groups if it holds for the simple groups. This final conjecture implies the previous ones in the series. Its proof will imply the validity of (10.4) and (10.5). In [55] and [54] other generalizations of (10.5) are presented. The paper [55] exhibits an "equivariant" form of Alperin's conjecture predicting compatibility with the action of groups of automorphisms.

It has been proved in [29] that the conjectures (10.2) and (10.4) imply (10.3). Moreover for *abelian* defect groups it was shown in [28] that any two of (10.4) and (10.5) and the $\Leftarrow$-part of (10.2) imply the third.

Attempts in another direction to understand some of the phanomenae in block theory (for instance the "reduction theorems" for blocks of symmetric and other groups as well as some local-global observations) have been made by M. Broué. Some of his ideas are exhibited in the survey article [11]. The main idea is to formulate various concepts of "equivalence" of blocks, where of course the blocks are considered as algebras. The block invariants described above as well as the decomposition and Cartan matrices are expressed using the block algebra. Broué describes the wellknown Morita equivalence as well as "Rickard equivalence" and "stable equivalence". Morita equivalence preserves all the relevant block invariants. The weaker Rickard equivalence preserves the $k_i(B)$'s, $l(B)$ and the defect as well as invariant factors of the decomposition and Cartan matrices. Rickard equivalence seems to be very difficult to verify or check in concrete cases. It is conjectured that the principal blocks of G and $N_G(P)$ are Rickard equivalent if P is an abelian p-Sylow subgroup in G. A weaker form of Rickard equivalence on the level of characters is "perfect isometric". A lot of interesting perfect isometries between blocks have been constructed. The stable equivalence is weaker than Rickard equivalence and preserves differences between some of the block invariants.

11 Blocks of S_n

In section 3 we studied cores and quotients of partitions and in section 6 the degrees of irreducible characters in S_n and A_n. This will be relevant for the study of p-blocks of characters in these groups.

As mentioned in section 6 the irreducible characters $[\lambda]$ of S_n are labelled canonically by the partitions $\lambda \in \mathcal{P}(n)$. Let p be any prime. Then a famous result (still referred to as the "Nakayama conjecture") states:

Theorem (11.1). Let $\lambda, \mu \in \mathcal{P}(n)$. Then $[\lambda]$ and $[\mu]$ are in the same p-block if and only if $\lambda_{(p)} = \mu_{(p)}$, i.e. they have the same p-core.

Many proofs have been given for this result, see [25]. In section 3 we denoted the set of partitions of n with p-weight w and p-core κ by $\mathcal{B}(w,\kappa)$. If $\lambda \in \mathcal{B}(w,\kappa)$ is a partition of n then w p-hooks have to be removed to go from λ to $\kappa = \lambda_{(p)}$, so that $\kappa \in \mathcal{P}(n-wp)$. Then the characters of S_n in the same p-block as $[\lambda]$ are exactly the characters $[\mu]$, $\mu \in \mathcal{B}(w,\kappa)$. Obviously the integer w is an invariant of the p-block B containing $[\lambda]$. This invariant is denoted $w(B)$ and is called the *weight* of the block B. (Of course this concept has nothing to do with the modular weights of general blocks described in the previous section.)

Let us assume that B is a p-block of S_n of weight $w = w(B)$, and that the irreducible characters of B are labelled by the partitions in $\mathcal{B}(w,\kappa)$ (we call κ the *core* of B). The defect $d(B)$ of B is the minimum of $d_p(\lambda)$, $\lambda \in \mathcal{B}(w,\kappa)$ (see section 10). Using the p-adic decomposition of w the formulae (1) (applied to $n = wp$) and (4) of section 6 shows that

Lemma (11.2). If $w(B) = w$, then $d(B) = \nu_p((wp)!) = w + \nu_p(w!)$.

In fact we have for the defect group:

Proposition (11.3). If $w(B) = w$, then $\Delta(B)$ is conjugate in S_n to a p-Sylow subgroup of $S_{pw} \subseteq S_n$.

There is a very nice proof of (11.3), due essentially to R. Brauer. Variations of this proof are applicable in other connections and we include it here.

Proof of (11.3). If $w(B) = 0$, then $d(B) = 0$ and then of course $\Delta(B) = 1$. Assume $w(B) = w > 0$. Let $D = \Delta(B)$ a defect group of B. Choose an element $\pi \in D$ subject to the following conditions:

(i) π is a central element of order p in D

(ii) If π' is another central element of order p in D then it has at least as many fixpoints as π

We assume that π is a product of w_0 cycles of length p (and thus has $n - pw_0$ fixpoints). Thus $C_{S_n}(\pi) = C^0(\pi) \times C^1(\pi)$ where $C^0(\pi)$ is a wreath product $C^0(\pi) \cong Z_p \mathrm{wr}\, S_{w_0}$ and $C^1(\pi) \cong S_{n-pw_0}$. Let $b \in \mathrm{Bl}\,(C_{S_n}(\pi), B)$ and write $b = b^0 \times b^1$ where $b^i \in \mathrm{Bl}\,(C^i(\pi))$, $i = 0, 1$. The group $C^0(\pi)$ has a selfcentralizing normal p-subgroup, generated by the w_0 p-cycles occurring in π (i.e. the base subgroup of the wreath product). Therefore $C^0(\pi)$ has only block, the principal block, e.g. by [19], V.3.11., i.e. b^0 is the principal block of $C^0(\pi)$. Let D_0 be a p-Sylow subgroup of $C^0(\pi)$. Since $\nu_p(|C^0(\pi)|) = \nu_p(|S_{pw_0}|)$ (e.g. by (1) of section 6), D_0 is a p-Sylow subgroup of S_{pw_0}. Since $DC_{S_n}(D) \subseteq C_{S_n}(\pi)$ trivially, we get by the transitivity of block induction that D is conjugate to $\Delta(b)$. From (11.2) we get $w_0 \leq w$ since $|D_0| \leq |D|$. Moreover, b^1 has defect 0. Otherwise choose $\pi_1 \in Z(\Delta(b'))$ of order p. Then $\pi\pi_1$ is central of order p in $\Delta(b)$ and has fewer fixpoints than π, contrary to assumption. Thus $\Delta(b) = D_0$ and we get $|D_0| = |D|$. This forces $w = w_0$ and (11.3) follows.

Since the ordinary irreducible characters in a p-block B with weight w and core κ are labelled by the partitions in $\mathcal{B}(w,\kappa)$ we get $k(B) = |\mathcal{B}(w,\kappa)|$. Using (3.10) we get

Proposition (11.4). Let B be a p-block of weight w. Then $k(B) = k(p,w)$ where $k(p,w)$ is given by (3.11). In particular, $k(B)$ is independent of the core of B.

We proceed to compute the height of a character of S_n in the block containing it. To do so we need again the p-core tower of a partition λ introduced in section 6. As was the case there $\beta_i(p,\lambda)$ is the sum of the cardinalities of the partitions in the i-th row of the p-core tower of λ. From the definitions we get:

Proposition (11.5). If $\lambda \in \mathcal{P}(n)$, $[\lambda] \in B$, $w(B) = w$, then

$$w = \sum_{i\geq 1} \beta_i(p,\lambda)p^{i-1}$$

and if $\sum_{i\geq 0} a_i p^i = w$ is the p-adic decomposition of w then

$$h_B([\lambda]) = \left(\sum_{i\geq 1} \beta_i(p,\lambda) - \sum_{j\geq 0} a_j\right) /(p-1).$$

In particular, since the first summation is only on $i > 0$, $h_B([\lambda])$ does not depend on the p-core of λ (i.e. the core of B).

Proof. Since $\beta_0(p,\lambda)$ is the cardinality of the p-core of λ and

$$n = \sum_{i\geq 0} \beta_i(p,\lambda)p^i$$

by (6.3) we get the formula for w from the equation $n = \beta_0(p,\lambda) + wp$. By definition $h_B([\lambda]) = d(B) - d_p(\lambda)$. By (1) of section 6 applied to $n = wp$

$$(1) \qquad d(B) = \nu_p((wp)! = \left(wp - \sum_{i\geq 0} a_i\right) /(p-1)$$

Moreover by (4) of section 6

$$(2) \qquad d_p(\lambda) = \left(n - \sum_{i\geq 0} \beta_i(p,\lambda)\right) /(p-1) = \left(wp - \sum_{i\geq 1} \beta_i(p,\lambda)\right) /(p-1).$$

Subtracting (2) from (1) yields the formula for $h_B([\lambda])$.

Remark (11.6) (On Reduction theorems, equivalences etc.) As we have seen, if B is a block of weight w and core κ then the ordinary irreducible characters in B are labelled by the partitions in $\mathcal{B}(w,\kappa)$. Therefore, the map Θ^w_κ described in (3.9) induces a map between the set of irreducible characters in B and the corresponding set of the *principal* block B_0 of S_{pw}. (Clearly, this principal block has empty core, since it contains the trivial character $[pw]$.) Moreover, this map is *height-preserving* by (11.5). The blocks B and B_0 have more in common than the numbers $k_i(B) = k_i(B_0)$. As we shall see in (11.14) also $l(B) = l(B_0)$. Moreover the Cartan matrices of B and B_0 have the same elementary divisors. This is a special case of the fact that B and B_0 have the same lower defect group multiplicities (subpair multiplicities) ([47], [12]). The question arises whether there is some "equivalence" between B and B_0. Enguehard [16] has shown that the blocks are "perfectly isometric" (see also section 10). As consequence there is also an equivalence of the decomposition matrices. On the level of characters the perfect isometry is a "bijection with signs". The bijection is indeed the one described above and the signs are as follows: If $\Theta^w_\kappa(\lambda) = \lambda_0$ then $[\lambda]$ is mapped onto $\delta_p(\lambda)\delta_p(\lambda_0)[\lambda_0]$ where as in section 3 δ_p is the (relative) p- sign map. It is still an open question whether B and B_0 are "Rickard equivalent" (see section 10 or [12]). They are in general *not* Morita-equivalent, since their decomposition matrices may be different. However, there is a very interesting result of J. Scopes [59] stating that there is only a finite number of blocks of symmetric groups of a given weight w up to Morita equivalence. In fact, the number of such Morita equivalence classes of blocks of weight w is at most

$$\prod_{i=1}^{p} [(i-1)(w-1)+1] .$$

On the local level the situation is different. It was shown in [34] that the Brauer correspondents b and b_0 of B and B_0 are indeed Morita-equivalent (see (11.12)).

Using Proposition (11.5) we may derive formulas for $k_i(B)$. If $w(B) = w$ we may write $k_i(B) = k_i(p,w)$ since the k_i's are independent of the cores of B. The formula for the $k_i(p,w)$ has a striking similarity to the McKay number formula of (6.6) and we use the notation introduced there.

Proposition (11.7). Notation as above. Let $i, w \geq 0$. Then

$$k_i(p,w) = \sum_{\alpha} c_p(p,\alpha_0)c_p(p^2,\alpha_1)\ldots$$

where the summation is over all $\alpha \in E_i(p,w)$.

Proof. Analogous to that of (6.6).

Since $E_0(p,w)$ has only one element we get

Corollary (11.8). Let $w \geq 0$ have p-adic decomposition $w = \sum_{i\geq 0} a_i p^i$. Then

$$\begin{aligned} k_0(p,w) &= \prod_{i\geq 0} c_p\left(p^{i+1}, a_i\right) \\ &= k(p,a_0)k(p^2,a_1)k(p^3,a_2)\ldots . \end{aligned}$$

We also have

Corollary (11.9). Let $w(B) = w$ have p-adic decomposition $w = \sum_{i\geq 0} a_i p^i$. Then the *maximal* possible height of character in B is

$$h = (w - a_0 - a_1 - \ldots)/(p-1).$$

Moreover, $k_h(p,w) = c_p(p,w)$.

Proof. Trivially $E_a(p,w) = p$ if $a > h$, h as above, and $E_h(p,w) = \{(w,0,0,\ldots)\}$.

Note. There are no characters of height $h = 7$ in a 2-block of weight 8, since $c_2(2,8) = 0$. In fact, $c_2(2,w) = 0$ for infinitely many values of w. Presumably $c_p(p,w) > 0$ for $p \geq 3$, $w \geq 0$. This is a question about p-cores and related to (and easier than) (3.15) and (3.16).

Note. The formulas of (11.7) and (11.3) may easily be used to prove the following. Let B be a block of S_n of weight w. Then the following statements are equivalent:

(i) $\Delta(B)$ non abelian

(ii) $w \geq p$

(iii) $k_1(B) \neq 0$.

In particular Conjecture (10.2) is verified for all blocks of symmetric groups.

If B is a block of weight w, then by (11.2) $d(B) = w + \nu_p(w!)$. Since $\Delta(B)$ is a p-Sylow group of S_{pw} it is a direct product of wreath products of cyclic groups or order p. Indeed if

$$X_i = (\mathbb{Z}_p \text{wr}\, \mathbb{Z}_p)\text{wr}\, \mathbb{Z}_p \ldots \quad (i \text{ times})$$

and $w = \sum_{i\geq 0} a_i p^i$ p-adically, then $\Delta(B) \cong \Pi_{i\geq 0} X_{i+1}^{a_i}$, whence

$$\nu_p|\Delta(B) : \Delta(B)'| = \sum_{i\geq 0} a_i(i+1).$$

Therefore the proofs of the Conjectures (10.1) and (10.3) for blocks of S_n are equivalent to

Proposition (11.10). Let $w \geq 0$ be written $w = \sum a_i p^i$ p-adically. Then

(i) $k(p,w) \leq p^{w+\nu_p(w!)}$

(ii) $k_0(p,w) \leq p^{\Sigma a_i(i+1)}$

Proof. This is an easy consequence of (11.4), (11.8) and the following:

Lemma (11.11). If $s \geq 3$ or $s = 2$ and $t \neq 2,3,4,5,6$, then $k(s,t) \leq s^t$.

Proof. Generally, as is easily seen

$$p(n) \leq p(n-1) + p(n-2) \quad \text{for} \quad n \geq 2.$$

Therefore it is easy to prove by induction that $p(n) \leq \rho^n$ where $\rho = \frac{1+\sqrt{5}}{2}$ satisfies $\rho^2 = \rho+1$. Consider the statement for $s \in \mathbb{N}$

$$T(s) : k(s,t) \leq s^t \quad \text{for all} \quad t \geq 0.$$

If $T(s_1)$ and $T(s_2)$ are true then

$$\begin{aligned} k(s_1 + s_2, t) &= \sum_{i=0}^{t} k(s_1, i) k(s_2, t-i) \\ &\leq \sum_{i=0}^{t} s_1^i s_2^{t-i} \leq (s_1 + s_2)^t \end{aligned}$$

so that $T(s_1 + s_2)$ holds. If we prove $T(3)$, $T(4)$ and $T(5)$ then it follows that $T(s)$ is true for $s \geq 3$. Now for $s \geq 1$

$$k(s,t) = \sum_{(t_1, t_2, \ldots, t_s)} p(t_1)p(t_2)\ldots p(t_s)$$

where the summation is over s-tuples of non-negative integers adding up to t. Thus, as is wellknown, the number of summands is

$$\binom{s+t-1}{t} = \binom{s+t-1}{s-1},$$

a binomial coefficient. Moreover each summand is less than ρ^t where ρ is as above. Thus for $s \geq 1$, $t \geq 0$

$$k(s,t) \leq \binom{s+t-1}{s-1} \rho^t.$$

Consider the case $2 \leq s \leq 5$. Since $\frac{s}{\rho} > 1$ we have that $\binom{s+t-1}{s-1} \leq (\frac{s}{\rho})^t$ for t sufficiently large. In these cases $k(s,t) \leq (\frac{s}{\rho})^t \rho^t = s^t$ as desired. For the remaining few values of t the result is checked directly.

Let as above $X_i = (\mathbb{Z}_p \mathrm{wr}\, \mathbb{Z}_p)\mathrm{wr} \ldots \mathrm{wr}\, \mathbb{Z}_p$ (i times), so that X_i may be considered as a p-Sylow subgroup of S_{p^i}. Moreover, if $w = \sum_{i \geq 0} a_i p^i$ p-adically, then for a block B of weight w in S_n

$$\Delta(B) = \Pi_{i \geq 0} X_i^{a_i}.$$

Considering $\Delta(B)$ as a p-Sylow subgroup of S_{pw} contained in the Young subgroup $S_{pw} \times S_{n-pw}$ of S_n we get

$$N_{S_n}(\Delta(B)) = N_{S_{pw}}(\Delta(B)) \times S_{n-pw}$$

If b is the Brauer correspondent of B in this normalizer, then arguing as in the proof of (11.3) we get that $b = b^0 \times b^1$, where b^0 is the principal block of $N_{S_{pw}}(\Delta(B))$ and b^1 has defect 0. Therefore the blocks b and b^0 have the same invariants $k(b) = k(b^0)$, $k_i(b) = k_i(b^0)$, $l(b) = l(b^0)$. We have shown:

Lemma (11.12). Let B be a block of weight $w(B) = w$. Then the invariants $k(b)$, $k_i(b)$, $l(b)$ of its Brauer correspondent depend only on w and not on the core of B. (It can be seen that b and b_0 are Morita-equivalent blocks.)

We are in the position to describe why the Alperin-McKay conjecture (10.4) is true for S_n:

Proposition (11.13). Let b be the Brauer correspondent of the block B of weight w in S_n. Then $k_0(B) = k_0(b)$.

Proof. By (11.5) and (11.12) we may assume that B is the principal block of S_{pw}. If $w = \sum_{i\geq 0} a_i p^i$ p-adically, then $\Delta(B)$ is a p-Sylow subgroup in S_{pw} and it is wellknown that

$$N_{S_{pw}}(\Delta(B)) \simeq \Pi_{i\geq 0} N_i \mathrm{wr}\, S_{a_i}$$

where N_i is the normalizer of a p-Sylow subgroup in S_{p^i}. Since $N_{S_{pw}}(\Delta(B))$ has only one block we get

$$k_0(b) = m_0\left(p, N_{S_{pw}}(\Delta(B))\right),$$

(the 0-th McKay number). Now (11.13) follows from (11.8) and the representation theory of wreath products (see e.g. [25], chapter 4).

We now explain why $l(B)$, B a block of S_n, only depends on $w(B)$ and not on the core.

Proposition (11.14). Let B be a block of weight w in S_n. Then $l(B) = k(p-1, w)$.

Proof (Sketch). We use Brauer's formula

$$k(B) = \sum_{\pi} \sum_{b \in \mathrm{Bl}(C(\pi), B)} l(b)$$

where π runs through a set of representatives for the conjugacy classes of p-elements in S_n. We define the weight $w(\pi)$ of π to be w' if π has exactly $n-pw'$ fixpoints. If $\mathrm{Bl}\,(C(\pi), B) \neq \emptyset$ then obviously $w' \leq w$ since π has to be conjugate to an element of $\Delta(B)$. Now if $w(\pi) = w'$ write $C(\pi) = C_{S_{pw'}}(\pi) \times S_{n-pw'}$. Then $b_\pi \in \mathrm{Bl}\,(C(\pi), B)$ may be written $b_\pi = b^0 \times b^1$ where b^0 is the principal block of $C_{S_{pw'}}(\pi)$ and b^1 is a block of S_{n-pw} of weight $w - w'$. Moreover, b^0 is the unique block in $C_{S_{pw'}}(\pi)$. Therefore, it is easily seen that for a given π the set $\mathrm{Bl}\,(C(\pi), B)$ contains a unique block if $w(\pi) \leq w$ and is empty otherwise. The statement of (11.14) is proved by induction on w. For $w = 0$ the result is trivially true. Suppose it has been proved for blocks of weight $0, 1, \ldots, w-1$. We get from the above and the induction hypothesis that

$$k(B) - l(B) = \sum_{w'=1}^{w} \sum_{\{\pi | w(\pi) = w'\}} l\,(C_{pw'}(\pi))\, k(p-1, w-w')$$

where the π's again are conjugacy class representatives. Since $k(B)$ and the right hand side of this equation are independent of the core of B, so is $l(B)$. An easy combinatorial argument shows that

$$\sum_{\{\pi | w(\pi) = w'\}} l\,(C_{pw'}(\pi)) = p(w')$$

for any w' (see e.g. [47], section 2). Thus

$$k(B) - l(B) = \sum_{w'=1}^{w} p(w')k(p-1, w-w')$$

and then (11.4) shows that $l(B) = k(p-1, w)$, since

$$k(p, w) = \sum_{w'=0}^{w} p(w')k(p-1, w-w').$$

Note. Alperin and Fong [2] have verified Alperin's weight conjecture (10.5) for blocks of S_n using of course the formula of (11.14). Very recently, Olsson and Uno [52] verified Dade's conjecture (10.7) for blocks of S_n using the formula of (11.7). We have to refer to the papers quoted for further details.

Remark (11.15) (On the Mullineux conjecture). As mentioned earlier, multiplying the ordinary irreducible character $[\lambda]$ of S_n by the sign character yields the irreducible character $[\lambda^0]$ where λ^0 is the partition conjugate to λ. This is fairly easy to show.

Let p be a prime number and consider the modular representations of S_n in characteristic p.

The modular irreducible characters $[[\lambda]]$ of S_n are labelled by *p-regular* partitons λ of n, a partition being p- regular if no part is repeated p (or more) times (see [25], chapter 6).

Multiplying the modular character $[[\lambda]]$ of S_n by the sign character of S_n gives another modular irreducible character, labelled by a p-regular partition λ^P. We are then faced with the following problem.

Describe the involutory map $\lambda \longrightarrow \lambda^P$ on the set of p-regular partitions.

A conjectured answer to this problem is the Mullineux map. We describe this map $\lambda \longrightarrow \lambda^M$ on the set $\mathcal{P}_p(n)$ of p-regular partitions of n.

Let $\lambda \in \mathcal{P}_p(n)$. The *p-rim* of λ is a part of the rim of λ (i.e. the (1,1)-rim of λ, see section 1) which is composed of *p-segments.* Each p-segment except possibly the last contains p points. The first p-segment consists of the first p points of the rim of λ, starting with the longest row. (If the rim contains at most p points it is the entire rim.) The next segment is obtained by starting in the row next below the previous p-segment. This process is continued until the final row is reached. We let a_1 be the number of nodes in the p-rim of $\lambda = \lambda^{(1)}$ and let r_1 be the number of rows in λ. Removing the p-rim of $\lambda = \lambda^{(1)}$ we get a new p-regular partition $\lambda^{(2)} \in \mathcal{P}_p(n - a_1)$. We let a_2, r_2 be the length of the p-rim and the number of parts of $\lambda^{(2)}$ respectively. Continuing this way we get a sequence of partitions $\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ where $\lambda^{(k)} \neq 0$ and $\lambda^{(k+1)} = 0$, and a corresponding *Mullineux symbol* of λ

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ r_1 & r_2 & \ldots & r_k \end{pmatrix}.$$

It is easy to recover λ from its Mullineux symbol $G_p(\lambda)$. (Start with $\lambda^{(k)}$, given by a_k, r_k and work backwards. In placing each p-rim it is convenient to start from below.) Moreover, it can be shown that the entries of $G_p(\lambda)$ satisfy

$$\begin{cases} \text{(i)} & \epsilon_i \le r_i - r_{i+1} < p + \epsilon_i,\ 1 \le i \le k-1;\ 1 \le r_k < p \\ \text{(ii)} & r_i - r_{i+1} + \epsilon_{i+1} \le a_i - a_{i+1} < p + r_i - r_{i+1} + \epsilon_{i+1};\ 1 \le i \le k-1;\ r_k \le a_k < p + r_k \\ \text{(iii)} & \sum_i a_i = n, \end{cases}$$

where $\epsilon_i = 1$ if $p \nmid a_i$, $\epsilon_i = 0$ if $p \mid a_i$.

If $G_p(\lambda)$ is as above then $\lambda^M \in \mathcal{P}_p(n)$ is by definition the partition satisfying

$$G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \dots & a_k \\ s_1 & s_2 & \dots & s_k \end{pmatrix} \quad \text{where} \quad s_i = a_i - r_i + \epsilon_i.$$

Example. $p = 5$, $\lambda = (8, 6, 5^2)$

$$\begin{array}{cccccccc} 4 & 4 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 1 & 1 & & \\ 3 & 3 & 2 & 2 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & & \end{array} \qquad G_5(\lambda) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 4 & 4 & 3 & 2 \end{pmatrix}$$

$$\begin{array}{cccccccccc} 4 & 4 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & & \\ 2 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ 1 & & & & & & & & & \end{array} \qquad G_5(\lambda^M) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 6 & 3 & 2 & 2 \end{pmatrix}$$

(In both cases the nodes of the successive 5-rims are numbered 1,2,3,4). Thus $(8, 6, 5^2)^M = (10, 8, 2^2, 1^2)$.

The Mullineux conjecture is that the maps P and M above coincide (see [39]). This conjecture seems inaccessible at the present moment, but there is evidence that it is a very reasonable conjecture. In [40] it was shown that the conjecture is compatible with the block distribution of the modular characters. The conjecture has been verified in some "small" cases [32]. In [4] it was shown that the maps P and M have the same number of fixpoints. This demanded the proof of a rather involved partition identity. There is also compatibility between the map M and another conjecture about the restriction of modular characters [7].

12 Blocks of A_n

In section 6 (see (6.7) and the remarks following it) the following was stated:

Lemma (12.1). Let $\lambda \in \mathcal{P}(n)$.

(i) If $\lambda = \lambda^0$ then $[\lambda]|_{A_n} = \{\lambda\} + \{\lambda\}'$, where $\{\lambda\}$ and $\{\lambda\}'$ are different irreducible characters of A_n (which are conjugate in S_n).

(ii) If $\lambda \neq \lambda^0$ then $[\lambda]|_{A_n} = [\lambda^0]|_{A_n} = \{\lambda\}$ is an irreducible character of A_n.

If $\lambda \in \mathcal{P}(n)$ we let $\{\lambda\}$ denote an irreducible character of A_n, such that $([\lambda]_{A_n}, \{\lambda\}) \neq 0$.

In this section we consider p-blocks of S_n and A_n for arbitrary primes p.

Proposition (12.2). Let B be a block of S_n of positive defect (weight). Then B covers a unique block (called $\tilde{B}$) of A_n.

Proof. Since B has positive defect, the weight w of B is nonzero. Suppose that B covers two blocks $\tilde{B}$ and $\tilde{\tilde{B}}$ of A_n. Then each character $[\lambda] \in B$ has to be reducible when restricted to A_n, since $[\lambda]|_{A_n}$ must have one constinutent in $\tilde{B}$ and one in $\tilde{\tilde{B}}$. By (12.1), $\lambda = \lambda^0$ for all $[\lambda] \in B$. This forces $\lambda^{(p)}$ to be self-conjugate for all $[\lambda] \in B$ (by (3.5)). However, each p-quotient of weight w occurs as a p-quotient of some λ, $[\lambda] \in B$ and $((w), 0, \ldots, 0)$ is a non-self-conjugate p-quotient of weight w. This is a contradiction.

If B is a block of S_n of positive weight w we also associate the same *weight* w to the block $\tilde{B}$ of A_n covered by B. Similarly, if μ is the core of B (i.e. $\mu = \lambda_{(p)}$ for some $[\lambda] \in B$) we also call μ the *core of* $\tilde{B}$. Then μ^0 is also a p-core and the corresponding block of S_n is denoted B^0. Thus

$$B^0 = \left\{[\lambda^0] \mid [\lambda] \in B\right\} = [1^n]B.$$

It is clear by the above that $\tilde{B} = \tilde{B}^0$ and that if B_1 and B_2 are different blocks of positive defect with $\tilde{B}_1 = \tilde{B}_2$ then $B_1 = B_2^0$. If $B = B^0$ we call B (and $\tilde{B}$) *sself-conjugate.* Otherwise the blocks are called *non-self-conjugate.* If $p = 2$ all blocks of S_n are self-conjugate (e.g., since all 2-cores are self-conjugate, see for example (3.15)).

As in section 9 $k^s(p, w)$ and $k^n(p, w)$ denote the number of self-conjugate and non-self-conjugate p-quotients of weight w.

Proposition (12.3). Let p be odd and $\tilde{B}$ a block of A_n of positive weight covered by the block B of S_n.

(i) If $\tilde{B}$ is self-conjugate, then

$$k(\tilde{B}) = \tilde{k}^s(p, w) = 2k^s(p, w) + \frac{1}{2}k^n(p, w).$$

If $[\lambda] \in B$ then $\{\lambda\} \in \tilde{B}$ and $\mathrm{ht}_B([\lambda]) = \mathrm{ht}_{\tilde{B}}(\{\lambda\})$.

(ii) If $\tilde{B}$ is non-self-conjugate, then

$$k(\tilde{B}) = k(B) = k(p, w)$$

and restriction gives a height-preserving bijection between the characters of B and the characters of $\tilde{B}$.

Proof. Follows from the above and the fact that B and $\tilde{B}$ have a common defect group [19], V.3.

It is of course possible to decompose $k(\tilde{B})$ according to the heights and our approach allows us to give explicit formulas for $k_i(\tilde{B})$. These numbers are again independent of the core of $\tilde{B}$. The principle is the same as in the previous section, so we omit the details. (The height of $[\lambda]$ and thus of $\{\lambda\}$ depend on the p-core tower of λ only.) If B is non-self-conjugate then $k_i(B) = k_i(\tilde{B}) = k_i(p, w)$, where $k_i(p, w)$ is computed in the previous section. We state here only the formulas for $k_0(\tilde{B})$.

Proposition (12.4). Let p be odd and let $w > 0$ have the p-adic decomposition $w = \sum_{i=0}^{r} a_i' p^i$. Then let

$$\tilde{k}_0^s(p, w) = \frac{3}{2} k_0^s(p, w) + \frac{1}{2} k_0(p, w),$$

where

$$k_0^s(p, w) = \Pi_{i=0}^{r} k^s \left(p^{i+1}, a_i'\right)$$

and

$$k_0(p, w) = \Pi_{i=0}^{r} k \left(p^{i+1}, a_i'\right).$$

If $\tilde{B}$ has weight w, then $k_0(\tilde{B}) = \tilde{k}_0^s(p, w)$ if $\tilde{B}$ is self-conjugate and $k_0(\tilde{B}) = k_0(p, w)$ otherwise.

Next we consider the case $p = 2$.

Proposition (12.5). Let $p = 2$ and let $\tilde{B}$ be block of A_n of weight $w > 0$ covered by the block B of S_n.

(i) If $w = 2w'$ is even then

$$k(\tilde{B}) = \tilde{k}(2, w) = \frac{3}{2} p(w') + \frac{1}{2} k(2, 3).$$

Let $\{\lambda\} \in \tilde{B}$. If λ is self-conjugate then $\text{ht}_{\tilde{B}}(\{\lambda\}) = \text{ht}_B([\lambda]) - 1$. If λ is non-self-conjugate then $\text{ht}_{\tilde{B}}(\{\lambda\}) = \text{ht}_B([\lambda])$.

(ii) If w is odd then

$$k(\tilde{B}) = \tilde{k}(2, w) = \frac{1}{2} k(2, w) \doteq \frac{1}{2} k(B)$$

and restriction gives a height-preserving surjective map from the set of characters of B to the set of characters of $\tilde{B}$.

Proof. The 2-cores of partitions are all self-conjugate. Moreover, a self-conjugate 2-quotient has the form (μ, μ^0) for some partition μ and this is only possible when the weight is $2w'$, where $\mu \vdash w'$. Hence there are $p(w')$ symmetric 2-quotients of weight $w = 2w'$ and none when w is odd. For the defects of the blocks we have $d(\tilde{B}) = d(B) - 1$. From this the result follows easily.

Again explicit formulae may be given for $k_i(\tilde{B})$ for all $i \geqq 0$. For $i = 0$ we have the following result whose proof is omitted:

Proposition (12.6). Let $\tilde{B}$ be a 2-block of weight $w = \sum a_i' 2^i$ in A_n. Then

$$k_0(\tilde{B}) = \tilde{k}_0(2,w) = \left\{ \begin{array}{ll} k_0(2,w) = 2^{j+1} & \text{if } w = 2^j \text{ is a power of 2} \\ \frac{1}{2}k_0(2,w) & \text{if } w \text{ is not a power of 2.} \end{array} \right\}$$

From the above results the following may be proved fairly easily. It verifies Conjecture (10.2) for blocks of A_n.

Proposition (12.7). Let $\tilde{B}$ be a block of weight w in A_n. The following conditions are equivalent:

(i) The defect group of $\tilde{B}$ is nonabelian

(ii) $k(\tilde{B}) \neq k_0(\tilde{B})$

(iii) $k_1(\tilde{B}) \neq 0$

(iv) $\left\{ \begin{array}{ll} w \geq p & \text{if } p \text{ is odd} \\ w \geq 3 & \text{if } p = 2. \end{array} \right.$

Note. The verification of the conjectures (10.1) and (10.3) for blocks of A_n is (for p odd) somewhat similar to the verification for blocks of S_n. For $p = 2$ there are complications as may be expected from (12.6). We refer to [49] for further details. The Alperin-McKay conjecture (10.4) for A_n was verified in [35], also for $p = 2$.

We next describe the number of modular characters in blocks of A_n which was unknown until recently.

Proposition (12.8). Let p be odd and $\tilde{B}$ a block of A_n of weight $w > 0$ covered by the block B of S_n.

(i) If $\tilde{B}$ is non-self-conjugate then $l(\tilde{B}) = l(B) = k(p-1, w)$.

(ii) If $\tilde{B}$ is self-conjugate, $l(\tilde{B}) = \frac{1}{2}(k(p-1,w) + 3k^{(s)}(p-1,w))$.

Proof. See [50]. The non-self-conjugate case is obvious. In the self-conjugate case a refined version of Brauer's formula (see the proof of (11.14)) has to be applied.

For the prime $p = 2$ we have

Proposition (12.9). Let $\tilde{B}$ be a 2-block of A_n of weight $w > 0$ covered by the block B of S_n.

(i) If w is odd then $l(\tilde{B}) = l(B) = p(w)$.

(ii) If $w = 2w'$ is even, then $l(\tilde{B}) = l(B) + p(w') = p(w) + p(w')$.

Proof. In the case of w odd, restriction to A_n yields a bijection between the modular characters of B and $\tilde{B}$. The w even case the proof involves again Brauer's formula, a reduction result and a calculation with some of the generating functions of section 9. See [50] for details.

Remark (12.10) (A combinatorial question). As in section 3 let $K(r,s)$ be the set of r-quotients of s, i.e. the set of r-tuples $(\lambda_1, \dots, \lambda_r)$ of partitions such that $|\lambda_1|+\cdots+|\lambda_r| = s$. Then $k(r,s) = |K(r,s)|$. In particular, if B is a p-block of S_n of weight w, then $l(B) = |K(p-1,w)|$. If the core of B is κ then the modular characters in B are labelled by the set $\mathcal{B}_1(w,\kappa)$ of p-regular partitions of n with core κ. ($n = |\kappa| + pw$.) On $K(p-1,w)$ we have a conjugation map (say C) (see also section 6) mapping $(\lambda_1, \dots, \lambda_{p-1})$ onto $(\lambda_{p-1}^0, \dots, \lambda_1^0)$, and the Mullineux map M described in (11.15) induces a bijection

$$M : \mathcal{B}_1(w,\kappa) \to \mathcal{B}_1(w,\kappa^0).$$

The number of fixpoints for C is $k^{(s)}(p-1,w)$. Therefore, (12.8) suggests the following question: Does there exist for each p-core κ and each $w \geq 0$ and (explicit describable) map

$$m_{w,\kappa} : \kappa(p-1,w) \to \mathcal{B}_1(w,\kappa)$$

so that the following diagram of maps is commutative?

$$\begin{array}{ccc} K(p-1,w) & \stackrel{m_{w,\kappa}}{\longrightarrow} & \mathcal{B}_1(w,\kappa) \\ \downarrow C & & \downarrow M \\ K(p-1,w) & \stackrel{m_{w,\kappa^0}}{\longrightarrow} & \mathcal{B}_1(w,\kappa^0) \end{array}$$

Note. To the knowledge of the author no work has been done on the conjectures (10.5) to (10.7) in the case of alternating groups.

13 Blocks of $\hat{S}_n$ and $\hat{A}_n$

In section 4 we studied cores and quotients of bar partitions and in section 7 the degrees of spin characters in $\hat{S}_n$ and $\hat{A}_n$. This will be relevant for the study of p-blocks of spin characters in these groups for *p odd*.

As mentioned in section 7 the associate classes of spin characters in $\hat{S}_n$ are labelled canonically by the bar partitions of n.

When p is an odd prime then a p-block of $\hat{S}_n$ cannot contain ordinary and spin characters at the same time, as is easily seen (evaluate the central characters on the central element z). It was conjectured by Morris how the spin characters should distribute into p-blocks, p odd. Morris' conjecture was proved in [24], [13]:

Theorem (13.1). Let $\lambda, \mu \in \bar{\mathcal{P}}(n)$.

(i) If $\lambda = \lambda_{(\bar{p})}$, then $\langle\lambda\rangle$ has p-defect 0.

(ii) If $\lambda \neq \lambda_{(\bar{p})}$, then $\langle\lambda\rangle$ and $\langle\mu\rangle$ are in the same p-block if and only if $\lambda_{(\bar{p})} = \mu_{(\bar{p})}$, i.e. they have the same p-core.

(Another proof of this result, partly inspired by the original proof of Nakayama conjecture, is available from the author.)

In section 4 we denoted the set of bar partitions of $\bar{p}$-weight w with $\bar{p}$-core κ by $\bar{\mathcal{B}}(w,\kappa)$. If $\lambda \in \bar{\mathcal{B}}(w,\kappa)$ is a partition of n, then $\kappa \in \bar{\mathcal{P}}(n-wp)$. If $w > 0$ then the spin characters of $\hat{S}_n$ in the same block as $\langle\lambda\rangle$ are exactly the characters $\langle\mu\rangle$ (and their associates) with $\mu \in \bar{\mathcal{B}}(w,\kappa)$. Thus the integer w is an invariant of the p-block $\hat{B}$ containing $\langle\lambda\rangle$ and is called the *weight* $\bar{w}(\hat{B})$ of $\hat{B}$. If $\lambda = \lambda_{(\bar{p})}$ then $\langle\lambda\rangle$ is the unique character in a block $\hat{B}$ of defect 0 (see formula (3) of section 7) and in this case we of course put $\bar{w}(\hat{B}) = 0$. We call a block consisting of spin characters a *spin block.*

Analogous to the results (11.2) and (11.3) we have

Lemma (13.2). (p odd) Let $\hat{B}$ be a spin block and $\bar{w}(\hat{B}) = w$. Then $d(\hat{B}) = \nu_p((wp)!) = w + \nu_p(w!)$.

Proposition (13.3). (p odd) Let $\hat{B}$ be a spin block and $\bar{w}(\hat{B}) = w$. Then $\Delta(\hat{B})$ is conjugate in $\hat{S}_n$ to a p-Sylow subgroup of $\hat{S}_{pw} \subseteq \hat{S}_n$.

The proof of (13.2) is based on the formula (3) of section 7 and a variation of the proof of (11.3) yields (13.3). We omit the details.

The above may lead the reader to expect that the number $k(\hat{B})$ of characters in a spin block $\hat{B}$ depends only on $\bar{w}(\hat{B})$. However, this is not quite correct. For positive weights there are two possibilities for $k(\hat{B})$ and $k_i(\hat{B})$, depending also on a sign. If the characterss in $\hat{B}$ are labelled by the bar partitions in $\bar{\mathcal{B}}(w,\kappa)$, then κ is called the *core* of $\hat{B}$ and the sign $\sigma(\kappa)$ of κ (see section 7) is called the *sign* $\sigma(\hat{B})$ of $\hat{B}$.

Proposition (13.4) (p odd). Let $\hat{B}$ be a spin block of sign σ and weight $w > 0$. Then $k(\hat{B})$ depend only on σ and w and we have

$$k(\hat{B}) = k^{\sigma}(\bar{p},w) = q^{\sigma}(\bar{p},w) + 2q^{-\sigma}(\bar{p},w),$$

where $q^{\sigma}(\bar{p},w)$ is defined in section 7 (see also section 9).

Proof. Let κ be the core of $\hat{B}$. In section 7 the sign of a $\bar{p}$-quotient was defined such that generally $\sigma(\lambda) = \sigma(\lambda^{(\bar{p})})\sigma(\lambda_{(\bar{p})})$. Therefore the set $\bar{B}(w,\kappa)$ contains $q^{\sigma}(\bar{p},w)$ bar partitions with sign 1 and $q^{-\sigma}(\bar{p},w)$ bar partitions with sign -1. Since the bar partitions with sign -1 label *pairs* of associate spin characters, the result follows from (13.1).

If $w(\hat{B}) = w > 0$ we thus have two possibilities for $k(\hat{B})$. We discuss briefly the relation between the two values $\hat{k}^{+}(\bar{p},w)$ and $\hat{k}^{-}(\bar{p},w)$ of $k(\hat{B})$. As in sections 2, 7 and 9 $t = (p-1S)/2$. We have

Proposition (13.5) (p odd). For all $w \geq 1$

$$k^{(-1)^{w+1}}(\bar{p}, w) - k^{(-1)^w}(\bar{p}, w) = k(t-1, w).$$

Proof. Using (13.4) and the notation of section 9 we have $k^{(-1)^{w+1}}(\bar{p}, w) - k^{(-1)^w}(\bar{p}, w) = \tilde{q}^+(\bar{p}, w) - \tilde{q}^-(\bar{p}, w)$. The generating function for $\tilde{q}^\sigma(\bar{p}, w)$ is denoted $\tilde{Q}^\sigma_{\bar{p}}(x)$ in section 9 and by (9.6) (v) we have $\tilde{Q}^+_{\bar{p}}(x) - \tilde{Q}^-_{\bar{p}}(x) = P(x)^{t-1}$, proving our result.

Corollary (13.6). For all $w \geq 1$

$$k^+(\bar{3}, w) = k^-(\bar{3}, w).$$

This corollary shows that the two possibilities for $k(\hat{B})$ coincide for $p = 3$. For $p \geq 5$ this never happens. Since $k(t-1, w) > 0$ when $t - 1 \geq 1$ we have

Corollary (13.7). For $p \geq 5$ and $w \geq 0$

$$k^{(-1)^{w+1}}(\bar{p}, w) > k^{(-1)^w}(\bar{p}, w).$$

Special cases.

(i) $k^{(-1)^{w+1}}(\bar{5}, w) - k^{(-1)^w}(\bar{5}, w) = p(w)$.

(ii) $k^+(\bar{p}, 0) = 1, k^-(\bar{p}, 0) = 2$.

(iii) $k^+(\bar{p}, 1) = p, k^-(\bar{p}, 1) = 2 + (p-1)/2$ (corresponding to blocks with a cyclic defect group of order p).

(iv) $k^+(\bar{p}, 2) = 2p + \binom{t}{2}, k^-(\bar{p}, 2) = 2p - 1 + t^2$.

Next we describe the heights $h_{\hat{B}}(\langle\lambda\rangle)$ of a character $\langle\lambda\rangle \in \hat{B}$, $\hat{B}$ a spin block. This involves the $\bar{p}$-core tower of a bar partition λ defined in section 7. Then $\beta_i(\bar{p}, w)$ is defined as the sum of the cardinalities of the partitions in the i-th row of the $\bar{p}$-core tower of λ. A proof completely analogous to the proof of (11.5) (the relevant facts are all in section 7) yields:

Proposition (13.8). If $\lambda \in \bar{\mathcal{P}}(n)$, $\langle\lambda\rangle \in \hat{B}$, $\bar{w}(\hat{B}) = w$ then

$$w = \sum_{i \geq 1} \beta_i(\bar{p}, w) p^{i-1}$$

and if $\sum_{j \geq 0} a_j p^j$ is the p-adic decomposition of w then

$$h_{\hat{B}}(\langle\lambda\rangle) = \left(\sum_{i \geq 1} \beta_i(\bar{p}, \lambda) - \sum_{j \geq 0} a_j \right) / (p-1).$$

In particular, $h_{\hat{B}}(\langle\lambda\rangle)$ does not depend on the core of $\hat{B}$.

In order to give a formula for $k_a(\hat{B})$ we decompose

$$q^\sigma(\bar{p},w) = \sum_{a\geq 0} q_a^\sigma(\bar{p},w)$$

such that

$$k_a(\hat{B}) = k_a(\bar{p},w) = q_a^\sigma(\bar{p},w) + 2q_a^{-\sigma}(\bar{p},w).$$

if $\hat{B}$ has sign σ. Moreover $q_a^\sigma(\bar{p},w)$ will enumerate $\bar{p}$-core towers with special properties. As in section 6, $E_a(p,w)$ is the set of sequences $(\alpha_0,\alpha_1,\dots,\alpha_k,\dots)$ satisfying

$$\alpha_i \in \mathbb{N}_0, \quad \sum_{i\geq 0}\alpha_i p^i = w, \quad \left(\sum_{i\geq 0}\alpha_i - \sum_{j\geq 0} a_j\right)/(p-1) = a.$$

We need the integers $\bar{c}_p^\sigma(qr+1,w)$ defined after (7.3). Their generating functions may be expressed as follows:

(13.9) $$\tilde{F}_{\bar{p}}^\sigma(x)F(x)^r = \sum_{w\geq 0}\bar{c}_p^\sigma(2r+1,w)x^w$$

where $\tilde{F}_{\bar{p}}^\sigma(x)$ and $F_p(x)$ are given by (9.9) (iv) and (9.14) (2). Then if

$$q_a^\sigma(\bar{p},w) = \sum_{\substack{\alpha=(\alpha_0,\alpha_1,\dots)\\ \sigma=(\sigma=(\sigma_0,\sigma_1,\dots)}} \prod_{i\geq 0}\bar{c}_p^{\sigma_i}(p^i,\alpha_i)$$

where α runs through $E_a(p,w)$ and σ runs through all sequences of signs satisfying $\prod\sigma_i = \sigma$, we get

Proposition (13.10) (p odd). In the above notation, if $\hat{B}$ is a spin block of sign σ and weight w then for $a \geq 0$

$$k_a(\hat{B}) = k_a^\sigma(\bar{p},w) = q_a^\sigma(\bar{p},w) + 2q_a^{-\sigma}(\bar{p},w).$$

Proof. Analogous to previous proofs (see (7.4), (11.7)).

Corollary (13.11) (p odd). In the above notation

$$k_0(\hat{B}) = q_0^\sigma(\bar{p},w) + 2q_0^{-\sigma}(p,w)$$

where

$$q_0^\sigma(\bar{p},w) = \sum_{\sigma=(\sigma_0,\sigma_1,\dots)}\prod_{i\geq 0}\bar{c}_p^{\sigma_i}(p^i,a_i),$$

(σ as above).

Remark (13.12) (p odd).

(i) If $q_a(\bar{p},w) := q_a^+(\bar{p},w) + q_a^-(\bar{p},w)$ then

$$\hat{k}_a^\sigma(p,w) \neq 0 \Leftrightarrow q_a(\bar{p},w) \neq 0.$$

Moreover, if $c_{\bar{p}}(p^i, n) = c_{\bar{p}}^+(p^i, n) + c_{\bar{p}}^-(p^i, n)$ is defined by the equation

$$F_{\bar{p}}(x)F_p(x)^{(p^i-1)/2} = \sum_{n\geq 0} c_{\bar{p}}(p^i, n)x^n$$

then

$$q_a(\bar{p}, w) = \sum_{(\alpha_j)\in E_a(p,w)} \Pi_{i\geq 0} c_{\bar{p}}(p^{i+1}, \alpha_i).$$

(ii) We have $c_{\bar{p}}^\sigma(p^i, n) = q^\sigma(\bar{p}^i, n)$ for $i \geq 1$, $0 \leq n \leq p-1$, σ a sign.

This is because all (bar) partitions of cardinality $\leq p-1$ are p- (bar) cores. This fact also implies that

$$c_{\bar{p}}(p^i, n) \neq 0 \quad \text{if} \quad n \leq (p-1)(1 + (p^i - 1)/2).$$

Lemma (13.13) (p odd). If $w \geq p$ then $k^\sigma(\bar{p}, w) \neq 0$.

Proof. If $w = \sum_{i=0}^{s} a_i p^i$ is the p-adic decomposition of w (where $a_s \neq 0$) then $(a_0, a_1, \dots, a_{s-1} + p, a_s - 1, 0, \dots) \in E_1(p, w)$. Using (13.12) we see that it suffices to show $c_{\bar{p}}(p^s, a_{s-1} + p) \neq 0$, because then $q_1(\bar{p}, w) \neq 0$ forcing $\hat{k}_1^\sigma(p, w) \neq 0$. Suppose $c_p(p^s, a_{s-1} + p) = 0$. Then by (13.12) (ii) we get

$$(p-1)(1 + (p^s - 1)/2) < a_{s-1} + p \leq 2p - 1$$

(since $a_{s-1} \leq p-1$). Thus

$$\left[\frac{2p-1}{p-1}\right] = 2 > (1 + (p^s - 1)/2)$$

forcing $p^s = 3$. We get $a_{s-1} + p = 5$. But $c_{\bar{3}}^-(3, 5) \neq 0$, since (1) and (3, 1) is a pair consisting of a 3-core and a $\bar{3}$-core whose cardinalities add to 5. This is a contradiction.

Note. Using (13.13) and (13.3) we get the following. Let $\hat{B}$ be a spin block of $\hat{S}_n$ of weight w. Then the following statements are equivalent:

(i) $\Delta(\hat{B})$ is non-abelian.

(ii) $w \geq p$.

(iii) $k_1(\hat{B}) \neq 0$.

Thus Conjecture (10.2) is verified for all spin blocks of $\hat{S}_n$ and thus for all blocks of $\hat{S}_n$ when p is odd (see a note following (11.9)).

It turns out that the conjectures (10.1) and (10.3) for spin blocks are weaker than the corresponding conjectures for ordinary blocks. In fact the following holds:

Proposition (13.14) (p odd). Let $k(p,w)$ and $k_0(p,w)$ be defined as in section 11 (i.e. they are block invariants in S_n). Then if σ is a sign and $w \geq 1$ then

$$\begin{aligned} k^{\sigma}(\bar{p},w) &\leq k(p,w) \\ k_0^{\sigma}(\bar{p},w) &\leq k_0(p,w). \end{aligned}$$

We omit the proof of this which is not very difficult (see [49], section 3). Now a spin block of weight w and a block of ordinary characters in S_n of weight w have isomorphic defect groups (by (11.3) and (13.3)). Since the conjectures (10.1) and (10.3) have been verified for all blocks of S_n, (13.14) shows that they are also valid for all spin blocks.

Remark (13.15). (On possible equivalences) In Remark (11.6) a discussion was given of various equivalences between blocks of symmetric groups. Since there are also reduction results for spin blocks (including also the numbers $l(\hat{B})$ as we shall see) it seems reasonable to ask:

Let p be odd and $\hat{B}$ be a spin block of $\hat{S}_n$ of weight w and sign σ. Is $\hat{B}$ perfectly isometric (or even Rickard equivalent) to the principal spin block of $\hat{S}_{pw+1-\sigma}$ (i.e. the block containing the basic spin character $\langle pw+1-\sigma\rangle$)?

The answer to this question is still unknown, but at least the prospective ingredients in a perfect isometry (as in [16]) are available. The map $\bar{\Theta}^{w}_{\kappa\langle 1-\sigma\rangle}$ (where κ is the core of $\hat{B}$) described in (4.7) yields a bijection of characters and the sign may be obtained using the $\bar{p}$-sign of bar partitions described towards the end of section 4.

Another intriguing question is whether a result like Scopes' [59] is valid for spin blocks, i.e.: Are there only finitely many spin blocks of a given weight up to Morita equivalence?

This may be quite difficult to answer. At least analogues of the ingredients in Scopes' proof for S_n are not available (see below).

Remark (13.16). (Labels of modular spin characters?) As mentioned in (11.15) the modular irreducible characters of S_n are labelled canonically by the set of p-regular partitions of n. Using this labelling it is possible to describe generally the arrangement of rows and columns in the decomposition matrices, so that these matrices are lower unitriangular ([25], 6.3.60). This fact is an important ingredient in the proof of the main result of [59] (see (11.6) and (13.15)). Examples seem to indicate that decomposition matrices of spin blocks may be arranged in an "almost"lower unitriangular form (up to the appearance of pairs of associate characters in the diagonal). However, it is not known what a reasonable labelling of the modular spin characters should be. It is clear that new and difficult combinatorial problems involving partition identities would need to be solved to deal with the case of arbitrary (odd) p. In [6] a class of label partitions for the modular spin characters when $p=3$ was determined leading to a proof for an almost triangular shape of the decomposition matrices. The proof of the fact that the number of labels was correct involved (surprisingly) a partition identity proved by Schur [58] in 1926. The corresponding question about the number of labels for $p=5$ turned out to be equivalent to a conjecture of Andrews from 1976. This conjecture was then proved in [3]. It seems that for $p \geq 7$ the method suggested in [6] does not even produce the right number of labels! However it is possible for arbitrary p to determine the number $l(\hat{B})$ of modular spin characters in a spin block.

In [50] the principle of proof presented in (11.14) was modified to give a formula for $l(\hat{B})$ when $\hat{B}$ is a spin block. The details were much more involved then in (11.14), but the following was proved: Let $l^s(\hat{b})(l^n(\hat{B}))$ denote the number of selfassociate (the number of associate pairs of) modular irreducible characters in a spin block $\hat{B}$, so that $l(\hat{B}) = l^s(\hat{B}) + 2l^n(\hat{B})$.

Proposition (13.17) (p odd). Let $\hat{B}$ be a spin block of $\hat{S}_n$ with $w = \bar{w}(\hat{B}) > 0$. Let $t = (p-1)/2$ (as above). Let $\sigma = \sigma(\hat{B})$ be the sign of $\hat{B}$.

If $\sigma = 1$ then

$$l^s(\hat{B}) = \begin{cases} k(t,w) & \text{for } w \text{ even,} \\ 0 & \text{for } w \text{ odd,} \end{cases}$$

$$l^n(\hat{B}) = \begin{cases} 0 & \text{for } w \text{ even,} \\ k(t,w) & \text{for } w \text{ odd,} \end{cases}$$

If $\sigma = -1$ then

$$l^s(\hat{B}) = \begin{cases} 0 & \text{for } w \text{ even,} \\ k(t,w) & \text{for } w \text{ odd,} \end{cases}$$

$$l^n(\hat{B}) = \begin{cases} k(t,w) & \text{for } w \text{ even,} \\ 0 & \text{for } w \text{ odd.} \end{cases}$$

In any case $l(\hat{B}) = l^s(\hat{B}) + 2l^n(\hat{B})$.

Note. The above shows that in a spin block of $\hat{S}(n)$ either all modular characters are self-associate or all modular characters are non self-associate. (This is not the case for ordinary blocks, as we have seen in section 12.)

Note. In [35] the Alperin-McKay conjecture (10.4) and in [36] Alperin's weight conjecture (10.5) were verified for spin blocks of $\hat{S}_n$ (and $\hat{A}_n$), using (13.17). The papers involve complicated analysis both on the local and the global level and it is not possible to discuss the details here. An outline is given in [33], section 6.

Remark (13.18). There is a remarkable "duality" between the spin characters of $\hat{S}_n$ and the spin characters of $\hat{A}_n$ which is reflected naturally in the blocks for odd primes.

Let $\lambda \in \bar{\mathcal{P}}(n)$. If λ is odd there are two spin characters of $\hat{S}_n$ indexed by λ. These characters have the same restriction to $\hat{A}_n$. If λ is even then $\langle\lambda\rangle\mid_{\hat{A}_n}$ is a sum of two irreducible characters in $\hat{A}_n$ which are conjugate in $\hat{S}_n$. Thus the irreducible spin characters of $\hat{A}_n$ are indexed canonically by the bar partitions λ of n, such that λ indexes one spin character when λ is odd and two spin characters when λ is even.

Let $\hat{B}$ be a p-block of spin characters in $\hat{S}_n$. If the weight of $\hat{B}$ is nonzero then it is easy to see that there exists a $\lambda \in \bar{\mathcal{P}}(n)$, $\langle\lambda\rangle \in \hat{B}$, and λ odd. Then $\langle\lambda\rangle \mid_{\hat{A}_n}$ is irreducible so $\hat{B}$ covers only one block of $\hat{A}_n$. Thus we have:

Let $\hat{B}$ be a spin block of $\hat{S}_n$ of positive defect. Then $\hat{B}$ covers a unique block (called $\hat{B}'$) of $\hat{A}_n$. In the notation above we had the same core and sign to $\hat{B}$ and $\hat{B}'$. Note that $\hat{B}$ and $\hat{B}'$ have the same defect. As an immediate consequence of (13.4), (13.9) and (13.17) we get the following:

Proposition (13.19) (p odd). Let $\hat{B}'$ be a spin block of weight $w > 0$ and sign σ in $\hat{A}_n$ (covered by the spin block $\hat{B}$ of $\hat{S}_n$). Then for $a \geq 0$

$$\begin{aligned} k(\hat{B}') &= k^{-\sigma}(\bar{p}, w) \\ k_a(\hat{B}') &= k_a^{-\sigma}(\bar{p}, w) \\ l(\hat{B}') &= k^n(\hat{B}) + 2k^s(\hat{B}). \end{aligned}$$

The above shows that the proofs of the conjectures (10.1) – (10.5) for spin blocks of $\hat{A}_n$ are virtually identical to the proofs for spin blocks of $\hat{S}_n$ and thus (10.1) – (10.5) are also verified for $\hat{A}_n$ for p odd.

Until now we considered only the case p odd and exploited the fact that a p-block of $\hat{S}_n$ cannot contain ordinary and spin characters at the same time (so the concept "spin block" made sense). In the case $p = 2$, the characters in a 2-block B of S_n may be considered as the ordinary characters in a unique 2-block $\hat{B}$ of $\hat{S}_n$ (see [19], V, 4.5). Then $\hat{B}$ also contains some spin characters. But neither the 2-core nor the $\bar{2}$-core (which may indeed be defined) of a $\lambda \in \bar{\mathcal{P}}(n)$ tells you in which block $\langle\lambda\rangle$ is contained. In fact the spin character $\langle 7\rangle$ is not in the principal 2-block of $\hat{S}_7$. More generally, $\langle 4n+3\rangle$ is not in the principal 2-block of $\hat{S}_{4n+3}$ ([5]). A conjecture about how the spin characters should distribute into 2-blocks due to R. Knörr and the author was stated in [48] and proved recently in [8].

Given a partition $\lambda = (l_1, \ldots, l_m) \in \bar{\mathcal{P}}(n)$ we may replace each part l_i of λ by the parts $\left[\frac{l_i+1}{2}\right]$, $\left[\frac{l_i}{2}\right]$, where $[\ldots]$ denotes "integral part of". Thus $l_i = 2t+1$ is replaced by $t+1, t$ and $l_i = 2t$ replaced by t, t. The resulting partition is called the *doubling* of λ and denoted by $\mathrm{dbl}(\lambda)$. Thus

$$\mathrm{dbl}(\lambda) = \left(\left[\frac{l_1+1}{2}\right], \left[\frac{l_1}{2}\right], \left[\frac{l_2+1}{2}\right], \left[\frac{l_2}{2}\right], \ldots, \left[\frac{l_m}{2}\right]\right).$$

(For $\lambda = (13, 11, 8, 5, 2)$ we obtain $\mathrm{dbl}(\lambda) = (7, 6^2, 5, 4^2, 3, 2, 1^2)$.)

We have seen in section 3 that the 2-cores were exactly the partitions $\kappa_k = (k, k-1, \ldots, 2, 1)$, $k \geq 0$. By section 11 the 2-blocks of S_n (and thus the 2-blocks of $\hat{S}_n$) are labelled canonically by the 2-cores of partitions of n. The distribution of spin characters into 2-blocks is as follows:

Theorem (13.20). Let $\lambda \in \bar{\mathcal{P}}(n)$. Then $\langle\lambda\rangle$ is in the 2-block of $\hat{S}_n$ labelled by the 2-core of $\mathrm{dbl}(\lambda)$. This means that for $\lambda, \mu \in \bar{\mathcal{P}}(n)$, $\langle\lambda\rangle$ and $\langle\mu\rangle$ are in the same 2-block if and only if $\mathrm{dbl}(\lambda)_{(2)} = \mathrm{dbl}(\mu)_{(2)}$.

Although the statement of Theorem (13.20) thus bears resemblance to the Nakayama conjecture (11.1) and the analogous statements for spin characters in odd characteristic or for unipotent characters of certain classes of finite linear groups in the non-defining characteristic [20], [21], the methods applied to prove it were essentially different. Indeed the principles of proof applied in the other cases cannot possibly work here. They involve in an essential way iterated versions of the (analogues of the) Murnaghan-Nakayama formula. These are used to evaluate character values on special p-singular elements, whose p-part has a large "weight". However, spin characters vanish on almost all 2-singular elements except those whose 2-part is z. The only other exception is one more 2-singular class where a given non self-associate spin character does not vanish. This fact was indeed quite important in the proof in [8] of (13.20).

The first step in the proof was to use Brauer's formula (see (11.14)) to determine the number of ordinary characters in a 2-block $\hat{B}$ of $\hat{S}_n$ using that if B is the 2-block of S_n contained in $\hat{B}$, then $l(B) = l(\hat{B})$ (!). Let

$$\tilde{p}^+(n) = \begin{cases} p^+(n) & n \text{ even} \\ p^-(n) & n \text{ odd} \end{cases}$$

$$\tilde{p}^-(n) = \begin{cases} p^-(n) & n \text{ even} \\ p^+(n) & n \text{ odd} \end{cases}$$

where $p^+(n)$ ($p^-(n)$) is the number of conjugacy classes of S_n of even (odd) elements. Then

Proposition (13.21). Let B be a 2-block of S_n of weight w and let $\hat{B}$ be the 2-block of $\hat{S}_n$ containing B. Then $\hat{B}$ contains $\tilde{p}^+(w)$ self-associate spin characters and $\tilde{p}^-(w)$ pairs of non self-associate spin characters. Thus

$$k(\hat{B}) - k(B) = p(w) + \tilde{p}^-(w).$$

As a corollary to the proof of (13.21) a result relating an invariant of a given $\lambda \in \bar{\mathcal{P}}^-(n)$ to the weight of the 2-block containing may be proved. Then the proof of (13.20) is obtained by combining this surprisingly strong corollary with some branching techniques in a very delicate way. We refer to [8] for further details.

In spite of (13.20) it does not seem to be easy to compute the numbers $k_a(\hat{B})$ for 2-blocks of $\hat{S}_n$ and none of the conjectures (10.1) – (10.4) have been verified for these 2-blocks.

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