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The LLPT Notes

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The LLPT Notes.

In 1991, a project on cohomology and algebraic geometry was initiated by Dan Laksov (Stockholm), Alain Lascoux (Paris), Piotr Pragacz (Torun) and Anders Thorup (Copenhagen). Part of the project was a presentation of the necessary theory of symmetric polynomials. This set of notes is essentially the first, very preliminary, draft of this part of the project. It was developed in close cooperation between the four writers. The draft in its present form was edited by A. Thorup, and used in a course at the University of Copenhagen in 1995; it has never been approved by the three other writers.

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Anders Thorup

Permutations and symmetric functions

1. The length function.

(1.1) Setup. Fix an *alphabet* A, that is, a finite totally ordered set; the elements of A a called *letters*. Every letter a which is not the last letter has a successor, denoted a'. It is often convenient to enumerate the letters, say $A = \{a_1, \dots, a_n\}$ where n is the number of letters. Since A is assumed to be totally ordered, there is a unique enumeration such that the a_i are in increasing order. In this enumeration, the successor of a_i is $a_i' = a_{i+1}$ for $i = 1, \dots, n-1$.

Unless otherwise specified, a *permutation* will mean a permutation of the letters. The group of all permutations will be denoted $\mathfrak{S}(A)$. The identity map is the unique order preserving permutation of A, denoted 1_A or simply 1. We denote by $\omega = \omega_A$ the unique order reversing permutation of A. Obviously, ω is an involution, that is, $\omega^2 = 1$. Sometimes ω is called the *maximal permutation* of A.

The transposition that interchanges two letters a and b will be denoted $\tau_{a,b}$. Transpositions that interchange two neighbors (with respect to the given order in A) are said to be *simple*. So the simple transposition are the transpositions $\tau_a := \tau_{a,a'}$ where a is not the last letter. The number of simple transpositions is one less than the number of letters.

Assume that the letters are indexed a_i in the natural order. It is easily seen that the product $\tau_{a_1}\tau_{a_2}\cdots\tau_{a_{n-1}}$ is the n-cycle $(a_1, \dots, a_n)$. It follows, for instance by an inductive argument, that

$$\omega_A = (\tau_{a_1}\cdots\tau_{a_{n-1}})\cdots(\tau_{a_1}\tau_{a_2})\tau_{a_1}.$$

(1.2) Definition. A pair (a, b) of letters will be called an *inversion* for the permutation μ if $a < b$ and $\mu a > \mu b$. The number of inversions of μ is denoted by $\ell(\mu)$ and called the *length* of μ.

(1.3) Lemma. *For any permutation μ, the following six assertions hold:*

(1) *We have equality $\ell(\mu) = 0$ if and only if $\mu = 1$. Moreover, if $\mu \neq 1$, then there is an inversion for μ of the form (a, a').*

(2) *We have equality $\ell(\mu) = 1$ if and only if μ is a simple transposition.*

(3) *For a simple transposition τ_a, we have that*

$$\ell(\mu\tau_a) = \begin{cases} \ell(\mu) + 1 & \text{if } \mu a < \mu a', \\ \ell(\mu) - 1 & \text{if } \mu a > \mu a'. \end{cases}$$

More precisely, if $\mu a < \mu a'$, then the inversions for $\mu\tau_a$ are the pairs $(\tau_a b, \tau_a c)$ where (b, c) is an inversion for μ and the pair (a, a'), and if $\mu a > \mu a'$, then the *inversions for $\mu\tau_a$ are the pairs $(\tau_a b, \tau_a c)$ where (b, c) is an inversion for μ different from (a, a').*

(4) *We have the inequality $\ell(\mu) \le n(n-1)/2$, and equality holds if and only if $\mu = \omega$.*
(5) *We have that $\ell(\mu) = \ell(\mu^{-1})$.*
(6) *We have that $\ell(\omega\mu) + \ell(\mu) = \ell(\omega)$.*

Proof. The five first assertions are easily checked. The sixth follows from the equation $\ell(\omega) = n(n-1)/2$, because each pair (a, b) of letters with $a < b$ is an inversion for exactly one of the permutations μ and $\omega\mu$. □

(1.4) Proposition. *The group $\mathfrak{S}(A)$ is generated by the simple transpositions. In fact, any permutation μ is a product of $\ell(\mu)$ simple transpositions.*

Proof. We prove the second assertion by induction on $\ell(\mu)$. If $\ell(\mu) = 0$, then $\mu = 1$. Hence the assertion holds when $\ell(\mu) = 0$. Assume that $\ell(\mu) = l > 0$ and that the assertion holds for permutations of length $l-1$. Since $\mu \ne 1$, there is an inversion for μ of the form (a, a'). It follows from (1.3)(3) that $\ell(\mu\tau_a) = l-1$. Hence $\mu\tau_a$ is a product $l-1$ simple transpositions. Therefore $\mu = (\mu\tau_a)\tau_a$ is a product of l simple transpositions. □

(1.5) Note. The proof of Proposition (1.4) is constructive. Let μ be a permutation, and consider the *direct representation* of μ:

$$\mu = (b_1 b_2 \dots b_n). \tag{1.5.1}$$

By definition, if $A = \{a_1, \dots, a_n\}$ with the letters a_i in increasing order, then μ is determined from the sequence in (1.5.1) by $\mu a_i = b_i$. If the b_i are in increasing order, then $b_i = a_i$ and $\mu = 1$. If $\mu \ne 1$, then there is an index $j < n$ such that $b_j > b_{j+1}$. Now interchange in the sequence b_j and b_{j+1}. The new sequence represents the permutation $\mu\tau_{a_j}$. Continue the process until the b_i's appear in increasing order. The sequence obtained at the end represents the identity permutation 1. Hence we have an equation $1 = \mu\tau_{a_{j_1}} \cdots \tau_{a_{j_r}}$. Thus $\mu = \tau_{a_{j_r}} \cdots \tau_{a_{j_1}}$. It follows from the proof of Proposition (1.4) that $r = \ell(\mu)$.

(1.6) Proposition. *For any presentation $\mu = \tau_1 \cdots \tau_r$ of μ as a product of r simple transpositions, we have that $\ell(\mu) \le r$ and $\ell(\mu) \equiv r \pmod 2$.*

Proof. We prove the Proposition by induction on the number r of factors in the presentation of μ. If $r = 0$, then $\mu = 1$ and the two assertions hold. Assume that $r > 0$ and that the two assertions hold for all presentations with $r-1$ factors. Consider the presentation $\nu := \tau_1 \cdots \tau_{r-1}$. Since $\mu = \nu\tau_r$, it follows from Lemma (1.3)(3) that $\ell(\mu) = \ell(\nu) \pm 1$. In particular, we have that $\ell(\mu) \le \ell(\nu) + 1$ and $\ell(\mu) \equiv \ell(\nu) + 1 \pmod 2$. By the induction hypothesis, we have that $\ell(\nu) \le r-1$ and $\ell(\nu) \equiv r-1 \pmod 2$. Therefore, the two assertions hold for the presentation of μ. □

(1.7) Corollary. *For any two permutations μ and ν in $\mathfrak{S}(A)$, we have the inequality $\ell(\mu) + \ell(\nu) \ge \ell(\mu\nu)$.*

Proof. The Corollary follows immediately from the two previous Propositions. □

(1.8) Definition. The *signature* of a permutation, denoted sign μ, is the number

$$\operatorname{sign} \mu := (-1)^{\ell(\mu)}.$$

It follows from Proposition (1.6) that the signature is a homomorphism of groups,

$$\mathfrak{S}(A) \to \{\pm 1\}.$$

By (1.3), the signature of a simple transposition is equal to -1. It is easy to see that the signature of any transposition is equal to -1. It follows that the map sign is independent of the given order of the letters of A.

2. The Coxeter–Moore relations.

(2.1) Definition. A sequence $(\tau_1,\dots,\tau_r)$ of simple transpositions will be called a *presentation* of the permutation μ in $\mathfrak{S}(A)$ if $\mu=\tau_1\cdots\tau_r$. By Proposition (1.6), for any presentation $(\tau_1,\dots,\tau_r)$ of μ, we have that $r\geq\ell(\mu)$. The presentation is said to be *minimal* if $r=\ell(\mu)$. Minimal presentations of a given permutation μ exist by Proposition (1.4).

(2.2) Lemma. *Let $(\tau_1,\dots,\tau_r)$ be a minimal presentation of a permutation μ. For $s=1,\dots,r$, let b_s be the letter for which $\tau_s=\tau_{b_s}$. Then the inversions for μ are the pairs $\bigl(\tau_r\cdots\tau_{s+1}(b_s),\tau_r\cdots\tau_{s+1}(b_s')\bigr)$ for $s=1,\dots,r$.*

In particular (when $r\geq 1$), the pair (b_r,b_r') is an inversion for μ.

Proof. We shall prove the Lemma by induction on $r=\ell(\mu)$. Clearly, the assertion holds for $r=0$. Assume that $r\geq 1$ and that the assertion holds for all permutations of length $r-1$. It follows from Lemma (1.3)(3) that $\nu:=\mu\tau_r=\tau_1\cdots\tau_{r-1}$ has length $r-1$. By the induction hypothesis, we have that the inversions of ν are the pairs $\bigl(\tau_{r-1}\cdots\tau_{s+1}(b_s),\tau_{r-1}\cdots\tau_{s+1}(b_s')\bigr)$ for $s=1,\dots,r-1$. Since $\ell(\nu\tau_{b_r})=\ell(\mu)=\ell(\nu)+1$, it follows from Lemma (1.3)(3) that the assertion holds for μ. ∎

(2.3) Lemma (The exchange property). *Let $(\tau_1,\dots,\tau_r)$ and $(\sigma_1,\dots,\sigma_r)$, for $r\geq 1$, be two minimal presentations of the same permutation μ. Then, for some $q=1,\dots,r$, there is a presentation of μ of the form $(\sigma_1,\tau_1,\dots,\widehat{\tau_q},\dots,\tau_r)$, where the hat indicates an omitted transposition.*

Proof. Assume $\sigma_1=\tau_a$ and $\tau_i=\tau_{b_i}$ for $i=1,\dots,r$. By Lemma (2.2), we have that (a,a') is an inversion for $\mu^{-1}=\sigma_r\cdots\sigma_1$. By the same Lemma, since $\mu^{-1}=\tau_r\cdots\tau_1$, there is a q such that $a=\tau_1\cdots\tau_{q-1}(b_q)$ and $a'=\tau_1\cdots\tau_{q-1}(b_q')$. It follows that the permutation $\tau:=(\tau_1\cdots\tau_{q-1})\tau_q(\tau_1\cdots\tau_{q-1})^{-1}$ interchanges a and a'. However, τ is conjugate to the transposition τ_q and hence τ is a transposition. Since τ interchanges a and a', it follows that $\tau=\tau_a=\sigma_1$. As a consequence, we have the equation,

$$\sigma_1\tau_1\cdots\tau_{q-1}=\tau_1\cdots\tau_{q-1}\tau_q.$$

Clearly, the assertion of the Lemma is obtained after multiplication by $\tau_{q+1}\cdots\tau_r$. ∎

(2.4) Remark. Every transposition τ is an involution. In particular, for every simple transposition τ_a we have that $\tau_a^2=1$. Consider a second simple transposition τ_b with $b\neq a$. Then we have the relations,

$$\begin{aligned}\tau_a\tau_b\tau_a&=\tau_b\tau_a\tau_b\quad\text{if } a \text{ and } b \text{ are neighbors,}\\ \tau_a\tau_b&=\tau_b\tau_a\quad\text{if } a \text{ and } b \text{ are not neighbors.}\end{aligned}$$

Indeed, if a and b are neighbors, we may assume that $b=a'$, and then $\tau_a\tau_b\tau_a=\tau_{a,b'}=\tau_b\tau_a\tau_b$. If a and b are not neighbors, then the permutations τ_a and τ_b are disjoint, and hence they commute.

In general, an ordered set $g_1, \dots, g_{n-1}$ of elements in a semi-group G are said to satisfy the *Coxeter–Moore relations* if

(1) $g_j g_k = g_k g_j \quad \text{if} \quad |k-j| > 1,$
(2) $g_j g_k g_j = g_k g_j g_k \quad \text{if} \quad |k-j| = 1.$

In particular, the simple transpositions τ_{a_j}, with the letters a_j in increasing order, satisfy the Coxeter–Moore relations.

(2.5) Definition. Two presentations are said to be *Coxeter–Moore equivalent* if one can be obtained from the other by a finite number (possibly none) of the following two *allowable replacements*: Given a presentation $(\tau_1, \dots, \tau_r)$. If two consecutive transpositions τ_i and τ_{i+1} are disjoint, then it is allowed to replace τ_i, τ_{i+1} by τ_{i+1}, τ_i. If for three consecutive transpositions τ_i, τ_{i+1}, and τ_{i+2}, we have that $\tau_i = \tau_{i+2} = \tau_a$ and $\tau_{i+1} = \tau_b$, where a and b are neighboring letters, then it is allowed to replace $\tau_i, \tau_{i+1}, \tau_{i+2}$ by $\tau_{i+1}\tau_i\tau_{i+1}$.

Clearly, two Coxeter–Moore equivalent presentations contain the same number of simple transpositions. As the simple transpositions satisfy the Coxeter–Moore relations, it follows that two Coxeter–Moore equivalent presentations are presentations of the same permutation.

(2.6) Proposition. *Any two minimal presentations of the same permutation are Coxeter–Moore equivalent.*

Any presentation which is not a minimal presentation is Coxeter–Moore equivalent to a presentation in which two consecutive transpositions are equal.

Proof. To prove the first assertion, consider two minimal presentations of the permutation μ:

$$\alpha = (\tau_1, \dots, \tau_r), \qquad \beta = (\sigma_1, \dots, \sigma_r).$$

Then $r = \ell(\mu)$. We have to prove β and α are (Coxeter–Moore) equivalent.

Clearly, the assertion holds when $r = 1$. Proceed by induction on $r = \ell(\mu)$. Assume that $r \geq 2$ and that the assertion holds for minimal presentations of permutations of length $r-1$.

Observe that the assertion holds if $\sigma_r = \tau_r$ as it follows by applying the induction hypothesis to the two minimal presentations $(\tau_1, \dots, \tau_{r-1})$ and $(\sigma_1, \dots, \sigma_{r-1})$ of $\mu\sigma_r$. Similarly, the assertion holds if $\sigma_1 = \tau_1$.

Now, by the exchange property, there is, for some $q = 1, \dots, r$ a presentation of μ of the form $\gamma = (\sigma_1, \tau_1, \dots, \widehat{\tau_q}, \dots, \tau_r)$. As observed above, the presentation γ is equivalent to β. Hence we may replace β by γ and assume that $\beta = (\sigma_1, \tau_1, \dots, \widehat{\tau_q}, \dots, \tau_r)$. Again, by the observation, the equivalence of α and β holds if $q < r$. Hence we may assume that $q = r$, that is, we may assume that

$$\beta = (\sigma_1, \tau_1, \dots, \tau_{r-1}).$$

Again, by the exchange property, there is a presentation of μ of the form $\gamma = (\tau_1, \dots)$ where the dots indicate the transpositions of β with one omitted. If it was the first transposition σ_1 that was omitted, then γ would have the form $\gamma = (\tau_1, \tau_1, \dots)$ and hence γ would not be a minimal presentation. Therefore, the presentation γ is of the form $\gamma = (\tau_1, \sigma_1, \tau_1, \dots, \widehat{\tau_s} \dots, \tau_{r-1})$

for some $s = 1, \dots, r-1$. As observed above, γ is equivalent to α. Hence we may replace α by γ, that is, we may assume that

$$\alpha = (\tau_1, \sigma_1, \tau_1, \dots, \widehat{\tau_s} \dots, \tau_{r-1}), \quad \beta = (\sigma_1, \tau_1, \dots, \tau_{r-1})$$

for some $s = 1, \dots, r-1$.

The assertion holds if $r = 2$. Indeed, if $r = 2$ then $\alpha = (\tau_1, \sigma_1)$ and $\beta = (\sigma_1, \tau_1)$ and we have the equation $\mu = \tau_1\sigma_1 = \sigma_1\tau_1$. Since $\ell(\mu) = 2$, we have that $\tau_1 \neq \sigma_1$. Clearly, then the equation $\tau_1\sigma_1 = \sigma_1\tau_1$ implies that τ_1 and σ_1 are disjoint. Thus β is obtained from α by an allowable replacement, and hence α and β are equivalent.

Thus we may assume that $r \geq 3$. As observed above, if $s < r-1$, then the assertion holds. So assume that $s = r-1$. Then $s \geq 2$, and so the presentation α has the form $\alpha = (\tau_1, \sigma_1, \tau_1, \dots)$. Since α is minimal, we have that $\tau_1 \neq \sigma_1$ and $\tau_1\sigma_1 \neq \sigma_1\tau_1$. Therefore σ_1 and τ_1 are simple transpositions associated to neighboring letters. Thus with an allowable replacement we may obtain from α a presentation of the form $(\sigma_1, \tau_1, \sigma_1, \dots)$. As observed above, the replaced presentation is equivalent to β. Therefore α and β are equivalent and the first assertion of the Proposition has been proved.

The second assertion is proved by induction on the number r of factors in the presentation. Clearly, the number of factors of a non-minimal presentation is at least 2. Moreover, a presentation (τ_1, τ_2) is minimal unless $\tau_1 = \tau_2$. Hence the assertion holds when $r = 2$. Assume that $r > 2$ and that the assertion holds for presentations with $r-1$ factors. Let $\alpha = (\tau_1, \dots, \tau_r)$ be a non-minimal presentation of μ. Then $r > \ell(\mu)$. Now $\beta := (\tau_1, \dots, \tau_{r-1})$ is a presentation of $\nu := \mu\tau_r$. Clearly, if β is non-minimal then, by the induction hypothesis, the assertion holds for α. So assume that the presentation β is minimal. Then $\ell(\nu) = r-1$. Since $\mu = \nu\tau_r$ is not of length r, it follows from Lemma (1.3)(3) that $\ell(\mu) = r-2$. Hence there is a minimal presentation $(\sigma_1, \dots, \sigma_{r-2})$ of μ. Since $\nu = \mu\tau_r$, it follows that $\delta := (\sigma_1, \dots, \sigma_{r-2}, \tau_r)$ is a presentation of ν. Moreover, the presentation δ is minimal because $\ell(\nu) = r-1$. Therefore, by the first part of the Proposition, the presentation β is Coxeter–Moore equivalent to δ. It follows that α is Coxeter–Moore equivalent to the presentation $(\sigma_1, \dots, \sigma_{r-2}, \tau_r, \tau_r)$. Hence the assertion holds for α.

Thus both assertions of the Proposition have been proved. $\square$

3. The Bruhat–Ehresman order.

4. Young subgroups.

5. Symmetric polynomials.

(5.1) Setup. Fix a commutative ring R. Consider the ring $R[A]$ of polynomials with coefficients in R in the letters of A. With the letters of A in increasing order $a_1, \dots, a_n$, the *monomials* in $R[A]$ are the products,

$$a_1^{j_1} \cdots a_n^{j_n},$$

where $(j_1, \dots, j_n)$ is a sequence of n nonnegative integers. The monomials form an R-basis of $R[A]$ since, by definition, every polynomial f is an R-linear combination,

$$f = \sum f_{j_1,\dots,j_n} a_1^{j_1} \cdots a_n^{j_n},$$

with uniquely determined coefficients $f_{j_1,\dots,j_n}$ in R.

The notation is simplified through the use of multi indices. A *multi index* J is a sequence $J = (j_1, \dots, j_n)$ of n nonnegative integers. Associate with J the monomial,

$$a^J := a_1^{j_1} \cdots a_n^{j_n}.$$

Then the coefficients of a polynomial f are the elements f_J of R, for all multi indices J. If $f_J \neq 0$, then the monomial a^J is said to *appear* in f. The *degree* of a multi index J is the sum of the entries, $\|J\| = j_1 + \cdots + j_n$, and the *degree* of monomial a^J is the sum of exponents, $\|J\|$. If $f \neq 0$, then the *degree* of f is the maximal degree of a monomial appearing in f. The polynomial f is said to be *homogeneous of degree* d if all monomials appearing in f are of degree d. According to this definition, the zero polynomial is homogeneous of every degree d.

Multi indices are ordered as follows: we write $I < J$ if either $\|I\| < \|J\|$ or $\|I\| = \|J\|$ and there is a $p = 1, \dots, n$ such that $i_q = j_q$ for $q = 1, \dots, p-1$ and $i_p < j_p$. Clearly, the order is a total order, and there is only a finite number of multi indices less than a given. The smallest multi index is the set $(0, \dots, 0)$. With respect to addition of multi indices, we have that if $I < J$ then $I + K < J + K$.

According to the order on the multi indices, there is a total order on the monomials: $a^I < a^J$ if $I < J$. (Note that the order on the a_i's as monomials of degree 1 is the reverse of the given order on the a_i's as letters.) The *leading monomial* of a non-zero polynomial f is the biggest monomial a^J appearing in f; the corresponding coefficient f_J is called the *leading coefficient* and $f_J a^J$ is called the *leading term*. Addition of multi indices corresponds to multiplication of monomials. Hence, if $a^I \le a^J$ and $a^L \le a^K$ and one of the inequalities is strict, then $a^I a^L < a^J a^K$. It follows that if f and g are nonzero polynomials with leading terms $f_J a^J$ and $g_K a^K$, then every monomial appearing in the product fg is at most equal to a^{J+K}; moreover, if $f_J g_K \neq 0$, then $f_J g_K a^{J+K}$ is the leading term of fg.

(5.2) Note. The number of monomials of degree d is equal to the binomial coefficient,

$$\binom{n+d-1}{n-1}.$$

Indeed, the number is equal to the number of multi indices $J = (j_1, \dots, j_n)$ such that $\|J\| = j_1 + \cdots + j_n = d$. Now, there is a bijective map $J \mapsto J'$ from the set of all multi indices onto the set of all strictly increasing multi indices, defined by

$$j'_p := j_1 + \cdots + j_p + (p-1) \quad \text{for } p = 1, \dots, n.$$

Under this map, we have that $\|J\| = d$ if and only if $j'_n = d + n - 1$. Hence, the number of multi indices J such that $\|J\| = d$ is equal to the number of strictly increasing multi indices $(k_1, \dots, k_n)$ such that $k_n = n + d - 1$. Clearly, the latter number is equal to the number of subsets with $n-1$ elements of the $d+n-1$ integers $0, 1, \dots, d+n-2$.

(5.3) Remark. The algebra $R[A]$ of polynomials has the following universal property: Given a homomorphism $R \to S$ of commutative rings and a set $\alpha = (\alpha_1, \dots, \alpha_n)$ of n elements of S. Then there is a unique homomorphism of R-algebras $R[A] \to S$ such that $a_i \mapsto \alpha_i$. It is called *evaluation* of polynomials at α, and the value of f, denoted

$$f(\alpha) = f(\alpha_1, \dots, \alpha_n),$$

is said to be obtained by the *substitution* $a_i \mapsto \alpha_i$ (or $a_i := \alpha_i$) for $i = 1, \dots, n$.

(5.4) Definition. The symmetric group $\mathfrak{S}(A)$ acts on the algebra $R[A]$ of polynomials. Indeed, any permutation σ can be viewed as a permutation of the variables of $R[A]$ and as such it extends uniquely to an R-algebra automorphism of $R[A]$, denoted $f \mapsto \sigma f$. Obviously, we have the equation $(\sigma\tau)f = \sigma(\tau f)$ for permutations σ and τ.

A polynomial f in $R[A]$ is called *symmetric* if it is invariant under the action of $\mathfrak{S}(A)$, that is, if

$$\sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}(A). \tag{5.4.1}$$

The symmetric polynomials in $R[A]$ form an R-subalgebra, denoted $\mathrm{Sym}_R[A]$.

A polynomial f is called *anti-symmetric* if it is semi-invariant under the action of $\mathfrak{S}(A)$ in the sense that

$$\sigma(f) = \mathrm{sign}(\sigma) f \text{ for all } \sigma \in \mathfrak{S}(A). \tag{5.4.2}$$

Clearly, the anti-symmetric polynomials in $R[A]$ form a module over the ring $\mathrm{Sym}_R[A]$ of symmetric polynomials. In particular, a product of a symmetric polynomial and an anti-symmetric polynomial is an anti-symmetric polynomial. Similarly, a product of two anti-symmetric polynomials is a symmetric polynomial.

The symmetric group is generated by the simple transpositions. It follows that a polynomial is symmetric if it is unchanged whenever two neighbor variables are interchanged. Similarly, a polynomial is anti-symmetric if it changes sign whenever two neighbor variables are interchanged.

(5.5). If σ is a permutation of $\mathfrak{S}(A)$, then $\sigma(a^J)$ is the monomial $a^{\sigma J}$, where σJ is the multi index obtained from J by permuting the entries as follows: With the given enumeration of A, we can identify $\mathfrak{S}(A)$ with the symmetric group $\mathfrak{S}_n$ of permutations of the numbers

$1, 2, \ldots, n$. Then σJ is obtained from J by moving, for $p = 1, \ldots, n$, the entry j_p from its position p to the position $\sigma(p)$. In other words,

$$\sigma(j_1, \ldots, j_n) = (j_{\sigma^{-1}1}, \ldots, j_{\sigma^{-1}n}).$$

Symmetry is detected on the coefficients: f is symmetric if and only if $f_{\sigma J} = f_J$ for all J and all $\sigma \in \mathfrak{S}(A)$; it suffices that f_J is unchanged whenever two neighbor entries in J are interchanged. Similarly, f is anti-symmetric if and only if $f_{\sigma J} = \operatorname{sign}(\sigma) f_J$; it suffices that f_J changes sign whenever to neighbor entries in J are interchanged.

(5.6) Example. Consider the following polynomial of $R[A]$:

$$\Delta(a_1, \ldots, a_n) := \prod_{a<b}(b-a) = \prod_{p<q}(a_q - a_p).$$

Each factor is homogeneous of degree 1, and so Δ is homogeneous of degree equal to the number, $n(n-1)/2$, of factors. The leading monomial in Δ is $a_1^{n-1}a_2^{n-2}\cdots a_n^0$ and the leading coefficient is $(-1)^{n(n-1)/2}$.

Let σ be a permutation. Then $\sigma\Delta$ is the product of the factors $\sigma b - \sigma a$ for $a < b$. Clearly, if (a, b) is not an inversion for σ, then $\sigma b - \sigma a$ is one of the factors of Δ and if (a, b) is an inversion for σ, the $\sigma b - \sigma a = -(\sigma a - \sigma b)$ is equal to -1 times a factor of Δ. It follows that

$$\sigma\Delta = (-1)^{\ell(\sigma)}\Delta.$$

Hence Δ is an anti-symmetric polynomial. As a consequence, the square Δ^2 is a symmetric polynomial. The square Δ^2 is called the *discriminant*.

(5.7) Definition. For each multi index J, define the *monomial symmetric polynomial* $m^J = m^J(A)$ as the following sum of monomials,

$$m^J = \sum{}' \sigma(a^J),$$

where the sum is over all *different* monomials of the form $\sigma(a^J)$ for $\sigma \in \mathfrak{S}(A)$. In other words, m^J is the sum of all monomials a^I where I can be obtained from J by a permutation of the entries. The polynomial m^J is obviously a symmetric polynomial, and homogeneous of degree $\|J\|$.

Clearly, if the multi index I is a permutation of J, then $m^I = m^J$. Among the multi indices that are permutations of J, the largest, K say, is characterized as being decreasing, that is, by the property that $k_1 \geq k_2 \geq \cdots \geq k_n$. Hence the monomial symmetric polynomials are naturally parametrized by the decreasing multi indices K. Note that if K is a decreasing multi index, then the leading term in m^K is the monomial a^K.

If K is a decreasing multi index, the trailing zeros in K are often omitted in the notation m^K. (However, for the smallest multi index $K = (0, \ldots, 0)$ we write $m^0 = m^{0\ldots 0} = 1$.) For instance, for the alphabet with $n = 3$ letters a, b, c, we have that

$$m^1 = a + b + c,$$

$$m^2 = a^2 + b^2 + c^2, \quad m^{11} = ab + ac + bc,$$

$$m^3 = a^3 + b^3 + c^3, \quad m^{21} = a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2, \quad m^{111} = abc.$$

(5.8) Lemma. *The monomial symmetric polynomials m^K, for all decreasing multi indices K, form an R-basis for* $\mathrm{Sym}_R[A]$.

Proof. The terms of the polynomial m^K are the monomials a^J where J is a permutation of K. Hence a polynomial f is an R-linear combination of the m^K if and only if, for all decreasing multi indices K we have that $f_J = f_K$ when J is a permutation of K, that is, if and only if f is a symmetric polynomial. □

(5.9) Definition. The d'th *elementary symmetric polynomial* $e_d = e_d(A)$ is the sum of all products of d different letters. Thus $e_0 = 1$ (the empty product is equal to 1) and $e_d = 0$ when $d > n$. Clearly, $e_1 = a_1 + \cdots + a_n$, and $e_n = a_1 \cdots a_n$. In general,

$$e_d = \sum_{1 \le i_1 < \cdots < i_d \le n} a_{i_1} \cdots a_{i_d}.$$

Obviously, e_d is a symmetric polynomial, and homogeneous of degree d. Note that e_d for $d \le n$ is the special monomial symmetric polynomial,

$$e_d = m^{1\ldots10\ldots0} = m^{1\ldots1},$$

with d occurrences of 1.

Equivalently, the e_d may be defined by the following expansion in the polynomial ring $R[A][T]$ in one variable T over $R[A]$:

$$\prod_{a\in A}(T-a) = T^n - e_1T^{n-1} + \cdots + (-1)^n e_n.$$

(5.10) Theorem. *The products $e^I := e_1^{i_1} \cdots e_n^{i_n}$, for all multi indices I, form an R-basis for* $\mathrm{Sym}_R[A]$.

Proof. The leading term in e_d is the monomial $a^{1\ldots10\ldots0}$. Hence the leading term of e^I is the monomial a^K, where

$$K = (i_1 + \cdots + i_n, i_2 + \cdots + i_n, \ldots, i_{n-1} + i_n, i_n). \tag{5.10.1}$$

To prove the Theorem, we have to prove that any symmetric polynomial f has an expansion $f = \sum_I r_I e^I$ with uniquely determined coefficients $r_I \in R$. To prove the existence, assume that $f \ne 0$, and consider the leading monomial, a^K say, of f. Since f is symmetric, it follows, for instance from (5.8), that K is a decreasing multi index. Hence there is a unique multi index I such that (5.10.1) holds. Then the polynomial f and the polynomial $f_K e^I$ have the same leading term, namely $f_K a^K$. Consequently, the difference $f - f_K e^I$ is either the zero polynomial or its leading monomial is strictly less that the leading monomial of f. If the difference $f - f_K e^I$ is non-zero, repeat the argument. By induction, in a finite number of steps, we obtain the required expansion of f.

By almost the same argument, unicity holds. Indeed, it follows for a sum $\sum_I r_I e^I$ that the non-zero terms have different leading monomials, and so a non-trivial sum is never the zero polynomial. □

(5.11) Remark. An important application of the theorem is the following: Consider a monic polynomial $P \in R[T]$, say,

$$P = T^n + r_1T^{n-1} + \cdots + r_{n-1}T + r_n. \qquad (5.11.1)$$

Let S be a commutative ring containing R as a subring, and assume that P as a polynomial in $S[T]$ has an expansion as a product,

$$P = (T - \alpha_1)\cdots(T - \alpha_n). \qquad (5.11.2)$$

Evaluation at $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the homomorphism of R-algebras $R[A] \to S$,

$$f \mapsto f(\alpha_1, \ldots, \alpha_n).$$

By expanding the product (5.11.2) it follows that

$$e_d(\alpha_1, \ldots, \alpha_n) = (-1)^d r_d.$$

So, up to a sign, evaluation of the elementary symmetric polynomials yield the coefficients of P. It follows from the Theorem that if f is a symmetric polynomial in $R[A]$, then the value $f(\alpha_1, \ldots, \alpha_n)$ is an R-linear combination of products $r_1^{i_1} \cdots r_n^{i_n}$. In other words, without knowing the "roots" α_i of P it is possible to express, for a symmetric polynomial $f \in R[A]$, the value $f(\alpha_1, \ldots, \alpha_n)$ as a polynomial in the coefficients of P.

(5.12) Example. The polynomial e^I is obviously a sum of monomials a^J. Hence, in the notation of the proof of theorem (5.10), there is an equation,

$$e^I = m^K + \sum_{L<K} \alpha_{I,L} m^L,$$

where the sum is over decreasing multi indices L less that K and the coefficients $\alpha_{I,L}$ are non-negative integers. The equation expresses the basis e^I in terms of the basis m^K, and it follows that the base change matrix is an upper triangular matrix with 1 in the diagonal. So it is easy to invert the matrix and express the m^K in terms of the basis e^I.

In degree 0, we have that $e^{0\ldots0} = 1 = m^0$, and in degrees 1, 2, and 3,

$$\begin{gathered}
e_1 = m^1, \\
e_2 = m^{11}, \quad e_1^2 = m^2 + 2m^{11}, \\
e_3 = m^{111}, \quad e_1e_2 = m^{21} + 3m^{111}, \quad e_1^3 = m^3 + 3m^{21} + 6m^{111}
\end{gathered}$$

By solving the equations, it follows that

$$\begin{gathered}
m^1 = e_1, \\
m^{11} = e_2, \quad m^2 = e_1^2 - 2e_2, \\
m^{111} = e_3, \quad m^{21} = e_1e_2 - 3e_3, \quad m^3 = e_1^3 - 3e_1e_2 + 3e_3.
\end{gathered}$$

(5.13) Definition. The d'th *power sum* $p_d = p_d(A)$, for $d \geq 1$, is the sum of the d'th powers of the variables, that is, $p_d = \sum_a a^d$. Equivalently,

$$p_d = m^d.$$

The *complete symmetric* polynomial $s_d = s_d(A)$ is the sum of all monomials of degree d, that is, $s_d = \sum_{\|J\|=d} a^J$. Equivalently,

$$s_d = \sum_{\|K\|=d}{}' m^K,$$

where the sum is over decreasing multi indices.

(5.14). The polynomials e_d, s_d, and p_d appear naturally as coefficients of power series. Indeed, in the power series ring $R[A][[T]]$ in one variable, we have the equations,

$$e(T) := \prod_{a\in A}(1+aT) = \sum_{d=0}^{\infty} e_d T^d, \tag{5.14.1}$$

$$s(T) := \prod_{a\in A}\frac{1}{1-aT} = \sum_{d=0}^{\infty} s_d T^d, \tag{5.14.2}$$

$$p(T) := \sum_{a\in A}\frac{a}{1-aT} = \sum_{d=0}^{\infty} p_{d+1} T^d. \tag{5.14.3}$$

It follows from the equations that the power series $s(T)$ is the inverse of the power series $e(-T)$, that is, we have the equation $e(-T)s(T) = 1$. Hence, since $e_0 = 1$, the coefficients s_d of $s(T)$ are determined recursively from the coefficients $(-1)^d e_d$ of $e(-T)$. For instance, in low degrees we obtain the formulas,

$$\begin{aligned}
e_0 s_0 = 1;&\quad s_0 = 1,\\
s_1 - e_1 = 0;&\quad s_1 = e_1,\\
s_2 - e_1 s_1 + e_2 = 0;&\quad s_2 = e_1^2 - e_2,\\
s_3 - e_1 s_2 + e_2 s_1 - e_3 = 0;&\quad s_3 = e_1^3 - 2e_1e_2 + e_3,\\
s_4 - e_1 s_3 + e_2 s_2 - e_3 s_1 + e_4 = 0;&\quad s_4 = e_1^4 - 3e_1^2 e_2 + 2e_1e_3 + e_2^2 - e_4.
\end{aligned}$$

Similarly, it follows from the equations that the power series $p(T)$ is equal to the logarithmic derivative of $s(T)$ and equal to -1 times the logarithmic derivative of $e(-T)$, that is, we have the equations $p(T) = s(T)'/s(T) = -(e(-T))'/e(-T)$ or, equivalently,

$$s(T)p(T) = s'(T), \qquad e(-T)p(T) = e'(-T). \tag{5.14.4}$$

Hence, the coefficients p_{d+1} of $p(T)$ are determined recursively from the coefficients e_d of $e(T)$. For instance, in low degrees we obtain the formulas,

$$\begin{aligned}
e_0 p_1 &= e_1; & p_1 &= e_1, \\
p_2 - e_1 p_1 &= -2e_2; & p_2 &= e_1^2 - 2e_2, \\
p_3 - e_1 p_2 + e_2 p_1 &= 3e_3; & p_3 &= e_1^3 - 3e_1 e_2 + 3e_3, \\
p_4 - e_1 p_3 + e_2 p_2 - e_3 p_1 &= -4e_4; & p_4 &= e_1^4 - 4e_1^2 e_2 + 4e_1 e_3 + 2e_2^2 - 4e_4.
\end{aligned}$$

6. Alternating polynomials.

(6.1) Lemma. *The following three conditions on a polynomial $f = \sum_J f_J a^J$ of $R[A]$ are equivalent:*

(i) *The coefficients f_J are alternating in the multi index J, that is, f_J changes sign when two entries at different positions in J are interchanged and f_J vanishes when two entries of different position in J are equal.*

(ii) *The polynomial f is anti-symmetric and divisible by the product $\prod_{a<b}(b-a)$.*

(iii) *The polynomial f is anti-symmetric and, for all $q > p$, the substitution $a_q := a_p$ in f yields the zero polynomial.*

Proof. As noted in (5.4), the polynomial f is anti-symmetric if and only if f_J changes sign when two entries of J are interchanged. Therefore, to prove the equivalence of the three conditions, we may assume that f is anti-symmetric.

Let $p < q$ be arbitrary integers between 1 and n, and let τ be the transposition of $\mathfrak{S}(A)$ that interchanges a_p and a_q. Then, to prove the equivalence of (i) and (iii), it suffices to prove the following assertion: the substitution $a_q := a_p$ in f yields zero if and only if $f_J = 0$ for all multi indices J such that $j_p = j_q$. To prove the latter assertion, decompose f into two sums of monomials,

$$f = \sum_{j_p = j_q} f_J a^J + \sum_{j_q < j_p} \big(f_J a^J + f_{\tau J} a^{\tau J}\big). \tag{6.1.1}$$

Clearly, the substitution $a_q := a_p$ in a^J and in $a^{\tau J}$ yield the same result. Moreover, since f is anti-symmetric, we have that $f_{\tau J} = -f_J$. Consequently, the substitution $a_q := a_p$ in the second sum of (6.1.1) yields zero. Therefore, the substitution $a_q := a_p$ in f yields zero, if and only if the substitution $a_q := a_p$ in the first sum yields zero. Obviously, the substitution $a_q := a_p$ in the first sum yields zero if and only if all the coefficients f_J in the first sum are equal to zero. Hence the equivalence of (i) and (iii) holds.

To prove the equivalence of (ii) and (iii), note that the substitution $a_q := a_p$ in f yields zero if and only if f is divisible by the difference $a_q - a_p$. Clearly, f is divisible by all differences $a_q - a_p$ for $p < q$ if and only if f is divisible by the product of all the differences. Hence the equivalence of (ii) and (iii) holds. $\square$

(6.2) Definition. A polynomial f is said to be *alternating* if it satisfies the equivalent conditions in Lemma (6.1). Clearly, the alternating polynomials form an R-submodule $\mathrm{Alt}_R[A]$ of $R[A]$. In fact, it follows from any of the characterizations (ii) or (iii) that $\mathrm{Alt}_R[A]$ is a $\mathrm{Sym}_R[A]$-submodule, that is, a product of a symmetric polynomial and an alternating polynomial is alternating.

If 2 is a regular element in R then the alternating polynomials are simply the anti-symmetric polynomials. Indeed, if f is anti-symmetric and J is a multi-index with two equal entries, then $f_J = -f_J$ and hence $2f_J = 0$.

(6.3) Definition. Consider the $\infty \times n$ matrix,

$$V(a_1, \dots, a_n) := \begin{pmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_n \\ \vdots & & \vdots \\ a_1^i & \dots & a_n^i \\ \vdots & & \vdots \end{pmatrix}, \tag{6.3.1}$$

whose rows are naturally indexed $0, 1, 2, \dots$. For any multi-index $J = (j_1, \dots, j_n)$, consider the $n \times n$ matrix obtained by selecting from the matrix V the n rows with indices $j_1, \dots, j_n$, and denote by Δ^J its determinant, that is,

$$\Delta^J = \Delta^{j_1,\dots,j_n}(a_1, \dots, a_n) := \begin{vmatrix} a_1^{j_1} & \dots & a_n^{j_1} \\ \vdots & & \vdots \\ a_1^{j_n} & \dots & a_n^{j_n} \end{vmatrix}. \tag{6.3.2}$$

The special determinant obtained when J is the sequence $0, 1, \dots, n-1$ will be called the *Vandermonde determinant* and denoted $\Delta(a_1, \dots, a_n)$, that is,

$$\Delta(a_1, \dots, a_n) := \begin{vmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_n \\ \vdots & & \vdots \\ a_1^{n-1} & \dots & a_n^{n-1} \end{vmatrix}. \tag{6.3.3}$$

The determinants Δ^J are polynomials in $R[A]$. Clearly, they are homogeneous of degree $\|J\|$. It follows from usual properties of determinants, as functions of the columns, that the condition (6.1)(iii) holds for Δ^J. Hence Δ^J is an alternating polynomial. Moreover, the determinants Δ^J are alternating in the entries of J, that is, if σ is a permutation in $\mathfrak{S}(A)$ then $\Delta^{\sigma J} = (\operatorname{sign}\sigma)\Delta^J$ and $\Delta^J = 0$ if the multi index J has two equal entries.

Note that, since Δ^J is an alternating polynomial, we have that Δ^J is divisible by the product $\prod_{p<q}(a_q - a_p)$.

(6.4) Example. For $n = 3$ and the alphabet with the letters $a, b, c,$ we have that

$$\Delta^{ijk}(a, b, c) = \begin{vmatrix} a^i & b^i & c^i \\ a^j & b^j & c^j \\ a^k & b^k & c^k \end{vmatrix} = a^i b^j c^k + a^k b^i c^j + a^j b^k c^i - a^i b^k c^j - a^j b^i c^k - a^k b^j c^i.$$

In particular,

$$\Delta(a, b, c) = -a^2 b + a^2 c + ab^2 - ac^2 - b^2 c + bc^2 = (b - a)(c - a)(c - b).$$

(6.5) Note. The Vandermonde determinant defined in (6.3) is the determinant used by Jacobi [1]. Up to a sign, the determinant is independent of the given ordering of the letters of A. Our choice of sign from that used by Macdonald [2] and others. Indeed, the Vandermonde determinant defined in [2] differs by the sign $(-1)^{n(n-1)/2}$ from ours.

(6.6) Proposition. *The determinant Δ^J is given by the following formula,*

$$\Delta^J = \sum_{\sigma \in \mathfrak{S}(A)} \operatorname{sign}(\sigma)\sigma(a^J). \tag{6.6.1}$$

Moreover, the determinants Δ^J, for all strictly increasing multi indices J, form an R-basis for the module $\mathrm{Alt}_R[A]$ *of alternating polynomials in $R[A]$.*

Proof. The formula (6.6.1) is just the usual expression for the determinant (6.3.2).

Let f be an alternating polynomial. By (6.1)(i), the only monomials appearing in f are of the form a^I where all entries in the multi index I are different. If all entries in a multi index I are different, then they may be arranged into strictly increasing order by a unique permutation, that is, we have that $I = \sigma J$ where σ is a permutation and J is a strictly increasing multi index. Moreover, since f is anti-symmetric, we have that $f_I = (\operatorname{sign}\sigma) f_J$, that is, the term $f_I a^I$ is equal to $f_J(\operatorname{sign}\sigma)a^{\sigma J}$. It follows that polynomials on the right side of (6.6.1), for all strictly increasing multi indices J, form an R-basis for $\mathrm{Alt}_R[A]$. ∎

(6.7) Corollary. *For the Vandermonde determinant we have the equation,*

$$\Delta = \Delta(a_1, \dots, a_n) = \prod_{p<q} (a_q - a_p). \tag{6.7.1}$$

Moreover, multiplication by Δ is an isomorphism from the R-submodule $\mathrm{Sym}_R[A]$ *of symmetric polynomials onto the R-submodule of* $\mathrm{Alt}_R[A]$ *of alternating polynomials. Finally, the symmetric polynomials,*

$$s^J(A) := \Delta^J/\Delta,$$

for all strictly increasing multi indices J, form an R-basis for the module $\mathrm{Sym}_R[A]$ *of symmetric polynomials.*

Proof. In (6.7.1) the two polynomials have the same degree, namely $n(n-1)/2$. The Vandermonde determinant is alternating and hence divisible by the right side. Hence it suffices to compare the coefficient of the monomial $a_1^0 a_2^1 \cdots a_n^{n-1}$ in the two polynomials. Clearly, both coefficients are equal to 1.

By Lemma (6.1), every alternating polynomial is divisible by Δ. Clearly, if a polynomial f is divisible by Δ, then f is alternating if and only if f/Δ is symmetric. Hence the second assertion of the Corollary holds. The final assertion follows from the second and the description of the R-basis for $\mathrm{Alt}_R[A]$ in Proposition (6.6). ∎

(6.8) Definition. The symmetric polynomials $s^J = \Delta^J/\Delta$ of (6.7) are called *Schur polynomials*. Since Δ^J is homogeneous of degree $\|J\|$ and Δ is homogeneous of degree $n(n-1)/2$, it follows that s^J is homogeneous of degree $\|J\| - n(n-1)/2$.

The polynomials s^J for strictly increasing multi indices J will be called *proper Schur polynomials*. They form an R-basis for $\mathrm{Sym}_R[A]$. For the smallest strictly increasing multi index $J = (0, 1, \dots, n-1)$, we have that

$$s^{0,1,\dots,n-1} = 1.$$

The Schur polynomials s^J are alternating in the multi index J, that is, $s^J = 0$ if J has two equal entries and s^J changes sign when two entries at different position in J are interchanged. In particular, any Schur polynomial s^J is either equal to 0 or, up to sign, equal to a proper Schur polynomial.

In general, it is hard to compute the Schur polynomials directly as the determinants divided by Δ, and later we will prove other relations involving the Schur polynomials. As an example, let us prove here the following formulas, for $p = 0, \dots, n$:

$$s^{0,1,\dots,\hat{p},\dots,n} = e_{n-p} . \tag{6.8.1}$$

Consider the polynomial $D(T)$ in $R[A][T]$ defined as the Vandermonde determinant

$$D(T) := \Delta(a_1, \dots, a_n, T) = \begin{vmatrix} 1 & \dots & 1 & 1 \\ a_1 & \dots & a_n & T \\ \vdots & & \vdots & \vdots \\ a_1^n & \dots & a_n^n & T^n \end{vmatrix} .$$

By (6.7.1), applied to the alphabet $\{a_1, \dots, a_n, T\}$, we have the product expansion $D(T) = \Delta \prod_i (T - a_i)$. On the other hand, by developing the determinant $D(T)$ along its last column, we obtain the equation $D(T) = \sum_p (-1)^{n-p} \Delta^{0,\dots,\hat{p},\dots,n} T^p$. Hence we have the equation,

$$\Delta \prod_{i=1}^{n} (T - a_i) = \sum_{p=0}^{n} (-1)^{n-p} \Delta^{0,\dots,\hat{p},\dots,n} T^p .$$

As noted in (5.9), the Formula (6.8.1) is a consequence.

(6.9) Definition. If f is any polynomial in $R[A]$, then the sum,

$$\sum_{\sigma \in \mathfrak{S}(A)} (\operatorname{sign} \sigma) \sigma f,$$

is an alternating polynomial. Indeed, the sum is R-linear as a function of f and if f is a monomial a^J, then, by (6.6.1), the sum is equal to the determinant Δ^J which is an alternating polynomial. Therefore, it follows from Corollary (6.7) that the equation,

$$\delta(f) := \frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}(A)} (\operatorname{sign} \sigma) \sigma f , \tag{6.9.1}$$

defines a map $\delta = \delta^A \colon R[A] \to \operatorname{Sym}_R[A]$. Obviously, the map δ is an R-linear operator. It is called the *symmetrization operator*. As noted above, the Schur polynomial s^J is the result of symmetrizing the monomial a^J,

$$s^J = \delta(a^J). \tag{6.9.2}$$

It is obvious from (6.9.1) that symmetrization is a $\mathrm{Sym}_R[A]$-linear operator, that is, if f and g are polynomials and g is symmetric, then

$$\delta(gf) = g\delta(f). \tag{6.9.2}$$

As a consequence, we obtain for a symmetric polynomial g and any multi index L the formula,

$$gs^L = \sum_I g_I s^{I+L}. \tag{6.9.3}$$

Indeed, the two sides of (6.9.3) are the results of symmetrizing $ga^L = \sum_I g_I a^{I+L}$.

(6.10) Note. The formula (6.9.3) expresses the product gs^L of a symmetric polynomial g and a Schur polynomial s^L as an R-linear combination of Schur polynomials. To get the expansion of gs^L in the basis s^J consisting of proper Schur polynomials we have to consider the non-zero terms in (6.9.3), that is, the terms for which the multi index $I + L$ has all entries different, and then, for the non-zero terms we have to collect the coefficients for which $I + L$ is a permutation of a given strictly increasing multi index J. This collection of terms is often of combinatorial nature.

For instance, let m^K be the monomial symmetric polynomial. Then

$$m^K s^L = \sum{}' s^{I+L}, \tag{6.10.1}$$

where the sum is over all *different* permutations of the entries in K. Indeed, the formula follows from (6.9.3) since $m^K = \sum' a^I$.

In particular, since the d'th elementary symmetric polynomial e_d, for $0 \le d \le n$, is the monomial symmetric polynomial $m^{1\dots10\dots0}$ (with 1 occurring d times), it follows that $e_d s^L$ is the sum (6.10.1) over all the $\binom{n}{d}$ permutations I of $(1, \dots, 1, 0, \dots, 0)$. Take $L := (0, 1, \dots, n-1)$. Then $s^L = 1$ and the formula is the expansion of e_d in terms of Schur polynomials. Clearly, $I + L$ has two equal entries unless $I = (0, \dots, 0, 1, \dots, 1)$. So the formula reduces to the formula of (6.8.1),

$$e_d = s^{0,\dots,n-d-1,n-d+1,\dots,n}. \tag{6.10.2}$$

Similarly, the d'th power sum p_d, for $d \ge 1$, is the monomial symmetric polynomial $m^{d0\dots0}$. Hence $p_d s^L$ is the sum (6.10.1) over the n permutations I of $(d, 0, \dots, 0)$. Take $L := (0, 1, \dots, n-1)$ to obtain the following expansion of p_d:

$$p_d = \sum_i s^{0,1,\dots,i-1,d+i,i+1,\dots,n-1} = \sum_{i \ge n-d} (-1)^{n-i-1} s^{0,\dots,\hat{\imath},\dots,n-1,d+i}. \tag{6.10.3}$$

(6.11) Note. Take $L := (0, 1, \dots, n-1)$ in (6.10.1). Then (6.10.1) is the expansion of the monomial symmetric polynomial m^K in the basis of Schur polynomials. Clearly, among the multi indices $I + L$ appearing in formula, the smallest, $\tilde{K}$ say, is the strictly increasing multi index $I + L$ obtained when I is the strictly increasing permutation of K. So the expansion obtained has the form

$$m^K = s^{\tilde{K}} + \sum_{J > \tilde{K}} \gamma_{K,J} s^J,$$

where the coefficients $\gamma_{K,J}$ are integers. In particular, the base change matrix is a lower triangular matrix, and hence it can be used to obtain expansions of the Schur polynomials s^J in terms of the basis m^K. For $n = 4$, we have in degrees 1, 2, and 3 the equations,

$$m^1 = s^{0124}$$
$$m^{11} = s^{0134}, \quad m^2 = s^{0125} - s^{0134}$$
$$m^{111} = s^{0234}, \quad m^{21} = s^{0135} - 2s^{0234}, \quad m^3 = s^{0126} - s^{0135} + s^{0234},$$

and in degree 4,

$$m^{1111} = s^{1234}, \quad m^{211} = s^{0235} - 3s^{1234}, \quad m^{22} = s^{0145} - s^{0235} + s^{1234},$$
$$m^{31} = s^{0136} - s^{0145} - s^{0235} + 2s^{1234}, \quad m^4 = s^{0127} - s^{0136} + s^{0235} - s^{1234}.$$

By solving the equations, it follows in degrees 1, 2, and 3 that

$$s^{0124} = m^1$$
$$s^{0134} = m^{11}, \quad s^{0125} = m^2 + m^{11}$$
$$s^{0234} = m^{111}, \quad s^{0135} = m^{21} + 2m^{111}, \quad s^{0126} = m^3 + m^{21} + m^{111}.$$

and in degree 4,

$$s^{1234} = m^{1111}, \quad s^{0235} = m^{211} + 3m^{1111}, \quad s^{0145} = m^{22} + m^{211} + 2m^{1111},$$
$$s^{0136} = m^{31} + m^{22} + 2m^{211} + 3m^{1111}, \quad s^{0127} = m^4 + m^{31} + m^{22} + m^{211} + m^{1111}.$$

In fact, the coefficients of the proper Schur polynomials in terms of the basis m^K are always non-negative, and we will later give a combinatorial expression for the coefficients.

(6.12) Pieri's Formula. *Let s_d be the d'th complete symmetric polynomial. Then, for every strictly increasing multi index L, we have the expansion,*

$$s_d s^L = \sum{}' s^J, \tag{6.12.1}$$

where the sum is over all strictly increasing multi indices $J = (j_1, \dots, j_n)$ satisfying the inequalities,

$$l_1 \le j_1 < l_2 \le j_2 < \cdots < l_{n-1} \le j_{n-1} < l_n \le j_n,$$

and the equality $\|J\| = \|L\| + d$. *In particular, we have the equation,*

$$s_d = s^{0,\dots,n-2,n-1+d}. \tag{6.12.2}$$

Proof. Write $J \supset L$ if every entry in the multi index J is at least equal to the corresponding entry in L. Denote by $\mathcal{J}$ the set of multi indices J such that $J \supset L$ and $\|J\| = \|L\| + d$. Since $s_d = \sum_{\|I\|=d} a^I$, it follows from (6.9.3) that $s_d s^L = \sum_{\|I\|=d} s^{I+L}$, or, equivalently,

$$s_d s^L = \sum_{J \in \mathcal{J}} s^J. \tag{6.12.3}$$

In (6.12.1), the sum is over all $J \in \mathcal{J}$ such that the following inequality holds for $p = 1, \dots, n-1$:

$$j_p < l_{p+1}. \tag{*}$$

Therefore, to prove the Formula (6.12.1), we have to prove that the sum of the s^J, over those $J \in \mathcal{J}$ for which one of the inequalities (*) is false, is equal to zero.

For each $q = 1, \dots, n-1$, let $\mathcal{J}_q$ be the subset of $\mathcal{J}$ consisting of multi indices J such that the inequality (*) holds for all $p < q$ but not for $p = q$. It suffices to prove that the sum of the s^J for $J \in \mathcal{J}_q$ is equal to zero.

To prove the latter assertion, let τ be the simple transposition that interchanges in a multi index J the q'th and the $(q+1)$'st entry. It J belongs to $\mathcal{J}_q$, then $j_q \geq l_{q+1}$ and $j_{q+1} \geq l_{q+1}$ since $J \supset L$. It follows that τJ belongs to $\mathcal{J}_q$. Hence τ defines an involution of the set $\mathcal{J}_q$. If $\tau J = J$, then J has two equal entries, and then $s^J = 0$. If $\tau J \neq J$, then $s^J + s^{\tau J} = 0$, since s^J is alternating in J. It follows that the sum $\sum_{J \in \mathcal{J}_q} s^J$ is equal to zero.

Thus Formula (6.12.1) holds. Clearly, the Formula (6.12.2) is the special case obtained when $L = (0, 1, \dots, n-1)$. ∎

(6.13) Notation. Consider the three bases for the R-module $\mathrm{Sym}_R[A]$ of symmetric polynomials: the *monomial basis* of monomial symmetric polynomials m^K, indexed by decreasing multi indices K, the *elementary basis* of products $e^I = e_1^{i_1} \cdots e_n^{i_n}$ of the elementary symmetric polynomials, indexed by arbitrary multi indices I, and the *Schur basis* of proper Schur polynomials s^J, indexed by strictly increasing multi indices J. In all three cases, the multi indices are assumed to be of size equal to the number n of letters of A. It will be convenient to introduce a notation where multi indices of arbitrary sizes are allowed.

First, as noted in (5.7), it is common for a decreasing multi index K to omit (some of) the trailing zeros in the notation m^K. More precisely, we define, for any decreasing multi index $K = (k_1, \dots, k_r)$, the polynomial m^K as follows: If $r < n$, then $m^K := m^{k_1,\dots,k_r,0,\dots,0}$ with $n - r$ trailing zeros. If $r > n$ and $k_{n+1} = \cdots = k_r = 0$, then $m^K := m^{k_1,\dots,k_n}$. Finally, if $r > n$ and some entry k_q with $q > n$ is positive, then $m^K := 0$. In all cases, m^K is homogeneous of degree $\|K\|$. In this notation, the monomial basis consists of the polynomials m^K where K is a decreasing multi index of size at most n and with no trailing zeros.

Next, for an arbitrary multi index $I = (i_1, \dots, i_r)$ of size r, we define $e^I := e_1^{i_1} \dots e_r^{i_r}$. The elementary symmetric polynomials e_d are defined for all d, and they vanish when $d > n$. So, $e^I = 0$ if and only if $r > n$ and some entry i_q for $q > n$ is positive. In all cases, e^I is homogeneous of degree equal to $i_1 + 2i_2 + \cdots + ri_r$. In this notation, the elementary basis consists of the products e^I where I is a multi index of size at most n and with no trailing zeros.

Consider finally a strictly increasing multi index $J = (j_1, \dots, j_r)$. It will be convenient to say that multi indices of the following form, for some $t \geq 0$, are *extensions* of J:

$$\hat{J} := (0, 1, \dots, t-1, t+j_1, \dots, t+j_r),$$

If $r \leq n$, define $s^J := s^{\hat{J}}$, where $\hat{J}$ is the extension to a multi index of size n. If $r > n$ and J is the extension $\hat{J}_0$ of a multi index J_0 of size n, define $s^J := s^{J_0}$. Finally, if $r > n$ and J is not the extension of a multi index of size n, define $s^J = 0$. Note that s^J is a proper Schur polynomial except in the last case. In all cases, s^J is homogeneous of degree $\|J\| - r(r-1)/2$. In this notation, the Schur basis consists of the Schur polynomials s^J where J is a strictly increasing multi index of size at most n and with $j_1 > 0$.

The empty sequence ( ) is allowed in all three cases, and, according to the definitions, $m^{()} = e^{()} = s^{()} = 1$.

For instance, in degree 4, the elements of the three bases are the following:

$$\begin{array}{ccccc} m^{1111}, & m^{211}, & m^{22}, & m^{31}, & m^4; \\ e^{0001}, & e^{101}, & e^{02}, & e^{21}, & e^4; \\ s^{1234}, & s^{124}, & s^{23}, & s^{14}, & s^4; \end{array}$$

except that, when the number of variables is less than 4, some of the polynomials vanish and have to be discarded in the list.

Note that, in the extended notation for the proper Schur polynomials, the formulas of (6.10.2), (6.12.2), and (6.10.3), simplify to the following:

$$e_d = s^{1,\dots,d}, \quad s_d = s^d, \quad p_d = \sum_{j=0}^{d-1} (-1)^j s^{1,\dots,j,d}.$$

(6.14) Notation. There is another natural way to index the products $e^I = e_1^{i_1} \cdots e_n^{i_n}$ of the elementary basis. If $L = (l_1, \dots, l_r)$ is multi index of size r, let

$$e_L := e_{l_1} \cdots e_{l_r}.$$

Note that e_L is homogeneous of degree $\|L\|$. The products e_L are symmetric in the entries of L, and we will usually index them by decreasing multi indices. If $l_1 > n$, then $e_L = 0$.

Clearly, $e^I = e_L$, where

$$L := (\overbrace{n, \dots, n}^{i_n}, \dots, \overbrace{2, \dots, 2}^{i_2}, \overbrace{1, \dots, 1}^{i_1}).$$

Hence the elementary basis consists of the products e_L where L is a decreasing multi index with $l_1 \leq n$.

For instance, in degree 4 we have that

$$e^{0001} = e_4,\ e^{101} = e_1 e_3 = e_{31},\ e^{02} = e_2^2 = e_{22},\ e^{21} = e_1^2 e_2 = e_{211},\ e^4 = e_1^4 = e_{1111}.$$

(6.15) Remark. It is easy to express, for each decreasing multi index L of size r, the coefficients $\alpha_{L,K}$ in the expansion $e_L = \sum_K \alpha_{L,K} m^K$ of e_L in the monomial basis. Namely, *$\alpha_{L,K}$ is the number of $r \times n$ matrices with entries 0 or 1 and row sums $l_1, \dots, l_r$ and column sums $k_1, \dots, k_n$.* Indeed, the terms of e_l are the monomials $a_1^{p_1} \dots a_n^{p_n}$ where the p_j are 0 or 1 and $\sum_j p_j = l$. Hence each matrix of the said form corresponds to the selection of a term in each e_{l_i} for $i = 1, \dots, r$ such that the product of the selected terms is equal to a^K.

Similarly, if we define $s_L := s_{l_1} \cdots s_{l_r}$, then, in the expansion $s_L = \sum_K \beta_{L,K} m^K$, the coefficient *$\beta_{L,K}$ is the number of $r \times n$ matrices with non-negative integer entries and row sums $l_1, \dots, l_r$ and column sums $k_1, \dots, k_n$.*

As a consequence, the matrix of the $\alpha_{L,K}$, indexed by all decreasing multi indices of size n and degree d, is a symmetric matrix. Consider, for $d \leq n$, the products e_L for all decreasing multi indices L of size n and degree d. They are simply the elementary symmetric polynomials e^I of degree at most d. Hence they form a basis for the R-module of symmetric polynomials of degree at most d, and the matrix $\alpha_{L,K}$ is the base change matrix from the basis of the e_L to the basis of the m^K for $\|K\| \leq d$.

7. Determinantal Methods.

(7.1) Setup. Work with matrices and power series over a given commutative ground ring. We will follow a classical convention for matrices M. If I is an ordered set of row indices, we denote by M^I the matrix obtained from M by selecting its rows with indices in I, and if J is an ordered set of column indices, we denote by M_J the matrix obtained from M by selecting its columns with indices in J. In this notation, M^i is the i'th row of M, and M_j is the j'th column. In particular, M_j^i is the ij'th entry in M.

In general, if u is a power series we denote by u_i the coefficient of T^i, that is,

$$u = u_0 + u_1 T + u_2 T^2 + \cdots .$$

If $I = (i_1, \dots, i_r)$ is a multi index of size r, we denote by u_I the product,

$$u_I := u_{i_1} \cdots u_{i_r}.$$

The products u_I are symmetric in the entries of I. In particular, if K is the decreasing permutation of I, then $u_K = u_I$.

(7.2) Notation. For a power series u, denote by $\langle u\rangle$ the infinite column of coefficients of u. More generally, for any finite or infinite sequence of r ($1 \le r \le \infty$) power series $u, v, w, \dots$, denote by $\langle u, v, w, \dots \rangle$ the $\infty \times r$ matrix with columns $\langle u\rangle, \langle v\rangle, \langle w\rangle, \dots$.

In this notation, associate with a given power series u the $\infty \times \infty$ matrix,

$$M(u) := \langle u, Tu, T^2u, \dots \rangle = \begin{pmatrix} u_0 & 0 & 0 & 0 & \dots \\ u_1 & u_0 & 0 & 0 & \dots \\ u_2 & u_1 & u_0 & 0 & \dots \\ u_3 & u_2 & u_1 & u_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The rows and columns in $M(u)$ are naturally indexed $0, 1, 2, \dots$. In particular, the ij'th entry in $M(u)$ is equal to u_{i-j} where, by convention, $u_k = 0$ if $k < 0$.

Clearly, for power series u, v, and w, the equation $uv = w$ is equivalent to any of the following two matrix equations:

$$M(u)\langle v\rangle = \langle w\rangle, \qquad M(u)M(v) = M(w). \tag{7.2.1}$$

(7.3) Lemma. *Let u be a power series with $u_0 = 1$. Assume for power series v and w that $uv = w$. Then, for $d = 0, 1, \dots$, the following equation holds:*

$$(-1)^d v_d = \begin{vmatrix} w_0 & u_0 & 0 & \dots & 0 \\ w_1 & u_1 & u_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{d-1} & u_{d-1} & u_{d-2} & \dots & u_0 \\ w_d & u_d & u_{d-1} & \dots & u_1 \end{vmatrix} \tag{7.3.1}$$

In particular, if $uv = 1$, then

$$(-1)^d v_d = \begin{vmatrix} u_1 & u_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{d-1} & u_{d-2} & \dots & u_0 \\ u_d & u_{d-1} & \dots & u_1 \end{vmatrix}. \tag{7.3.2}$$

Proof. Let U be the $(d+1) \times (d+1)$ matrix consisting of the first $d+1$ rows and columns of $M(u)$. The columns of U are $U_0, \dots, U_d$. In particular, the column U_0 consists of the first $(d+1)$ coefficients $u_0, \dots, u_d$ of u. Define the columns W_0 and V_0 similarly. The matrix $M(u)$ is a lower triangular matrix. Hence, from the first equation of (7.2.1), or directly, we obtain the matrix equation $UV_0 = W_0$. The latter equation, with the v_i for $i = 0, \dots, d$ as unknowns, is solved by Cramer's formula: Since $\det U = 1$, it follows that v_i is equal to the determinant of the matrix obtained from U by replacing the column U_i by W_0. In particular,

$$v_d = \det(U_0, \dots, U_{d-1}, W_0) = (-1)^d \det(W_0, U_0, \dots, U_{d-1}),$$

which is the asserted Formula (7.3.1). Clearly, Formula (7.3.2) is a special case. $\square$

(7.4) Definition. Let $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ be multi indices of the same size r. Then $M^I(u)$ is an $r \times \infty$ matrix, and $M_J(u)$ is an $\infty \times r$ matrix, and $M^I_J(u)$ is an $r \times r$ matrix. Denote by u^I_J the determinant of $M^I_J(u)$, that is,

$$u^I_J := \begin{vmatrix} u_{i_1 - j_1} & \dots & u_{i_1 - j_r} \\ \vdots & & \vdots \\ u_{i_r - j_1} & \dots & u_{i_r - j_r} \end{vmatrix}$$

Subsets with r elements of the non-negative integers will be identified with strictly increasing multi indices of size r. In particular, the interval $[r] := \{0, 1, \dots, r-1\}$ consisting of the first r non-negative integers will be identified with the multi index $(0, 1, \dots, r-1)$. For any multi index I of size r, we will normally write u^I for the determinant $u^I_{[r]} = \det M^I_{[r]}(u)$. Note that $M^I_{[r]}(u)$ is the matrix obtained from $M(u)$ by selecting the first r columns and the rows with indices in I, and u^I is the determinant,

$$u^I := \begin{vmatrix} u_{i_1} & u_{i_1 - 1} & \dots & u_{i_1 - r + 1} \\ \vdots & \vdots & & \vdots \\ u_{i_r} & u_{i_r - 1} & \dots & u_{i_r - r + 1} \end{vmatrix}$$

In this notation, the 1×1 determinant u^i, where the superscript i is an index, is the i'th coefficient u_i of u.

Note that the determinant u^I_J is alternating in I and alternating in J. In particular, the special determinant u^I is alternating in the multi index I. As a consequence, general properties of the determinants u^I_J may be deduced from properties valid for strictly increasing multi indices I and J.

The special determinants u^I should not be confused with the products u_I defined in (7.1)

(7.5) Proposition. *Let u be a power series and let I and J be strictly increasing multi indices of the same size r. View I and J as subsets of the interval $[N] = \{0, 1, \dots, N-1\}$ for some $N \gg 0$. Then:*

(1) (Vanishing) *The determinant u^I_J is non-zero only when $I \supset J$, that is, when $i_p \geq j_p$ for $p = 1, \dots, r$. If u is a polynomial of degree at most n, then u^I_J is non-zero only when $n + j_p \geq i_p \geq j_p$ for $p = 1, \dots, r$.*

(2) (Extension) *The determinant u^I_J is unchanged if the same number is subtracted from all entries in I and in J. If $u_0 = 1$ and $i_p = j_p$ for $p = 1, \dots, t$, then $u^I_J = u^{I_0}_{J_0}$, where $I_0 = (i_{t+1}, \dots, i_r)$ and $J_0 = (j_{t+1}, \dots, j_r)$.*

(3) (Homogeneity) *If λ is an element in the ground ring and w is the power series defined by $w(T) = u(\lambda T)$, then*

$$w^I_J = \lambda^{\|I\| - \|J\|} u^I_J. \tag{7.5.1}$$

(4) (Symmetry) *Denote by $x \mapsto x^* = N - 1 - x$ the order reversing involution of the interval, and by I^* and J^* the images of I and J, as subsets of $[N]$. Then,*

$$u^I_J = u^{J^*}_{I^*}. \tag{7.5.2}$$

(5) (Duality) *Denote by $\tilde{I}$ and $\tilde{J}$ the complements with respect to the interval $[N]$, and denote by I' and J' the images of $\tilde{I}$ and $\tilde{J}$ under the involution $x \mapsto x^*$. If $u_0 = 1$ and v is the power series defined by the equation $u(T)v(-T) = 1$, then*

$$u^I_J = v^{I'}_{J'}. \tag{7.5.3}$$

(6) (Multiplication) *If v and w are power series such that $uv = w$, then*

$$w^I_J = \sum_{I \supset K \supset J} u^I_K v^K_J, \tag{7.5.4}$$

where the sum is over strictly increasing multi indices K of size r.

Proof. (1) Consider the matrix $U := M^I_J(u)$. By definition, the pq'th entry in U is the coefficient $U^p_q = u_k$ where $k = i_p - j_q$. Entries of U above U^p_q and to the right of U^p_q are coefficients u_l with $l \leq k$. Assume that $i_p < j_p$ for some p. The diagonal element U^p_p is equal to u_k with $k < 0$. It follows that the largest rectangular block of U with the diagonal element U^p_p as its lower left corner is equal to zero. Hence the determinant $u^I_J = \det U$ vanishes.

Similarly, assume that u is a polynomial of degree at most n, and that $i_p > j_p + n$. Then the diagonal element U^p_p is equal to u_k with $k > n$. It follows that the largest rectangular block of U with the diagonal element U^p_p as its upper right corner is equal to zero. Hence the determinant $u^I_J = \det U$ vanishes.

(2) If the same number is subtracted from all entries of I and J, then the differences $i_p - j_q$ are unchanged. Hence the matrix $U = M^I_J(u)$ and its determinant are unchanged.

If $i_p = j_p$ for $p = 1, \ldots, t$, then U is a block matrix,

$$U = \begin{pmatrix} T & 0 \\ * & U_0 \end{pmatrix},$$

where T is an lower triangular $(r-t) \times (r-t)$ matrix with the element u_0 in the diagonal, and $U_0 := M^{I_0}_{J_0}(u)$. Hence, if $u_0 = 1$, we have that $\det U = \det U_0$, and hence $u^I_J = u^{I_0}_{J_0}$.

(3) Since $w_i = \lambda^i u_i$, it follows that if the pq'th entry in $M^I_J(w)$ is non-zero, then it is equal to $\lambda^{i_p - j_q}$ times the pq'th entry of $M^I_J(u)$. Hence, in the usual expansion of a determinant as a signed sum of products, any non-zero product in the expansion of $M^I_J(w)$ is equal to $\lambda^{\|I\|-\|J\|}$ times the corresponding product in the expansion of $M^I_J(u)$. Thus Equation (7.5.1) holds.

(4) Denote by U' the matrix $M^{j_1^*,\ldots,j_r^*}_{i_1^*,\ldots,i_r^*}(u)$. Then the matrix $M^{J^*}_{I^*}(u)$ is obtained from U' by reversing first the order of the rows and next the order of the columns. Hence the two matrices have the same determinant, that is $u^{J^*}_{I^*} = \det U'$. The pq'th entry in U' is $u_{j_p^* - i_q^*}$. As $j_p^* - i_q^* = i_q - j_p$, it follows that U' is the transpose of the matrix $M^I_J(u)$. Hence $\det U' = u^I_J$. Thus Equation (7.5.2) holds.

(5) Let w be the power series defined by $w(T) = v(-T)$ so that, by hypothesis, $uw = 1$. Let U be the matrix consisting of the first N rows and columns of $M(u)$. It is a lower triangular matrix with $u_0 = 1$ in the diagonal. In particular, $\det U = 1$. Define W and V similarly from $M(w)$ and $M(v)$.

First, from the equation $uw = 1$ of power series, we obtain the matrix equation,

$$UW = 1.$$

The determinant u^I_J is the minor $\det U^I_J$ of U corresponding to the rows in I and the columns in J. It is the complementary minor to the determinant $\det U^{\tilde{I}}_{\tilde{J}}$. Now, since $\det U = 1$, it is well known that the complementary minor u^I_J is equal to the minor $\det W^{\tilde{J}}_{\tilde{I}}$ of the inverse matrix W multiplied by the signatures of the permutations $(I\,\tilde{I})$ and $(J\,\tilde{J})$, that is,

$$u^I_J = \operatorname{sign}(I\,\tilde{I}) \operatorname{sign}(J\,\tilde{J}) w^{\tilde{J}}_{\tilde{I}}. \tag{7.5.5}$$

The permutation $(I\,\tilde{I})$ of $(0, 1, \ldots, N-1)$ may be brought into strictly increasing order using $i_r - (r-1) + i_{r-1} - (r-2) + \cdots + i_2 - 1 + i_1$ simple transpositions. Hence the length of the permutation $(I\,\tilde{I})$ is equal to $\|I\| - r(r-1)/2$. Thus we obtain for the product of signatures in (7.5.5) the equation,

$$\operatorname{sign}(I\,\tilde{I}) \operatorname{sign}(J\,\tilde{J}) = (-1)^{\|I\|-\|J\|} = (-1)^{\|\tilde{J}\|-\|\tilde{I}\|}.$$

Therefore, by the homogeneity (7.5.1) with $\lambda := -1$, it follows from (7.5.5) that

$$u^I_J = v^{\tilde{J}}_{\tilde{I}}. \tag{7.5.6}$$

Finally, I' and J' are the images of $\tilde{I}$ and $\tilde{J}$ under the involution $x \mapsto x^*$. So, by the symmetry (7.5.2), the equation (7.5.3) follows from (7.5.6).

(6) Assume that an $r \times r$ matrix W is a product $W = UV$, where U is an $r \times N$ matrix and V is an $N \times r$ matrix. Then it is well known that the following formula holds for the determinant:

$$\det W = \sum_K \det U_K \det V^K, \tag{7.5.7}$$

where the sum is over all subsets K with r elements of the common set $[N]$ of indices for the columns of U and the rows of V.

Since $w = uv$, we have by (7.2.1) the matrix equation $M(w) = M(u)M(v)$. By extracting the equations for the rows in I and the columns in J, it follows that $M^I_J(w) = M^I(u)M_J(v)$. Moreover, in $M^I(u)$ only the first N columns are non-zero since I is a subset of $[N]$. It follows that $M^I_J(w) = M^I_{[N]}(u)M^{[N]}_J(v)$. Apply (7.5.7). The equation (7.5.4) is a consequence, since the product $u^I_K v^K_J$ is only non-zero when $I \supset K \supset J$. ∎

(7.6) Definition. Recall that if $I = (i_1, \dots, i_r)$ is a strictly increasing multi index of size r, then the extensions of I are the multi indices of the form,

$$\hat{I} := (0, 1, \dots, t-1, t+i_1, \dots, t+i_r).$$

Consider a second strictly increasing multi index J of the same size r, and the extension $\hat{J}$ of J (with the same t). If u is a power series with $u_0 = 1$, then it follows from (7.5)(2) that

$$u^I_J = u^{\hat{I}}_{\hat{J}}. \tag{7.6.1}$$

In particular, since the extension of $[r]$ is $[t+r]$, it follows that $u^I = u^{\hat{I}}$.

Assume that I is a subset of the interval $[N]$. The complement $\tilde{I}$ of I with respect to the interval is a strictly increasing multi index $\tilde{I} = (\tilde{i}_1, \dots, \tilde{i}_t)$ with $t := N - r$. The multi index I' defined in (7.5)(5) is $(i'_1, \dots, i'_t) = (\tilde{i}_t{}^*, \dots, \tilde{i}_1{}^*)$. It is said to be *conjugate* to I. The conjugate of the multi index $[r]$ is equal to $[t]$. Hence, under the conditions of (7.5)(5), we have the equation,

$$u^I = v^{I'}. \tag{7.6.2}$$

Note that the definition of the conjugate depends on the choice of N. However, if N is enlarged, then the new conjugate multi index is an extension of the old. In particular, if u is a power series with $u_0 = 1$, then the determinant $u^{I'}$ is independent of the choice of N.

(7.7) Notation. Let $(\alpha_1, \dots, \alpha_n)$ be a set of n elements of the ground ring. Form, in the notation of (7.2), the $n \times \infty$ matrix,

$$V = V(\alpha_1, \dots, \alpha_n) := \left\langle \frac{1}{1-\alpha_1 T}, \dots, \frac{1}{1-\alpha_n T} \right\rangle = \begin{pmatrix} 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_n \\ \vdots & & \vdots \\ \alpha_1^i & \dots & \alpha_n^i \\ \vdots & & \vdots \end{pmatrix}.$$

The rows of V are naturally indexed $0, 1, 2 \dots$. The matrix V is the evaluation at $(\alpha_1, \dots, \alpha_n)$ of the matrix of (6.3.1). Hence, for any multi index $I = (i_1, \dots, i_n)$ of size n, the determinant $\det V^I(\alpha_1, \dots, \alpha_n)$ is the evaluation of the polynomial Δ^I,

$$\Delta^I(\alpha_1, \dots, \alpha_n) = \det V^I(\alpha_1, \dots, \alpha_n). \tag{7.7.1}$$

In particular, the determinant of the matrix $V^{[n]}$ consisting of the first n rows of V is the Vandermonde determinant $\Delta(\alpha_1, \dots, \alpha_n)$.

In addition, form the two power series,

$$s = s(\alpha_1, \dots, \alpha_n) := \prod(1 - \alpha_i T)^{-1}, \quad e = e(\alpha_1, \dots, \alpha_n) := \prod(1 + \alpha_i T)$$

(of which e is a polynomial of degree at most n). Finally, for multi indices I and J of the same size r, form the determinants,

$$s_J^I = s_J^I(\alpha_1, \dots, \alpha_n) := \det M_J^I\big(s(\alpha_1, \dots, \alpha_n)\big),$$
$$e_J^I = e_J^I(\alpha_1, \dots, \alpha_n) := \det M_J^I\big(e(\alpha_1, \dots, \alpha_n)\big).$$

Note that the determinants s_J^I and e_J^I are alternating in I and J.

(7.8) Corollary. *Assume that I and J are strictly increasing multi indices of the same size r. Then:*

(1) (Vanishing) *The determinant $s_J^I = s_J^I(\alpha_1, \dots, \alpha_n)$ is non-zero only when $I \supset J$ and the following inequalities hold for all the entries in the conjugate multi indices I' and J':*

$$i'_q \le j'_q + n. \tag{7.8.1}$$

In particular, if $r > n$, then $s_{[r]}^I$ is non-zero only if I is an extension of a strictly increasing multi index I_0 of size n; moreover, if I is an extension of I_0, then $s_{[r]}^I = s_{[n]}^{I_0}$.

(2) *For $n = 1$, the determinant $s_J^I(\alpha)$ is non-zero only when the following inequalities hold:*

$$j_1 \le i_1 < j_2 \le i_2 < \dots \le i_{r-1} < j_r \le i_r. \tag{7.8.2}$$

Moreover, if the inequalities (7.8.2) *hold, then $s_J^I(\alpha) = \alpha^{\|I\| - \|J\|}$.*

(3) (Duality) *The following equation holds,*

$$s_J^I(\alpha_1, \dots, \alpha_n) = e_{J'}^{I'}(\alpha_1, \dots, \alpha_n), \tag{7.8.3}$$

where I' and J' are the conjugate multi indices.

(4) (Multiplication) *If $(\beta_1, \dots, \beta_m)$ is a second set of elements of the ground ring, then*

$$s_J^I(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m) = \sum_{I \supset K \supset J} s_K^I(\alpha_1, \dots, \alpha_n) s_J^K(\beta_1, \dots, \beta_m), \tag{7.8.4}$$

where the sum is over strictly increasing multi indices K of size r.

(5) (Jacobi–Trudi's Formula) *If I a multi index of size equal to the number n of the α_i, then*

$$s^I_{[n]}(\alpha_1, \dots, \alpha_n)\,\Delta = \Delta^I(\alpha_1, \dots, \alpha_n), \tag{7.8.5}$$

where $\Delta = \Delta(\alpha_1, \dots, \alpha_n)$ is the Vandermonde determinant.

Proof. The duality formula (3) follows from (7.5)(5) since $s(T)e(-T) = 1$. Similarly, the multiplication formula (4) follows from (7.5)(6) since $s(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m) = s(\alpha_1, \dots, \alpha_n)s(\beta_1, \dots, \beta_m)$.

Given the duality formula (3), the first vanishing statement in (1) follows from (7.5)(1) since e is a polynomial of degree at most n. Consider the special case $J = [r]$. Then $I \supset [r]$ since I is assumed to be strictly increasing. Assume that I is a subset of the interval $[N]$, where $N = r+t$. Then $J' = [t]$. So the inequalities (7.8.1) are the inequalities $i'_q \le q-1+n$ for $q = 1, \dots, t$. Obviously, they hold for all q if and only if $i'_t \le t-1+n$, that is, if and only if

$$r - n \le \tilde{i}_1, \tag{7.8.6}$$

where $\tilde{i}_1$ is the first entry in the complement $\tilde{I}$. The condition (7.8.6) is vacuous if $r \le n$. If $r > n$, then (7.8.6) holds if and only if I is an extension of a strictly increasing multi index I_0 of size n. Moreover, if I is an extension of I_0 then $s^I_{[r]} = s^{I_0}_{[n]}$ by (7.5)(2). Hence the special vanishing assertion in (1) holds.

To prove (2) assume that $n = 1$. Then $s = s(\alpha) = 1 + \alpha T + \alpha^2 T^2 + \cdots$. Hence $M(s)$ is the matrix whose ij'th entry is α^{i-j} with the (strange) convention that $\alpha^k = 0$ if the exponent k is negative. Assume that $I \supset J$ and consider the matrix $S := M^I_J(s)$. Its pp'th diagonal entry is the power $\alpha^{i_p - j_p}$ since $i_p - j_p$ is non-negative. The inequalities (7.8.2) implies that all entries above the diagonal are zero. Thus, if the inequalities (7.8.2) hold, then $s^I_J(\alpha)$ is the product of the diagonal entries and hence $s^I_J(\alpha) = \alpha^{\|I\| - \|J\|}$. Assume that the inequalities (7.8.2) do not hold, and let $q < r$ be the first index for which $i_q \ge j_{q+1}$. Consider the p'th entries in the q'th and the $(q+1)$'st column in S. By the choice of q, both entries vanish if $p < q$ and if $p \ge q$ then the entries are the powers $\alpha^{i_p - j_q}$ and $\alpha^{i_p - j_{q+1}}$. Hence the q'th column is equal to $\alpha^{j_{q+1} - j_q}$ times the $(q+1)$'th column. Therefore, the determinant $s^I_J(\alpha) = \det S$ vanishes. Hence (2) has been proved.

Finally, to prove (5), consider for $j = 1, \dots, n$ the product $d^{(j)} := \prod_{i \ne j}(1 - \alpha_i T)$. Then $d^{(j)}$ is a polynomial of degree at most $n-1$, and we have the equation of power series $sd^{(j)} = (1 - \alpha_j T)^{-1}$. Hence it follows from (7.2.1) that we have the matrix equation,

$$M(s)\big\langle d^{(j)} \big\rangle = \Big\langle \frac{1}{1 - \alpha_j T} \Big\rangle.$$

Therefore, by definition (7.7) of the matrix $V = V(\alpha_1, \dots, \alpha_n)$, we have the equation,

$$M(s)\big\langle d^{(1)}, \dots, d^{(n)} \big\rangle = V. \tag{7.8.7}$$

Let D be the $n\times n$ matrix consisting of the first n rows of the $\infty\times n$ matrix $\langle d^{(1)},\dots,d^{(n)}\rangle$. The matrix D contains all the non-zero rows, since each polynomial $d^{(j)}$ is of degree at most $n-1$. Therefore, from (7.8.7) we obtain the matrix equation,

$$M_{[n]}(s)D = V. \tag{7.8.8}$$

In (7.8.8), extract the equation corresponding to the rows in I and take determinants. Since $\det V^I = \Delta^I$ by (7.7.1), we obtain the equation,

$$s^I_{[n]}(\alpha_1,\dots,\alpha_n)\det D = \Delta^I(\alpha_1,\dots,\alpha_n). \tag{7.8.9}$$

Take $I := [n]$ in (7.8.9). On the left the determinant $s^{[n]}_{[n]}$ is equal to 1, and on the right the determinant is the Vandermonde determinant $\Delta = \Delta(\alpha_1,\dots,\alpha_n)$. It follows that $\det D = \Delta$. Now (7.8.5), for any multi index I of size n, follows from (7.8.9). □

(7.9) Corollary. *For any strictly increasing multi index J of size r contained in an interval $[r+t]$, we have the equalities,*

$$s^J(\alpha_1,\dots,\alpha_n) = s^J_{[r]}(\alpha_1,\dots,\alpha_n) = e^{J'}_{[t]}(\alpha_1,\dots,\alpha_n), \tag{7.9.1}$$

where the left side is the Schur polynomial of (6.13) *evaluated at α and the right sides are the determinants of* (7.7).

Proof. The second equation is the duality formula of (7.8.3) for $J=(0,1,\dots,r-1)$.

Clearly, to prove the first equation, we may assume that the ground ring is the polynomial ring $R[A]$ and $\alpha_i = a_i$. When the size r is equal to n, the equation follows from Jacobi–Trudi's formula, since $s^J = \Delta^J/\Delta$ by definition of the Schur polynomials. If $r<n$, then J has an extension $\hat J$ to a multi index of size n, and it follows from (7.6.1) and (6.13) that $s^J_{[r]} = s^{\hat J}_{[n]} = s^{\hat J} = s^J$. Similarly, if $r>n$ and J is an extension of a multi index J_0 of size n, then $s^J_{[r]} = s^{J_0}_{[n]} = s^{J_0} = s^J$. Finally, if $r>n$ and J is not an extension of a multi index of size n, then $s^J_{[r]} = 0 = s^J$ by (7.8)(1). □

(7.10) Application. Assume that the base ring is the polynomial ring $R[A]=R[a_1,\dots,a_n]$ over the alphabet A. The definitions of (7.7) and the results of (7.8) apply to any set of polynomials α_i, and not necessarily with the number of α_i equal to the number n of letters of A. However, a natural choice is $(\alpha_1,\dots,\alpha_n) := (a_1,\dots,a_n)$. Then the power series $s(a_1,\dots,a_n)$ and $e(a_1,\dots,a_n)$ of (7.7) are the power series $s(A)$ and $e(A)$ of (5.14). Their d'th coefficients are, respectively, the complete symmetric polynomial $s_d = s_d(A)$ and the elementary symmetric polynomial $e_d = e_d(A)$. Consequently, the matrix $M\big(s(a_1,\dots,a_n)\big)$ has as ij'th entry the complete symmetric polynomial s_{i-j} (equal to zero if $i<j$), and the matrix $M\big(e(a_1,\dots,a_n)\big)$ has as ij'th entry the elementary symmetric polynomial e_{i-j} (equal to zero if $i<j$ or $i>j+n$). It follows that the determinants $s^I_J = s^I_J(A)$ and $e^I_J = e^I_J(A)$, for multi indices I and J of the same size r, are symmetric polynomials in the letters of A.

They are alternating in I and in J. Moreover, since s_d and e_d are homogeneous of degree d, it follows that the polynomials s^I_J and e^I_J are homogeneous of degree $\|I\| - \|J\|$.

The polynomials s^I_J are called *skew Schur polynomials*. By (7.9.1), the skew Schur polynomial s^I_r, for a strictly increasing multi index I, is equal to the Schur polynomial s^I of (6.13). In particular, for the special skew Schur polynomials $s^I_{[r]}$ we may omit the subscript $[r]$ according to the notation introduced at the end of (7.4). It should be noted, however, that the corresponding determinant $e^I_{[r]}$ is not equal to the power product $e^I = e_1^{i_1} \cdots e_r^{i_r}$; we will never omit the subscript on the determinants $e^I_{[r]}$.

Note finally that the notation u_L of (7.1) for the product of the coefficients of a power series u in the cases $u = e$ and $u = s$ is in accordance with the notations e_L and s_L of (6.14) and (6.15).

The duality formula of (7.8) is valid for the polynomials s^I_J and e^I_J. The formula is an explicit expression of the skew Schur polynomial s^I_J as a determinant in the elementary symmetric polynomials e_d. In particular, with $J = [r]$ and with the conjugate I' of size t, duality is the following formula for the Schur polynomial s^I:

$$s^I = e^{I'}_{[t]} = \begin{vmatrix} e_{i'_1} & e_{i'_1-1} & \cdots & e_{i'_1-t+1} \\ \vdots & \vdots & & \vdots \\ e_{i'_t} & e_{i'_t-1} & \cdots & e_{i'_t-t+1} \end{vmatrix}. \tag{7.10.1}$$

(7.11) Note. As noted above, the determinant $s^I_J(\alpha_1, \dots, \alpha_m)$ of (7.7) is defined for any set of polynomials α_j in $R[A]$. Of course, for an arbitrary set of polynomials α_j, the determinant is not a symmetric polynomial.

As an example, consider an alphabet $(A, B) = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ obtained as the union of A and the letters b_i of a second alphabet B. Take $R[A, B] = R[A][B]$ as ground ring. Consider for strictly increasing multi indices I and J of size r the skew Schur polynomial $s^I_J(A, B)$. It is a symmetric polynomial in the letters of (A, B). In particular, it is symmetric in the letters of A and in the letters of B. By (7.8)(4), we have the formula,

$$s^I_J(A, B) = \sum_{I \supset K \supset J} s^I_K(A) s^K_J(B), \tag{7.11.1}$$

where the sum is over strictly increasing multi indices K of size r. For $J := [r]$, it follows that

$$s^I(A, B) = \sum_{I \supset K} s^I_K(A) s^K(B). \tag{7.11.2}$$

Take as r the number m of letters of B. View the two sides of (7.11.2) as polynomials in $R[A][B]$. They are symmetric polynomials in the letters of B, with coefficients that are polynomials in the letters of A. The polynomials $s^K(B)$ on the right side are the proper Schur polynomials, and they form an $R[A]$-basis for $\mathrm{Sym}_{R[A]}[B]$. Therefore, the equation (7.11.2) is the expansion of $s^I(A, B)$ in terms of the Schur basis. In other words, the skew Schur polynomials $s^I_K(A)$ may be defined as the coefficients of the Schur polynomial $s^I(A, B)$ expanded in the basis $S^K(B)$.

As a second example, consider for a fixed letter a_k of A the polynomial $S^I_J(a_k)$. It follows from (7.8)(2) that $S^I_J(a_k)$ is non-zero only if the following inequalities hold,

$$j_1 \le i_1 < j_2 \le i_2 < \cdots \le i_{r-1} < j_r \le i_r. \tag{7.11.3}$$

Moreover, if the inequalities (7.11.3) hold, then $s^I_J(a_k) = a_k^{\|I\|-\|J\|}$. For reasons that will become more transparent later we will say that I/J is a *horizontal strip* if the inequalities (7.11.3) hold.

Clearly, for strictly increasing multi indices I and J of the same size r, we obtain by repeated application of (7.11.1) the formula,

$$s^I_J(A) = \sum_{I=K_0 \supset K_1 \supset \cdots \supset K_n = J} S^I_{K_1}(a_1) \cdots S^{K_{n-1}}_J(a_n), \tag{7.11.4}$$

where the sum is over strictly increasing multi indices K_p of size r. As just observed, the sum may be restricted by the condition that each K_{p-1}/K_p is a horizontal strip, and then the corresponding term in the sum is the monomial,

$$a_1^{k_1} \cdots a_n^{k_n} \quad \text{where } k_p := \|K_{p-1} - K_p\| \text{ for } p = 1, \ldots, n. \tag{7.11.5}$$

If $I \supset J$ are multi indices of the same size r, then a *tableau of shape I/J and biggest entry n* is a sequence $T = (K_0, \ldots, K_n)$ of strictly increasing multi indices such that $I = K_0 \supset K_1 \supset \cdots \supset K_n = J$ and such that K_{p-1}/K_p is a horizontal strip for $p = 1, \ldots, n$. With each tableau T there is an *associated monomial* a^T defined as the monomial (7.11.5). The following formula is simply a fancy rewriting of (7.11.4):

$$s^I_J(A) = \sum_T a^T, \tag{7.11.6}$$

where the sum is over all tableaux T of shape I/J and biggest entry n. It is a consequence of the formula that *the skew Schur polynomial s^I_J is a sum of monomials.* Equivalently, if s^I_J is expanded in the basis of monomial symmetric polynomials m^K, then the coefficients are non-negative. More precisely, the coefficient to m^K is the number of tableaux T for which $a^T = a^K$.

(7.12) Special cases. Consider the complete symmetric polynomial s_d. It is equal to the skew Schur polynomial s^d, where d is considered as a multi index (d) of size 1. The equation $s_d = s^d$ is the equation of (6.13). It should be noted, however, that the results in this section essentially provide an alternative proof of the equation of (6.13). Indeed, by the extension property, we have the equation $s^d = s^{0,\ldots,n-1,n+d}$ for skew Schur polynomials and by Jacobi–Trudi's formula, the skew Schur polynomial $s^{0,\ldots,n-1,n+d}$ is equal to the proper

Schur polynomial $s^{0,\dots,n-1,n+d}$. Finally, in the notation of (6.13), the proper Schur polynomial s^d is simply an abbreviated notation for the proper Schur polynomial $s^{0,\dots,n-1,n+d}$.

The multi index $I := (d)$ is contained in the interval $[d+1] = \{0, 1, \dots, d\}$, and the conjugate multi index is $I' = (1, \dots, d)$. Hence, by duality,

$$s_d = e^{1,2,\dots,d}_{0,1,\dots,d-1} = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{d-1} & e_{d-2} & e_{d-3} & \dots & 1 \\ e_d & e_{d-1} & e_{d-2} & \dots & e_1 \end{vmatrix}. \tag{7.12.1}$$

For instance,

$$s_4 = \begin{vmatrix} e_1 & 1 & 0 & 0 \\ e_2 & e_1 & 1 & 0 \\ e_3 & e_2 & e_1 & 1 \\ e_4 & e_3 & e_2 & e_1 \end{vmatrix} = e_1^4 - 3e_1^2 e_2 + 2e_1 e_3 + e_2^2 - e_4\,.$$

Similarly, since $I'' = I$, we obtain the formula, equivalent to the formula in (6.13),

$$e_d = s^{1,2,\dots,d}. \tag{7.12.2}$$

Since $s(T)e(-T) = 1$, the formulas (7.12.1) and (7.12.2) could have be deduced directly from (7.3). As a direct application of (7.3), consider the power series $p = p(A)$ defined in (5.14). The d'th coefficient is the power sum $p_{d+1} = p_{d+1}(A)$. Since $e(T)p(-T) = e'(T)$, we obtain from (7.3.1) the formula,

$$p_d = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (d-1)e_{d-1} & e_{d-2} & e_{d-3} & \dots & 1 \\ de_d & e_{d-1} & e_{d-2} & \dots & e_1 \end{vmatrix}. \tag{7.12.3}$$

For instance,

$$p_4 = \begin{vmatrix} e_1 & 1 & 0 & 0 \\ 2e_2 & e_1 & 1 & 0 \\ 3e_3 & e_2 & e_1 & 1 \\ 4e_4 & e_3 & e_2 & e_1 \end{vmatrix} = e_1^4 - 4e_1^2 e_2 + 4e_1 e_3 + 2e_2^2 - 4e_4.$$

8. Base change.

(8.1) Setup. Several identities of symmetric polynomials are most easily expressed as equations in the power series ring $R[[A]]$ over the letters of the alphabet A. Recall that an element f of $R[[A]]$ is an infinite sequence $f = (f_0, f_1, \dots)$ such that f_i is a homogeneous polynomial of degree i in $R[A]$. The polynomial f_i is the i'th *term* in f. If $f_i = 0$ for all $i < d$, then f is said to have *order* at least d. A formal series,

$$\sum_\iota f_\iota, \tag{8.1.1}$$

over any set of indices ι, of elements f_ι in $R[[A]]$ is called *convergent* if, for any d, all but a finite number of f_ι have order greater than d. When the series is convergent, we may view the sum (8.1.1) as the element in $R[[A]]$ whose d'th term is the sum of the d'th terms in the f_ι.

(8.2) Proposition. *Let u be a power series in $R[[T]]$. Consider the monomial symmetric polynomials $m^K(A)$ for decreasing multi indices K of size n, and the proper Schur polynomials $s^I(A)$ for strictly decreasing multi indices of size n. Let J be a multi index of size n. Then the following two formulas hold in $R[[A]]$:*

$$\prod_{a\in A} u(a) = \sum_K u_K m^K(A), \tag{8.2.1}$$

$$s^J(A)\prod_{a\in A} u(a) = \sum_I u^I_J s^I(A). \tag{8.2.2}$$

Proof. The first formula is obtained by a simple multiplication of series,

$$\prod_{a\in A} u(a) = \prod_{q=1}^{n}\sum_{i=0}^{\infty} u_i a_q^i = \sum u_{i_1}\cdots u_{i_n} a_1^{i_1}\cdots a_n^{i_n},$$

where the last sum is over all multi indices $I = (i_1, \dots, i_n)$ of size n. Each multi index I is a permutation of a unique decreasing multi index K, and $u_I = u_K$. Hence the last sum is the right hand side of (8.2.1). Thus the first formula holds.

To prove the second formula, consider an element α in $R[[A]]$ without constant term. Then the evaluation $u(\alpha) = \sum_{i=0}^{\infty} u_i\alpha^i$ can be obtained by multiplying the infinite row $\langle u\rangle^{\mathrm{tr}} = (u_0, u_1, u_2, \dots)$ and the infinite column $V(\alpha)$ with entries $1, \alpha, \alpha^2, \dots$. Applied with $u := T^i u$, it follows that $\langle T^i u\rangle^{\mathrm{tr}} V(\alpha) = \alpha^i u(\alpha)$. Hence, in the notation of (7.2), we have the matrix equation,

$$M(u)^{\mathrm{tr}} V(\alpha) = V(\alpha)u(\alpha).$$

Applied with $\alpha := a_q$ for $q = 1, \dots, n$, we obtain the matrix equation,

$$M(u)^{\mathrm{tr}} V(a_1, \dots, a_n) = V(a_1, \dots, a_n)\,\mathrm{diag}\big(u(a_1), \dots, u(a_n)\big). \tag{8.2.3}$$

Extract the equations corresponding to the rows in J to obtain the matrix equation,

$$M_J(u)^{\mathrm{tr}} V(a_1,\dots,a_n) = V^J(a_1,\dots,a_n)\operatorname{diag}\bigl(u(a_1),\dots,u(a_n)\bigr). \tag{8.2.4}$$

Take determinants and use the formula mentioned in the proof of (7.5)(6) for the product on the left hand side. The result is the equation in $R[[A]]$,

$$\sum_I u_J^I \Delta^I(a_1,\dots,a_n) = \Delta^J(a_1,\dots,a_n)\prod_{q=1}^n u(a_q), \tag{8.2.5}$$

where the sum is over strictly increasing multi indices I of size n. The polynomials, Δ^I on the left side and Δ^J on the right side, are divisible by the Vandermonde determinant Δ, and $\Delta^I = s^I\Delta$ and $\Delta^J = s^J\Delta$. Moreover, the Vandermonde determinant Δ is a regular element of $R[[A]]$. Therefore, formula (8.2.2) follows from (8.2.5) after division by Δ. □

(8.3) Corollary. *Let B be a second alphabet with m letters. Let J be a multi index of size n. Then the following two formulas hold in $R[B][[A]]$:*

$$\prod_{a\in A,\, b\in B}\frac{1}{1-ab} = \sum_K s_K(B)m^K(A), \tag{8.3.1}$$

$$s^J(A)\prod_{a\in A,\, b\in B}\frac{1}{1-ab} = \sum_I s_J^I(B)s^I(A), \tag{8.3.2}$$

where the first sum is over all decreasing multi indices K of size n and the second sum is over all strictly increasing multi indices I of size n. In addition, the following formula holds in $R[A,B]$:

$$\prod_{a\in A,\, b\in B}(a+b) = \sum_{I\subseteq[n+m]} s^I(A)s^{\tilde I}(B)\,. \tag{8.3.3}$$

Proof. Replace in (8.2) R by $R[B]$. Clearly, the first two formulas follow from the Proposition by taking $u := s(B) = \prod_{b\in B}(1-bT)^{-1}$.

To prove the third formula, take $u := \prod_{b\in B}(T+b)$ and $J := (0,1,\dots,n-1)$ in (8.2.2). We obtain the equation,

$$\prod_{a\in A,\, b\in B}(a+b) = \sum_I u_{[n]}^I s^I(A)\,. \tag{8.3.4}$$

The pq'th entry in the determinant $u_{[n]}^I$ is the polynomial

$$u_{i_p-(q-1)} = e_{m-i_p+(q-1)}(B).$$

In particular, $u_{[n]}^I = 0$ if $i_n \ge n+m$. Assume that $i_n \le n+m-1$ and identify I with a subset of the interval $[n+m]$. Since

$$m - i_p + (q-1) = (m+n-1-i_p) - (n-q),$$

it follows that $u_{[n]}^I = e_{[n]}^{I^*}(B)$. Moreover, by duality (7.9), $e_{[n]}^{I^*}(B) = s^{\tilde I}(B)$. Hence (8.3.3) follows from (8.3.4). □

(8.4) Definition. Define in $\mathrm{Sym}_R[A]$ an R-bilinear form, denoted $(g, h) \mapsto (g \mid h)$, by the equations, for strictly increasing multi indices I and J of size n,

$$(s^I \mid s^J) := \delta_{I,J},$$

where $\delta_{I,J}$ is Kronecker's δ. In other words, if symmetric polynomials g and h are expanded in the basis of the Schur polynomials s^J, say $g = \sum \alpha_J s^J$ and $h = \sum \beta_I s^I$, then

$$(g \mid h) = \sum_J \alpha_J \beta_J.$$

Clearly, the bilinear form is symmetric. It is called the *inner product* in $\mathrm{Sym}_R[A]$. It follows from the definition that the inner product $(g \mid s^J)$ is equal to the coefficient to s^J when the symmetric polynomial g is expanded in the Schur basis.

(8.5) Proposition. (1) *If K and L are decreasing multi indices of size n, then*

$$(s_L \mid m^K) = \delta_{L,K}. \tag{8.5.1}$$

(2) *If I, J, and K are strictly increasing multi indices of size n, then*

$$(s_J^I \mid s^K) = (s^I \mid s^J s^K). \tag{8.5.2}$$

Proof. (1) Consider the expansions of $s_L(A)$ and $m^K(A)$ in the basis of proper Schur polynomials,

$$s_L(A) = \sum_J \lambda_{L,J} s^J(A), \quad m^K(A) = \sum_J \mu_{K,J} s^J(A). \tag{8.5.3}$$

Then $(s_L \mid m^K) = \sum_J \lambda_{L,J} \mu_{K,J}$. In other words, if λ and μ denote the matrices of the $\lambda_{L,J}$ and $\mu_{L,J}$, for all L, J, then (1) holds if and only if the product matrix $\lambda\mu^{\mathrm{tr}}$ is the unit matrix 1. Hence (1) holds if and only if $\mu^{\mathrm{tr}}\lambda = 1$, that is, for all strictly increasing I and J,

$$\sum_K \mu_{K,I} \lambda_{K,J} = \delta_{I,J}. \tag{8.5.4}$$

To prove (8.5.4), let B be a second alphabet with n letters. Take $J := [n]$ in (8.3). It follows from the two equations (8.3.1) and (8.3.2) that

$$\sum_K s_K(B) m^K(A) = \sum_I s^I(B) s^I(A). \tag{8.5.5}$$

The expansions (8.5.3) hold when A is replaced by B. Insert the expansions of $s_K(B)$ and $m^K(A)$ in (8.5.5). The result is the equation in $R[[B, A]]$,

$$\sum_{K,I,J} \lambda_{K,J} \mu_{K,I} s^J(B) s^I(A) = \sum_I s^I(B) s^I(A). \tag{8.5.6}$$

It follows from (8.5.6), since the Schur polynomials $s^J(A)$ form a basis for the polynomials that are symmetric in the letters of A, that for any fixed strictly increasing multi index I we have the equation in $R[[B]]$,

$$\sum_{K,J} \lambda_{K,J}\mu_{K,I}s^J(B) = s^I(B). \tag{8.5.7}$$

Again, since the $s^I(B)$ form a basis for the polynomials that are symmetric in the letters of B, it follows from (8.5.7) that (8.5.4) holds. Hence (1) has been proved.

(2) The proof of (8.5.2) is similar. Consider the expansions of s^I_J and $s^J s^K$ in the basis of proper Schur polynomials,

$$s^I_J = \sum_K \lambda_{I,J,K}s^K, \quad s^J s^K = \sum_I \mu_{J,K,I}s^I, \tag{8.5.8}$$

where both sums are over strictly increasing multi indices of size n. Then $\lambda_{I,J,K} = (s^I_J \mid s^K)$ and $(s^I \mid s^J s^K) = \mu_{J,K,I}$. Thus (8.5.2) is the equation $\lambda_{I,J,K} = \mu_{J,K,I}$.

From the equation (8.3.1) multiplied by $s^J(A)$ and the equation (8.3.2) it follows that

$$\sum_I s^I_J(B)s^I(A) = s^J(A)\sum_K s^K(B)s^K(A).$$

Insert the expansions (8.5.8) to obtain the equation,

$$\sum_{I,K} \lambda_{I,J,K}s^K(B)s^I(A) = \sum_{I,K} \mu_{J,K,I}s^K(B)s^I(A). \tag{8.5.9}$$

As in the proof of (1), it follows from (8.5.9) that $\lambda_{I,J,K} = \mu_{J,K,I}$, which is the asserted equation (8.5.2). ∎

(8.6) Note. It follows from (8.5.2) that problem of determining the coefficients in the expansions of all skew Schur polynomials s^I_J in terms of the Schur basis is the same as the problem of determining the coefficients in the expansions of all products $s^J s^K$. The coefficients are in fact non-negative, and given by a combinatorial rule, called the Littlewood–Richardson rule.

(8.7) Corollary. *The products $s_L = s_L(A)$, for all decreasing multi indices L of size n, form an R-basis for* $\mathrm{Sym}_R[A]$.

Proof. The assertion follows from (8.5.1) since the m^K form a basis. More precisely, if a symmetric polynomial g is an R-linear combination of the products s_L, say

$$g = \sum_L \alpha_L s_L, \tag{8.7.1}$$

then it follows from (8.5.1) that $\alpha_K = (g \mid m^K)$. Hence the coefficients in (8.7.1) are uniquely determined by g. To prove the existence, consider the inner products $\alpha_K := (g \mid m^K)$ for decreasing multi indices K of size n. The expansion of g in the basis of Schur polynomials involves only Schur polynomials s^J of degree at most equal to the degree of g. The expansion of m^K involves only Schur polynomials s^J of degree equal to the degree $\|K\|$. Hence the inner product α_K vanishes when $\|K\|$ is bigger than the degree of g. In particular, only a finite number of α_K are non-zero. We claim that the equation (8.7.1) holds. To prove it, consider the difference, $\tilde{g} := g - \sum_L \alpha_L s_L$. It follows from (8.5.1) that $(\tilde{g} \mid m^K) = 0$ for all m^K. Since the m^K form an R-basis for $\mathrm{Sym}_R[A]$, it follows that $(\tilde{g} \mid h) = 0$ for all symmetric polynomials h. In particular $(\tilde{g} \mid s^J) = 0$ for all Schur polynomials s^J. As a consequence, $\tilde{g} = 0$. Hence the equation (8.7.1) holds. ∎

(8.8) Definition. It follows from Theorem (5.10) that the R-algebra $\mathrm{Sym}_R[A]$ of symmetric polynomials is the free polynomial ring over R in the elementary symmetric polynomials $e_1, \dots, e_n$. In particular, there is a unique endomorphism of $\mathrm{Sym}_R[A]$ such that $e_d \mapsto s_d$ for $d = 1, \dots, n$. By definition, if I is a multi index of size n, we have that

$$e^I = e_1^{i_1} \cdots e_n^{i_n} \mapsto s_1^{i_1} \cdots s_n^{i_n}. \tag{8.8.1}$$

Note that s^I is the notation of a Schur polynomial, and hence it can not be used as a notation for the right hand side of (8.8.1). In this context, it is common to denote, for $d = 0, 1, \dots,$ the d'th complete symmetric polynomial s_d also by h_d. Accordingly, we define, for a multi index I of arbitrary size r,

$$h^I := h_1^{i_1} \cdots h_r^{i_r}, \qquad h_I := h_{i_1} \cdots h_{i_r}. \tag{8.8.2}$$

With this notation, the endomorphism of $\mathrm{Sym}_R[A]$ is given by $e^I \mapsto h^I$ for multi indices I of size n. The endomorphism is denoted $g \mapsto g^*$. Clearly, if I is a multi index of size r and $i_p = 0$ for $p = n+1, \dots, r$, then $\left(e^I\right)^* = h^I$.

(8.9) Lemma. *The endomorphism $g \mapsto g^*$ is an involution of* $\mathrm{Sym}_R[A]$. *Let K be a decreasing multi index of size r and let J be a strictly increasing multi index of size r. Consider the following two equations:*

$$(e_K)^* = h_K, \quad \left(s^J\right)^* = s^{J'}. \tag{8.9.1}$$

If $k_1 \le n$ then the first equation holds. If, for some conjugate J' of J of size t, we have that $j'_t \le n$, then the second equation holds. In particular, both equations hold if the left hand sides are non-zero and of degree at most n. Moreover, if f and g are symmetric polynomials of degree at most n, then

$$(f^* \mid g^*) = (f \mid g). \tag{8.9.2}$$

Proof. Recall that the power series $e = \prod_a (1 + aT)$ and $s = \prod_a (1 - aT)^{-1}$ are related by the equation $e(T)s(-T) = 1$. In particular, the complete symmetric polynomials s_d, for $d = 1, \dots, n$, are determined from the elementary symmetric polynomials e_d by the congruence,

$$(1 + e_1 T + \cdots + e_n T^n)(1 + s_1(-T) + \cdots + s_n(-T)^n) \equiv 1 \pmod{T^{n+1}}.$$

Substitute $T := -T$ in the congruence, and apply the endomorphism $g \mapsto g^*$. On the left, the first factor is changed to the second. Therefore, since the right side of the congruence is unchanged, the second factor is changed to the first. Hence we have, for $d = 1, \dots, n$, the equation $s_d^* = e_d$, that is, $e_d^{**} = e_d$. Since $\mathrm{Sym}_R[A]$ is generated as an R-algebra by $e_1, \dots, e_n$ and $g \mapsto g^*$ is an R-algebra endomorphism, it follows that $g^{**} = g$. Thus the endomorphism is an involution.

In the first equation in (8.9.1), the multi index K is decreasing. Hence, if $k_1 \le n$, then each factor in e_K is of the form e_k with $k \le n$, and hence $e_k^* = h_k$. Thus the first equation holds.

In the second equation, the multi index J is strictly increasing. By duality, we have the equation,

$$s^J = \det M_{[t]}^{J'}(e)\,. \tag{8.9.3}$$

In the matrix on the right side, the pq'th entry is e_d where $d = j'_p - (q-1)$. The largest possible d, obtained for $p = t$ and $q = 1$, is $d = j'_t$. Assume that $j'_t \le n$. Then every entry of the matrix on the right side is of the form e_d where $d \le n$. Therefore, if the endomorphism $g \mapsto g^*$ is applied to (8.9.3), we obtain the equation,

$$\big(s^J\big)^* = \det M_{[t]}^{J'}(s) = s^{J'}\,.$$

Thus the second equation of (8.9.1) holds.

If the product $e_K = e_{k_1} \cdots e_{k_r}$ is non-zero, then $k_1 \le n$, and so the first equation of (8.9.1) holds.

Assume similarly that Schur polynomial s^J is non-zero. To prove the equation in (8.9.1), we may replace J by any strictly increasing multi index of smaller size of which J is an extension. Thus we may assume J is not an extension of a multi index of smaller size, that is, we may assume that $j_1 > 0$. Assume that the degree $\|J\| - r(r-1)/2$ of s^J is at most n. Since $j_1 \ge 1$, it follows that $j_q \ge q$ for $q = 1, \dots, r$. Hence, from

$$n \ge \|J\| - r(r-1)/2 = j_1 + (j_2{-}1) + \cdots + (j_r{-}(r{-}1)) \ge (r-1) + j_r - (r-1),$$

it follows that $j_r \le n$. Thus, if J' is the conjugate of J determined with respect to the interval $[n+1]$, it follows that $j'_t \le n$. Hence the second equation of (8.9.1) holds.

To prove (8.9.2), note that the proper Schur polynomials s^J of degree at most n form a basis for the module of symmetric polynomials of degree at most n. It follows from the second equation of (8.9.1) that in this basis, the involution $g \mapsto g^*$ is a permutation of the basis elements. By definition, the proper Schur polynomials form an orthonormal basis with respect to the inner product. Therefore (8.9.2) holds. □

(8.10) Definition. Up to now we have found several bases for the R-module $\mathrm{Sym}_R[A]$ of symmetric polynomials. The bases $m^\bullet = \{m^K\}$ of (5.8) and $h_\bullet = \{h_K\}$ of (8.6) are indexed by decreasing multi indices K of size n. The basis $e^\bullet = \{e^I\}$ of (5.10) is indexed by arbitrary multi indices I of size n. The basis $s^\bullet = \{s^J\}$ of (6.7) is indexed by strictly increasing multi indices of size n. Since $g \mapsto g^*$ is an automorphism of $\mathrm{Sym}_R[A]$, it follows that the products $h^I = (e^I)^*$, for multi indices I of size n, form a basis $h^\bullet$.

Consider an arbitrary basis $g^\bullet = \{g^I\}$ (where I runs through some suitable index set) of $\mathrm{Sym}_R[A]$. For any symmetric polynomial f, denote by $C(f, g^\bullet)$ the row of coefficients in the expansion of f in the basis $g^\bullet$. In a similar notation, denote by $(f \mid g^\bullet)$ the row of inner products $(f \mid g^I)$. For instance, with respect to the basis $s^\bullet$ of proper Schur polynomials we have, as noted in (8.4),

$$C(f, s^\bullet) = (f \mid s^\bullet). \tag{8.10.1}$$

Similarly, if $f^\bullet = \{f^K\}$ is a second basis, denote by $C(f^\bullet, g^\bullet)$ the matrix whose K'th row is $C(f^K, g^\bullet)$, and denote by $(f^\bullet \mid g^\bullet)$ the matrix whose K'th row if $(f^K \mid g^\bullet)$. For instance, it follows from (8.10.1) that

$$C(f^\bullet, s^\bullet) = (f^\bullet \mid s^\bullet). \tag{8.10.2}$$

Moreover, it follows from (8.5.1) that

$$C(f^\bullet, m^\bullet) = (f^\bullet \mid h_\bullet), \quad C(f^\bullet, h_\bullet) = (f^\bullet \mid m^\bullet). \tag{8.10.3}$$

The matrices $C(f^\bullet, g^\bullet)$ are infinite matrices. In general, it is assumed that the symmetric polynomials of a basis are homogeneous. Then the part of the basis consisting of polynomials of fixed degree d is a finite basis for the R-module of homogeneous symmetric polynomials of degree d. Accordingly, the matrix $C(f^\bullet, g^\bullet)$ may be viewed as a sequence of quadratic matrices where the part in degree d is obtained from the parts of $f^\bullet$ and $g^\bullet$ in degree d.

Note that the two bases, $h_\bullet = \{h_K\}$ indexed by decreasing multi indices K of size n and $h^\bullet = \{h^I\}$ indexed by arbitrary multi indices of size n, agree in degree at most n, but not in degree bigger than n. For instance, h_{n+1} is part of the first basis and not of the second, and h_1^{n+1} is part of the second and not of the first.

Note also that the products e_K, for decreasing multi indices K of size n, do not form a basis, since $e_K = 0$ if $k_1 > n$. However, the products e_K, for decreasing multi indices K of size n and $\|K\| \leq n$, form a basis $e_\bullet$ for the symmetric polynomials of degree at most n, equal to the part of degree at most n of the basis $e^\bullet$.

(8.11) Definition. Consider in particular the matrix $C := C(s^\bullet, m^\bullet)$. Its IL'th entry C_{IL}, for a strictly increasing multi index I and a decreasing multi index L, both of size n, is determined by the expansion,

$$s^I = \sum_L C_{IL} m^L, \tag{8.11.1}$$

of the Schur polynomial s^I in terms of the basis of monomial symmetric functions. It follows from (7.10) that the entries C_{IL} are non-negative integers, determined combinatorically as a number of tableaux with certain properties. The numbers C_{IL} are called the *Kostka numbers*.

The Kostka number $C_{I,L}$ and, more generally, the coefficient $C_{I,L}$ in the expansion (8.11.1) of s^I for an arbitrary strictly increasing multi index of size r may be determined as follows:

There is a bijective correspondence between strictly increasing multi indices of size r and weakly decreasing multi indices of size r, given by $J \mapsto \bar{J}$, where $\bar{J} = (j_r - (r-1), \dots, j_2 -$

$1, j_1)$. Hence a sequence $K_0, \dots, K_n$ of strictly increasing multi indices of size r corresponds to an $r \times (n+1)$ matrix T of non-negative integers whose q'th column is the weakly decreasing sequence $\bar{K}_q$. The relations $K_0 \supset \cdots \supset K_n$ correspond to the condition that the entries in each row of T are weakly decreasing. The condition that each K_{q-1}/K_q is a horizontal strip corresponds to the condition that the entries in each skew diagonal of T (southwest to northeast) are weakly decreasing. The two conditions on T, on the rows and on the skew diagonals, are called the *tableau conditions*. Note that the tableau conditions imply that the entries in each column of T, which is not the last column, are weakly increasing. Hence a tableau of shape $I/[r]$, as defined in (7.11), can be identified with a matrix T satisfying the tableau conditions and such that the first column of T is equal to $\bar{I}$ and the last column consists of zeros. For each tableau, let t_q denote the sum of the entries in the q'th column of the matrix. Clearly, $\|K_{q-1} - K_q\| = t_{q-1} - t_q$. Hence it follows from (7.11.1) that

$$s^I = \sum_T a_1^{t_0 - t_1} \cdots a_n^{t_{n-1} - t_n}.$$

In particular, the coefficient C_{IL} is equal to the number of matrices T satisfying the above conditions and the equation

$$(t_0 - t_1, \dots, t_{n-1} - t_n) = L. \tag{8.11.2}$$

(8.12) Lemma. *Let $I = (i_1, \dots, i_n)$ be a strictly increasing multi index and let $L = (l_1, \dots, l_n)$ be a decreasing multi index. Form the decreasing multi index $\bar{I} = (i_n - (n-1), \dots, i_2 - 1, i_1)$. Then the Kostka number C_{IL} vanishes unless $\|\bar{I}\| = \|L\|$ and the following inequalities hold:*

$$\bar{i}_1 + \cdots + \bar{i}_q \ge l_1 + \cdots + l_q \quad \textit{for } q = 1, \dots, n. \tag{8.12.1}$$

Moreover, if all the inequalities are equalities, that is, if $L = \bar{I}$, then $C_{IL} = 1$.

Proof. The Kostka number C_{IL} is the number of $n \times (n+1)$ matrices T satisfying the conditions of (8.11).

Assume that $C_{IL} \neq 0$. Then there is a matrix $T = (t_{pq})$ satisfying the conditions. The first column T_0 of T is $\bar{I}$, and the last column T_n of T consist of zeros. Clearly, the tableau conditions on the skew diagonals of T imply that the last q entries in the q'th column are equal to 0. The column sum t_0 is equal to $\|\bar{I}\|$. Hence it follows from (8.11.2) that

$$l_1 + \cdots + l_q = t_0 - t_q = \sum_{p=1}^{n} \bar{i}_p - \sum_{p=1}^{n-q} t_{pq}. \tag{8.12.2}$$

Since the entries along the skew diagonal are weakly increasing, it follows for a term t_{pq} in the last sum that

$$t_{pq} \ge t_{p+1,q-1} \ge \cdots \ge t_{p+q,0} = \bar{i}_{p+q}. \tag{8.12.3}$$

Hence the inequalities (8.12.1) follow from (8.12.2) and (8.12.3).

Clearly, if the inequalities are equalities, then T is unique and determined by equalities in (8.12.3). Hence the last assertion of the Lemma holds. □

(8.13) Proposition. *Let $C = C(s^\bullet, m^\bullet)$ be the matrix of Kostka numbers. Let S be the quadratic matrix indexed by strictly increasing multi indices I, J of size n given by*

$$S_{IJ} = \begin{cases} 1 & \text{if } I \text{ and } J' \text{ are extensions of the same multi index,} \\ 0 & \text{otherwise.} \end{cases}$$

Then S is a symmetric matrix. Moreover, the following formulas hold:

$$C(s^\bullet, m^\bullet) = C, \tag{8.13.1}$$

$$C(h_\bullet, s^\bullet) = C^{\mathrm{tr}}, \tag{8.13.2}$$

$$C(h_\bullet, m^\bullet) = C^{\mathrm{tr}} C, \tag{8.13.3}$$

and, in degree at most n,

$$C(e_\bullet, s^\bullet) = C^{\mathrm{tr}} S, \tag{8.13.4}$$

$$C(e_\bullet, m^\bullet) = C^{\mathrm{tr}} S C. \tag{8.13.5}$$

In particular, in degree at most n, the matrices $C(h_\bullet, m^\bullet)$ and $C(e_\bullet, m^\bullet)$ are symmetric.

Proof. The matrix S is symmetric because conjugation $J \mapsto J'$ is an involution. The formula (8.13.1) is the definition of the matrix C. As noted in (8.10), we have that

$$C(h_\bullet, s^\bullet) = (h_\bullet \mid s^\bullet) = (s^\bullet \mid h_\bullet)^{\mathrm{tr}} = C(s^\bullet, m^\bullet)^{\mathrm{tr}}.$$

Hence (8.13.2) holds.

Clearly, $C(h_\bullet, m^\bullet) = C(h_\bullet, s^\bullet)C(s^\bullet, m^\bullet)$. Hence (8.13.3) follows from (8.13.1) and (8.13.2).

Restrict to the parts of degree at most n. Then it follows from (8.9) that $S = C(s^\bullet, (s^\bullet)^*)$. Therefore, again by (8.9), we have that

$$C(e_\bullet, s^\bullet) = C((e_\bullet)^*, (s^\bullet)^*) = C(h_\bullet, s^\bullet)S.$$

Hence (8.13.4) follows from (8.13.2). Finally, (8.13.5) follows from (8.13.1) and (8.13.4) since $C(e_\bullet, m^\bullet) = C(e_\bullet, s^\bullet)C(s^\bullet, m^\bullet)$. □

9. Partitions.

(9.1) Definition. A *partition* is a decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers containing only a finite number of positive terms. The term λ_i is called the *i'th part* of the partition λ.

The number of positive parts is called the *length* of the partition, the sum of the parts is called the *degree* of the partition and denoted $\|\lambda\|$. The *biggest part* of the partition is the first part λ_1, since the sequence is decreasing.

There are several convenient notations for partitions. First, we may indicate a partition by giving any finite subsequence containing all the positive parts, in particular the finite sequence containing only the positive parts. For instance, each of the sequences $(7, 7, 3, 3, 3, 1)$ and $(7, 7, 3, 3, 3, 1, 0, 0)$ represent the partition,

$$\lambda = (7, 7, 3, 3, 3, 1, 0, \dots). \tag{9.1.1}$$

The length of λ is 6, the degree of λ is 24, and the biggest part of λ is 7.

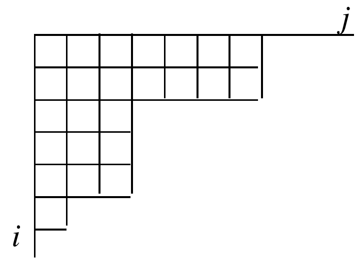
In this notation, the *zero-partition* $(0, 0, \dots)$ is represented by any finite sequence of zeros, in particular by the empty sequence $(\ )$.

Next, a partition λ may be given by its *type*, that is, by the numbers $m_p = m_p(\lambda)$ counting, for $p = 1, 2, \dots$, the number of parts of λ that are equal to p. The type is often indicated by the "formal" product $1^{m_1}2^{m_2}\cdots$. For instance, the partition λ of (9.1.1) may be given by its type $1^1 3^3 7^2$ (or $7^2 3^3 1^1$).

Note that a positive integer d, both as the sequence (d) and as the type d^1, represents the partition $(d, 0, 0, \dots)$.

Third, a partition λ may be given by its *Ferrers diagram* D_λ. The diagram D_λ consists of the set of points $(i, j) \in \mathbb{N}^2$ such that $1 \le j \le \lambda_i$. The diagram will always be pictured in a system of matrix coordinates where the first index i is a row index and the second j is a column index. Moreover, the point (i, j) will be pictured as the unit box with (i, j) as the lower right vertex.

For instance, the diagram of the partition (9.1.1) is the following:



(9.2) Definition. There are several natural order relations among partitions. First, we write $\lambda < \mu$ if either $\|\lambda\| < \|\mu\|$ or if $\|\lambda\| = \|\mu\|$ and, for the first i for which $\lambda_i \ne \mu_i$ we have that $\lambda_i < \mu_i$. The relation is a total order on the set of partitions.

Next, we write $\lambda \subset \mu$ if $\lambda_i \le \mu_i$ for all i. The relation is a partial order. With respect to the Ferrers diagrams, we have that $\lambda \subset \mu$ if and only $D_\lambda \subset D_\mu$. If $\mu \subset \lambda$, then the difference set $D_\mu - D_\lambda$ (of boxes) is called the *skew diagram* of μ/λ.

Finally, we write $\lambda \ll \mu$ if $\|\lambda\| = \|\mu\|$ and, for $q = 1, 2, \dots,$

$$\lambda_1 + \dots + \lambda_q \le \mu_1 + \cdots \mu_q.$$

Clearly, if $\lambda \ll \mu$, then $\lambda \le \mu$.

(9.3) Definition. For any partition λ, define the *conjugate partition* λ' by

$$\lambda'_p := \#\{i \mid \lambda_i \ge p\}.$$

For instance, for the partition λ of (9.1.1), the conjugate is the partition $\lambda' = (6, 5, 5, 2, 2, 2, 2) = 2^4 5^2 6^1$.

Note that the Ferrers diagram of λ' is obtained by reflecting the diagram of λ in the diagonal $i = j$, that is, the diagram $D_{\lambda'}$ is the transpose, D_λ^{tr}, of D_λ.

Clearly, we have that $\lambda'' = \lambda$. The biggest part of λ' is the length of λ, and the length of λ' is the biggest part of λ.

(9.4). There are several natural ways to associate partitions with multi indices.

(1) To a decreasing multi index K of size r, associate the partition given by the map,

$$(k_1, \dots, k_r) \mapsto (k_1, \dots, k_r, 0, 0, \dots) \tag{1}$$

Clearly, the map (1) defines a bijective correspondence between decreasing multi indices of size r and partitions of length at most r. Moreover, two decreasing multi indices of different size define the same partition if and only if the longer is obtained from the shorter by adding a trailing sequence of zeros.

Obviously, if λ and κ are the partitions associated via the map (1) to decreasing multi indices L and K of the same size, then $\lambda \le \kappa$ if and only if $L \le K$ and $\lambda \subset \kappa$ if and only if $L \subset K$.

(2) To an arbitrary multi index I of size r, associate the partition given by the map,

$$(i_1, \dots, i_r) \mapsto (i_1 + \dots + i_r, \dots, i_{r-1} + i_r, i_r, 0, 0, \dots). \tag{2}$$

Clearly, the map (2) defines a bijective correspondence between multi indices of size r and partitions of length at most r. Moreover, two multi indices of different size define the same partition if and only if the longer is obtained from the shorter by adding a trailing sequence of zeros.

(2') To an arbitrary multi index I of size r, associate the partition given by the map,

$$(i_1, \dots, i_r) \mapsto (\overbrace{r, \dots, r}^{i_r}, \dots, \overbrace{2, \dots, 2}^{i_2}, \overbrace{1, \dots, 1}^{i_1}, 0, \dots). \tag{2'}$$

In other words, the associated partition is given by the type $1^{i_1} \cdots r^{i_r}$. Clearly, the map (2') defines a bijective correspondence between multi indices of size r and partitions of biggest part at most r. Moreover, two multi indices of different size define the same partition if and only if the longer is obtained from the shorter by adding a trailing sequence of zeros.

Clearly, if I is a multi index, then the two partitions associated with I via the maps (2) and (2') are conjugate.

(3) To a strictly increasing multi index J of size r, associate the partition given by the map,

$$(j_1, \dots, j_r) \mapsto (j_r - (r-1), \dots, j_2 - 1, j_1, 0, 0, \dots). \tag{3}$$

Clearly, the map (3) defines a bijective correspondence between strictly increasing multi indices of size r and partitions of length at most r. Moreover, two strictly increasing multi indices of different size define the same partition if and only if the longer is an extension of the shorter.

(9.5) Lemma. *If λ is the partition associated to a strictly increasing multi index J of size r via the map* (3)*, then λ' is associated to the conjugate strictly increasing multi index J'.*

Proof. Indeed, assume that J is a subsequence of $[r+t]$. Clearly, the conjugate of the partition λ is given by

$$\lambda'_p = \#\{q \in [1, r] \mid j_q - (q-1) \geq p\}.$$

Since $j_q - (q-1) \leq j_r - (r-1) < (r+t) - (r-1) = t-1$, it follows that $\lambda'_p = 0$ for $p \geq t-1$. Assume $p \leq t$. Now, for any non-negative integer n, the following relations are equivalent:

$$j_q \geq n, \quad \#J \cap [n] \leq q-1, \quad \#\tilde{J} \cap [n] \leq n - (q-1), \quad \tilde{j}_{n-(q-1)} < n$$

It follows in particular, with $n := p+q-1$, that $j_q - (q-1) \geq p$ if and only if $\tilde{j}_p < p+q-1$. Hence λ'_p is the number of $q = 1, \dots, r$ such that $\tilde{j}_p - (p-1) < q$, that is,

$$\lambda'_p = r - \big(\tilde{j}_p - (p-1)\big). \tag{9.5.1}$$

Clearly, the right hand side of (9.5.1) is the partition defined by $(j'_1, \dots, j'_t)$. ∎

(9.6) Definition. For any partition λ of length at most n, define

$$m_\lambda := m^K,$$

where K is the decreasing multi index of size n corresponding to λ via the map (1). In other words,

$$m_\lambda := m^{\lambda_1, \dots, \lambda_n}.$$

Define $m^\lambda := 0$ if the length of λ is bigger than n. Note that $m_d = p_d$ is the d'th power sum, and $m_{1\dots1} = e_d$, or, in the type notation, $m_{1^d} = e_d$.

For any partition λ, define
$$e_\lambda := e_K, \quad h_\lambda := h_K$$
where K is a decreasing multi index corresponding to λ via the map (1). In other words,
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots .$$
Note that the products are finite since $e_0 = h_0 = 1$. The products are defined for arbitrary partitions λ. The product e_λ vanishes if the biggest part of λ is strictly greater than n, since $e_d = 0$ when $d > n$. Clearly, if I is any multi index and λ is associated to I via the map (2'), then $e^I = e_\lambda$.

For any partition λ of length at most n, define
$$\Delta_\lambda := \Delta^J, \qquad s_\lambda := s^J,$$
where J is the strictly increasing multi index of size n corresponding to λ via the map (3). In other words,
$$\Delta_\lambda = \Delta^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+(n-1)}. \qquad s_\lambda = s^{\lambda_n, \lambda_{n-1}+1, \dots, \lambda_1+(n-1)}.$$
Note that the empty partition ( ) corresponds to the sequence $(0, 1, \dots, n-1)$. Hence $\Delta_{()}$ is the Vandermonde determinant, and $s_{()} = s^{0,1,\dots,n-1} = 1$. Under the correspondence, the partition d corresponds to the sequence $(0, 1, \dots, n-2, n-1+d)$. Hence it follows that, with d as partition, we have that s_d is the d'th complete symmetric polynomial.

Finally, for any two partitions λ and μ of length at most n, define $e_{\lambda/\mu}$ and $s_{\lambda/\mu}$ as the determinants e^I_J and s^I_J where I and J strictly increasing multi indices of the same size associated via the map (3) to λ andμ.

(9.7). In the language of partitions, the notion of a tableau is the following: Denote by $J \mapsto \bar{J}$ the bijection (3) from strictly increasing multi indices of r to partition of length at most r. Under this correspondence, we have that $I \supset J$ if and only if $D_{\bar{I}} \supset D_{\bar{J}}$. Moreover, it $I \supset J$, then I/J is a horizontal strip as defined in (7.11) if and only if the skew diagram $D_{\bar{I}} - D_{\bar{J}}$ has no more than one box in each column.

Let $I \supset J$ be strictly increasing multi indices of size r corresponding, via the map (3), to partitions λ and μ. Then a sequence $I = K_0 \supset K_1 \supset \cdots \supset K_n = J$ of strictly increasing multi indices K_r corresponds to a sequence of partitions, $\lambda = \kappa_0 \supset \kappa_1 \supset \cdots \supset \kappa_n = \mu$. The sequence of partitions may be represented by the skew diagrams D_q of κ_{q-1}/κ_q for $q = 1, \dots, n$. The skew diagram D_q is a subdiagram of the skew diagram D of λ/μ, and it may be visualized by inserting the number q in all boxes of D_q. Note that the inserted number increase along the rows and along the columns of D. Moreover, the condition that each κ_{q-1}/κ_q is a horizontal strip means that no number q can occur more than once in each column. I other words, the condition means that the inserted numbers are strictly decreasing in each column of D. It follows that a tableau of shape I/J and biggest entry n, as defined in (7.11), can be identified with an insertion of numbers from $\{1, \dots, n\}$ in the boxes of the diagram of λ/μ such that the inserted numbers increase weakly in each row and increase strictly in each column.

Under the identifications, the degree $\|K_{q-1} - K_q\|$ is equal to the degree $\|\kappa_{q-1} - \kappa_q\|$, and hence equal to the number of boxes in the skew diagram D_q. Hence the degree $\|K_{q-1} - K_q\|$ is equal to the number, denoted $k_q(T)$ of times the number q occurs in the tableau T. The following formula is therefore a rewriting of (7.11.6):

$$s_{\lambda/\mu} = \sum_T a_1^{k_1(T)} \cdots a_n^{k_n(T)},$$

where the sum is over all tableaux T of shape λ/μ and biggest entry n. In particular, if K is a given decreasing multi index of size at most n, then the coefficient to m^K in the expansion of $s_{\lambda/\mu}$ in the basis of monomial symmetric polynomials is equal to the number of tableaux T of shape λ/μ for which

$$K = (k_1(T), \dots, k_n(T)).$$

10. Applications of Determinantal Methods.

(10.1). In the proof of Jacobi–Trudi's formula (7.8)(5), we proved, for a multi index I of size n the formula (7.8.8):

$$S_{[n]}(\alpha_1, \dots, \alpha_n)D = V(\alpha_1, \dots, \alpha_n), \tag{10.1.1}$$

where $D = D(\alpha_1, \dots, \alpha_n)$ is an $n \times n$ matrix with determinant equal to the Vandermonde determinant $\Delta = \Delta(\alpha_1, \dots, \alpha_n)$. The formula has far reaching consequences.

Consider first a power series u which is a polynomial. Then, for every element α in the ground ring, the value $u(\alpha)$ is a well defined. Recall that $V(\alpha) = \langle (1 - \alpha T)^{-1} >$ is the infinite column with entries $1, \alpha, \alpha^2, \dots$. Hence, if the column $V(\alpha)$ is multiplied from the left by the infinite row whose entries are the coefficients of $T^i u$, the result is $\alpha^i u(\alpha)$. In other words, we have the following matrix equation,

$$M(u)^{\mathrm{tr}} V(\alpha) = V(\alpha) u(\alpha).$$

As a consequence, by multiplying (10.1.1) from the left by the matrix $M(u)^{\mathrm{tr}}$ we obtain the equation,

$$M(u)^{\mathrm{tr}} S_{[n]}(\alpha_1, \dots, \alpha_n)D = V(\alpha_1, \dots, \alpha_n)\mathrm{diag}(u(\alpha_1), \dots, u(\alpha_n)). \tag{10.1.2}$$

Let J be a multi index of size n, and extract from (10.1.2) the equations corresponding to the rows in J. The result is the equation,

$$M_J(u)^{\mathrm{tr}} S_{[n]}(\alpha_1, \dots, \alpha_n)D = V^I(\alpha_1, \dots, \alpha_n)\mathrm{diag}(u(\alpha_1), \dots, u(\alpha_n)). \tag{10.1.3}$$

Finally, take determinants in the equation (10.1.3). The determinant of D is the Vandermonde determinant Δ. The determinant of the product $M_J(u)^{\mathrm{tr}} S_{[n]}$ is developed as in the proof of the multiplication formula (7.5)(6). The result is the equation,

$$\Delta \sum_{I \supset J} u^I_J s^I_{[n]}(\alpha_1, \dots, \alpha_n) = \Delta^J(\alpha_1, \dots, \alpha_n) \prod_i u(\alpha_i). \tag{10.1.4}$$

From (10.1.4) it follows, since $\Delta^J = \Delta s^J$ by definition of the Schur polynomials, that

$$\sum_{I \supset J} u^I_J s^I(\alpha_1, \dots, \alpha_n) = s^J(\alpha_1, \dots, \alpha_n). \prod_i u(\alpha_i) \tag{10.1.5}$$

Indeed, the equation follows from (10.1.4) when the ground ring is the polynomial ring $R[A]$ and $\alpha_i := a_i$, because then the Vandermonde determinant is a regular element. Therefore, since (10.1.5) is of universal nature, it holds for general ground rings.

Equation (10.1.5) holds when u is a polynomial. For some ground rings, the equation holds even when u is a power series. Assume for simplicity that the ground ring is the power series ring $R[[A]]$ over the alphabet A. If α is an element of $R[[A]]$ of positive order (that is, without constant term), then evaluation $u(\alpha)$ is the element of $R[[A]]$ defined formally by

$$u(\alpha) := u_0 + u_1\alpha + u_2\alpha^2 + \dots,$$

In fact, in the sum only finitely many terms have order less than a given number, and so the infinite sum is a well defined element of $R[[A]]$.

In this setup, if $\alpha_1, \dots, \alpha_n$ are power series in $R[[A]]$ without constant terms, then (10.1.5) holds. Indeed, (10.1.5) is an equation of power series in $R[[A]]$, and so it suffices to show that its two sides agree in every degree. However, in a fixed degree, the equation involves only finitely many terms of u. Hence, since the equation holds for polynomials u, it holds for arbitrary power series.

As a standard application, replace R by a power series ring $R[[B]]$ over a second alphabet B. Take $u := s(B)$ and $\alpha_i := a_i$. Then formula (10.1.5), for a strictly increasing multi index J of size n equal to the number of letters of A, is the following equation in $R[[A, B]]$:

$$\sum_{I \supset J} s_J^I(B) s^I(A) = s^J(A) \prod_{a \in A,\, b \in B} \frac{1}{1 - ab}\,, \tag{10.1.6}$$

where the sum is over strictly increasing multi indices I of size n. In particular, with $J = [n]$, we obtain the formula,

$$\sum_I s^I(B) s^I(A) = \prod_{a \in A,\, b \in B} \frac{1}{1 - ab}\,. \tag{10.1.7}$$

Difference operators

1. Operators on rational functions.

(1.1) Setup. Fix a commutative ring R and a finite alphabet $A = \{a_1, \dots, a_n\}$ of $n \geq 0$ letters. Denote by $R(A)$ the ring of *rational functions* in the letters of A, that is, the ring of all fractions f/g where f and g are polynomials in $R[A]$ and g is not a zero divisor in $R[A]$.

Clearly, the action of the symmetric group $\mathfrak{S}(A)$ on $R[A]$ extends to an action on $R(A)$. A rational function f in $R(A)$ is called *symmetric* if it is invariant under the action of $\mathfrak{S}(A)$, that is, if

$$\sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}(A). \tag{1.1.1}$$

The symmetric rational functions in $R(A)$ form an R-subalgebra, denoted $\mathrm{Sym}_R(A)$.

A rational function f is called *anti-symmetric* if it is semi-invariant under the action of $\mathfrak{S}(A)$ in the sense that

$$\sigma(f) = \mathrm{sign}(\sigma) f \text{ for all } \sigma \in \mathfrak{S}(A). \tag{1.1.2}$$

Clearly, the anti-symmetric rational functions in $R(A)$ form a module over the ring $\mathrm{Sym}_R(A)$ of symmetric rational functions. In particular, a product of a symmetric rational function and an anti-symmetric rational function is an anti-symmetric rational function. Similarly, a product of two anti-symmetric rational functions is a symmetric rational function.

It follows that if h is an anti-symmetric polynomial and not a zero divisor in $R[A]$ (for instance, h could be the Vandermonde determinant $\Delta = \prod_{a<b}(b - a)$), then multiplication by h defines an isomorphism from the R-module $\mathrm{Sym}_R(A)$ of symmetric rational functions onto the R-module of anti-symmetric rational functions.

(1.2) Remark. If g is any polynomial in $R[A]$, then by multiplying g by the product of the polynomials $\sigma(g)$ for all permutations $\sigma \neq 1$, we obtain a symmetric polynomial. It follows easily that any rational function in $R(A)$ can be written as a fraction f/g where g is a symmetric polynomial. Moreover, it follows that the ring of symmetric functions $\mathrm{Sym}_R(A)$ is the total fraction ring of its subring $\mathrm{Sym}_R[A]$, that is, the symmetric functions are the fractions of the form f/g where f and g are symmetric polynomials and g is not a zero divisor in $\mathrm{Sym}_R[A]$.

(1.3) Proposition. *Let E be the sequence $(0, 1, \dots, n-1)$. Then the monomials a^J for $J \subset E$ form a basis both for the algebra $R(A)$ of rational functions as a module over its subring $\mathrm{Sym}_R(A)$ of symmetric functions, and for the algebra of $R[A]$ of polynomials as a module over its subring $\mathrm{Sym}_R[A]$ of symmetric polynomials.*

Proof. As noted in (1.2), every rational function in $R(A)$ is a fraction f/g where g is a symmetric polynomial. As a consequence, the first assertion of the Lemma follows from the second.

The second assertion will be proved by induction on the number n of letters of A. Clearly, the assertion holds when $n = 1$. Assume that $n > 1$ and consider the alphabet $\bar{A} := \{a_1, \dots, a_{n-1}\}$. Then $R[A] = R[a_n][\bar{A}]$. The subring $\mathrm{Sym}_{R[a_n]}[\bar{A}]$ consists of the polynomials of $R[A]$ that are symmetric in the first $n-1$ letters. By induction, $R[A]$ is a free module over its subring $\mathrm{Sym}_{R[a_n]}[\bar{A}]$, with a basis formed by the monomials $\bar{a}^{\bar{J}}$ for $\bar{J} \subset \bar{E}$, where $\bar{E} = (0, 1, \dots, n-2)$. Therefore, it suffices to prove that the ring $\mathrm{Sym}_{R[a_n]}[\bar{A}]$ is a free module over its subring $\mathrm{Sym}_R[A]$, with a basis formed by the powers $1, a_n, \dots, a_n^{n-1}$.

In other words, it suffices to prove that any polynomial p in $\mathrm{Sym}_{R[a_n]}[\bar{A}]$ has a unique expansion,

$$p = q_0 + q_1 a_n + \cdots + q_{n-1} a_n^{n-1}, \tag{1}$$

where $q_i \in \mathrm{Sym}_R[A]$.

Assume that p is a polynomial in $\mathrm{Sym}_{R[a_n]}[\bar{A}]$. For $i = 1, \dots, n$, let $p_i := \mu(p)$ where μ is any permutation of $\mathfrak{S}(A)$ such that $\mu(a_n) = a_i$. The polynomial p_i is independent of the choice of μ, because p is symmetric in the letters $a_1, \dots, a_{n-1}$. Consider the following system of n equations,

$$p_i = q_0 + q_1 a_i + \cdots + q_{n-1} a_i^{n-1} \quad \text{for } i = 1, \dots, n, \tag{2}$$

with unknown functions q_i in $R(A)$. The determinant of the matrix of coefficients is the Vandermonde determinant Δ. Hence the system (2) has a unique solution $(q_0, \dots, q_{n-1})$ with q_i in $R(A)$.

The n'th equation in (2) is the equation (1), because $p_n = p$. Clearly, if the equation (1) holds with polynomials q_i that are symmetric in the letters of A, then the equations (2) hold. Hence it suffices to prove for the solutions q_i to the system of equations (2) that each q_i is a polynomial and symmetric in the letters of A. By Cramer's rule, the solution q_i is the fraction Q_i/Δ, where the denominator is the Vandermonde determinant and the numerator is the determinant,

$$Q_i := \begin{vmatrix} 1 & \dots & a_1^{i-1} & p_1 & a_1^{i+1} & \dots & a_1^{n-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \dots & a_n^{i-1} & p_n & a_n^{i+1} & \dots & a_n^{n-1} \end{vmatrix}.$$

Let τ be the transposition that interchanges two different letters a_j and a_k of A. It follows from the definition of the polynomials p_i that $\tau(p_i) = p_i$ when i is different from k and j and $\tau(p_k) = p_j$. Hence, when τ is applied to the determinant Q_i, the k'th and the j'th row of the determinant are interchanged, and if we substitute $a_k = a_j$ in the determinant then its j'th and k'th row become equal. Therefore, the determinant Q_i is an alternating polynomial. As a consequence, the quotient $q_i = Q_i/\Delta$ is a polynomial and symmetric in the letters of A.

Thus the Lemma has been proved. ∎

(1.4) Observation. The number of elements in the basis of (1.3) is the number, $n!$, of permutations in $\mathfrak{S}(A)$.

2. The simple difference operators.

(2.1) Definition. Consider the *(twisted) group algebra* $R(A)[\mathfrak{S}(A)]$ of the ring of rational functions $R(A)$. As a left-$R(A)$-module, the group algebra is freely generated by the permutations of $\mathfrak{S}(A)$, that is, the elements of the algebra are $R(A)$-linear combinations of permutations,

$$\alpha = \sum_{\sigma} f_\sigma \sigma, \tag{2.1.1}$$

and the multiplication in the algebra is given by the rule,

$$\sigma \cdot f = \sigma(f)\sigma. \tag{2.1.2}$$

The twisted group algebra $R(A)[\mathfrak{S}(A)]$ contains the ring of rational functions $R(A)$, and it contains the group $\mathfrak{S}(A)$. Note that σf, for $\sigma \in \mathfrak{S}(A)$ and $f \in R(A)$ can be interpreted both as the product of σ and f in the group algebra and as the function obtained from f be the action of σ. When the interpretation is not clear from the context, we write $\sigma \cdot f$ for the product in the group algebra (defined by (2.1.2)) and $\sigma(f)$ for the function obtained from the action.

A permutation μ in $\mathfrak{S}(A)$ is invertible in the group algebra. Hence it induces an inner automorphism $\alpha \mapsto \mu\alpha\mu^{-1}$ of the group algebra, called *conjugation* by μ. On the subring of rational functions, conjugation is the map $f \mapsto \mu(f)$.

In addition, the group algebra has a *canonical involution* $\alpha \mapsto \alpha^*$. It is the anti-automorphism of the group algebra defined by

$$(f\sigma)^* := \operatorname{sign}(\sigma)\sigma^{-1} \cdot f. \tag{2.1.3}$$

Note that the involution is an anti-automorphism, that is, it reverses the order of the factors in a product. It is equal to the identity on the subring $R(A)$ of rational functions. In particular, $(\sigma f)^* = f\sigma^* = \operatorname{sign}(\sigma) f \sigma^{-1}$.

(2.2) Definition. The group algebra $R(A)[\mathfrak{S}(A)]$ acts naturally on the R-module $R(A)$. More precisely, to the element α of (2.1.1) we associate the operator on $R(A)$ defined by

$$\alpha(g) := \sum_{\sigma} f_\sigma \sigma(g). \tag{2.2.1}$$

The action is faithful, that is, if an element α of the group algebra operates as the zero map on $R(A)$, then $\alpha = 0$. Equivalently, if a sum $\alpha = \sum_{i=1}^k f_i \sigma_i$, where the σ_i are k different permutations, defines the zero operator (2.2.1), then the functions f_i are equal to zero. Indeed, by a standard argument of Galois theory, the assertion is proved by induction on k. It holds when $k = 1$. Assume that $k > 1$ and that α defines the zero operator. Then, for every function $f \in R(A)$, we have that $\sigma_k(f)\alpha$ and $\alpha \cdot f$ defines the zero operator. Hence the difference $\sigma_k(f)\alpha - \alpha \cdot f$ defines the zero operator. Clearly, the difference is the sum $\sum_{i=1}^{k-1} [\sigma_k(f) - \sigma_i(f)] f_i \sigma_i$. Hence, by induction, we have that $[\sigma_k(f) - \sigma_i(f)] f_i$ for

$i = 1, \dots, k-1$. Applied with each letter of A as f, it follows that $f_i = 0$ for $i = 1, \dots, k-1$. Clearly, then also $f_k = 0$.

The algebra of all operators on $R(A)$ of the form $g \mapsto \alpha(g)$ where α is an element of the twisted group algebra will be denoted $\mathcal{E}_R(A)$. The elements of the algebra $\mathcal{E}_R(A)$ will simply be called *operators* on $R(A)$. By the result just proved, operators can be identified with elements of the twisted group algebra. Note that all operators are $\mathrm{Sym}_R(A)$-linear. The subalgebra of $\mathcal{E}_R(A)$ consisting of operators that map the subring $R[A]$ into itself will be denoted $\mathcal{E}_R[A]$. Obviously, the algebra $\mathcal{E}_R[A]$ contains the ring of polynomials $R[A]$ and the group of permutations $\mathfrak{S}(A)$, that is, the algebra contains the twisted group algebra $R[A][\mathfrak{S}(A)]$.

It will be proved in Chapter SCHUB that the algebra of operators $\mathcal{E}_R(A)$ is the full ring of all $\mathrm{Sym}_R(A)$-linear endomorphisms of $R(A)$.

Clearly, the subalgebra $\mathcal{E}_R[A]$ is invariant under the conjugations. It will be proved in Chapter SCHUB that $\mathcal{E}_R[A]$ is invariant under the canonical involution.

(2.3) Definition. The *symmetrization operator* $\delta = \delta^A$ of $R(A)$ is the following operator:

$$\delta^A := \sum_{\sigma\in\mathfrak{S}(A)} \sigma\cdot\frac{1}{\Delta} = \frac{1}{\Delta}\sum_{\sigma\in\mathfrak{S}(A)} (\mathrm{sign}\,\sigma)\sigma.$$

The two expressions are equal, because $\sigma(\Delta) = (-1)^{\ell(\sigma)}$. It is clear from the first expression that the values of the operator δ^A are symmetric functions. When the sum in the second expression is applied to a polynomial, the result is an alternating polynomial and hence divisible by Δ. Therefore the operator δ^A belongs to $\mathcal{E}_R[A]$.

Clearly, the first sum is transformed into the second by the canonical involution. Hence the operator δ^A is invariant under the canonical involution. Moreover, under conjugation by a permutation μ of $\mathfrak{S}(A)$ we have that $\mu\delta^A\mu^{-1} = (\mathrm{sign}\,\mu)\delta^A = \delta^A\mu^{-1}$.

(2.4) Definition. For $1 \le p < n$, let τ_p is the simple transposition that interchanges a_p and a_{p+1}. Define the *simple difference operator*,

$$\partial^p := \frac{1}{a_{p+1} - a_p}(1 - \tau_p), \tag{2.4.1}$$

Clearly, for each polynomial f in $R[A]$ we have that the difference $f - \tau_p(f)$ vanishes when we substitute $a_{p+1} = a_p$. Hence the difference is divisible by $a_{p+1} - a_p$. Therefore the operator ∂^p belongs to the subring $\mathcal{E}_R[A]$. Moreover, the operator ∂^p is invariant under the canonical involution:

$$(\partial^p)^* = \partial^p. \tag{2.4.2}$$

Indeed, if we let $\Delta_p := a_{p+1} - a_p$, then $\partial^p = 1/\Delta_p - (1/\Delta_p)\tau_p$. Under the canonical involution, the function $1/\Delta_p$ is invariant and τ_p is changed into $-\tau_p$. Hence $(1/\Delta_p)\tau_p$ is changed to $-\tau_p\cdot(1/\Delta_p) = (1/\Delta_p)\tau_p$, and consequently the equation (2.4.2) holds.

It will be convenient to define $\partial_p := -\partial^p$. In addition, define operators π^p and ψ^p, and π_p and ψ_p, by the equations,

$$\pi^p = \partial^p \cdot a_{p+1}, \quad \psi^p = a_p \partial^p, \quad \pi_p = \partial_p \cdot a_p, \quad \psi_p = a_{p+1}\partial_p. \tag{2.4.3}$$

Clearly, under conjugation by τ_p the function $1/\Delta_p$ changes sign and the operator $1 - \tau_p$ is invariant. Therefore, under conjugation by τ_p the three operators ∂^p, π^p and ψ^p are changed into ∂_p, π_p and ψ_p. Similarly, under conjugation by ω the functions a_p and a_{p+1} are mapped to a_{n-p+1} and a_{n-p}, and τ_p is mapped to τ_{n-p}. Hence we obtain the formulas,

$$\omega\partial^p\omega = \partial_{n-p}, \quad \omega\pi^p\omega = \pi_{n-p}, \quad \omega\psi^p\omega = \psi_{n-p}. \tag{2.4.4}$$

Since the canonical involution reverses the order of the factors, we obtain from (2.4.2) and (2.4.3) the following formulas:

$$(\partial^p)^* = -\partial_p, \quad (\pi^p)^* = -\psi_p, \quad (\psi^p)^* = -\pi_p. \tag{2.4.5}$$

Finally, by combining the previous two sets of formulas we obtain the following:

$$\omega(\partial^p)^*\omega = -\partial^{n-p}, \quad \omega(\pi^p)^*\omega = -\psi^{n-p}, \quad \omega(\psi^p)^*\omega = -\pi^{n-p}. \tag{2.4.6}$$

(2.5) Observation. It is immediate from the definition that $\partial^p(a_{p+1}) = 1$ and $\partial^p(a_p) = -1$. As a consequence, $\pi^p(1) = 1$. Note that the operator ∂^p is of degree -1 and π^p and ψ^p are of degree 0 in the variables of A.

(2.6) Lemma. *The operators of* (2.4) *are linear with respect to polynomials that are symmetric in the variables a_p and a_{p+1}. Moreover, the image of ∂^p is symmetric in these variables, and ∂^p vanishes on polynomials that are symmetric in these variables. Finally, the image of π^p is symmetric in the variables a_p and a_{p+1} and $\pi^p(1) = 1$.*

Proof. All assertions result directly from the definition. ∎

(2.7) The Leibnitz Formula. *The operator ∂^p is a τ_p-derivation, that is, for rational functions f and g in $R(A)$ we have that*

$$\partial^p(gf) = \partial^p(g)f + \tau_p(g)\partial^p(f).$$

Proof. The assertion follows by a direct calculation. ∎

(2.8) Lemma. *The following equations hold:*

$$1 = \pi^p - \psi^p, \quad \partial^p\partial^p = 0, \quad \pi^p\pi^p = \pi^p, \quad \psi^p\psi^p = -\psi^p.$$

Proof. It follows from the Leibnitz formula that $\partial^p(a_{p+1}f) = f + a_p\partial^p(f)$. Hence the first equation of the Lemma holds. The second equation follows from the second assertion of Lemma (2.6). The third equation follows from the third assertion of Lemma (2.6). Finally, the last equation is a consequence of the first and the third. ∎

(2.9) Lemma. *Let* $E := (0, 1, 2, \ldots, n-1)$ *and* $E_1 := (0, 0, 1, \ldots, n-2)$*. Then the following equation of operators holds:*

$$a^{E_1}\pi^1 \cdots \pi^{n-1} = \partial^1 \cdots \partial^{n-1} \cdot a^E.$$

Proof. The monomial a^{E_1} is equal to $a^E/(a_2 \cdots a_n)$. Define more generally, for $p = 1, \ldots, n$, the monomial $a^{E_p} := a^E/(a_{p+1} \cdots a_n)$. Then the following equation holds for $p < n$:

$$a^{E_p}\pi^p = \partial^p \cdot a^{E_{p+1}}. \tag{1}$$

Indeed, the left side of the equation is equal to $\pi^p \cdot a^{E_p}$ because a^{E_p} is symmetric in the variables a_p and a_{p+1}, and $\pi^p a^{E_p}$ is equal to the right side by definition of π^p and a^{E_p}.

Clearly, the equations (1) for $p < n$ imply that the following product of operators, for $p = 1, \ldots, n$, is independent of p:

$$\partial^1 \cdots \partial^{p-1} a^{E_p} \pi^p \cdots \pi^{n-1}.$$

Finally, the asserted equation of the Lemma is the equality of the latter products for $p = 1$ and $p = n$. ∎

3. General difference operators.

(3.1) Definition. Define, by induction on the number of letters n of the alphabet A, two operators ∂^A and π^A as follows. If $n = 1$, then both operators are equal to 1. If $n > 1$, define

$$\partial^A := \partial^{A_1}\partial^1 \cdots \partial^{n-1} \text{ and } \pi^A := \pi^{A_1}\pi^1 \cdots \pi^{n-1},$$

where A_1 is the alphabet with the $n-1$ letters $a_2, \dots, a_n$. In addition, define the operator ψ^A by the equation,

$$\psi^A := (-1)^{n(n-1)/2}\omega(\pi^A)^*\omega.$$

In (3.5) we will give a more flexible definition of all three operators, and we show in particular that the operator ψ^A satisfies the equation $\psi^A = \psi^{A_1}\psi^1 \cdots \psi^{n-1}$, analogous to the formulas used for the inductive definition of ∂^A and π^A.

(3.2) Example. If $n = 3$, then

$$\partial^A = \partial^2\partial^1\partial^2, \quad \pi^A = \pi^2\pi^1\pi^2, \quad \psi^A = \psi^1\psi^2\psi^1.$$

Indeed, the first two equations follow from the inductive definition in (3.1). By the middle equation of (DIFF.2.4.6), the third asserted equation follows by applying the involution $\alpha \mapsto \omega\alpha^*\omega$ to the second equation.

(3.3) Theorem. *Consider the symmetrization operator δ^A of* (DIFF.2.3),

$$\delta^A = \sum_{\sigma \in \mathfrak{S}(A)} \sigma \cdot \frac{1}{\Delta} = \frac{1}{\Delta} \sum_{\sigma \in \mathfrak{S}(A)} (\operatorname{sign} \sigma)\sigma. \tag{3.3.1}$$

Then, for any monomial a^J we have that

$$\delta^A(a^J) = \Delta^J/\Delta\,. \tag{3.3.2}$$

In particular, if $E = (0, 1, \dots, n-1)$, then $\delta^A(a^E) = 1$ and $\delta^A(a^J) = 0$ when $J \subset E$ and $J \neq E$. Finally, the following three operator equations hold:

$$\partial^A = \delta^A, \quad \pi^A = \partial^A \cdot a^E, \quad \psi^A = \omega(a^E)\partial^A. \tag{3.3.3}$$

Proof. The equality of the expressions in (3.3.1) was observed in the definition (DIFF.2.3) of ∂^A. Clearly, when the second sum in (3.3.1) is applied to a monomial a^J the result is the determinant Δ^J, see (SYM.6.6). Hence the equation (3.3.2) follows from the second expression for δ^A.

By definition of the Vandermonde determinant we have that $\Delta^E = \Delta$. Moreover, if $J \subset E$, then J has two equal entries, and consequently $\Delta^J = 0$. Hence we have obtained for $\delta^A(a^J)$ the special values given in the Theorem.

Consider the three operator equations of (3.3.3). The second equation follows by induction on the number of letters of A from Lemma (DIFF.2.9) and the recursive definitions of ∂^A and π^A in (3.1). Indeed, let A_1 be the alphabet of (3.1), and let E_1 by the sequence of (DIFF.2.9). Clearly, by induction, we may assume that $\pi^{A_1} = \partial^{A_1} a^{E_1}$. Hence, by the recursive definitions of ∂^A and π^A, it suffices to prove the following equation,

$$\partial^{A_1} a^{E_1} \pi^1 \cdots \pi^{n-1} = \partial^{A_1} \partial^1 \cdots \partial^{n-1} a^E.$$

The latter equation follows immediately from Lemma (DIFF.2.9).

Consider next the first equation of (3.3.3). The two sides are operators of $\mathcal{E}_R(A)$. In particular, they are $\mathrm{Sym}_R(A)$-linear operators. Therefore, by Proposition (DIFF.1.3), it suffices to prove that the two operators yield the same value when they are evaluated on a monomial a^J, where $J \le E$. The values $\delta^A(a^J)$ were found in the first part of the Theorem. The other operator ∂^A is a product of $n(n-1)/2$ simple operators ∂^p and each operator ∂^p lowers the degree by 1. Therefore, if $J \subset E$ and $J \neq E$, then $\partial^A(a^J) = 0$, because the degree of a^J is strictly less than $n(n-1)/2$. Moreover, for $J = E$ we have that $\partial^A(a^E) = \pi^A(1)$ by the second equation of (3.3.3), and $\pi^A(1) = 1$, because π^A is a composition of simple operators π^p and $\pi^p(1) = 1$ for all p. Hence, the equality $\partial^A(a^J) = \delta^A(a^J)$ holds for $J \subset E$. Consequently, the two operators are equal and the first equation of (3.3.3) has been proved.

It remains to prove the third equation of (3.3.3). The following equation holds:

$$\omega(\partial^A)^* \omega^{-1} = (-1)^{n(n-1)/2} \partial^A. \tag{3.3.4}$$

Indeed, it follows from the observations in Definition (2.3) that the equation holds for the operator δ^A and we have proved that $\partial^A = \delta^A$. Apply the involution $\alpha \mapsto (-1)^{n(n-1)/2} \omega \alpha^* \omega^{-1}$ to the second equation of (3.3.3). We obtain the equation,

$$(-1)^{n(n-1)/2} \omega(\pi^A)^* \omega^{-1} = (-1)^{n(n-1)/2} \omega(a^E) \omega(\partial^A)^* \omega^{-1}.$$

The left hand side is ψ^A by definition and the right hand side is $\omega(a^E)\partial^A$ by (3.3.4). Therefore the third equation of (3.3.3) holds.

Thus all the assertions of the Theorem have been proved. ∎

(3.4) Corollary. *Each of the three sets of operators,*

$$\{\partial^1, \dots, \partial^{n-1}\}, \qquad \{\pi^1, \dots, \pi^{n-1}\}, \qquad \{\psi^1, \dots, \psi^{n-1}\},$$

satisfies the Coxeter–Moore relations.

Proof. The first of the Coxeter–Moore relations, for any of the three sets, are satisfied because the operator ∂^p commutes with ∂^q and with a_q and a_{q+1} when $|p-q| > 1$.

Clearly, to verify the second Coxeter–Moore relation, it suffices to consider an alphabet A with 3 letters a_1, a_2, a_3. The equations, for the three sets of operators, are the following:

$$\partial^2\partial^1\partial^2 = \partial^1\partial^2\partial^1, \quad \pi^2\pi^1\pi^2 = \pi^1\pi^2\pi^1, \quad \psi^2\psi^1\psi^2 = \psi^1\psi^2\psi^1. \tag{1}$$

Consider the first equation. The left side is equal to ∂^A. Hence, by the Theorem, we obtain the equation $\partial^2\partial^1\partial^2 = \delta^A$. Now apply the involution $\alpha \mapsto \omega\alpha^*\omega$ to the latter equation. On the left, the result is $-\partial^1\partial^2\partial^1$ by the first equation of (DIFF.2.4.6) and on the right the result is $-\delta^A$ as observed in (DIFF.2.3). Hence we obtain the equation $\partial^1\partial^2\partial^1 = \delta^A$. So we have proved that both sides of the first equation of (1) are equal to δ^A. In particular, the first equation holds.

Consider the second equation in (1). The left side is the operator π^A and hence, by the Theorem, the left side is equal to the operator $\partial^2\partial^1\partial^2 a_2 a_3^2$. So, by the Coxeter relations proved for ∂^p, the left side is equal to the operator $\partial^1\partial^2\partial^1 a_2 a_3^2$. Hence it suffices to prove that the latter operator is equal to the right side, that is, it suffices to prove the following equation:

$$\partial^1\partial^2\partial^1 a_2 a_3^2 = \partial^1 a_2 \partial^2 a_3 \partial^1 a_2.$$

The latter equation is easily verified. Indeed, on the right side, a_3 commutes with ∂^1. Next apply the equation $a_2\partial^2 = \partial^2 a_3 - 1$ of Lemma (DIFF.2.8). Finally, use the equation $\partial^1\partial^1 = 0$ of Lemma (DIFF.2.8) to obtain the left hand side. Hence the second equation in (1) has been proved.

Finally, the third equation in (1) follows by applying the involution $\alpha \mapsto \omega\alpha^*\omega^{-1}$ to the second equation, since $\omega(\pi^p)^*\omega^{-1} = -\psi^{n-p}$ by Formula (DIFF.2.4.3).

Thus the second set of relations have been verified for all three sets of operators, and the proof is completed. $\square$

(3.5) Definition. Let μ be a permutation of A. Define the corresponding *difference operator* ∂^μ by the following equation:

$$\partial^\mu := \partial^{i_1} \cdots \partial^{i_r}, \tag{3.5.1}$$

where $(\tau_{i_1}, \dots, \tau_{i_r})$ is any minimal presentation of μ. Since the operators ∂^p satisfy the Coxeter-Moore relations by Corollary (3.4), it follows from Proposition (SYM.2.6) that the operator ∂^μ is well defined, that is, the right hand side of the equation is independent of the choice of the minimal presentation of μ.

Define similarly operators π^μ and ψ^μ by the equations,

$$\pi^\mu := \pi^{i_1} \cdots \pi^{i_r} \quad \text{and} \quad \psi^\mu := \psi^{i_1} \cdots \psi^{i_r}.$$

Again, it follows from Corollary (3.4) that the operators are well defined. Finally, using the equations (3.4.4) it follows from Corollary (3.4) that each of the three sets of operators, $\{\partial_1, \dots, \partial_{n-1}\}$, $\{\pi_1, \dots, \pi_{n-1}\}$, and $\{\psi_1, \dots, \psi_{n-1}\}$, satisfies the Coxeter-Moore relations. Hence we obtain operators ∂_μ, π_μ, and ψ_μ, defined by the equations,

$$\partial_\mu := \partial_{i_1} \cdots \partial_{i_r}, \quad \pi_\mu := \pi_{i_1} \cdots \pi_{i_r}, \quad \text{and} \quad \psi_\mu := \psi_{i_1} \cdots \psi_{i_r}.$$

When $\mu = \omega$, we obtain the operators defined in (3.1):

$$\partial^A = \partial^\omega, \quad \pi^A = \pi^\omega, \quad \psi^A = \psi^\omega.$$

Indeed, the first two equations follow immediately from the recursive definitions of (3.1). The third equation is a consequence of the definition of ψ^A and the general formula $\omega(\pi^\mu)^*\omega = (-1)^{\ell(\mu)}\psi^{\omega\mu^{-1}\omega}$ proved in the following Lemma.

(3.6) Lemma. *Let μ be a permutation of A. Then*

$$\partial^\mu = (-1)^{\ell(\mu)}\partial_\mu \text{ and } (\partial^\mu)^* = \partial^{\mu^{-1}}. \tag{3.6.1}$$

Moreover, the following nine formulas hold:

$$\begin{aligned}
(\partial^\mu)^* &= (-1)^{\ell(\mu)}\partial_{\mu^{-1}}, & \omega\partial^\mu\omega &= \partial_{\omega\mu\omega}, & \omega(\partial^\mu)^*\omega &= (-1)^{\ell(\mu)}\partial^{\omega\mu^{-1}\omega},\\
(\pi^\mu)^* &= (-1)^{\ell(\mu)}\psi_{\mu^{-1}}, & \omega\pi^\mu\omega &= \pi_{\omega\mu\omega}, & \omega(\pi^\mu)^*\omega &= (-1)^{\ell(\mu)}\psi^{\omega\mu^{-1}\omega},\\
(\psi^\mu)^* &= (-1)^{\ell(\mu)}\pi_{\mu^{-1}}, & \omega\psi^\mu\omega &= \psi_{\omega\mu\omega}, & \omega(\psi^\mu)^*\omega &= (-1)^{\ell(\mu)}\pi^{\omega\mu^{-1}\omega}.
\end{aligned}$$

Proof. The first equation in (3.6.1) follows from the definition of ∂^μ and ∂_μ, because $\partial^p = -\partial_p$. The second equation in (3.6.1) follows from the first equation and the first of the nine formulas.

To prove the nine formulas, let $(\tau_{i_1}, \dots, \tau_{i_r})$ be a minimal presentation of μ. Consider the equation of Definition (3.5):

$$\partial^\mu = \partial^{i_1} \cdots \partial^{i_r}. \tag{1}$$

Apply the canonical involution $\alpha \mapsto \alpha^*$. The canonical involution reverses the order of the factors in a product, and $(\partial^p)^* = -\partial_p$ by the first equation of (DIFF.2.4.5). Hence, from Equation (1) we obtain the equation,

$$(\partial^\mu)^* = (-1)^r \partial_{i_r} \cdots \partial_{i_1}. \tag{2}$$

As the reversed sequence $(\tau_{i_r}, \dots, \tau_{i_1})$ is a minimal presentation of μ^{-1}, it follows that the right side (2) is equal to $(-1)^{\ell(\mu)}\partial_{\mu^{-1}}$. Hence the first of the nine formulas holds.

To prove the second of the nine formulas, apply the conjugation $\alpha \mapsto \omega\alpha\omega$ to Equation (1). Conjugation is a homomorphism, and $\omega\partial^p\omega = \partial_{n-p}$ by the first equation of (DIFF.2.4.4). Hence from Equation (1) we obtain the equation,

$$\omega\partial^\mu\omega = \partial_{n-i_1} \cdots \partial_{n-i_r}. \tag{3}$$

As the sequence $(\tau_{n-i_1}, \dots, \tau_{n-i_r})$ is a minimal presentation of $\omega\mu\omega$, it follows that the right side of (3) is equal to $\partial_{\omega\mu\omega}$. Hence the second of the nine formulas holds.

Clearly, the third of the nine formulas is consequence of the first two formulas. Finally, the proofs of remaining two sets of three formulas are entirely analogous to the proof of the first three formulas. ∎

(3.7) Proposition. *Let μ and ν be permutations. Then:*

(1) *If $\ell(\mu) + \ell(\nu) = \ell(\mu\nu)$, then $\partial^\mu\partial^\nu = \partial^{\mu\nu}$, $\pi^\mu\pi^\nu = \pi^{\mu\nu}$ and $\psi^\mu\psi^\nu = \psi^{\mu\nu}$.*
(2) *If $\ell(\mu) + \ell(\nu) > \ell(\mu\nu)$, then $\partial^\mu\partial^\nu = 0$.*

Proof. Let $(\tau_{i_1}, \dots, \tau_{i_r})$ and $(\tau_{j_1}, \dots, \tau_{j_s})$ be minimal presentations of μ and ν. Then the concatenated sequence $(\tau_{i_1}, \dots, \tau_{i_r}, \tau_{j_1}, \dots, \tau_{j_s})$ is a presentation of $\mu\nu$. Moreover, the latter presentation is minimal if and only if $\ell(\mu) + \ell(\nu) = \ell(\mu\nu)$. Hence assertion (1) follows from the Definition (3.5).

If the concatenated sequence is not minimal, then, by Proposition (SYM.2.6), it is Coxeter–Moore equivalent to a sequence $(\tau_{k_1}, \dots, \tau_{k_t})$ where two consecutive k_i's are equal. It follows from Lemma (3.4) that $\partial^\mu \partial^\nu$ is equal to the product $\partial^{k_1} \cdots \partial^{k_t}$. The latter product vanishes by the second equation of Lemma (DIFF.2.8). Therefore assertion (2) holds. ∎

(3.8) Corollary. *For all μ in $\mathfrak{S}(A)$ we have that*

$$\partial^\omega = \partial^{\omega\mu^{-1}} \partial^\mu = \partial^\mu \partial^{\mu^{-1}\omega},$$

$$\pi^\omega = \pi^{\omega\mu^{-1}} \pi^\mu = \pi^\mu \pi^{\mu^{-1}\omega}, \quad \psi^\omega = \psi^{\omega\mu^{-1}} \psi^\mu = \psi^\mu \psi^{\mu^{-1}\omega}.$$

Proof. It follows from assertions (5) and (6) in Lemma (SYM.1.3) that $\ell(\omega) = \ell(\omega\mu^{-1}) + \ell(\mu)$. Hence the formulas of the Corollary follow from the first assertion of the Proposition. ∎

(3.9) Lemma. *For a multi index $K = (k_1, \dots, k_n)$, let $R[A]_{\subset K}$ denote the R-submodule of $R[A]$ generated by the monomials a^J for $J \subset K$. Assume that $|k_{p+1} - k_p| \le 1$ for all $p < n$. Then the R-submodule $R[A]_{\subset K}$ is invariant under the operators ∂^μ for all permutations μ of $\mathfrak{S}(A)$.*

Proof. The operator ∂^μ is a composition of the simple difference operators ∂^p. Therefore we may assume that $\mu = \tau_p$ for some $p < n$. It suffices to prove for a given multi index $J \subset K$ that the value $\partial^p(a^J)$ is an R-linear combination of monomials a^I for $I \subset K$. Moreover, since ∂^p is linear with respect to polynomials that do not depend on the variables a_p and a_{p+1}, we may assume that a_p and a_{p+1} are the only letters of the alphabet, that is, we may assume that $n = 2$ and $\mu = \tau_1$. Then $K = (k_1, k_2)$, and $J = (j_1, j_2) \subset K$, and we consider the monomials $a^I = a_1^{i_1} a_2^{i_2}$ occurring in the value $\partial^1(a^J)$.

Let k denote the larger of k_1 and k_2, and let j denote the larger of j_1 and j_2. By the hypothesis on K, we have that $(k-1, k-1) \subset K$, and by the assumption on J we have that $j \le k$. Hence $(j-1, j-1) \subset K$. Therefore, to prove the assertion, it suffices to prove for any of the monomials a^I occurring in $\partial^1(a^J)$ that $I \subset (j-1, j-1)$.

Clearly, the expansion of $\partial^1(a^J)$ is given by the formula,

$$\partial^1 a^J = \frac{a_1^{j_1} a_2^{j_2} - a_2^{j_1} a_1^{j_2}}{a_2 - a_1} = \pm \sum_{i_1, i_2}{}' a_1^{i_1} a_2^{i_2},$$

where the sum is over all pairs $I = (i_1, i_2)$ such that i_1 and i_2 are less than or equal to $j-1$, and $i_1 + i_2 = j_1 + j_2 - 1$. In particular, for each of the occurring monomials a^I we have that $I \subset (j-1, j-1)$, as asserted.

Thus we have proved the Lemma. ∎

(3.10) Lemma. *Given invertible rational functions $f_1, \dots, f_n$ and $g_1, \dots, g_n$ of $R(A)$. Define operators ξ_i for $i = 1, \dots, n$ in $\mathcal{E}_R(A)$ by $\xi_i = f_i(1-\tau_i)g_i$. Then, for every permutation μ in $\mathfrak{S}(A)$ and every minimal presentation $(\tau_{i_1}, \dots, \tau_{i_r})$ of μ we have an expansion of operators,*

$$\xi_{i_1} \cdots \xi_{i_r} = h_\mu \mu + \sum_{\ell(\nu)<\ell(\mu)} h_\nu \nu,$$

with rational functions h_ν. Moreover, the coefficient h_μ to μ in the expansion is an invertible function in $R(A)$.

As a consequence, each of the three sets of operators, $\{\partial^\mu\}$, $\{\pi^\mu\}$, and $\{\psi^\mu\}$ for μ in $\mathfrak{S}(A)$, is an $R(A)$-basis for the algebra $\mathcal{E}_R(A)$ of all operators.

Proof. Use the rule $\tau \cdot g = \tau(g)\tau$ in the algebra $\mathcal{E}_R(A)$ to develop the expression $\xi_{i_1} \cdots \xi_{i_r} = f_{i_1}(1-\tau_{i_1})g_{i_1} \cdots f_{i_r}(1-\tau_{i_r})g_{i_r}$ as an $R(A)$-linear combination of monomials in $\tau_1, \dots, \tau_n$. Clearly, each occurring monomial is of the form $\tau_{j_1} \cdots \tau_{j_q}$ where $(j_1, \dots, j_q)$ is a subsequence of $(i_1, \dots, i_r)$. In particular, the product $\tau_{j_1} \cdots \tau_{j_q}$ is either equal to μ or it has length strictly less than $\ell(\mu)$. [In fact, the product is less than or equal to μ in the Bruhat–Ehresman order by Proposition (SYM.3.7?).] Moreover, the coefficient h_μ is equal to the sign $(-1)^r$ multiplied by a product of functions obtained by applying suitable permutations to the f_i and g_i. Hence h_μ is invertible in $R(A)$. Thus the first two assertions of the Lemma holds.

Since the operators ∂^μ, π^μ and ψ^μ are all of the form $\xi_{i_1} \cdots \xi_{i_r}$ for particular choices of invertible f_i and g_i, it follows from the expansion, by induction on the length, that every permutation μ belongs to the $R(A)$-module generated by any of these sets of operators. However, the $R(A)$-module of operators can be identified with the twisted group algebra $R(A)[\mathfrak{S}(A)]$ and so the permutations μ in $\mathfrak{S}(A)$ form a basis for $\mathcal{E}(A)$ as an $R(A)$–module. Moreover, the number of operators in any of the three sets is equal to the number of permutations. Hence any of the three sets is a basis. □

(3.11) Note. We will prove in Section (SCHUB.2.4) that the operators ∂^μ for μ in $\mathfrak{S}(A)$ form a basis for the algebra of operators $\mathcal{E}_R[A]$ as a module over the ring of polynomials $R[A]$.

(3.12) Lemma. *Let μ be a permutation in $\mathfrak{S}(A)$ and let g be a rational function which is symmetric in the letters $a_2, \dots, a_n$. Then the derivative $\partial^\mu(g)$ vanishes unless $\mu = \tau_{q-1} \cdots \tau_1$ for some $q = 1, \dots, n$. Moreover, the derivative $\partial^{q-1} \cdots \partial^1(g)$ is symmetric in the letters $a_1, \dots, a_q$ and in the letters $a_{q+1}, \dots, a_n$.*

Proof. Clearly, a function f is symmetric if and only if $\partial^p(f) = 0$ for $p = 1, \dots, n-1$.

Consider, for $q = 1, \dots, n$, the permutation $\sigma_q := \tau_{q-1} \cdots \tau_1$. [Thus σ_q is the identity if $q = 1$ and the q-cycle $(q, \dots, 2, 1)$ for $q > 1$.] Then, for $p = 1, \dots, n-1$ and $p \neq q-1$, we have the equation,

$$\tau_p \sigma_q = \begin{cases} \sigma_q \tau_p & \text{if } p > q, \\ \sigma_{q+1} & \text{if } p = q, \\ \sigma_q \tau_{p+1} & \text{if } p < q-1. \end{cases}$$

The equations are easily verified. For instance, the equation $\tau_q \sigma_q = \sigma_{q+1}$ follows from the definition of the σ_q, and the remaining equations follow from the Coxeter–Moore relations.

Consider a minimal presentation of μ. If μ is not of the form σ_q, then there is a unique $q \geq 1$ so that the presentation is of the form,

$$\mu = \tau_{p_1} \cdots \tau_{p_s} \tau_{q-1} \cdots \tau_1,$$

where $p_s \neq q$. Since the presentation is minimal, it follows that $p_s \neq q-1$. Hence, by the equations proved above, there is a second minimal presentation of the form $\mu = \tau_{q_1} \cdots \tau_{q_r}$ where $q_r \geq 2$. Therefore, since g is symmetric in the letters $a_2, \dots, a_n$, it follows that $\partial^\mu(g) = 0$.

To prove the second assertion of the Lemma, let $h := \partial^{\sigma_q}(g)$. Consider, for $p = 1, \dots, n$, the derivative $\partial^p(h)$. For $p = q-1$ we have that $\partial^{q-1}(h) = 0$, since $\partial^{q-1}\partial^{q-1} = 0$. For $p < q-1$, it follows from the equations above that $\partial^p(h) = \partial^{\sigma_q}\partial^{p+1}(g)$, and $\partial^{p+1}(g) = 0$ since g is symmetric in $a_2, \dots, a_n$. Thus $\partial^p(h) = 0$ for $p = 1, \dots, q-1$, and hence h is symmetric in the letters $a_1, \dots, a_q$. Similarly, if $p > q$, it follows that $\partial^p(h) = \partial^{\sigma_q}\partial^p(g) = 0$, and hence h is symmetric in the letters $a_{q+1}, \dots, a_n$. □

(3.13) Definition. Every operator α in $\mathcal{E}_R[A]$ acts naturally on the power series ring $R[[A]]$. Indeed, α is an $R(A)$-linear combination of permutations σ. Hence, if $h \in R[A]$ is a common denominator for the coefficients, then there is an expansion, $\alpha = \sum_\sigma (h_\sigma/h)\sigma$ where the h_σ are polynomials. Let k be the order of the polynomial h. If f is a polynomial of order at least d, then the sum $\sum_\sigma h_\sigma \sigma(f)$ has order at least d. Hence the value $\alpha(f)$, which is a obtained by dividing the sum by h, is a polynomial of order at least $d-k$.

It follows that if $f = \sum f_i$ is a power series in $R[[A]]$, then there is a well defined series,

$$\alpha(f) := \sum \alpha(f_i).$$

In particular, the difference operators ∂^μ act on the power series ring $R[[A]]$. Note that the operator ∂^μ is homogeneous and lowers the degree by $d = \ell(\mu)$, that is, the homogeneous term of degree i in $\partial^\mu(f)$ is equal to $\partial^\mu(f_{i+d})$.

Clearly, the Leibnitz formula of (DIFF.2.7) holds for power series f, g in $R[[A]]$.

(3.14) Example. For $1 \leq q < n$, we have in $R(A)$ and in $R[[A]]$ the equations,

$$\partial_q\Big(\frac{1}{1-a_q}\Big) = \frac{1}{(1-a_q)(1-a_{q+1})}, \tag{3.14.1}$$

$$\partial_q \cdots \partial_1\Big(\frac{1}{1-a_1}\Big) = \frac{1}{(1-a_1)\cdots(1-a_{q+1})}. \tag{3.14.2}$$

Indeed, the first formula is obtained by applying the Leibnitz rule to the equation $(1-a_q)^{-1}(1-a_q) = 1$, and the second formula follows by induction on q from the first.

(3.15) Example. For a polynomial g depending only on the first letter a_1, it follows from (3.12) that the derivatives $\partial_\mu(g)$ vanish unless $\mu = \tau_{q-1} \cdots \tau_1$ for some $q = 1, \dots, n$. For a polynomial g of the form,

$$g := (a_1 - b_1) \cdots (a_1 - b_m), \tag{3.15.1}$$

a formula for the derivative $\partial_{q-1}\cdots\partial_1(g)$ may be obtained as follows:

Assume first that the b_i are variables, that is, let $B=\{b_1,\dots,b_m\}$ be a second alphabet with m letters, and replace the ground ring with the polynomial ring $R[B]$. Form, for $q=1,\dots,n$, the power series in $R[[A,B]]$,

$$S^{(q)}=S(a_1,\dots,a_q;B):=\frac{(1-b_1)\cdots(1-b_m)}{(1-a_1)\cdots(1-a_q)}, \tag{3.15.2}$$

and denote by $s_d(a_1,\dots,a_q;B)$ the homogeneous term of degree d in $S^{(q)}$. Then

$$\partial_{q-1}\cdots\partial_1(g)=s_{m-q+1}(a_1,\dots,a_q;B). \tag{3.15.3}$$

Indeed, the numerator in $S^{(q)}$ is the polynomial,

$$(1-b_1)\cdots(1-b_m)=1-e_1+\cdots+(-1)^m e_m,$$

where $e_i:=e_i(B)$. Hence, for $q=1$, we have the equation,

$$S^{(1)}=(1-e_1+\cdots+(-1)^m e_m)(1+a_1+a_1^2+\cdots).$$

It follows that the homogeneous term of degree d in $S^{(1)}$, for $d\ge m$, is the polynomial,

$$a_1^d-e_1a_1^{d-1}+\cdots+(-1)^m e_m a_1^{d-m}=a_1^{d-m}(a_1-b_1)\cdots(a_1-b_m).$$

In particular, for $d=m$, it follows that g is the homogeneous term of degree m in $S^{(1)}$. It follows from (3.14.2) that

$$\partial_{q-1}\cdots\partial_1 S^{(1)}=S^{(q)}.$$

Therefore, by taking the homogeneous terms of degree $m-q+1$, we obtain the asserted formula (3.15.3).

In (3.15.3), the b_i are assumed to be variables. The formula for the general case when the b_i are arbitrary elements of R is obtained by specializing. Note however, that in the collection of the homogeneous terms in the $S^{(q)}$, the b_i have to be considered as homogeneous of degree 1.

For instance, let $b_1=\cdots=b_m=0$. Then $g=a_1^m$ and the power series $S^{(q)}$ of (3.15.2) specializes to the series,

$$S(a_1,\dots,a_q)=\frac{1}{(1-a_1)\cdots(1-a_q)}.$$

It follows that

$$\partial_{q-1}\cdots\partial_1(a_1^m)=s_{m-q+1}(a_1,\dots,a_q)$$

is the $(m-q+1)$'th complete symmetric polynomial in the letters $a_1,\dots,a_q$.

4. The bilinear form.

(4.1) Definition. Define a bilinear form on the algebra $R(A)$ of rational functions by the following equation:

$$\langle f, g\rangle := \sum_{\sigma\in\mathfrak{S}(A)} \sigma\Big(\frac{fg}{\Delta}\Big) = \frac{1}{\Delta}\sum_{\sigma\in\mathfrak{S}(A)} (\operatorname{sign}\sigma)\sigma(fg). \tag{4.1.1}$$

By Definition (DIFF.2.3) of the symmetrization operator δ^A, the two sums are equal, and equal to the value $\delta^A(fg)$. Moreover, by Theorem (DIFF.3.3) and Definition (DIFF.3.5), the following equation holds:

$$\langle f, g\rangle = \partial^\omega(fg). \tag{4.1.2}$$

The bilinear form is called the *inner product* on $R(A)$.

(4.2) Lemma. *The values of the inner product* (4.1) *are symmetric functions in* $R(A)$. *Moreover, the inner product is symmetric and* $\operatorname{Sym}_R(A)$*-bilinear. Furthermore, if* α *is any operator in* $\mathcal{E}_R(A)$, *then*

$$\langle\alpha(f), g\rangle = \langle f, \alpha^*(g)\rangle, \tag{4.2.1}$$

where $\alpha\mapsto\alpha^*$ *is the canonical involution of* (DIFF.2.1). *In particular, for any permutation* μ *in* $\mathfrak{S}(A)$, *we have that*

$$\langle\mu(f), g\rangle = (\operatorname{sign}\mu)\langle f, \mu^{-1}(g)\rangle \quad \textit{and} \quad \langle\partial^\mu(f), g\rangle = \langle f, \partial^{\mu^{-1}}(g)\rangle. \tag{4.2.2}$$

Finally, if f *and* g *are polynomials in* $R[A]$, *then* $\langle f, g\rangle$ *belongs to* $\operatorname{Sym}_R[A]$.

Proof. We have observed in (DIFF.2.3) the value $\langle f, g\rangle = \delta^A(fg)$ is a symmetric function. Moreover, it is clear that the inner product is symmetric in f and g, and $\operatorname{Sym}_R(A)$-bilinear.

By additivity of the inner product it suffices to prove equation (4.2.1) when α is a product $h\mu$ of a function h in $R(A)$ and a permutation μ in $\mathfrak{S}(A)$. Moreover, since the canonical involution reverses the orders of the factors in a product, it suffices to treat separately the two cases: $\alpha = h$ and $\alpha = \mu$. In the first case the equation is obvious: $\langle hf, g\rangle = \delta^A(hfg) = \langle f, hg\rangle$. In the second case the equation is the first equation of (4.2.2), and it follows immediately by rearranging the terms in second sum in (4.1.1).

By the second equation in (DIFF.3.6.1), we have that $(\partial^\mu)^* = \partial^{\mu^{-1}}$. Hence the second equation of (4.2.2) is a special case of the general equation (4.2.1).

The final assertion of the Lemma is a consequence of the equation $\langle f, g\rangle = \delta^A(fg)$, since the symmetrization operator δ^A belongs to $\mathcal{E}_R[A]$, see Definition (DIFF.2.3). ∎

(4.3) Note. The inner product is non-degenerate, that is, if $\langle f, g\rangle = 0$ for all g, then $f = 0$. Indeed, it follows from Equation (4.4.1) that $\langle f, g\rangle$ is the result of evaluating the operator $\sum_\sigma \sigma f = \sum_\sigma \sigma(f)\sigma$ on the function g/Δ. If the result is 0 for all g, then the operator is zero and consequently, by (DIFF.2.2), zero as an element in the twisted group algebra $R(A)[\mathfrak{S}(A)]$. Hence $f = 0$.

We prove in (SCHUB.2.4) that if $\langle f, g\rangle$ is a polynomial for all polynomials g, then f is a polynomial.

(4.4) Lemma. *Let I and J be multi indices of size n such that $I \subset (n-1, \dots, 1, 0)$ and $J \subset (0, 1, \dots, n-1)$. Then the inner product $\langle a^I, a^J \rangle$ is equal to 0 unless all entries in the sequence $I + J$ are different. In the exceptional case, the sequence $I + J$ is a permutation of the sequence $(0, 1, \dots, n-1)$, and $\langle a^I, a^J \rangle$ is equal to the signature of the latter permutation.*

Proof. By definition we have that $\langle a^I, a^J \rangle = \delta^A(a^{I+J})$. Moreover, as observed in Theorem (DIFF.3.3), we have that $\delta^A(a^K) = \Delta^K/\Delta$. From the assumptions on I and J it follows that all entries in the sequence $I + J$ are non-negative integers between 0 and $n-1$. The assertion of the Lemma is a consequence, because Δ^K is alternating in K and equal to Δ when $K = (0, 1, \dots, n-1)$. $\square$

5. The Möbius transformation.

Schubert polynomials

1. Double Schubert polynomials.

(1.1) Setup. Fix an alphabet $A = \{a_1, \dots, a_n\}$ with $n \geq 0$ letters, and consider the algebras $R[A]$ of polynomials and $R(A)$ of rational functions in the letters of A. For a multi index I of size n, we denote by $R[A]_{\subseteq I}$ the R-module of polynomials generated by the monomials a^J for $J \subseteq I$. It follows from Lemma (DIFF.3.9) that the two R-modules, $R[A]_{\subseteq 0,1,\dots,n-1}$ and $R[A]_{\subseteq n-1,\dots,1,0}$, are invariant under the operators ∂^μ for $\mu \in \mathfrak{S}(A)$. Recall that $\partial_\mu = (\operatorname{sign}\mu)\partial^\mu$.

Fix a set $B = (b_1, \dots, b_n)$ of n elements in the ground ring R. If f is a polynomial in $R[A]$, we denote by $f(B)$ the value obtained by specializing the letters of A to the elements of B.

Consider the following two polynomials in $R[A]$:

$$X^B := \prod_{p+q \leq n} (a_p - b_q) \quad \text{and} \quad Y^B := \prod_{p>q} (a_p - b_q). \tag{1.1.1}$$

Note that $Y^B = \omega(X^B)$.

When the ground ring is a ring of polynomials over an alphabet B with the n letters b_i, we write $X(A, B) := X^B$ and $Y(A, B) := Y^B$.

(1.2) Lemma. *The following assertions hold:*

1. *The value $Y^B(B)$ is equal to $\Delta(B)$. Moreover, for a permutation $\mu \neq 1$ in $\mathfrak{S}(A)$, the value $\mu(Y^B)(B)$ is equal to* 0.
2. *The polynomials $\partial^\omega(Y^B)$ and $\partial_\omega(X^B)$ are equal to* 1. *Moreover, for a permutation $\mu \neq \omega$ in $\mathfrak{S}(A)$, the value $\partial_\mu(X^B)(B)$ is equal to* 0.
3. *If f is a polynomial in $R[A]_{\subseteq n-1,\dots,1,0}$, then*

$$\langle Y^B, f \rangle = f(B).$$

Proof. In (3) we may, by linearity of the inner product, assume that f is a monomial a^I for $I \subseteq (n-1, \dots, 1, 0)$ in which case the equation asserts that $\langle Y^B, a^I \rangle = b^I$. It suffices to prove the latter equation and the equations of (1) and (2) in the special case where the ground ring is a polynomial ring $R[B]$ over an alphabet B with n letters. Indeed, if the equations hold in the special case, then the equations in the general case follow by specializing the letters of

B to the given sequence of elements in R. In the polynomial ring $R[B]$, the Vandermonde determinant $\Delta(B)$ is not a zero divisor. Therefore, in the remaining part of the proof we may assume that the value $\Delta(B)$ is a not a zero divisor in R.

Consider the assertions of (1). It is obvious from the definition (1.1.1) of Y^B that $Y^B(B) = \Delta(B)$. The polynomial $\mu(Y^B)$ for a permutation μ is the product of the factors $\mu(a_p)-b_q$ for $p > q$. Assume that $\mu \neq 1$. Then there exist indices $p > q$ such that $\mu(a_p) = a_q$. Clearly, the corresponding factor $\mu(a_p) - b_q$ specializes to 0, and consequently $\mu(Y^B)(B) = 0$. Hence the assertions of (1) have been proved.

Consider the first assertion of (2). The operator ∂^ω is the symmetrization operator $\partial^A = \delta^A$. Hence $\partial^\omega = (\operatorname{sign}\omega)\partial^\omega\omega = \partial_\omega\omega$. As $X^B = \omega(Y^B)$, it follows that $\partial_\omega(X^B) = \partial^\omega(Y^B)$. Hence it suffices to prove that $\partial^\omega(Y^B) = 1$. Let $E := (0, 1, \dots, n-1)$. Clearly, by expanding the product defining Y^B we obtain an R-linear combination of monomials a^J where $J \subseteq E$, and the coefficient of a^E is equal to 1. The values of the operator $\partial^\omega = \partial^A$ on the monomials a^J for $J \subseteq E$ were determined in Theorem (DIFF.3.3). It follows that $\partial^\omega(Y^B) = 1$.

To prove the second assertion of (2), assume that $\mu \neq \omega$. It follows from Lemma (DIFF.3.10) that there is an expansion of operators,

$$\partial_\mu = \sum_{\ell(\nu)\leq\ell(\mu)} h_\nu \nu,$$

with rational functions h_ν. Since $\mu \neq \omega$, it follows that all permutations ν in the expansion are different from ω. Clearly, the denominators in the rational functions h_ν are products of factors of the form $a_p - a_q$ for $p \neq q$. Therefore, when N is sufficiently big, the operator $\Delta^N\partial_\mu$ is an $R[A]$-linear combination of permutations $\nu \neq \omega$. It follows from part (1) that if $\nu \neq \omega$, then $\nu(X^B)(B) = \nu\omega(Y^B)(B) = 0$. Consequently, when the operator $\Delta^N\partial_\mu$ is applied to X^B and A is specialized to B, we obtain the equation $\Delta^N(B)\partial_\mu(X^B)(B) = 0$. Since $\Delta(B)$ is assumed to be a non-zero divisor, the latter equation implies that $\partial_\mu(X^B)(B) = 0$. Hence the assertions of (2) have been proved.

Consider assertion (3). The polynomial Y^B is an R-linear combination of monomials a^J, where $J \subseteq (0, 1, \dots, n-1)$, and f is an R-linear combination of monomials a^I, where $I \subseteq (n-1, \dots 1, 0)$. From the latter conditions on I and J, it follows from Lemma (DIFF.4.4) that the inner product $\langle a^J, a^I\rangle$ is either 0 or ± 1. As a consequence, the inner product $\langle Y^B, f\rangle$ belongs to R.

On the other hand, from the definition of the inner product we obtain the equation,

$$\Delta\langle Y^B, f\rangle = \sum_{\mu\in\mathfrak{S}(A)} (\operatorname{sign}\mu)\mu(Y^B f).$$

Specialize A to B in the latter equation. On the left hand side, the first factor Δ specializes to $\Delta(B)$ and the second factor is left unchanged since it belongs to R. On the right hand side, by (1), the terms $\mu(Y^B f) = \mu(Y^B)\mu(f)$ corresponding to $\mu \neq 1$ specialize to 0 and the term corresponding to $\mu = 1$ specializes to $\Delta(B)f(B)$. Hence the specialization yields the following equation:

$$\Delta(B)\langle Y^B, f\rangle = \Delta(B)f(B).$$

As $\Delta(B)$ is assumed to be a non-zero divisor in R, the latter equation implies the equation of Assertion (3).

Hence all assertions of the Lemma have been proved. □

(1.3) Definition. In the setup of (1.1), define a family of polynomials X_μ^B indexed by permutations μ in $\mathfrak{S}(A)$, by the equation,

$$X_\mu^B = \partial_{\mu^{-1}\omega}(X^B). \tag{1.3.1}$$

When the ground ring is of the form $R[B]$, where B is a second alphabet with n letters, the resulting polynomials are called *double Schubert polynomials*, and they are often denoted $X_\mu(A, B)$. The double Schubert polynomials belong to the ring of polynomials $R[A, B]$.

It will be convenient to define an auxiliary family of polynomials,

$$Y_\mu^B = \partial_{\mu^{-1}}(Y^B). \tag{1.3.2}$$

By Lemma (DIFF.3.6) we have that $\omega\partial_\mu\omega = (\operatorname{sign}\mu)\partial_{\omega\mu\omega}$. Therefore the two families are related by the equation,

$$Y_\mu^B = (\operatorname{sign}\mu)\omega(X_{\mu\omega}^B). \tag{1.3.3}$$

Clearly, $X_\omega^B = X^B$ and $Y_1^B = Y^B$. It follows from Lemma (1.2)(2) that $X_1^B = Y_\omega^B = 1$.

(1.4) Remark. Note the following rule for the calculation of X_μ^B: represent μ as the sequence of indices $(i_1 \dots i_n)$ where $\mu(a_p) = a_{i_p}$. Rearrange by simple transpositions the elements in sequence so that the sequence becomes strictly decreasing, that is, solve the equation $\mu\tau_{p_1}\cdots\tau_{p_r} = \omega$ with a minimal number r. Then $X_\mu^B = \partial_{p_1}\cdots\partial_{p_r}(X^B)$.

(1.5) Example. For $n = 3$ we obtain, omitting the superscript B, the following polynomials:

$$\begin{array}{ll}
(321) = \omega, & X_{321} = X = (a_1 - b_1)(a_1 - b_2)(a_2 - b_1), \\
(312)\tau_2 = \omega, & X_{312} = \partial_2 X = (a_1 - b_1)(a_1 - b_2), \\
(231)\tau_1 = \omega, & X_{231} = \partial_1 X = (a_1 - b_1)(a_2 - b_1), \\
(132)\tau_1\tau_2 = \omega, & X_{132} = \partial_1\partial_2 X = a_1 - b_1 + a_2 - b_2, \\
(213)\tau_2\tau_1 = \omega, & X_{213} = \partial_2\partial_1 X = a_1 - b_1, \\
(123)\tau_2\tau_1\tau_2 = \omega, & X_{123} = \partial_2\partial_1\partial_2 X = 1.
\end{array}$$

(1.6) Lemma. *The polynomial X_μ^B belongs to the R-module $R[A]_{\subseteq n-1,\dots,1,0}$. Its degree is degree is equal to $\ell(\mu)$. Moreover, the polynomial X_1^B is equal to 1, and if $\mu \neq 1$, then the value $X_\mu^B(B)$ is equal to 0. Finally, if μ and ν are any permutations in $\mathfrak{S}(A)$, then the following equation holds:*

$$\partial_\nu(X_\mu^B) = \begin{cases} X_{\mu\nu^{-1}}^B & \text{if } \ell(\mu) = \ell(\mu\nu^{-1}) + \ell(\nu), \\ 0 & \text{otherwise}. \end{cases} \tag{1.6.1}$$

Proof. It is obvious from the definition that the polynomial X^B belongs to the submodule $R[A]_{\subseteq n-1,\dots,1,0}$. As noted in (1.1), the submodule is invariant under the operators ∂_ν. Therefore, the first assertion of the Lemma holds.

The polynomial X^B has degree equal to $\ell(\omega)$ and the difference operator ∂_ν lowers the degree by $\ell(\nu)$. Therefore the degree of X^B_μ is at most equal to $\ell(\omega) - \ell(\mu^{-1}\omega) = \ell(\mu)$. It follows from the last equation of the Lemma, applied with $\nu = \mu$, that $\partial_\mu(X^B_\mu) = X^B_1 = 1$. Hence, when the last equation of the Lemma has been proved, it follows that $\partial_\mu X^B_\mu \neq 0$ and therefore, the degree of X^B_μ is equal to $\ell(\mu)$.

The equation $X^B_\mu(B) = 0$ for $\mu \neq 1$ follows from the definition of X^B_μ and Lemma (1.2)(2).

Consider finally Equation (1.6.1). By definition, the left hand side is equal to the result of applying the operator $\partial_\nu\partial_{\mu^{-1}\omega}$ to the polynomial X^B. By Proposition (DIFF.3.7), the latter operator is equal to zero unless the following condition holds:

$$\ell(\nu) + \ell(\mu^{-1}\omega) = \ell(\nu\mu^{-1}\omega). \tag{1.6.2}$$

When condition (1.6.2) holds, the operator is equal to $\partial_{\nu\mu^{-1}\omega}$, and consequently, when applied to X^B the result is the polynomial $X^B_{\mu\nu^{-1}}$. Moreover, the condition for the first case in Equation (1.6.1) is equivalent to the condition (1.6.2), as it follows from Lemma (SYM.1.3). Hence the final equation of the Lemma holds. ∎

(1.7) Theorem. *The polynomials X^B_μ for permutations μ in $\mathfrak{S}(A)$ form a basis for the ring $R[A]$ of polynomials as a module over the subring* $\mathrm{Sym}_R[A]$ *of symmetric polynomials, and a basis for the ring $R(A)$ of rational functions as a module over the subring* $\mathrm{Sym}_R(A)$ *of symmetric functions. In the expansion of a function f in the latter basis, the coefficient to X^B_μ is given by the inner products,*

$$\langle Y^B_\mu, f\rangle = \langle \partial_{\mu^{-1}}(Y^B), f\rangle = \langle Y^B, \partial_\mu(f)\rangle. \tag{1.7.1}$$

Moreover, if μ and ν are permutations in $\mathfrak{S}(A)$, then we have the equation,

$$\langle Y^B_\mu, X^B_\nu\rangle = \begin{cases} 1 & \textit{if } \mu = \nu, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.7.2}$$

Finally, the polynomials X^B_μ form a basis for the R-module $R[A]_{\subseteq n-1,\dots,1,0}$ and, for every polynomial f in the latter module, the coefficient (1.7.1) *of X^B_μ satisfies the equation,*

$$\langle Y^B_\mu, f\rangle = \partial_\mu(f)(B), \tag{1.7.3}$$

that is, we have the Newton interpolation formula,

$$f = \sum_{\mu\in\mathfrak{S}(A)} \partial_\mu(f)(B) X^B_\mu. \tag{1.7.4}$$

Proof. In (1.7.1), the first equation follows from the definition (1.3.2), and the second equation holds, because the operators $\partial_{\mu^{-1}}$ and ∂_μ are adjoint with respect to the inner product by Proposition (DIFF.4.2). Hence the inner products in (1.7.1) are equal.

Let N be the R-submodule $N := R[A]_{\subseteq n-1,\dots,1,0}$ of $R[A]$. If f belongs N, then it follows from Lemma (1.2)(3) and the equations of (1.7.1) that $\langle Y^B_\mu, f\rangle = \langle Y^B, \partial_\mu(f)\rangle = \partial_\mu(f)(B)$. Hence Equation (1.7.3) holds.

By Lemma (1.6) we can apply Equation (1.7.3) with $f := X^B_\nu$. As a consequence,

$$\langle Y^M_\mu, X^B_\nu\rangle = \partial_\mu(X^B_\nu)(B). \tag{1.7.5}$$

Assume first that $\mu = \nu$. Then, by Lemma (1.6), the polynomial $\partial_\mu(X^B_\nu)$ is equal to 1. In particular, the value on the right hand side of (1.7.5) is equal to 1. Assume next that $\mu \neq \nu$. Then, again by Lemma (1.6), the polynomial $\partial_\mu(X^B_\nu)$ is either equal to 0 or it is of the form X^B_τ, with $\tau \neq 1$. In the latter case, the value $X^B_\tau(B)$ is equal to 0 by Lemma (1.6). Therefore, for $\mu \neq \nu$, the value on the right side of (1.7.5) is equal to 0. Hence Equation (1.7.2) follows from (1.7.5).

By a standard argument of linear algebra, the remaining assertions of the Theorem are consequences of the equations (1.7.2). Indeed, let $d = n!$ be the cardinality of $\mathfrak{S}(A)$. Let M denote the algebra of rational functions $R(A)$ as a module over the ring $S := \mathrm{Sym}_R(A)$. Define, for each of the d elements ν in $\mathfrak{S}(A)$, an S-linear form $\check{X}_\nu$ on the module M:

$$\check{X}_\nu(f) := \langle Y^B_\nu, f\rangle.$$

Now, the d elements X^B_μ of the S-module M define an S-linear map $X\colon S^d \to M$, and the d linear forms $\check{X}_\nu$ define an S-linear map $\check{X}\colon M \to S^d$. The equation (1.7.2) asserts that $\check{X}X = 1$. By Proposition (DIFF.1.3), the S-module M can be identified with S^d via the basis formed by the d monomials a^I for $I \subseteq (n-1,\dots,1,0)$. Under the latter identification, the linear maps X and $\check{X}$ are $d \times d$ matrices. Consequently, the equation $\check{X}X = 1$ implies that the matrix X is invertible. Hence the polynomials X^B_μ form an S-basis for M. Moreover, the linear forms $\check{X}_\nu$ form the dual basis.

Clearly, in the expansion of a rational function f of $R(A)$ in terms of the basis X^B_μ, the coefficient of X^B_μ is given by evaluation of the linear form $\check{X}_\mu$ on f. In other words, the coefficient is given by the expressions of (1.7.1). Hence we have proved the second assertion of the Theorem.

By the same standard argument, applied to the ring of polynomials $R[A]$ as a module over the subring $\mathrm{Sym}_R[A]$, it follows that the polynomials X^B_μ form a $\mathrm{Sym}_R[A]$-basis for $R[A]$. Hence the first assertion of the Theorem has been proved.

Consider finally the R-module N. It contains the polynomials X^B_μ, as noted in Lemma (1.6). Moreover, on the module N the forms $\check{X}_\nu$ take values in R, by (1.7.3). Therefore, again by the same standard argument, the polynomials X^B_μ form an R-basis for N and, in the expansion of a polynomial f of N, the coefficient to X^B_μ is given by the value (1.7.3). Therefore, the interpolation formula of Newton holds. ∎

(1.8) Example. For $1 \leq p < n$ we have the equation,

$$X_{\tau_p}(A, B) = a_1 + \cdots + a_p - b_1 - \cdots - b_p. \tag{1.8.1}$$

The equation follows by applying the interpolation formula to the polynomial f on the right hand side. Clearly, $f(B) = 0$. If $\mu \neq 1$, then $\partial_\mu f$ is equal to 0 unless $\mu = \tau_p$ in which case $\partial_{\tau_p}(f) = 1$.

(1.9) Example. For $0 \leq p < n$ we have the following equation,

$$X_{\tau_p \cdots \tau_1}(A, B) = (a_1 - b_1)(a_1 - b_2) \cdots (a_1 - b_p). \tag{1.9.1}$$

Indeed, we may assume that the ground ring is a polynomial ring $R[B]$, and that $b_1, \ldots, b_n$ are the letters of B. Denote by g the polynomial in $R[A, B]$ on the right hand side of (1.9.1).

It follows from Example (DIFF.3.15) that $\partial_\mu(g) = 0$ unless μ is of the form $\tau_q \cdots \tau_1$. Moreover, $\partial_q \cdots \partial_1(g)$ is the $(p - q)$'th homogeneous term in the power series in $R[[A, B]]$,

$$S(a_1, \ldots, a_{q+1}; B) = \frac{(1 - b_1) \cdots (1 - b_p)}{(1 - a_1) \cdots (1 - a_{q+1})}. \tag{1.9.2}$$

If $q > p$, then $p - q$ is negative, and the homogeneous term vanishes. If $q = p$, then the term is equal to 1. Assume that $q < p$. Then it follows that the value $\partial_q \cdots \partial_1(g)(B)$ is the $(p - q)$'th homogeneous term in the series obtained from (1.9.2) by substituting $a_i := b_i$. The series obtained is the polynomial,

$$(1 - b_{q+2}) \cdots (1 - b_p),$$

which is of degree $p - q - 1$. In particular, its term of degree $p - q$ vanishes.

Hence, the value $\partial_\mu(g)(B)$ is equal to 1 for $\mu = \tau_p \cdots \tau_1$, and equal to zero otherwise. Therefore, the asserted equation $g = X_{\tau_p \cdots \tau_1}$ follows from (1.7.4).

(1.10) Note. *The Newton interpolation in one variable.* The monomial a_1^p, for $p < n$, is obtained from the polynomial g of (1.9) by specializing the letters of B to 0. The power series (1.9.2) specializes to the series $S(a_1, \ldots, a_{q+1})$. It follows from (1.9) that $\partial_\mu(a_1^p)$ vanishes unless μ is of the form $\tau_q \cdots \tau_1$. Moreover,

$$\partial_q \cdots \partial_1(a_1^p) = s_{p-q}(a_1, \ldots, a_{q+1})$$

is the $(p - q)$'th complete symmetric function in the letters $a_1, \ldots, a_{q+1}$.

Now, let f be a polynomial in the variable a_1 only, and of degree at most n. Then, by R-linearity of ∂_μ, it follows that $\partial_\mu(f)$ vanishes unless the permutation μ is of the form $\mu = \tau_q \cdots \tau_1$ for some $q = 0, \ldots, n - 1$. Therefore, for any sequence B of elements in the ground ring, we obtain from the Newton interpolation formula (1.7.4) and Equation (1.9.1) the following formula,

$$f = \sum_{q=0}^{n-1} \partial_q \cdots \partial_1(f)(B)\,(a_1 - b_1) \cdots (a_1 - b_q). \tag{1.10.1}$$

(1.11) Remark. Assume that the ground ring is $R[B]$, where $B = \{b_1, \dots, b_n\}$ is a second alphabet. Then the polynomials $X(A, B)$ and $Y(A, B)$ can be expressed as multi Schur functions. In fact, with the words $A_p := a_1 + \cdots + a_p$ and $B_p := b_1 + \cdots + b_p$ for $p = 1, \dots, n$, we have the formulas,

$$s^{1,3,\dots,2n-3}(A_{n-1} - B_1, \dots, A_1 - B_{n-1}) = X(A, B) \tag{1.11.1}$$

$$s^{n-1,\dots,n-1}(A_n - B_{n-1}, \dots, A_2 - B_1, A_1) = Y(A, B). \tag{1.11.2}$$

Indeed, from the Factorization formula (SCHUR.2.8), applied with $r := n-1$ to the sequence $a_{n-1}, \dots, a_2, a_1$ we obtain for the left side of (1.11.1) the factorization,

$$s_{n-1}(a_n - B_{n-1}) \cdots s_1(a_2 - B_1) s_0(a_1) = \prod_{p>q} (a_p - b_q),$$

and, by the same formula, applied with $r := n$ to the sequence $a_n, \dots, a_1$, we obtain for the left side of (1.11.2) the factorization,

$$s_{n-1}(a_n - B_{n-1}) \cdots s_1(a_2 - B_1) s_0(a_1) = \prod_{p>q} (a_p - b_q).$$

2. Simple Schubert polynomials.

(2.1) Definition. We denote by X_μ and Y_μ the polynomials X_μ^B and Y_μ^B corresponding to the sequence $B = (0, \dots, 0)$. We call the polynomials X_μ the *Schubert polynomials*.

By Definition (SCHUB.1.3) we have that $X_\mu = \partial_{\mu^{-1}\omega} X = \operatorname{sign}(\mu\omega)\omega(Y_{\mu\omega})$, where X and Y are the monomials,

$$X := \prod_{i+j\le n} a_i = a_1^{n-1}\cdots a_n^0 \quad \text{and} \quad Y := \prod_{i>j} a_i = a_1^0 \cdots a_n^{n-1}. \tag{2.1.1}$$

(2.2) Example. For $n = 3$, it follows from the computation in (SCHUB.1.5) that $X_{321} = a_1^2 a_2$, $X_{312} = a_1^2$, $X_{231} = a_1 a_2$, $X_{132} = a_1 + a_2$, $X_{213} = a_1$, and $X_{123} = 1$.

(2.3) Remark. Note that Theorem (SCHUB.1.7) applies to the special sequence $B = (0, \dots, 0)$. In particular, the Schubert polynomials X_μ for μ in $\mathfrak{S}(A)$ form a basis for the algebra of rational functions $R(A)$ as a module over the ring of symmetric functions $\operatorname{Sym}_R(A)$, they form a basis for the algebra of polynomials $R[A]$ as a module over the ring of symmetric polynomials $\operatorname{Sym}_R[A]$, and they form an R-basis for the R-submodule generated by the monomials a^I for $I \subset (n-1, \dots, 1, 0)$. As it follows from the proofs of the following results, the basis of Schubert polynomials has extremely good properties with respect to the general difference operators ∂_μ and with respect to the inner product in $R(A)$.

(2.4) Proposition. *The algebra $\mathcal{E}_R(A)$ of operators on $R(A)$ is the full ring of endomorphisms of $R(A)$ as a module over the ring of symmetric functions* $\operatorname{Sym}_R(A)$. *Similarly, the subalgebra $\mathcal{E}_R[A]$ is the full ring of endomorphisms of $R[A]$ as a module over the ring of symmetric polynomials* $\operatorname{Sym}_R[A]$. *Moreover, the general difference operators ∂_μ for μ in $\mathfrak{S}(A)$ form a basis of $\mathcal{E}_R(A)$ as a module over $R(A)$, and a basis for $\mathcal{E}_R[A]$ as a module over $R[A]$. Furthermore, if f is a rational function in $R(A)$ such that the inner product $\langle g, f\rangle$ is a polynomial for every polynomial g in $R[A]$, then f is a polynomial in $R[A]$. Finally, the subring $\mathcal{E}_R[A]$ of $\mathcal{E}_R(A)$ is invariant under the canonical involution of* (DIFF.2.1).

Proof. Let M denote the algebra of rational functions $R(A)$ as a module over the ring $S := \operatorname{Sym}_R(A)$. The Schubert polynomials X_μ form an S-basis for M. Hence, if $\{\check{X}_\mu\}$ denotes the dual basis for the module of S-linear forms on M, every S-linear endomorphism of M is of the form $f \mapsto \sum_\mu \check{X}_\mu(f) h_\mu$ for rational functions h_μ in M. Therefore, to prove the first assertion of the Proposition, it suffices to prove for every permutation μ and every rational function h that the map $f \mapsto h\check{X}_\mu(f)$ belongs to the ring $\mathcal{E}_R(A)$. Now, by the equation (SCHUB.1.7.2), for the dual basis $\check{X}_\mu$ we have the equation,

$$\check{X}_\mu(f) = \langle Y_\mu, f\rangle.$$

Moreover, by Definition (DIFF.4.1) of the inner product, $\langle Y_\mu, f\rangle = \delta^A(Y_\mu f)$. Hence the map $f \mapsto h\check{X}_\mu(f)$ is the operator $h\delta^A\cdot Y_\mu$ belonging to $\mathcal{E}_R(A)$. Thus the first assertion of the Proposition has been proved. The proof of the second assertion entirely similar.

Consider the general difference operators ∂_μ. It was proved in Lemma (DIFF.3.10) that the ∂_μ form an $R(A)$-basis for $\mathcal{E}_R(A)$. Hence, every operator α of $\mathcal{E}_R(A)$ has a unique expansion,

$$\alpha = \sum_{\mu \in \mathfrak{S}(A)} \alpha_\mu \partial_\mu,$$

with uniquely determined coefficients α_μ in $R(A)$. If $\alpha \in \mathcal{E}_R[A]$, then the coefficients α_μ are polynomials. Indeed, to prove that $\alpha_\nu \in R[A]$ we may, by induction on $\ell(\nu)$, assume that $\alpha_\mu \in R[A]$ for all permutations μ with $\ell(\mu) < \ell(\nu)$. By subtracting from α the sum of the $\alpha_\mu \partial_\mu$ for $\ell(\mu) < \ell(\nu)$, we may assume that $\alpha_\mu = 0$ for $\ell(\mu) < \ell(\nu)$. By (SCHUB.1.6), if $\ell(\mu) \geq \ell(\nu)$, then $\partial_\mu(X_\nu) = 0$, unless $\mu = \nu$; moreover, if $\mu = \nu$, then $\partial_\mu(X_\nu) = X_1 = 1$. Therefore, since $\alpha_m = 0$ for $\mu \neq \nu$, it follows that $\alpha(X_\nu) = \alpha_\nu$. Hence the coefficient α_ν is a polynomial. Thus the third assertion of the proposition holds.

Let f be a rational function in $R(A)$ such that the inner product $\langle g, f\rangle$ is a polynomial for every polynomial g in $R[A]$. Expand f in terms of the Schubert polynomials, $f = \sum_\mu f_\mu X_\mu$, where the coefficients f_μ are symmetric rational functions. Now, by Theorem (SCHUB.1.7), the coefficient f_μ is equal to the inner product $\langle Y_\mu, f\rangle$, and hence the coefficient is a polynomial. Therefore, the function f is a polynomial.

Finally, the last assertion of the Proposition follows from the expansion of an operator α in terms of the ∂_μ. Indeed, if $\alpha = \sum_\mu \alpha_\mu \partial_\mu$ where the α_μ are rational functions in $R(A)$, then $\alpha^* = \sum_\mu (\partial_\mu)^* \alpha_\mu$. By Lemma (DIFF.3.6), we have that $(\partial_\mu)^* = \partial_{\mu^{-1}}$. As proved above, if α belongs to $\mathcal{E}_R[A]$ then the functions α_μ are a polynomials, and hence it follows from the expression for α^* that α^* belongs to $\mathcal{E}_R[A]$.

Hence all assertions of the Proposition have been proved. □

(2.5) The Cauchy Formula. *Assume that B and C are two sequences with n elements in the ground ring R. Then, for every permutation μ in $\mathfrak{S}(A)$ we have the equation of values,*

$$X^B_\mu(C) = (\operatorname{sign}\mu) X^C_{\mu^{-1}}(B). \tag{2.5.1}$$

Moreover, the expansion of X^B_μ in terms of the basis X^C_ν is given by the formula,

$$X^B_\mu = \sum_{\substack{\nu \in \mathfrak{S}(A) \\ \ell(\mu\nu^{-1}) + \ell(\nu) = \ell(\mu)}} X^B_{\mu\nu^{-1}}(C) X^C_\nu. \tag{2.5.2}$$

Proof. By Newton's interpolation formula (SCHUB.1.7.3), the left side of (2.5.1) is equal to $\langle Y^C, X^B_\mu\rangle$. The polynomial $X^C_{\mu^{-1}}$ is, by definition, equal to $\partial_{\mu\omega}(X^C)$. Hence, again by Newton's interpolation formula, the right side of (2.5.1) is equal to $(\operatorname{sign}\mu)\langle Y^B_{\mu\omega}, X^C\rangle$. Consequently, the equation (2.5.1) is equivalent to the following equation:

$$\langle Y^C, X^B_\mu\rangle = (\operatorname{sign}\mu)\langle Y^B_{\mu\omega}, X^C\rangle. \tag{2.5.3}$$

On the right side, $X^C = \omega(Y^C)$, and $(\operatorname{sign}\mu)Y^B_{\mu\omega} = (\operatorname{sign}\omega)\omega(X^B_\mu)$ by (SCHUB.1.3). Moreover, it follows from Lemma (DIFF.4.2) that $\langle\omega(f),\omega(g)\rangle = (\operatorname{sign}\omega)\langle f,g\rangle$. Therefore the right side of Equation (2.5.3) is equal to the left side. Hence Equation (2.5.1) holds.

To prove (2.5.2), we apply Newton's interpolation formula (SCHUB.1.7.4) to the function $f := X^B_\mu$ and obtain the equation,

$$X^B_\mu = \sum_{\nu\in\mathfrak{S}(A)} \partial_\nu(X^B_\mu)(C)X^C_\nu.$$

It follows from Proposition (SCHUB.1.6) that the polynomial $\partial_\nu(X^B_\mu)$ is equal to zero unless the condition for the summation index in (2.5.1) is satisfied; moreover, if the latter condition is satisfied, then the polynomial is equal to $X^B_{\mu\nu^{-1}}$. Therefore Equation (2.5.2) holds. ∎

(2.6) Corollary. *Let B be a set of n elements in R. Then the following formula holds:*

$$X^B = \sum_{\mu\in\mathfrak{S}(A)} (\operatorname{sign}\mu)X_\mu(B)X_{\mu\omega}. \tag{2.6.1}$$

Proof. Apply the Cauchy formula (2.5.2) with $\mu := \omega$ and $C := (0,\dots,0)$. In the sum on the right hand side, the condition $\ell(\omega\nu^{-1}) + \ell(\nu) = \ell(\omega)$ on the summation index ν is satisfied for all permutations ν, by (SYM.1.3). Hence we obtain the equation,

$$X^B_\omega = \sum_{\nu\in\mathfrak{S}(A)} X^B_{\omega\nu^{-1}}(0)X_\nu. \tag{2.6.2}$$

In the terms of the sum on the right hand side, let $\nu = \mu\omega$ and form the summation over μ. By Equation (2.5.1), we have that $X^B_{\mu^{-1}}(0) = (\operatorname{sign}\mu)X_\mu(B)$. Therefore, Equation (2.6.1) follows from Equation (2.6.2). ∎

(2.7) Example. Let $n = 3$. The right side of (2.6.1) is the sum,

$$\begin{aligned}
&X_\omega(A)X_1(B) - X_{\tau_2\tau_1}(A)X_{\tau_1}(B) - X_{\tau_1\tau_2}X_{\tau_2}(B)\\
&+ X_{\tau_2}(A)X_{\tau_1\tau_2}(B) + X_{\tau_1}(A)X_{\tau_2\tau_1}(B) - X_1(A)X_\omega(B)\\
&= a_1^2a_2 - a_1a_2(b_1+b_2) - a_1^2b_1 + a_1b_1^2 + (a_1+a_2)b_1b_2 - b_1^2b_2.
\end{aligned}$$

The formula is the reduction of this sum to

$$X(A,B) = (a_1-b_1)(a_1-b_2)(a_2-b_1).$$

Partially symmetric functions

1. Partially symmetric functions.

(1.1) Setup. Assume in this chapter that the alphabet A is *partitioned*, that is, assume that A is the union of a set $\mathbf{A} = (A_1|\dots|A_r)$ of r disjoint alphabets A_i such that, for all $i < j$, if $a \in A_i$ and $b \in A_j$, that $a < b$. Equivalently, the partitioning may be given by listing, for $i = 1, \dots, r-1$, the index p_i of the last letter in A_i:

$$A = \{\overbrace{a_1, \dots, a_{p_1}}^{A_1}, \overbrace{a_{p_1+1}, \dots, a_{p_2}}^{A_2}, \ \dots\ , \overbrace{a_{p_{r-1}+1}, \dots, a_n}^{A_r}\}.$$

The *Young subgroup* $\mathfrak{S}(\mathbf{A}) = \mathfrak{S}(A_1|\dots|A_r)$ corresponding to $\mathbf{A}$ is the subgroup of permutations in $\mathfrak{S}(A)$ that leaves every A_i invariant. The group $\mathfrak{S}(A_i)$ will be identified with the subgroup of $\mathfrak{S}(\mathbf{A})$ consisting of the permutations that are equal to the identity on letters outside A_i. Clearly, any permutation μ in $\mathfrak{S}(\mathbf{A})$ is a product $\mu = \mu_1 \cdots \mu_r$ with uniquely determined factors $\mu_i \in \mathfrak{S}(A_i)$. Moreover, as different factors μ_i commute, it follows that the Young subgroup is the product,

$$\mathfrak{S}(\mathbf{A}) = \mathfrak{S}(A_1) \times \cdots \times \mathfrak{S}(A_r).$$

The Young group $\mathfrak{S}(\mathbf{A})$ is generated by the simple transpositions τ_p for p different from the p_i.

Denote by $\mathfrak{T}(\mathbf{A})$ the subset of $\mathfrak{S}(A)$ consisting of the permutations that are increasing on every interval A_i. The subset $\mathfrak{T}(\mathbf{A})$ is a system of representatives for the set $\mathfrak{S}(A)/\mathfrak{S}(\mathbf{A})$ of left cosets modulo the Young group since, obviously, for every permutation μ there are unique permutations μ_i in $\mathfrak{S}(A_i)$ such that $\mu\mu_1 \cdots \mu_r$ is increasing on each interval A_i. Note also that if $\sigma \in \mathfrak{T}(\mathbf{A})$ and $\nu \in \mathfrak{S}(\mathbf{A})$, then we have the equation,

$$\ell(\sigma\nu) = \ell(\sigma) + \ell(\nu). \tag{1.1.1}$$

Note that the simple transpositions τ_{p_i} belong to $\mathfrak{T}(\mathbf{A})$.

Clearly, the Young subgroup $\mathfrak{S}(\mathbf{A})$ has order $n_1! \cdots n_r!$, where n_i is the number of letters in the subalphabet A_i. It follows that the subset $\mathfrak{T}(\mathbf{A})$ is of order equal to the multinomial coefficient,

$$\binom{n}{n_1, \dots, n_r} = \frac{n!}{n_1! \cdots n_r!}\,.$$

Denote by $\omega = \omega_A$ the order reversing permutation of A and by $\omega_i = \omega_{A_i}$ the order reversing permutation of A_i. It is obvious that a permutation σ of $\mathfrak{S}(A)$ belongs to $\mathfrak{T}(\mathbf{A})$ if and only if $\omega\sigma\omega_1\cdots\omega_r$ belongs to $\mathfrak{T}(\mathbf{A})$. Clearly, the permutation $\omega_{\mathbf{A}}$ defined by the equation,

$$\omega = \omega_{\mathbf{A}}\omega_1\cdots\omega_r,$$

or equivalently, $\omega_{\mathbf{A}} := \omega\omega_1\cdots\omega_r$, belongs to $T(\mathbf{A})$. The map ω defines an order reversing bijection $A_i \to \omega A_i$ and $\omega_{\mathbf{A}}$ defines an order preserving bijection $A_i \to \omega A_i$.

Note that there is a second partitioning $\omega\mathbf{A} := (\omega A_r|\ldots|\omega A_1)$ of A, and that $\mathfrak{S}(\omega\mathbf{A}) = \omega\mathfrak{S}(\mathbf{A})\omega^{-1}$. Moreover, a permutation σ of $\mathfrak{S}(A)$ belongs to $\mathfrak{T}(\omega\mathbf{A})$ if and only if $\omega\sigma\omega$ belongs to $\mathfrak{T}(\mathbf{A})$.

(1.2) Definition. A rational function of $R(A)$ invariant under the Young subgroup $\mathfrak{S}(\mathbf{A})$ will be called *partially symmetric*. We denote by $\mathrm{Sym}_R(\mathbf{A})$ and $\mathrm{Sym}_R[\mathbf{A}]$ the subalgebras of $R(A)$ of functions and polynomials respectively that are partially symmetric.

Clearly, a function f is partially symmetric if and only if $\tau_p(f) = f$ for all p different from the p_i.

(1.3) Lemma. *A function f of $R(A)$ is partially symmetric if and only if $\partial_p(f) = 0$ for all p different from the p_i. In addition, if f is partially symmetric, then $\partial_\mu(f) = 0$ for all permutations μ outside the subset $\mathfrak{T}(\mathbf{A})$.*

Proof. Since $(a_p - a_{p+1})\partial_p(f) = f - \tau_p(f)$, we have that $\partial_p(f) = 0$ if and only if $\tau_p(f) = f$. The first assertion is a consequence, because the Young group $\mathfrak{S}(\mathbf{A})$ is generated by the transpositions τ_p for p different from the p_i.

To prove the second assertion, assume that f is partially symmetric. Then clearly $\partial_\nu(f) = 0$ for every $\nu \neq 1$ in $\mathfrak{S}(\mathbf{A})$. Consider for any permutation μ the factorization $\mu = \sigma\nu$ where $\sigma \in \mathfrak{T}(\mathbf{A})$ and $\nu \in \mathfrak{S}(\mathbf{A})$. By Proposition (DIFF.3.7), it follows from the equality (1.1.1) that $\partial_\mu = \partial_\sigma\partial_\nu$. If μ does not belong to $\mathfrak{T}(\mathbf{A})$, then $\nu \neq 1$, and hence $\partial_\mu(f) = \partial_\sigma\partial_\nu(f) = 0$.

(1.4) Lemma. *Let σ be a permutation in $\mathfrak{S}(A)$. Then, for any of the three polynomials X_σ, $Y_{\omega\sigma}$, and $Y_{\sigma\omega_1\cdots\omega_r}$ defined in* (SCHUB.2.1)*, we have that the polynomial is partially symmetric if and only if σ belongs to $\mathfrak{T}(\mathbf{A})$.*

Proof. Consider first the Schubert polynomial X_σ. By Lemma (SCHUB.1.6) we have that $\partial_\sigma(X_\sigma) = 1$. Hence it follows from Lemma (1.3) that if X_σ is partially symmetric, then σ belongs to $\mathfrak{T}(\mathbf{A})$. Assume conversely that σ belongs to $\mathfrak{T}(\mathbf{A})$. To prove that X_σ is partially symmetric, we have to show that $\partial_p X_\sigma = 0$ for p different from the p_i. The polynomial $\partial_p X_\sigma = \partial_{\tau_p} X_\sigma$ is determined by Lemma (SCHUB.1.6). If p is different from the p_i, then, by (1.1.1), $\ell(\sigma\tau_p) = \ell(\sigma) + \ell(\tau_p)$. In particular, $\ell(\sigma\tau_p^{-1}) \neq \ell(\sigma) - \ell(\tau_p)$. Therefore, it follows from Lemma (SCHUB.1.6) that $\partial_p(X_\sigma) = 0$. Hence X_σ is partially symmetric.

By (SCHUB.1.3.3), we have that $Y_{\omega\sigma} = \mathrm{sign}(\omega\sigma)\omega(X_{\omega\sigma\omega})$. Hence $Y_{\omega\sigma}$ is partially symmetric if and only if $\omega(X_{\omega\sigma\omega})$ is partially symmetric. Moreover, $\omega(X_{\omega\sigma\omega})$ is partially symmetric if and only if $X_{\omega\sigma\omega}$ is invariant under the conjugate Young group $\omega\mathfrak{S}(\mathbf{A})\omega^{-1} = \mathfrak{S}(\omega\mathbf{A})$. So, by the assertion for the Schubert polynomials X_σ, we have that $Y_{\omega\sigma}$ is partially symmetric, if and only if $\omega\sigma\omega \in \mathfrak{T}(\omega\mathbf{A})$, that is, if and only $\sigma \in \mathfrak{T}(\mathbf{A})$.

Finally, the assertion about the polynomials $Y_{\sigma\omega_1\cdots\omega_r}$ follows from the assertion about the polynomials $Y_{\omega\sigma}$, since σ belongs to $\mathfrak{T}(\mathbf{A})$ if and only if $\omega\sigma\omega_1\cdots\omega_r$ belongs to $\mathfrak{T}(\mathbf{A})$. ◻

(1.5) Proposition. *The Schubert polynomials X_σ for $\sigma \in \mathfrak{T}(\mathbf{A})$ form a basis for the algebra* $\mathrm{Sym}_R(\mathbf{A})$ *of partially symmetric functions as a module of the ring* $\mathrm{Sym}_R(A)$ *of symmetric functions, and a basis for the algebra* $\mathrm{Sym}_R[\mathbf{A}]$ *of partially symmetric polynomials as a module over the ring* $\mathrm{Sym}_R[A]$ *of symmetric polynomials.*

Proof. By Theorem (SCHUB.1.7), the Schubert polynomials X_μ for μ in $\mathfrak{S}(A)$ form a $\mathrm{Sym}_R(A)$-basis for $R(A)$. Moreover, in the expansion of a function f in the basis the coefficient to X_μ is equal to the inner product,

$$\langle Y, \partial_\mu(f)\rangle \tag{1.5.1}$$

The monomials X_σ for σ in $\mathfrak{T}(\mathbf{A})$ are partially symmetric by Lemma (1.4). Therefore, to prove the assertion of the Proposition, it suffices to prove that if f is partially symmetric and μ does not belong to $\mathfrak{T}(\mathbf{A})$, then the inner product (1.5.1) vanishes. The latter assertion follows immediately from the second assertion of Lemma (1.3). ◻

2. Partial symmetrization.

(2.1) Definition. Keep the setup of (PARTL.1). Consider the following special polynomial:

$$\Delta(\mathbf{A}) := \prod_{i>j} \prod_{b\in A_i,\, a\in A_j} (b-a).$$

Clearly, the polynomial $\Delta(\mathbf{A})$ is partially symmetric. Moreover, we have the following factorization of the Vandermonde determinant:

$$\Delta(A) = \Delta(\mathbf{A})\Delta(A_1)\cdots\Delta(A_r). \tag{2.1.1}$$

Define the *partial symmetrization operator* $\delta^{\mathbf{A}}$ as the following sum,

$$\delta^{\mathbf{A}} := \sum_{\sigma\in\mathfrak{T}(\mathbf{A})} \sigma\cdot\frac{1}{\Delta(\mathbf{A})}\,. \tag{2.1.2}$$

By definition, we have that $\delta^{\mathbf{A}}$ is an operator, and as such it is defined on functions f of $R(A)$, and linear with respect to symmetric functions. Most often, we will consider values of $\delta^{\mathbf{A}}$ on partially symmetric functions f. Assume that f is partially symmetric. Then, clearly, to obtain the value $\delta^{\mathbf{A}}(f)$, the summation over $\mathfrak{T}(\mathbf{A})$ in (2.1.2) can be replaced by the summation over any set of representatives for the set of left cosets $\mathfrak{S}(A)/\mathfrak{S}(\mathbf{A})$. In particular, if μ is a given permutation of $\mathfrak{S}(A)$, we may replace σ in the terms of (2.1.2) by $\mu\sigma$. It follows that the value $\delta^{\mathbf{A}}(f)$ is a symmetric function. Hence the partial symmetrization operator may be viewed as a $\mathrm{Sym}_R(A)$-linear map,

$$\delta^{\mathbf{A}}\colon\ \mathrm{Sym}_R(\mathbf{A}) \to \mathrm{Sym}_R(A).$$

(2.2) Proposition. *The following equations of operators hold:* $\delta^A = \partial^\omega$ *and* $\delta^{A_i} = \partial^{\omega_i}$*, and*

$$\delta^A = \partial^{\omega_{\mathbf{A}}}\partial^{\omega_1}\cdots\partial^{\omega_r} = \delta^{\mathbf{A}}\delta^{A_1}\cdots\delta^{A_r}. \tag{2.2.1}$$

Proof. The first equations, $\delta^A = \partial^\omega$ and $\delta^{A_i} = \partial^{\omega_i}$, were proved in section (DIFF.3). Since $\omega = \omega_{\mathbf{A}}\omega_1\cdots\omega_r$, it follows from (1.1.1) and (DIFF.3.7) that the first equation in (2.2.1) holds.

Recall that δ^A is the total symmetrization operator, defined in (DIFF.2.3) as the sum $\sum_\mu \mu\cdot(1/\Delta(A))$ where the sum is over all permutations μ in $\mathfrak{S}(A)$, and δ^{A_i} is defined similarly using $\Delta(A_i)$ and permutations of $\mathfrak{S}(A_i)$. By Theorem (DIFF.3.3), we have that $\delta^A = \partial^\omega$ and $\delta^{A_i} = \partial^{\omega_i}$.

In the summation defining δ^A, each permutation μ is uniquely a product $\mu = \sigma\mu_1\ldots\mu_r$ where σ belongs to $\mathfrak{T}(\mathbf{A})$ and μ_i belongs to $\mathfrak{S}(A_i)$. From the factorization (2.1.1) we obtain the equation of operators,

$$\mu\cdot\frac{1}{\Delta(A)} = \sigma\cdot\frac{1}{\Delta(\mathbf{A})}\mu_1\cdot\frac{1}{\Delta(A_1)}\cdots\mu_r\cdot\frac{1}{\Delta(A_r)}\,,$$

because $\Delta(\mathbf{A})$ and $\Delta(A_j)$ for $j\neq i$ are symmetric with respect to the letters of A_i. Therefore, from the definitions of δ^A and δ^{A_i}, we obtain the factorization,

$$\delta^A = \delta^{\mathbf{A}}\delta^{A_1}\cdots\delta^{A_r}. \tag{2.2.2}$$

Hence the equations of (2.2.1) have been verified. ∎

(2.3) Corollary. *Let n_i be the cardinality of A_i, and let a^{E_i} the monomial in $R[A]$ defined as the product of the letters of A_i raised to the powers $0, 1, \dots, n_i - 1$. Then, on the subalgebra of partially symmetric functions, the operator $\partial^{\omega_1 \cdots \omega_r} \cdot a^{E_1} \cdots a^{E_r}$ is the identity and the following three operators are equal:*

$$\delta^A \cdot a^{E_1} \cdots a^{E_r}, \qquad \delta^{\mathbf{A}}, \qquad \partial^{\omega_{\mathbf{A}}}. \tag{2.3.1}$$

Moreover, on the subalgebra, the values taken by the three operators are symmetric functions, and the values taken on partially symmetric polynomials are symmetric polynomials.

Proof. To prove the first assertion, note that $\partial^{\omega_1 \cdots \omega_r} = \partial^{\omega_1} \cdots \partial^{\omega_r}$ and ∂^{ω_i} commutes with the operator a^{E_j} for $i \neq j$. Hence $\partial^{\omega_1 \cdots \omega_r} \cdot a^{E_1} \cdots a^{E_r} = \partial^{\omega_1} \cdot a^{E_1} \cdots \partial^{\omega_r} \cdot a^{E_r}$. Therefore, by Theorem (DIFF.3.3), we have the equation of operators,

$$\partial^{\omega_1 \cdots \omega_r} \cdot a^{E_1} \cdots a^{E_r} = \pi^{A_1} \cdots \pi^{A_r}. \tag{2.3.2}$$

Assume that f is a partially symmetric function. The operator π^{A_i} is linear with respect to functions symmetric in the letters of A_i and $\pi^{A_i}(1) = 1$. In particular, we have that $\pi^{A_i}(f) = f$. Therefore, by (2.3.2), we have the equation,

$$\partial^{\omega_1 \cdots \omega_r} \cdot a^{E_1} \cdots a^{E_r}(f) = f. \tag{2.3.3}$$

Clearly, the equality of the three operators in (2.3.1) on f follows from (2.3.3) and the equality of the three operators of (2.2) on the function $a^{E_1} \cdots a^{E_r} f$. Thus the first assertion of the Corollary has been proved.

That the value $\delta^{\mathbf{A}}(f)$ is a symmetric function was noted in (2.1). It follows also from the equality $\delta^{\mathbf{A}}(f) = \delta^A(a^{E_1} \cdots a^{E_r} f)$ since all values of the symmetrization operator δ^A are symmetric functions. From the same equality it follows that $\delta^{\mathbf{A}}(f)$ is a polynomial if f is a polynomial.

Hence the assertions of the Corollary have been proved. ▯

(2.4) Definition. Define for partially symmetric functions f and g their *partial inner product* as the sum:

$$\langle f, g \rangle^{\mathbf{A}} := \sum_{\sigma \in \mathfrak{T}(\mathbf{A})} \sigma\Big(\frac{fg}{\Delta(\mathbf{A})}\Big) = \delta^{\mathbf{A}}(fg). \tag{2.4.1}$$

Although the right hand side is defined for all rational functions f and g in $R(A)$, we will always assume that f and g are partially symmetric. It follows from Corollary (2.3) that

$$\langle f, g \rangle^{\mathbf{A}} = \partial^{\omega_{\mathbf{A}}}(fg). \tag{2.4.2}$$

(2.5) Lemma. *The values of the partial inner product* (2.4) *are symmetric functions. Moreover, the inner product is symmetric, and* $\mathrm{Sym}_R(A)$*-bilinear. Finally, if f and g are partially symmetric polynomials, then their inner product $\langle f, g \rangle^{\mathbf{A}}$ is a polynomial.*

Proof. The assertions follow immediately from Corollary (2.3). ▯

(2.6) Proposition. *Let f be a partially symmetric function. Then, in the expansion of f in the basis of Schubert polynomials X_σ for $\sigma \in \mathfrak{T}(\mathbf{A})$* (see Proposition (1.5)), *the coefficient to X_σ is equal to the partial inner product,*

$$\operatorname{sign}(\omega_1 \cdots \omega_r)\langle Y_{\sigma\omega_1\cdots\omega_r}, f\rangle^{\mathbf{A}}. \tag{2.6.1}$$

In particular, if σ and τ are permutations in $\mathfrak{T}(\mathbf{A})$, then

$$\operatorname{sign}(\omega_1 \cdots \omega_r)\langle Y_{\sigma\omega_1\cdots\omega_r}, X_\tau\rangle^{\mathbf{A}} = \begin{cases} 1 & \text{if } \sigma = \tau, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.2}$$

Proof. By Proposition (1.5), to prove the first assertion, we have to verify, for σ in $\mathfrak{T}(\mathbf{A})$, the equation,

$$\operatorname{sign}(\omega_1 \cdots \omega_r)\langle Y_{\sigma\omega_1\cdots\omega_r}, f\rangle^{\mathbf{A}} = \langle Y_\sigma, f\rangle.$$

Equivalently, by definition of the inner products, we have to prove that

$$\operatorname{sign}(\omega_1 \cdots \omega_r)\delta^{\mathbf{A}}(Y_{\sigma\omega_1\cdots\omega_r} f) = \delta^A(Y_\sigma f)\,. \tag{2.6.3}$$

By (SCHUB.1.3), we have that $Y_\mu = \partial_{\mu^{-1}} Y$. For $\mu = \sigma\omega_1 \cdots \omega_r$, we have, by (1.1) and (DIFF.3.7), that $\partial_{\mu^{-1}} = \partial_{\omega_1\cdots\omega_r}\partial_{\sigma^{-1}}$. Thus

$$\operatorname{sign}(\omega_1 \cdots \omega_r) Y_{\sigma\omega_1\cdots\omega_r} = \operatorname{sign}(\omega_1 \cdots \omega_r)\partial_{\omega_1\cdots\omega_r}\partial_{\sigma^{-1}} Y = \partial^{\omega_1\cdots\omega_r} Y_\sigma.$$

The operator $\partial^{\omega_1\cdots\omega_r}$ commutes with multiplication by f since f is partially symmetric. Hence we obtain for the left side of (2.6.3) the expression,

$$\delta^{\mathbf{A}}\partial^{\omega_1\cdots\omega_r}(Y_\sigma f),$$

which, by Proposition (2.2), is equal to the right side (2.6.3). Hence the first assertion of the Proposition holds.

Clearly, the second assertion of the Proposition is a particular case of the first. □

(2.7) Note. The total symmetrization operator δ^A vanishes on the partially symmetric functions except for the trivial partitioning $\mathbf{A}$ where each subinterval consists of a single letter. Indeed, assume that f is partially symmetric. The operator δ^{A_i} is linear with respect to functions that are symmetric in the letters of A_i. Therefore, by Proposition (2.2), we obtain the equation,

$$\delta^A(f) = \delta^{\mathbf{A}}\big(f\delta^{A_1} \cdots \delta^{A_1}(1)\big).$$

The operator δ^{A_i} lowers the degree by $\ell(\omega_i)$. Hence $\delta^{A_i}(1) = 0$ unless A_i consists of a single letter. Thus $\delta^A(f) = 0$ unless the partitioning is trivial.

Assume that the partitioning $\mathbf{A}$ is non-trivial. It follows in particular that the total inner product of (DIFF.4.1) vanishes identically on the module of partially symmetric functions. Note also that the polynomial Y_σ in (2.6.2) is not partially symmetric. Indeed, by Lemma (1.4), the polynomial Y_μ is partially symmetric if and only if $\omega\mu$ is increasing on each subinterval, and if $\sigma \in \mathfrak{T}(\mathbf{A})$, then $\omega\sigma$ is decreasing on each A_i.

(2.8) Note. *Lagrange interpolation in one variable.* Let x be an additional letter. Consider the alphabet $\{a_1, \dots, a_n, x\}$ and the partitioning $(A|x)$. Then $\Delta(A|x) = \prod_a (x-a)$ is a polynomial of degree n, and the partial symmetrization operator lowers the degree by n.

Let f be a polynomial depending only on x, and of degree strictly less than n. Since $\delta^{A|x}$ lowers the degree by n, it follows that partial symmetrization of f yields zero, that is, we obtain the following equation:

$$\sum_{\sigma \in \mathfrak{T}(A|x)} \sigma\Big(\frac{f}{\Delta(A|x)}\Big) = 0.$$

Consider the terms in the sum. The permutations σ in $\mathfrak{T}(A|x)$ are determined by the value $\sigma(x)$. If $\sigma(x) = x$, then $\sigma = 1$ and the corresponding term in the sum is the rational function $f/\Delta(A|x)$. If $\sigma(x) = b$ is a letter of A then, in the corresponding term, the numerator is $f(b)$ and the denominator is given by

$$\sigma(\Delta(A|x)) = (b-x)\prod_{a \neq b}(b-a)\,.$$

Therefore, by separating in the equation the terms corresponding to $\sigma = 1$ and $\sigma \neq 1$ and multiplying by $\Delta(A|x)$ we obtain the equation in $R(A)[x]$,

$$f = \sum_{b \in A} f(b) \prod_{a \neq b} \frac{(x-a)}{(b-a)}\,, \tag{2.8.1}$$

which is the usual form of Lagrange interpolation.

Note that (2.8.1) holds if the n letters a_i of A are replaced by any sequence of n elements α_i of R such that the differences $\alpha_p - \alpha_q$ for $p \neq q$ are invertible in R. In particular, replacing R by $R(A)$ we may apply the formula to the difference $f := \prod_a (x - ta) - \prod_a (x-a)$ for $t \in R$. Then $f(b) = (1-t)b\prod_{a \neq b}(b - ta)$ and we obtain the formula in $R(A)(x)$,

$$\prod_a \frac{x - ta}{x-a} = 1 + (1-t)\sum_b \frac{b}{x-b} \prod_{a \neq b} \frac{b - ta}{b-a}\,. \tag{2.8.2}$$

3. The Gysin formula.

(3.1) Setup. Assume in the setup of (1.1) that there is given, for each i, an R-basis $\{f_{i,J}\}$ for the module $\mathrm{Sym}_R[A_i]$ of polynomials symmetric in the letters of A_i. Then, clearly, the set of all products $f_{1,J_1}\cdots f_{r,J_r}$ form an R-basis for the module $\mathrm{Sym}_R[\mathbf{A}]$ of partially symmetric polynomials. For instance, the set of all products of monomial symmetric polynomials,

$$m^{K_1}(A_1)\cdots m^{K_r}(A_r),$$

where K_i is a weakly decreasing multi index of size n_i equal to the number of letters in A_i, form a basis. Similarly, the set of all products of Schur polynomials,

$$s^{J_1}(A_1)\cdots s^{J_r}(A_r),$$

where J_i is a strictly increasing multi index of size n_i, form a basis.

(3.2) Example. Consider a partitioning $\mathbf{A} = (A_1|A_2)$ into two subalphabets. Then we have the equation,

$$\prod_{a\in A_1, b\in A_2}(a+b) = \sum_{J\subseteq[n_1]} s^J(A_1)s^{\hat{J}}(A_2),$$

where the sum is over subsets J of size n_1. More generally, for $d\geq 0$, we have the equation,

$$\prod_{b\in A_2} b^d \prod_{a\in A_1, b\in A_2}(a+b) = \sum_{J\subseteq[n]} s^J(A_1)s^{\tilde{\hat{J}}}(A_2), \tag{3.2.1}$$

where the sum is over subsets J of size n_1, and $\hat{J}$ is the extension of J to a subset of size n_1+d of $[n+d]$.

Indeed, the first equation is the equation (SYM.8.3.3) with $A := A_1$ and $B := A_2$. To prove (3.2.1), extend A_1 to an alphabet $\hat{A}_1 = \{a_1,\dots,a_{n_1},\hat{a}_1,\dots,\hat{a}_d\}$ with n_1+d letters. Apply the first equation to the partitioning $(\hat{A}_1, A_2)$ and substitute $\hat{a}_i := 0$ for $i=1,\dots,d$. It follows that the left side of (3.2.1) is equal to the sum,

$$\sum_{I\subseteq[n+d]} s^I(A_1,0,\dots,0)s^{\tilde{I}}(A_2),$$

over subsets I of size n_1+d. Since $s^I(A_1,0,\dots,0) = 0$ if I is not an extension and $s^I(A_1,0,\dots,0) = s^J(A_1)$ if $I=\hat{J}$, it follows that equation (3.2.1) holds.

(3.3) Bott's Formula. *For $i=1,\dots,r$, let J_i be a strictly increasing multi index of size equal to the number of letters of A. Then the following equation holds:*

$$\delta^{\mathbf{A}}\Big(s^{J_1}(A_1)\cdots s^{J_r}(A_r)\Big) = s^{J_1\dots J_r}(A), \tag{3.3.1}$$

where $J_1 \dots J_r$ denotes the concatenated multi index.

Proof. With an abuse of notation, denote by a^{J_i} the monomial of $R[A]$ where the q'th letter of A_i appears with the exponent given by the q'th entry in J_i and all other letters appear with exponent zero.

In this notation, we have the equations,

$$s^{J_1}(A_1)\cdots s^{J_r}(A_r) = \delta^{A_1}(a^{J_1})\cdots\delta^{A_r}(a^{J_r}) = \delta^{A_1}\cdots\delta^{A_r}(a^{J_1\cdots J_r}). \tag{3.3.2}$$

Indeed, by the definition in (SYM.6.9.2) or by Jacobi–Trudi's Formula (SYM.7.8.5) or (SCHUR.1.11), we have the equation $s^J(A) = \delta^A(a^J)$ for the Schur polynomial. Hence the first equation in (3.3.2) holds. The polynomials a^{J_i} and $\delta^{A_i}(a^{J_i})$ depend only on the letters of A_i. Hence they are scalars with respect to the operator δ^{A_j} for $j \neq i$. Moreover, $a^{J_1\cdots J_r} = a^{J_1}\cdots a^{J_r}$. Hence the second equation in (3.3.2) holds.

By Proposition (2.2), the equation (3.3.1) follows by applying the operator $\delta^{\mathbf{A}}$ to the equation (3.3.2). ∎

(3.4) Setup. Assume that the number r of subalphabets is equal to 2. For convenience, set $B := A_1$ and $C := A_2$. Let m and k be the number of letters in B and C, so that $n = m + k$ is the number of letters of A. By definition of the partial inner product, the formula (3.3.1) is equivalent to the following *Gysin formula:*

$$\langle s^I(B), s^J(C)\rangle^{\mathbf{A}} = s^{IJ}(A), \tag{3.4.1}$$

where I and J are strictly increasing multi indices of sizes m and k.

(3.5) Corollary. *Consider the set of Schubert polynomials X_σ and the set of polynomials $Y_{\sigma\omega_1\omega_2}$ for $\sigma \in \mathfrak{T}(B|C)$. In addition, consider the set of Schur polynomials $s^I(B)$ for all strictly increasing multi indices I of size m with entries in the interval $[n] = [m+k]$, and the set of Schur polynomials $s^J(C)$ for all strictly increasing multi indices J of size k with entries in the same interval. Then any of the four sets is a basis for the algebra* $\mathrm{Sym}_R[B|C]$ *of partially symmetric polynomials as a module over its subring* $\mathrm{Sym}_R[A]$ *of symmetric polynomials. Moreover, for Schur polynomials of the two bases we have that*

$$\langle s^I(B), s^J(C)\rangle^{\mathbf{A}} = \mathrm{sign}(IJ), \tag{3.5.1}$$

where the right hand side is equal to the signature of IJ when the concatenated multi index IJ is a permutation of $[m+k]$ and equal to zero otherwise.

Finally, for any partially symmetric polynomial f, the expansion of f in the basis $s^I(B)$ is given by the formula,

$$f = \sum_I \mathrm{sign}(I\tilde{I})\delta^{\mathbf{A}}\big(fs^{\tilde{I}}(C)\big)\, s^I(B),$$

where $\tilde{I}$ denotes the complementary sequence of I with respect to the interval $[m+k]$. Similarly, the expansion of f in the basis $s^J(C)$ is given by the formula,

$$f = \sum_J \operatorname{sign}(\tilde{J}J)\delta^{\mathbf{A}}\bigl(fs^{\tilde{J}}(B)\bigr)\, s^J(C).$$

Proof. The equation (3.5.1) follows from (3.4.1) because the Schur polynomial $s^K(A)$ is alternating in K and equal to 1 when K is the sequence $(0,1,\dots,n-1)$.

Clearly, the four sets have the same number d of elements, namely $d = \binom{n}{m}$. That the X_σ form a basis was proved in (1.5). The remaining assertions of the Corollary follow, by the standard argument used in the proof of Theorem (SCHUB.1.7), from the equations (2.6.2) and (3.5.1). Indeed, since the X_σ form a basis with d elements for the module $M := \operatorname{Sym}_R[B|C]$ as the module over S, it follows from (3.5.1) that the d polynomials $s^I(B)$ form an S-basis for M and the dual basis is given by the d linear forms,

$$f \mapsto \operatorname{sign}(I\tilde{I})\langle f, s^{\tilde{I}}(C)\rangle^{\mathbf{A}}.$$

By the same argument, the $s^J(C)$ form a basis, and by (2.6.2) the $Y_{\sigma\omega_1\omega_2}$ form a basis.

Since $\langle f,g\rangle^{\mathbf{A}} = \delta^{\mathbf{A}}(fg)$, the first expansion of f given in the Corollary follows from the description of the basis dual to the $s^I(B)$. The proof of the second expansion is entirely similar. $\square$

4. Hall–Littlewood polynomials.

(4.1) Setup. Fix an element $t \in R$. Consider the following polynomial of $R[A]$:

$$\Delta_t(A) := \prod_{a<b} (b - ta),$$

where the product is over letters a and b of A. Clearly, $\Delta_t(A)$ has the same degree, $n(n-1)/2$, as the Vandermonde determinant $\Delta(A)$. For $t = 1$, we have that $\Delta_1(A) = \Delta(A)$. For $t = 0$, we have that $\Delta_0(A) = a^E$, where $E = (0, 1, \dots, n-1)$.

If a partitioning $\mathbf{A} = (A_1|\dots|A_r)$ is given, we write $a \ll b$ if $a < b$ and a and b belong to different subalphabets. In this notation, let

$$\Delta_t(\mathbf{A}) := \prod_{a \ll b} (b - ta).$$

Clearly, $\Delta_t(\mathbf{A})$ is a homogeneous partially symmetric polynomial. Moreover, we have the factorization,

$$\Delta_t(A) = \Delta_t(\mathbf{A})\Delta_t(A_1)\cdots\Delta_t(A_r). \tag{4.1.1}$$

For $t = 1$, the factorization is the factorization (2.1.1) of the Vandermonde determinant. For $t = 0$, we have in the notion of (2.3) that $\Delta_0(\mathbf{A})$ is the monomial determined by the equation,

$$a^E = \Delta_0(\mathbf{A})a^{E_1}\cdots a^{E_r}. \tag{4.1.2}$$

(4.2) Example. Consider a partitioning $\mathbf{A} = (B|x)$ where $B = \{b_1, \dots, b_m\}$ is the subalphabet consisting of the first $m = n - 1$ letters of A and $x = a_n$ is the last letter of A. By definition, we have that $\Delta_t(B|x) = \prod_{q=1}^m (x - tb_q)$. Moreover, we have the equation,

$$\delta^{\mathbf{A}} \prod_{q=1}^{m} (x - tb_q) = \sum_{j=0}^{m} t^j = \frac{t^n - 1}{t - 1}. \tag{4.2.1}$$

Indeed, by expanding the product, we obtain the equation,

$$\Delta_t(B|x) = \sum_{j=0}^{m} (-1)^{m-j} t^{m-j} e_{m-j}(B) x^j.$$

By (SYM.6.8.1), we have that $e_{m-j}(B) = s^{0,\dots,\hat{j},\dots,m}(B)$. As $x^j = s^j(x)$, it follows from Bott's Theorem that

$$\delta^{\mathbf{A}}\big(e_{m-j}(B)x^j\big) = s^{0,\dots,\hat{j},\dots,m,j}(B, x) = (-1)^{m-j}.$$

Therefore, partial symmetrization of $\Delta_t(B|x)$ yields the sum in (4.2.1).

Note that $\Delta(B|x) = \Delta_1(B|x)$ is the polynomial appearing as the denominator in the definition of $\delta^{\mathbf{A}}$. As in (2.7), a permutation σ in $\mathfrak{T}(B|x) = \mathfrak{T}(a_1, \dots, a_{n-1}|a_n)$ is determined by the index $p = 1, \dots, n$ such that $\sigma(a_n) = a_p$. Clearly, $\sigma(\Delta_t) = \prod_{q \neq p}(a_p - ta_q)$. Hence the formula (4.2.1) is equivalent to the following:

$$\sum_{p=1}^{n} \prod_{q \neq p} \frac{a_p - ta_q}{a_p - a_q} = \frac{t^n - 1}{t - 1}. \tag{4.2.2}$$

(4.3) Lemma. *The total symmetrization of the polynomial $\Delta_t(A)$ is the constant given by the equations,*

$$\delta^A\big(\Delta_t(A)\big) = \prod_{i=1}^{n} \frac{t^i - 1}{t - 1} = \sum_{\mu \in \mathfrak{S}(A)} t^{\ell(\mu)}. \tag{4.3.1}$$

Proof. The first equation in (4.3.1) is proved by induction on the number of n of letters of A. The two sides of the equation reduce to 1 when $n = 1$. If $n > 1$, then, in the notation of Example (4.2), we may assume that the equation holds for the alphabet B. As $\Delta_t(A) = \Delta_t(B|x)\Delta_t(B)$ by (4.1) and $\delta^A = \delta^{\mathbf{A}}\delta^B$ by Proposition (2.2), we have that

$$\delta^A\big(\Delta_t(A)\big) = \delta^{\mathbf{A}}\delta^B\big(\Delta_t(B|x)\Delta_t(B)\big).$$

The polynomial $\Delta_t(B|x)$ is symmetric in the letters of B and hence it commutes with the operator δ^B. Moreover, $\delta^B(\Delta_t(B))$ is a constant, given by the induction hypothesis. Hence $\delta^B(\Delta_t(B))$ commutes with the operator δ^A. Therefore, we obtain the equation,

$$\delta^A\big(\Delta_t(A)\big) = \delta^B(\Delta_t(B))\, \delta^{\mathbf{A}}(\Delta_t(B|x)).$$

By (4.2.1) and the induction hypothesis, the first equation of (4.3.1) is a consequence.

To obtain the alternative expression for the symmetrization, given by the right side of (4.3.1), recall the $\delta^A(\Delta_t(A))$ is the quotient obtained by dividing the following sum by the Vandermonde determinant $\Delta(A)$:

$$\sum_{\mu \in \mathfrak{S}(A)} (\operatorname{sign}\mu)\mu(\Delta_t(A)). \tag{4.3.2}$$

The sum (4.3.2) has the same degree, $n(n-1)/2$, as the Vandermonde determinant $\Delta(A)$. Therefore, to obtain the quotient, it suffices to compare the smallest terms of the two polynomials. For $\Delta(A)$, the smallest term is obtained as the product of the smallest terms of each factor $b - a$ for $a < b$. So the smallest term in $\Delta(A)$ is the monomial a^E obtained as the product of the letters b for all pairs (a, b) with $a < b$. Consider similarly the smallest term in $\mu(\Delta_t(A))$. The factors of $\mu(\Delta_t(A))$ are of the two forms,

$$b' - ta', \qquad -tb' + a', \qquad \text{with } a' < b',$$

where the factors of the first form correspond to factors $b - ta$ of $\Delta_t(A)$ such that (a, b) is not an inversion for μ, and the factors of the second form correspond to factors $b - ta$ such that (a, b) is an inversion for μ. It follows that the smallest term in $\mu(\Delta_t(A))$ is equal to $(-t)^{\ell(\mu)}a^E$. Hence the smallest term in the sum (4.3.2) is equal to the sum on the right hand side of (4.3.1) multiplied by a^E. Therefore, the alternative expression for $\delta^A(\Delta_t(A))$ holds. ∎

(4.4) Example. Consider, for a partially symmetric polynomial f, the partial symmetrization,

$$\delta^{\mathbf{A}}\bigl(f\Delta_t(\mathbf{A})\bigr) = \sum_{\sigma\in\mathfrak{T}(\mathbf{A})} \sigma\Bigl(f\prod_{a\ll b}\frac{b-ta}{b-a}\Bigr). \tag{4.4.1}$$

The partial symmetrization is a polynomial, symmetric in the letters of A, with coefficients depending on t. The fraction in (4.4.1) is the fraction $\Delta_t(\mathbf{A})/\Delta(\mathbf{A})$ of two homogeneous polynomials of the same degree. Hence, if f is homogeneous, then $\delta^{\mathbf{A}}\bigl(f\Delta_t(\mathbf{A})\bigr)$ is homogeneous of the same degree.

Denote by φ the following polynomial in t:

$$\varphi(t) := \prod_{i=1}^{r}\prod_{j=1}^{n_i}\frac{t^i-1}{t-1} = \sum_{\mu\in\mathfrak{S}(\mathbf{A})} t^{\ell(\mu)}, \tag{4.4.2}$$

where n_i is the number of letters of A_i. The second equation in (4.4.2) follows from the second equation in (4.3.1) since any permutation $\mu\in\mathfrak{S}(\mathbf{A})$ is a product $\mu=\mu_1\cdots\mu_r$ with uniquely determined factors $\mu_i\in\mathfrak{S}(A_i)$ and $\ell(\mu)=\ell(\mu_1)+\cdots+\ell(\mu_r)$.

Then we have the equations,

$$\delta^{A}\bigl(f\Delta_t(A)\bigr) = \sum_{\mu\in\mathfrak{S}(A)} \mu\Bigl(f\prod_{a<b}\frac{b-ta}{b-a}\Bigr) = \varphi(t)\delta^{\mathbf{A}}\bigl(f\Delta_t(\mathbf{A})\bigr). \tag{4.4.3}$$

Indeed, the first equation follows from the definition of the symmetrization operator δ^A. For the operator δ^A we have the factorization of $\delta^A=\delta^{\mathbf{A}}\delta^{A_1}\cdots\delta^{A_r}$ of (2.2) and for $\Delta_t(A)$ we have the factorization $\Delta_t(A)=\Delta_t(\mathbf{A})\Delta_t(A_1)\cdots\Delta_t(A_r)$ of (4.1.1). The polynomial $f\Delta_t(\mathbf{A})$ is partially symmetric and hence it commutes with the operators δ^{A_i}. Moreover, the polynomial $\delta^{A_i}(\Delta_t(A_i))$ is constant, and hence it commutes with any operator. Therefore, from the two factorizations, we obtain the equation,

$$\delta^A(f\Delta_t(A)) = \delta^{A_1}(\Delta_t(A_1))\cdots\delta^{A_r}(\Delta_t(A_r))\delta^{\mathbf{A}}(f\Delta_t(\mathbf{A})).$$

Hence the equation $\delta^A\bigl(f\Delta_t(A)\bigr)=\varphi(t)\delta^A\bigl(f\Delta_t(\mathbf{A})\bigr)$ follows from (4.3.1).

(4.5) Definition. Let J be a strictly increasing multi index of size n. Associate with J the partitioning $\mathbf{A}=(A_1|\dots|A_r)$ defined by the sequence $p_1<\cdots<p_{r-1}$ of indices $p<n$ such that $j_p+1<j_{p+1}$, cf. (1.1). Define the *Hall–Littlewood symmetric polynomial* $P^J(A;t)$ as the partial symmetrization,

$$P^J(A;t) := \delta^{\mathbf{A}}(a^{J-E}\Delta_t(\mathbf{A})) = \sum_{\sigma\in\mathfrak{T}(\mathbf{A})} \sigma\Bigl(a^{J-E}\frac{\Delta_t(\mathbf{A})}{\Delta(\mathbf{A})}\Bigr), \tag{4.5.1}$$

where $E=(0,1,\dots,n-1)$. Note that the sequence $J-E$ is weekly increasing; moreover, by the choice of the p_i, the monomial a^{J-E} is partially symmetric. Hence, by Corollary (2.3), the polynomial $P^J(A;t)$ is given as the total symmetrization,

$$P^J(A;t) = \delta^A\Bigl(\frac{a^J a^{E_1}\cdots a^{E_r}}{a^E}\Delta_t(\mathbf{A})\Bigr) \tag{4.5.2}$$

By the factorization (4.1.2) we have that

$$\frac{a^{E_1}\cdots a^{E_r}}{a^E}\Delta_t(\mathbf{A}) = \frac{\Delta_t(\mathbf{A})}{\Delta_0(\mathbf{A})} = \prod_{a\ll b}\frac{b-ta}{b}\,.$$

Hence it follows from (4.5.2) that

$$P^J(A;t) = \delta^A\Big(a^J\prod_{a\ll b}\Big(1-t\frac{a}{b}\Big)\Big). \tag{4.5.3}$$

Moreover, by (4.4.3) we have the equation,

$$\varphi_J(t)P^J(A;t) = \delta^A\big(a^{J-E}\Delta_t(A)\big) = \sum_{\mu\in\mathfrak{S}(A)}\mu\Big(a^{J-E}\prod_{a\ll b}\frac{b-ta}{b-a}\Big), \tag{4.5.4}$$

where $\varphi_J(t)$ is the polynomial in t defined by (4.4.2).

For $t=0$ we have the equation, $a^{E_1}\cdots a^{E_r}\Delta_0(\mathbf{A}) = a^E$. Hence it follows from (4.5.2) that $P^J(A;0) = \delta^A(a^J)$, that is, $P^J(A;0)$ is the Schur polynomial,

$$P^J(A;0) = s^J(A). \tag{4.5.5}$$

For $t=1$, we have that $\Delta_1(A) = \Delta(A)$. Hence it follows from (4.5.1) that

$$P^J(A;1) = \sum_{\sigma\in\mathfrak{T}(\mathbf{A})}\sigma(a^{J-E}).$$

By the choice of the p_i, the monomials $\sigma(a^{J-E})$ are exactly the different monomials of the form $\mu(a^{J-E})$ for $\mu\in\mathfrak{S}(A)$. Hence $P^J(A;1)$ is the monomial symmetric polynomial,

$$P^J(A;1) = m^{J-E}(A). \tag{4.5.6}$$

(4.6) Proposition. *In the basis of proper Schur polynomials, the expansion of the Hall–Littlewood polynomial $P^J(A;t)$ is of the form,*

$$P^J(A;t) = \sum_{I\gg J}\alpha_{JI}(t)s^I(A), \tag{4.6.1}$$

where the sum is over strictly increasing multi indices I of size n such that $\|I\| = \|J\|$ and $i_1+\cdots+i_s \geq j_1+\cdots+j_s$ for $s=1,\dots,n$. Moreover, the coefficient $\alpha_{JJ}(t)$ is equal to 1.

Proof. By (4.5.3), the polynomial $P^J(A;t)$ is obtained by applying the symmetrization operator δ^A to the product,

$$\Pi := a^J\prod_{a\ll b}\Big(1-t\frac{a}{b}\Big).$$

The product Π is a polynomial. Consider a monomial a^K appearing in Π. Then a^K is obtained from a^J by multiplication by a finite number of *different* factors a/b for $a \ll b$. Fix a set $l_1 < \cdots < l_s$ of s indices. Consider for monomials the sum of the exponents corresponding to the indices l_q. For a^K, the sum is $k_{l_1} + \cdots + k_{l_s}$. We claim that the following inequality holds:

$$k_{l_1} + \cdots + k_{l_s} \geq j_1 + \cdots + j_s. \tag{4.6.2}$$

Indeed, for a^J, the sum of the exponents is $j_{l_1} + \cdots + j_{l_s}$. Now, if a monomial is multiplied by a factor a/b for $a \ll b$, then the sum of the exponents is decreased by 1 if b is one of the a_{l_q} and a is not, it is increased by 1 if a is one of the a_{l_q} and b is not, and it is unchanged otherwise. Therefore, when a^J is multiplied by a number of different factors of the form a/b, the sum $j_{l_1} + \cdots + j_{l_s}$ is most decreased by the number of factors a/b such that $a \ll b$ and b is one of the a_{l_q} and a is not. Clearly, the latter number of factors is at most equal to $\sum_{q=1}^{s}(l_q - q)$. Hence we have the inequality,

$$k_{l_1} + \cdots + k_{l_s} \geq j_{l_1} + \cdots + j_{l_s} - \sum_{q=1}^{s}(l_q - q). \tag{4.6.3}$$

Since the l_q were increasing, we have that $l_q \geq q$. Moreover, the multi index J was strictly increasing. Hence, we have the inequalities,

$$j_{l_q} - l_q \geq j_q - q \quad \text{for } q = 1, \ldots, s. \tag{4.6.4}$$

Clearly (4.6.2) follows from the inequalities (4.6.3) and (4.6.4).

It follows from (4.6.2) that the expansion of $P^J(A;t)$ has the form asserted in (4.6.1). Indeed, from the expansion of Π as an R-linear combination of monomials a^K we obtain the expansion of $P^J(A;t)$ as the corresponding R-linear combination of $\delta^A(a^K) = s^K(A)$. The polynomial $s^K(A)$ is alternating in K. Hence a monomial a^K contributes with zero if two entries in K are equal. If all entries in K are different, then $s^K(A)$ is up to sign equal to the proper Schur polynomial $s^I(A)$, where I is the strictly increasing permutation of K. Take as l_q the indices such that $\{k_{l_1}, \ldots, k_{l_s}\} = \{i_1, \ldots, i_s\}$. Then it follows from (4.6.2) that

$$i_1 + \cdots + i_s \geq j_1 + \cdots + j_s.$$

Therefore, if $s^I(A)$ appears in the expansion of $P^J(A;t)$, we have that $\|I\| \gg \|J\|$.

Obviously, the monomial a^J appears with coefficient 1 in the expansion of Π. Therefore, to prove the last assertion of the Proposition, it suffices to prove that if the monomial a^K appears in the expansion of Π and K is a permutation of J, then $K = J$. By an inductive argument, it suffices to prove that if $k_q = j_q$ for $q = 1, \ldots, s-1$, then $k_s = j_s$. Let $l \geq s$ be the index such that $k_l = j_s$. Apply the reasoning leading to (4.6.2) with the set of l_q equal to $\{1, \ldots, s-1, l\}$. As (4.6.2) is an equality, it follows in particular that the last inequality in (4.6.4) is an equality, that is, $j_l - l = j_s - s$. Hence a_l and a_s belong to the same subalphabet

A_i. Therefore, there are no factors a/b where $a \ll b$ and b is one of the a_{l_q} and a is not. Hence we obtain in stead of (4.6.3) the inequality,

$$k_1 + \cdots + k_{s-1} + k_l \geq j_1 + \cdots + j_{s-1} + j_l.$$

Since $k_q = j_q$ for $q = 1, \dots, s-1$, it follows from the inequality that $k_l \geq j_l$. As $k_l = j_s$ and $l \geq s$, we conclude that $l = s$.

Hence both assertions of the Proposition have been proved. ∎

(4.7) Corollary. *For any $t \in R$, the Hall–Littlewood polynomials $P^J(A;t)$, for all strictly increasing multi indices J of size n, form an R-basis of* $\mathrm{Sym}_R[A]$.

Proof. It follows from the Proposition that the matrix expressing the $P^J(A;t)$ in terms of the basis $s^I(A)$ is a lower triangular matrix with 1 in the diagonal. In particular, the matrix is invertible. Hence the $P^J(A;t)$ form a basis. ∎

(4.8) Example. For $d \geq 0$ we have the equations,

$$P^{d,d+1,\dots,d+n-1}(A;t) = s^{d,d+1,\dots,d+n-1}(A) = e_n(A)^d. \tag{4.8.1}$$

Indeed, let $J := (d, d+1, \dots, d+n-1)$. Then, in the notation of (4.5), the corresponding partitioning is the trivial partitioning $\mathbf{A} = (A)$ with $r = 1$. As $a^{J-E} = (a_1 \cdots a_n)^d$, the asserted equation follows from (4.5.1).

(4.9) Example. For $d \geq 1$ we have the equations,

$$P^{0,1,\dots,n-2,n-1+d}(A;t) = \sum_b^n b^d \prod_{a \neq b} \frac{b - ta}{b - a} \tag{4.9.1}$$

$$= \sum_{n-d \leq j \leq n-1} (-t)^{n-1-j} s^{0,1,\dots,\hat{j},\dots,n-1,d+j}(A). \tag{4.9.2}$$

Indeed, let $J := (0, 1, \dots, n-2, n-1+d)$. Then, in the notation of (4.5), the corresponding partitioning is $\mathbf{A} = (a_1, \dots, a_{n-1}|a_n)$. Clearly, we have that $a^{J-E} = a_n^d$ and $\Delta_t(\mathbf{A}) = \prod_{q<n}(a_n - ta_q)$. In (4.5.1), the permutations σ are determined by the index p such that $\sigma(a_n) = a_p$. Hence the first expression for $P^J(A;t)$ is obtained from (4.5.1).

To obtain the second expression of $P^J(A;t)$, note that the monomials a^K appearing in the expansion of the product in (4.5.3) either have two equal exponents or appear as terms,

$$(-t)^{n-1-j} a^J (a_{j+1} \cdots a_{n-1}) a_n^{-(n-1-j)} = (-t)^{n-1-j} a^{0,1,\dots,\hat{j},\dots,n-2,d+j}.$$

The second expression for $P^J(A;t)$ is a consequence, since $\delta^A(a^K) = s^K(A)$.

It is customary to extend the notation $P^J(A;t)$ to strictly increasing multi indices J of arbitrary size in exactly the same way as the extension was defined in (SYM.6.13) for the Schur polynomials $s^J(A)$. In this extended notation, the second expression is the following:

$$P^d(A;t) = \sum_{k=0}^{d-1} (-t)^k s^{1,\dots,k,d}(A). \tag{4.9.3}$$

By replacing x by $1/T$ in the Lagrange interpolation formula (PARTL2.8.2), we obtain the equation in $R(A)[[T]]$,

$$\prod_a \frac{1-taT}{1-aT} = 1 + (1-t)\sum_b \frac{bT}{1-bT}\prod_{a\neq b}\frac{b-ta}{b-a}.$$

Therefore, by (4.9.1), it follows that

$$\prod_{a\in A}\frac{1-taT}{1-aT} = 1 + (1-t)\sum_{d\geq 1} P^d(A;t)T^d. \tag{4.9.4}$$

(4.10) Example. Assume that $d \geq 1$ and $1 \leq j \leq n-1$. Consider the polynomial $P^I(A;t)$ for $I := (0,\dots,j-1,j+d,\dots,n-1+d)$. In the notation of (4.5), the corresponding partitioning is $\mathbf{A} = (A_1|A_2)$, where $A_1 := \{a_1,\dots,a_j\}$ and $A_2 := \{a_{j+1},\dots,a_n\}$. By (4.5.1) we have that $P^I(A;t)$ is the partial symmetrization of the product,

$$a^{I-E}\Delta_t(\mathbf{A}) = \prod_{b\in A_2} b^d \prod_{a\in A_1,\, b\in A_2}(b-ta).$$

The product is given by the expansion obtained from (3.2.1) after the substitution $a_i := -ta_i$ for $i = 1,\dots,j$, that is, we have the equation,

$$a^{I-E}\Delta_t(\mathbf{A}) = \sum_{J\subseteq[n]}(-t)^{\|J-E_1\|}s^J(A_1)s^{\tilde{\hat{J}}}(A_2),$$

where the sum is over subsets J of size j, and $\hat{J}$ is the extension to a subset of size $j+d$ of $[n+d]$.

Therefore, by Bott's Formula, we obtain the following expression for the partial symmetrization,

$$P^I(A;t) = \sum_{J\subseteq[n]}(-t)^{\|J-E_1\|}s^{J,\tilde{\hat{J}}}(A), \tag{4.10.1}$$

where the sum is over subsets J of size j, and $\hat{J}$ denotes the extension to a subset of size $j+d$ of $[n+d]$.

Consider the special case $d = 1$. If $J \neq E_1$, then there exists an index $q = 1,\dots,j$ such that $j_q \geq 1$ and $j_q - 1 \notin J$. It follows that j_q belongs to the complement of $\hat{J}$. Hence $s^{J,\tilde{\hat{J}}}(A) = 0$. Therefore, the sum in (4.10.1) reduces to the term corresponding to $J = E_1$. Thus we obtain the equations,

$$P^{0,1,\dots,\hat{j},\dots,n}(A;t) = s^{0,\dots,\hat{j},\dots,n}(A) = e_{n-j}(A). \tag{4.10.2}$$

Note that the equations hold for $j = 0$ by (4.8) and trivially for $j = n$.

In the extended notation of Example (4.9), the polynomial $P^I(A;t)$ is equal to the polynomial $P^{d,d+1,\dots,d+n-1-j}(A;t)$. In particular, (4.10.2) is the following equation, for $0 \leq k \leq n$:

$$P^{1,\dots,k}(A;t) = s^{1,\dots,k}(A) = e_k(A). \tag{4.10.3}$$

(4.11) Proposition. *Let $\hat{A}$ be the alphabet obtained by adding a simple letter to A. For a strictly increasing multi index I of size $n+1$, denote by $P^I(A,0;t)$ the polynomial obtained by specializing the additional letter of $\hat{A}$ to 0. If I is not an extension of a multi index of size n, then $P^I(A,0;t) = 0$. Moreover, for an extension $I = \hat{J}$ of a strictly increasing multi index J of size n, we have the equation,*

$$P^{\hat{J}}(A,0;t) = P^J(A;t). \tag{4.11.1}$$

Proof. Note that, since $P^I(\hat{A};t)$ is symmetric in the letters of $\hat{A}$, we obtain the same polynomial $P^I(A,0;t)$ by specializing any letter of $\hat{A}$ to zero and specializing the remaining letters to the letters of A. We will denote the additional letter of $\hat{A}$ by a_0.

Let $\hat{\mathbf{A}}$ be the partitioning corresponding to I. Set $\hat{K} = I - \hat{E}$. Then $\hat{K} = (k_0, K)$, where K is a weakly increasing multi index of size n and $k_0 \leq k_1$. As $\hat{a}^{\hat{K}} = a_0^{k_0}a^K$, it follows from (4.5.1) that

$$P^I(\hat{A};t) = \sum_{\sigma \in \mathfrak{T}(\hat{\mathbf{A}})} \sigma\Big(a_0^{k_0}a^K \frac{\Delta_t(\hat{\mathbf{A}})}{\Delta(\hat{\mathbf{A}})}\Big). \tag{4.11.2}$$

It follows that the polynomial $P^I(A,0;t)$ is the sum of the rational functions obtained by substitution $a_0 := 0$ in each term of (4.11.2).

Assume that I is not an extension of a multi index of size n. Then $k_0 \geq 1$. Hence all exponents of the monomial $\hat{a}^{\hat{K}}$ are at least equal to 1. It follows that substitution $a_0 := 0$ in $\sigma(a_0^{k_0}a^K)$, for any $\sigma \in \mathfrak{S}(\hat{A})$, yields 0. Hence $P^I(A,0;t) = 0$.

Assume that I is an extension, say $I = \hat{J}$. Then $k_0 = 0$ and $J - E = K$. Let $\hat{A}_1$ be the first subalphabet in the partitioning $\hat{\mathbf{A}}$. Then, in the monomial $\hat{a}^{\hat{K}} = a^K$, the letter a has exponent 0 if and only if $a \in \hat{A}_1$. Hence the term in (4.11.2) corresponding to σ is non-zero after substitution $\sigma^{-1}(a_0) \in \hat{A}_1$. However, if $\sigma(a) = a_0$ for some $a \in A_1$ then $a = a_0$ since σ is assumed to increasing on the subinterval $\hat{A}_1$. Therefore, to determine the result of the substitution, we may in (4.11.2) restrict the index of summation to those $\sigma \in \mathfrak{T}(\hat{\mathbf{A}})$ for which $\sigma(a_0) = a_0$. Clearly, the latter set of permutations correspond to the permutations of $\mathfrak{T}(\mathbf{A})$, where $\mathbf{A}$ is the partitioning of A corresponding to I. Moreover, if $\sigma(a_0) = a_0$, then all factors of $\sigma(\Delta_t(\hat{\mathbf{A}}/\Delta(\hat{\mathbf{A}})$ containing a_0 are of the form $(b - ta_0)/(b - a_0)$, and they yield 1 after substitution of $a_0 := 0$. It follows that the result of substitution of $a_0 := 0$ in (4.11.2) is the sum

$$\sum_{\sigma \in \mathfrak{T}(\mathbf{A})} \sigma\Big(a^K \frac{\Delta_t(\mathbf{A})}{\Delta(\mathbf{A})}\Big),$$

and hence the result is equal to $P^I(A;t)$, as asserted. ∎

(4.12) Example. For any multi index K of size n, let $R_K(A;t)$ be the polynomial obtained as the total symmetrization,

$$R_K(A;t) := \delta^A(a^K\Delta_t(A)). \tag{4.12.1}$$

If K is weakly increasing, then $K+E$ is strictly increasing, and, by (4.5.4), we have the equation,

$$\varphi_{K+E}(t)P^{K+E}(A;t) = R_K(A;t). \tag{4.12.2}$$

Let $\mathbf{A} = (A_1|\dots|A_r)$ be any partitioning of A. Let K_i be the subsequence of K corresponding to the letters of A_i and, with an abuse of notation, let a^{K_i} be the product of the powers $a_p^{k_p}$ for $a_p \in A_i$. Then $a^K = a^{K_1}\cdots a^{K_r}$. Moreover,

$$\Delta_t(A) = \Delta_t(\mathbf{A})\Delta_t(A_1)\cdots\Delta_t(A_r).$$

Hence, we obtain the equation,

$$R_K(A;t) = \delta^{\mathbf{A}}\big(R_{K_1}(A_1;t)\cdots R_{K_r}(A_r;t)\Delta_t(\mathbf{A})\big). \tag{4.12.3}$$

Consider in particular the partitioning $(\bar{A}|a_n)$. For any letter b of a, let $\Pi_b = \Pi_b(A;t)$ be the rational function,

$$\Pi_b := \prod_{a\neq b}\frac{b-ta}{b-a}.$$

Then $\Delta_t(\mathbf{A})/\Delta(\mathbf{A}) = \Pi_{a_n}$, and we obtain an inductive definition of $R_K(A;t)$.

An inductive definition of the polynomials $P^J(A;t)$ is obtained as follows: In the notation of (4.5), let n_r be the number of letters in the last subalphabet A_r. Let $\bar{J} = (j_1,\dots,j_{n-1})$ denote the truncated multi index of size $n-1$. Then we have the equation,

$$\frac{t^{n_r}-1}{t-1}P^J(A;t) = \sum_{p=1}^{n} P^{\bar{J}}(a_1,\dots,\widehat{a_p},\dots,a_n;t)a_p^{j_n-(n-1)}\prod_{q\neq p}\frac{a_p-ta_q}{a_p-a_q}. \tag{4.12.4}$$

Indeed, by (4.5.4) we have that $\varphi_J(t)P^J(A;t)$ is the total symmetrization of the polynomial $a^{J-E}\Delta_t(A)$. Let $\bar{A} := \{a_1,\dots,a_{n-1}\}$ and consider the partitioning $\mathbf{A} := (\bar{A}|a_n)$ of A. Then $\delta^A = \delta^{\mathbf{A}}\delta^{\bar{A}}$. Clearly, we have that

$$a^{J-E}\Delta_t(A) = a^{\bar{J}-\bar{E}}a_n^{j_n-(n-1)}\Delta_t(\bar{A})\Delta_t(\mathbf{A}).$$

The factors $a_n^{j_n-(n-1)}$ and $\Delta_t(\mathbf{A})$ are scalars with respect to the operator $\delta^{\bar{A}}$. Moreover, we have that $\delta^{\bar{A}}\big(a^{\bar{J}-\bar{E}}\Delta_t(\bar{A})\big) = \varphi_{\bar{J}}(t)P^{\bar{J}}(\bar{A};t)$. Hence we obtain the equation,

$$\delta^A\big(a^{J-E}\Delta_t(A)\big) = \varphi_{\bar{J}}(t)\delta^{\mathbf{A}}\big(P^{\bar{J}}(\bar{A})a_n^{j_n-(n-1)}\Delta_t(\mathbf{A})\big). \tag{4.12.5}$$

Denote by $\Pi^J(A;t)$ the sum on the right hand side of (4.12.4). Then, clearly, $\Pi(A;t)$ is equal to the partial symmetrization on the right hand side of (4.12.5). Hence, from (4.12.5) we obtain the equation,

$$\varphi_J(t)P^J(A;t) = \varphi_{\bar{J}}(t)\Pi^J(A;t).$$

Since $\varphi_{\bar{J}}(t^{n_r}-1) = (t-1)\varphi_J(t)$, it follows that (4.12.4) holds when t is a variable. Hence (4.12.4) holds in general.

Schur functions

1. Schur functions.

(1.1) Setup. Fix a finite alphabet $A = \{a_1, \dots, a_n\}$ of $n \geq 0$ letters. In addition to the ring $R[A]$ of polynomials and the ring $R(A)$ of rational functions we consider the ring of formal power series $R[A]$. Clearly, the action of $\mathfrak{S}(A)$ on the polynomials extends canonically to an action on the power series. More generally, the action of the ring $\mathcal{E}_R[A]$ on $R[A]$ extends canonically to an action on $R[A]$.

For a power series p in $R[A]$, we denote by p_j its homogeneous term of degree j. By convention, $p_j = 0$ for $j < 0$.

(1.2) Definition. In the sequel we will consider matrices with coefficients in $R[A]$, possibly with an infinite number of rows or columns. Let s be a power series in $R[A]$. For any integer i we denote by $s[i]$ the infinite row of homogeneous terms of s shifted i places to the right, that is, $s[i]$ is the row whose j'th entry, for $j = 0, 1, \dots$, is equal to s_{j-i}. For any integer j we denote by $s^{j\geq}$ the infinite column whose i'th entry, for $i = 0, 1, \dots$, is equal to s_{j-i}. The infinite matrix S whose rows are $s[i]$ for $i = 0, 1, \dots$, or equivalently, whose columns are $s^{j\geq}$ for $j = 0, 1, \dots$, is an upper triangular matrix,

$$S = \begin{pmatrix} s[0] \\ s[1] \\ \vdots \end{pmatrix} = \left(s^{0\geq}, s^{1\geq}, \dots\right) = \begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ 0 & s_0 & s_1 & \dots \\ 0 & 0 & s_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ are sequences with the same number r of non-negative integers, we denote by $S^{I/J}$ the determinant of the $r \times r$ matrix obtained from S by selecting the rows with indices from I and the columns with indices from J, that is,

$$S^{I/J} = \begin{vmatrix} s_{j_1-i_1} & s_{j_2-i_1} & \dots s_{j_r-i_1} \\ s_{j_1-i_2} & s_{j_2-i_2} & \dots s_{j_r-i_2} \\ \vdots & \vdots & \vdots \\ s_{j_1-i_r} & s_{j_2-i_r} & \dots s_{j_r-i_r} \end{vmatrix}. \tag{1.2.1}$$

Clearly, if p is a second power series, then, for $i \geq 0$,

$$p[i]\, s^{j\geq} = (ps)_{j-i}, \tag{1.2.2}$$

where the left hand side is the product of a row and a column. In particular,

$$p[i]S = (ps)[i] \quad \text{and} \quad Sp^{j\geq} = (sp)^{j\geq}. \tag{1.2.3}$$

(1.3) Definition. Let W be a (commutative) word in the letters of A, that is, a formal sum,

$$W = z_1a_1 + \cdots + z_na_n,$$

where the z_k's are integers. The *degree* of the word is the sum of the coefficients, $|W| = z_1 + \cdots + z_n$. Associate with W the following power series in $R[A]$:

$$S(W) = \prod_{k=1}^{n}\Big(\frac{1}{1-a_k}\Big)^{z_k},$$

and denote by $S_j(W) = S(W)_j$ the homogeneous term of degree j in $S(W)$. Note that if the word is *positive*, that is, if the coefficients z_k are non-negative, then $S(-W)$ is a polynomial of degree equal to the degree of W. In particular, if W is positive, then $S_j(-W) = 0$ for $j > |W|$.

The special word $a_1 + \cdots + a_n$ will be denoted A. As a word, the degree of A is the cardinality of the alphabet A. Clearly $S_j(A)$ is the j'th *complete symmetric function* in the letters of A. The power series $S(-A)$ is the polynomial $\prod_{k=1}^{n}(1-a_k)$, and $(-1)^jS_j(-A)$ is the j'th *elementary symmetric function* in the letters of A.

(1.4) Remark. Enlarge the alphabet A with a single letter x, and consider the word $x - A$ in the letters of the enlarged alphabet. The power series $S(x - A)$ is the product $(1 - x)^{-1}\prod_{k=1}^{n}(1-a_k)$. It follows easily that the homogeneous term $S_j(x - A)$, for $j \geq n$, is equal to the polynomial $x^{j-n}\prod_{k=1}^{n}(x - a_k)$.

(1.5) Definition. Let W be a commutative word in the letters of A. For any two sequences I and J in $\mathbf{N}^r$, we define the associated *skew Schur function* $S^{I/J}(W)$ as the determinant $S^{I/J}$ of (1.2.1) associated with the power series $s := S(W)$. In other words, if we consider the matrices,

$$\begin{pmatrix} S(W)[i_1] \\ \vdots \\ S(W)[i_r] \end{pmatrix} \quad \text{and} \quad \Big(S(W)^{j_1\geq}, \dots, S(W)^{j_r\geq}\Big), \tag{1.5.1}$$

then the skew Schur function $S^{I/J}(W)$ is equal to the determinant of the matrix that is obtained either from the first matrix in (1.5.1) by selecting its columns from J or from the second matrix in (1.5.1) by selecting its rows from I.

The Schur function is a polynomial in $R[A]$. When I is the sequence $0, 1, \dots, r-1$ we write $S^J(W) := S^{I/J}(W)$. The polynomial $S^J(W)$ is the ordinary *Schur function* $S^J(W)$.

(1.6) Observation. Several properties of the skew Schur functions $S^{I/J}(W)$ are obvious from their definition as determinants. Clearly, the function is alternating in I, that is, if the entries in I are permuted, then the function is changed by the signature of the permutation and the function vanishes if two entries of I are equal. Similarly, the function is alternating in J.

The Schur function $S^{I/J}(W)$ is the determinant of an $r \times r$ matrix whose (p, q)'th entry is the homogeneous part of degree $j_q - i_p$ of the power series $S(W)$. In particular, the entry vanishes if $j_q < i_p$. As a consequence, if both sequences I and J are strictly increasing, then $S^{I/J}(W)$ vanishes unless $i_k \leq j_k$ for $k = 1, \dots, r$. Note that under the latter condition, each product in the expansion of the determinant $S^{I/J}(W)$ is homogeneous of degree $\sum j_k - \sum i_k = \|J\| - \|I\|$ in the letters of A. In particular, the ordinary Schur function $S^J(W)$ is homogeneous of degree $\|J\| - r(r-1)/2$.

(1.7) Remark. Consider the Schur function $S^{I/J}(W)$ corresponding to two strictly increasing sequences $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$. Assume that $i_1 = j_1$. Then the first column in the matrix defining the Schur function has 1 as its first entry and 0 as the remaining entries. Therefore, the Schur function is unchanged if the sequences I and J are replaced by $(i_2, \dots, i_r)$ and $(j_2, \dots, j_r)$. Clearly, the argument can be repeated if the first p entries in I and J agree.

Note in addition that the Schur function, as a determinant of a matrix of the form $\{s_{j_p - i_q}\}$, is unchanged if the same integer is added to all entries of I and J. It follows in particular that the ordinary Schur function $S^J(W)$ is unchanged if the sequence J is replaced for some p by the *extended* sequence

$$\tilde{J} = (0, 1, \dots, p-1, p + j_1, \dots, p + j_r),$$

of length $p + r$.

(1.8) Additivity Formula. *In the setup of* (1.5)*, let W' be a second word in the letters of A. Then,*

$$S^{I/J}(W' + W) = \sum_K S^{I/K}(W') S^{K/J}(W),$$

where the sum is over all strictly increasing sequences $K = (k_1, \dots, k_r)$.

Proof. It is clear from the definition of power series associated to words that $S(W' + W) = S(W')S(W)$. Therefore, the first formula of (1.2.3) yields the following matrix equation:

$$\begin{pmatrix} S(W' + W)[i_1] \\ \vdots \\ S(W' + W)[i_r] \end{pmatrix} = \begin{pmatrix} S(W')[i_1] \\ \vdots \\ S(W')[i_r] \end{pmatrix} \Big(S(W)^{0\geq}, S(W)^{1\geq}, \dots \Big).$$

From the matrix equation, extract the equation corresponding to the columns in J. The asserted additivity formula follows from the formula for the determinant of a product of matrices (the Cauchy-Binet Formula). $\square$

(1.9) Duality Formula. *Let W be a word in the letters of A, and let I and J be strictly increasing sequences of r non-negative integers. Then*

$$S^{I/J}(W) = (-1)^{\|J\|-\|I\|} S^{I'/J'}(-W), \tag{1.9.1}$$

where the primes indicate the dual sequences, defined as follows: Choose an integer N greater than all entries of I and J. Identify I with a subset of $\{0, 1, \dots, N-1\}$, and denote by I^c the complement of I in $\{0, 1, \dots, N-1\}$. Then I', as a strictly increasing sequence, is the image of the complement under the reflection $i \mapsto N-1-i$.

Proof. Consider the power series $s := S(W)$ and $t := s(-W)$. As $ts = 1$, the following matrix equation results from the Equations (1.2.2) for $i, j = 0, \dots, N-1$:

$$\begin{pmatrix} t[0] \\ \vdots \\ t[N-1] \end{pmatrix} \left(s^{0\geq}, \dots, s^{N-1\geq} \right) = 1,$$

where the right hand side is the $N \times N$ unit matrix. In the second factor on the left hand side, only the first N rows are nonzero. Therefore, if S denotes the $N \times N$ matrix consisting of the first N rows of the second factor, and T denotes the $N \times N$ matrix consisting of the first N columns of the first factor, then we obtain the matrix equation $TS = 1$. Moreover, the matrix T has determinant 1, since it is an upper triangular matrix with 1 in the diagonal.

Clearly, the two Schur functions $S^{I/J}(W)$ and $S^{I'/J'}(-W)$ of the Duality Formula are minors in the matrices S and T. Now, since $\det T = 1$, it is well known that the matrix equation $TS = 1$ implies that the (I, J)'th minor $S^{I/J}$ in S is equal to the algebraic complement of the (J, I)'th minor in T, that is,

$$S^{I/J} = \operatorname{sign}(I, I^c) \operatorname{sign}(J, J^c) T^{J^c/I^c}. \tag{1.9.2}$$

Indeed, from the matrix equation $TS = 1$ it follows that the matrix of r by r minors of S is the inverse of the matrix of r by r minors of T. On the other hand, from the equation $\det T = 1$ it follows by Laplace development of determinants that the transpose of matrix of algebraic complements of the r by r minors of T is the inverse of the matrix of r by r minors of T. Thus Equation (1.9.2) holds.

Clearly, the left side of Equation (1.9.2) is the Schur function on the left side of the Duality formula. The right hand side of Equation (1.9.2) is easily transformed to the right hand side of the Duality Formula. Indeed, if $I = (i_1, \dots, i_r)$, then the permutation (I, I^c) of the integers $0, \dots, N-1$ has length equal to $i_1 + i_2 - 1 + \dots + i_r - (r-1)$. Hence the products of the two signs on the right hand side of (1.9.2) is equal to $(-1)^{\|J\|-\|I\|}$. Moreover, the minor T^{J^c/I^c} on the right hand side of (1.9.2) is a minor in the matrix $\{t_{j-i}\}$. Therefore, T^{J^c/I^c} is equal T^{I^*/J^*} where I^* and J^* denote the images of the sequences I^c and J^c under the reflection $i \mapsto N-1-i$. Finally, the sequences I' and J' are obtained from I^* and J^* by reversing the order of the elements. The reversion of rows and columns does not change the determinant. Hence T^{I^*/J^*} is equal to the minor $T^{I'/J'}$ of T. Finally, since T was defined from the power series $t = S(-W)$, the minor $T^{I'/J'}$ is the Schur function $S^{I'/J'}(-W)$. Thus the Duality Formula has been proved. ∎

(1.10) Example. Take $I = (0, \dots, r-1)$ and $J = (1, \dots, r)$ (and $N := r+1$). Then $I' = (0)$ and $J' = (r)$. Hence $S^{I/J}(W) = (-1)^r S^r(-W)$ by the Duality Formula, and clearly $S^r(-W) = S_r(-W)$. In particular, when $W = A$, we obtain that the Schur function $S^{1,\dots,r}(A)$ is equal to the r'th elementary symmetric function in the letters of A, see (1.3).

(1.11) Jacobi–Trudi's Formula. *Let $a_1, \dots, a_n$ be the n letters of A, and let J be a sequence of n non-negative integers. Then,*

$$\Delta^J(a_1, \dots, a_n) = \Delta(a_1, \dots, a_n) S^J(A). \tag{1.11.1}$$

Moreover, if D is the $n \times n$ matrix whose k'th row is the ordered set of the n first homogeneous terms in the polynomial $\prod_{i \neq k}(1 - a_i)$, then $\det D = \Delta(a_1, \dots, a_n)$.

Proof. For $k = 1, \dots, n$, we have that $a_k = (a_k - A) + A$, and hence $S(a_k) = S(a_k - A)S(A)$. Consequently, the first equation in (1.2.3), applied with $p = S(a_k - A)$ for $k = 1, \dots, n$, yields the following matrix equation:

$$\begin{pmatrix} S(a_1)[0] \\ \vdots \\ S(a_n)[0] \end{pmatrix} = \begin{pmatrix} S(a_k - A)[0] \\ \vdots \\ S(a_n - A)[0] \end{pmatrix} \Big(S(A)^{0\geq}, S(A)^{1\geq}, \dots \Big).$$

In the first matrix on the right, the power series $S(a_k - A)$ is the polynomial $\prod_{i \neq k}(1 - a_i)$ of degree $n - 1$. Hence, in the first matrix on the right, the first n columns form the matrix D and the remaining columns are equal to zero. Therefore, the product on the right side is unchanged if the first matrix is replaced by D and the second matrix is replaced by its first n rows. Now, from the replaced equation select the columns corresponding to the elements of J, and take determinants. On the left side we obtain the determinant $\Delta^J = \Delta^J(a_1, \dots, a_n)$, and on the right side we obtain the product $(\det D)S^J(A)$. Hence we obtain the equation,

$$\Delta^J = (\det D) S^J(A). \tag{1.11.2}$$

Take $J = (0, 1, \dots, n-1)$ in (1.11.2). The left side becomes the Vandermonde determinant Δ. On the right side we have that $S^J(A) = 1$. Therefore, Equation (1.11.2) implies first that $\Delta = \det D$ and next that the Jacobi–Trudi Formula (1.11.1) holds. □

(1.12) Corollary. *The Schur functions $S^J(A)$, for all strictly increasing sequences J of n non-negative integers, form an R-basis for the algebra* $\mathrm{Sym}_R[A]$ *of symmetric polynomials.*

Proof. By Jacobi–Trudi's formula, $S^J = \Delta^J/\Delta$, and the polynomials Δ^J/Δ form an R-basis by Proposition (SYM.6.6). □

(1.13) Remark. Every ordinary Schur function $S^K(A)$, where $K = (k_1, \dots, k_r)$ is a sequence with an arbitrary number r of entries, is either equal to zero or up to a sign equal to one of the Schur functions $S^J(A)$ of Corollary (1.12), that is, a Schur function defined by a strictly increasing sequence J with n entries. Indeed, since $S^K(A)$ is alternating in K, we may assume that the sequence K is strictly increasing. If $r < n$, we have that $S^K(W) = S^J(W)$, where $J = \tilde{K}$ is the sequence obtained by extending K as in Remark (1.7). Assume that $r > n$. If K is an extension in the sense of Remark (1.7) of a sequence J with n entries, then $S^K(A) = S^J(A)$.

If K is not an extension of a sequence with n entries, then $S^K(A) = 0$. Indeed, proceeding by induction on r, we may assume that K is not an extension of a sequence with $r-1$ entries, that is, we may assume that $k_1 > 0$. Consider an alphabet $\tilde{A}$ with r letters obtained by adding $r-n$ letters $a_{n+1}, \dots, a_r$ to A. Clearly, the Schur function $S^K(A)$ is obtained from the Schur function $S^K(\tilde{A})$ by specializing the additional variables $a_{n+1}, \dots, a_r$ to 0. Now, if the last variable a_r is specialized to 0, then the Vandermonde determinant $\Delta(\tilde{A})$ specializes to a nonzero value and, since $k_1 > 0$, the determinant $\Delta^K(\tilde{A})$ specializes to 0. Therefore, by Jacobi–Trudi's Formula, the Schur function $S^K(\tilde{A})$ specializes to 0, and consequently $S^K(A) = 0$.

(1.14) Remark. Consider the symmetrization operator δ^A of Section (DIFF.2.3),

$$\delta^A(f) = \sum_{\sigma \in \mathfrak{S}(A)} \sigma\Big(\frac{f}{\Delta(A)}\Big) = \frac{1}{\Delta(A)} \sum_{\sigma \in \mathfrak{S}(A)} (\operatorname{sign} \sigma)\sigma(f).$$

The operator δ^A is $\mathrm{Sym}_R[A]$-linear. Let a^J be a monomial. It is elementary, see Theorem (DIFF.3.3), to prove that $\delta^A(a^J) = \Delta^J/\Delta$. Hence it follows from Jacobi–Trudi's Formula that $\delta^A(a^J) = S^J(A)$. As a consequence, if f is a symmetric polynomial, then $fS^J(A) = \delta^A(fa^J)$. In particular, if a symmetric polynomial f is given as a sum of monomials, $f = \sum_K f_K a^K$, then we obtain the formula

$$fS^J(A) = \sum_K f_K S^{K+J}(A).$$

(1.15) Example. *Pieri's Formula.* (1) Let I be a sequence of n non-negative integers, and denote by m^I the *symmetrized monomial* corresponding to I, that is,

$$m^I := \sum_\sigma{}' a^{\sigma I},$$

where the sum is over all *different* permutations σI of the sequence I of exponents. Then, for every sequence J of non-negative integers, we obtain from (1.14) the formula,

$$m^I S^J(A) = \sum_\sigma{}' S^{\sigma I + J}(A), \tag{1.15.1}$$

where the sum is over all different permutations σI of I. In particular, when J is the sequence $(0, 1, \dots, n-1)$ we have that $S^J(A) = 1$, and we obtain an explicit formula for m^I as a linear combination of Schur functions.

For the sequence $I = (1, \dots, 1, 0, \dots, 0)$ where $r \le n$ entries are equal to 1, the polynomial m^I is the r'th elementary symmetric function $(-1)^r S_r(-A)$, and we obtain *Pieri's formula*,

$$(-1)^r S_r(-A) S^J(A) = \sum_K S^{K+J}(A), \tag{1.15.2}$$

where the sum is over all sequences K of n integers where r entries are equal to 1 and the remaining entries are equal to 0. For $J = (0, 1, \ldots, n-1)$, we recover the formula of Example (1.10),

$$(-1)^r S_r(-A) = S^{1,\ldots,r}(A). \tag{1.15.3}$$

Indeed, let $J = (0, 1, \ldots, n-1)$. Then the left hand side of (1.15.2) is equal to the r'th elementary symmetric function since $S^J(A) = 1$. On the right hand side of (1.15.2), the sequence $K + J$ has two equal entries unless $K = (0, \ldots, 0, 1, \ldots, 1)$. Hence the only non-vanishing term on the right hand is the Schur function $S^{K+J}(A)$ where $K = (0, \ldots, 0, 1, \ldots, 1)$. Moreover, for $K = (0, \ldots, 0, 1, \ldots, 1)$ the sequence $K + J$ is an extension in the sense of (1.7) of the sequence $(1, \ldots, r)$. Therefore, the right hand side of (1.15.2) is the Schur function $S^{1,\ldots,r}(A)$.

(2) Similarly, the j'th complete symmetric function is the sum,

$$S_j(A) = \sum_{\|K\|=j} a^K,$$

where the sum is over sequences of n integers. Hence we obtain from (1.14) the formula,

$$S_j(A)S^J(A) = \sum_{\|K\|=j} S^{K+J}(A). \tag{1.15.3}$$

2. Multi Schur functions.

(2.1) Setup. In this section we consider simultaneously several alphabets. All alphabets are assumed to be subalphabets of a fixed (universal) alphabet. Letters and (commutative) words will be taken from the fixed alphabet. As in (SCHUR.1.3), if A is an alphabet, the word defined as the sum of the letters of A will also be denoted by A. If $\{a_1, \dots, a_n\}$ are the letters of A, we write $A_{\le p} := a_1 + \cdots + a_p$, and we define $A_{<p}$, $A_{\ge p}$, and $A_{>p}$ similarly.

(2.2) Definition. Let $\mathcal{W} = \{W_{pq}\}$ be an $r \times r$ matrix of words W_{pq}. Let I and J be sequences of r integers. Define the associated *multi skew Schur function* $S^{I/J}(\mathcal{W})$ as the determinant of the matrix whose (p, q)'th entry is $S_{j_q - i_p}(W_{pq})$, that is,

$$S^{I/J}\begin{pmatrix} W_{11} & \dots & W_{1r} \\ \vdots & & \vdots \\ W_{r1} & \dots & W_{rr} \end{pmatrix} = \begin{vmatrix} S_{j_1-i_1}(W_{11}) & \dots & S_{j_r-i_1}(W_{1r}) \\ \vdots & & \vdots \\ S_{j_1-i_r}(W_{r1}) & \dots & S_{j_r-i_r}(W_{rr}) \end{vmatrix}$$

As in Definition (SCHUR.1.5), when $I = (0, 1, \dots, r-1)$ we write $S^J(\mathcal{W}) := S^{I/J}(\mathcal{W})$ and obtain the ordinary *multi Schur function* $S^J(\mathcal{W})$. When all the words W_{pq} are equal to the same word W, we recover the Schur function $S^{I/J}(W)$ of (SCHUR.1.5).

Clearly, the Schur function is unchanged if the same integer is added to all entries of the two sequences I and J. In particular we will usually assume that the entries of the two sequences I and J are non-negative. Note also that the function $S^{I/J}(\mathcal{W})$ is alternating with respect to J and the columns of the matrix $\mathcal{W}$, and with respect to I and the rows of $\mathcal{W}$. Note finally, that the function is symmetric in the following sense: choose an integer N (say, greater than all elements in the two sequences I and J). Consider the sequences I^* and J^* obtained from I and J by the reflection $x \mapsto N - 1 - x$. Then $S^{I/J}(\mathcal{W}) = S^{J^*/I^*}(\mathcal{W}^{\mathrm{tr}})$, where $\mathcal{W}^{\mathrm{tr}}$ denotes the transposed matrix of $\mathcal{W}$.

Let $(W_1, \dots, W_r)$ be a sequence of r words in the letters of A. Then we denote by

$$S^{I/J}(W_1, \dots, W_r) \quad \text{and} \quad S^{I/J}\begin{pmatrix} W_1 \\ \vdots \\ W_r \end{pmatrix}$$

the two special functions defined as follows: The first function is the Schur function $S^{I/J}(\mathcal{W})$ obtained from the matrix $\{W_{pq}\}$ where $W_{pq} = W_q$ and the second function is defined similarly by the matrix where $W_{pq} = W_p$. Note that the two special Schur functions are maximal minors of the two matrices,

$$\big(S(W_1)^{j_1\ge}, \dots, S(W_r)^{j_r\ge}\big) \quad \text{and} \quad \begin{pmatrix} S(W_1)[i_1] \\ \vdots \\ S(W_r)[i_r] \end{pmatrix}.$$

The first function is the minor of the first matrix corresponding to the row indices in I, the second function is the minor of the second matrix corresponding to the column indices in J.

When $I = (0, 1, \dots, r-1)$ we write $S^J(W_1, \dots, W_r)$ and $S^J(W_1, \dots, W_r)^{\mathrm{tr}}$ for the functions $S^{I/J}(W_1, \dots, W_r)$ and $S^{I/J}(W_1, \dots, W_r)^{\mathrm{tr}}$.

(2.3) Additivity Formula. *Let $(W'_1, \dots, W'_r)$ and $(W_1, \dots, W_r)$ be two sequences of r words, and let I and J be two sequences of r non-negative integers. Then,*

$$S^{I/J}\begin{pmatrix} W'_1 + W_1 & \dots & W'_1 + W_r \\ \vdots & & \vdots \\ W'_r + W_1 & \dots & W'_r + W_r \end{pmatrix} = \sum_K S^{I/K}\begin{pmatrix} W'_1 \\ \vdots \\ W'_r \end{pmatrix} S^{K/J}(W_1, \dots, W_r),$$

where the summation is over all strictly increasing sequences K of r non-negative integers.

Proof. The argument is identical to the proof of (SCHUR.1.8). ∎

(2.4) Duality Formula. *Let $W_0, W_1, \dots$ be a sequence of words such that, for $j > 0$, the word $W_{j-1} - W_j$ is either equal to 0 or equal to a word formed by a single letter. Let I and J be strictly increasing sequences of r nonnegative integers. Finally, let N be an integer greater than all entries in the two sequences I and J. Then*

$$S^{I/J}(W_{j_1}, \dots, W_{j_r}) = (-1)^{\|J\|-\|I\|} S^{I'/J'}(-W_{j^c_t+1}, \dots, -W_{j^c_1+1}),$$

where the primes indicate the dual sequences with respect to N as in (SCHUR.1.9) *and $J^c = (j^c_1, \dots, j^c_t)$ is the complement of J in* $\{0, 1, \dots, N-1\}$.

Proof. The proof is similar to the proof of Duality (SCHUR.1.9). Consider the following product of matrices:

$$\begin{pmatrix} S(-W_1)[0] \\ \vdots \\ S(-W_N)[N-1] \end{pmatrix} \Big(S(W_0)^{0\geq}, \dots S(W_{N-1})^{N-1\geq} \Big). \tag{1}$$

It follows from Formula (SCHUR.1.2.2) that the (i, j)'th entry in the product, for $i, j = 0, 1, \dots N-1$, is equal to $S_{j-i}(W_j - W_{i+1})$. Clearly, the entry $S_{j-i}(W_j - W_{i+1})$ is equal to 0 for $j < i$ and equal to 1 for $j = i$. Moreover, the entry is equal to 0 for $j > i$, because, by hypothesis, the word $-(W_j - W_{i+1})$ a positive word of degree at most $j - i - 1$. Hence the product of matrices (1) is the $N \times N$ unit matrix 1.

From the two matrices in the product (1), let T be the submatrix of the first factor formed by the first N columns, and let S be the submatrix of the second factor formed by the first N rows. As in the proof of the Duality Formula (SCHUR.1.9), it follows that $TS = 1$ and $\det T = 1$. It is a consequence, as we saw in the proof of the Duality Formula (SCHUR.1.9), that the (I, J)'th minor $S^{I/J}$ in S is equal to the algebraic complement of the (J, I)'th minor in T, that is,

$$S^{I/J} = (-1)^{\|J\|-\|I\|} T^{J^c/I^c}. \tag{2}$$

Clearly, the left side of Equation (2) is the Schur function on the left side of the asserted Duality Formula. The right hand side of Equation (2) is easily transformed to the right hand side of the Duality Formula. Indeed, the minor T^{J^c/I^c} on the right hand side of (2) is the Schur function,

$$S^{J^c/I^c}\begin{pmatrix} W_{j_1^c+1} \\ \vdots \\ W_{j_t^c+1} \end{pmatrix}.$$

To obtain the Schur function on the right side of the Duality Formula, apply the reflection $i \mapsto N-1-i$ to I^c and J^c and transpose the matrix, and then reverse the order of row and columns.

Hence the Duality Formula is a consequence of Equation (2). ∎

(2.5) Corollary. *Let $A = \{a_1, \dots, a_n\}$ be an alphabet with n letters, and denote by $A_{\le j}$ the sum of the letters a_p for $p \le j$ (in particular, $A_{\le j} = A$ for $j \ge n$). Moreover, let I and J be strictly increasing sequences of r nonnegative integers, and set $\lambda_p := j_p - (p-1)$ for $p = 1, \dots, r$. Finally, let N be an integer greater than all entries of the two sequences I and J. Then, for any word W,*

$$S^{I/J}(W - A_{\le\lambda_1}, \dots, W - A_{\le\lambda_r}) = (-1)^{\|J\|-\|I\|} S^{I'/J'}(A_{\le t} - W, \dots, A_{\le 1} - W),$$

where the primes indicate the dual sequences with respect to N.

Proof. For any integer j, denote by $\mu(j)$ the number of elements in the complement J^c that are strictly less than j. Then the following equations hold:

$$\mu(j_k) = \lambda_k \quad \text{and} \quad \mu(j_k^c + 1) = k. \tag{1}$$

Indeed, the first equation holds because there are j_k nonnegative integers strictly less than j_k and of these exactly $k-1$ belong to J. The second equation holds, because the nonnegative integers in J^c that are strictly less than $j_k^c + 1$ are the k integers $j_1^c, \dots, j_k^c$.

Now apply Duality (2.4) with $W_j := W - A_{\le\mu(j)}$. It follows from the equations (1) that $W_{j_k} = W - A_{\le\lambda_k}$ and $-W_{j_k^c+1} = A_{\le k} - W$. Hence the asserted formula follows from the Duality Formula. ∎

(2.6) Jacobi's Lemma. *Let $(W_1, \dots, W_r)$ be a sequence of r words, and let $(C_1, \dots, C_r)$ be a sequence of r positive words. Moreover, let I and J be sequences of r non-negative integers. If $|C_p| + i_p < r$ for $p = 1, \dots, r$, then the following formula holds:*

$$S^{I/J}\begin{pmatrix} W_1 - C_1 & \dots & W_r - C_1 \\ \vdots & & \vdots \\ W_1 - C_r & \dots & W_r - C_r \end{pmatrix} = S^{I/0,1,\dots,r-1}\begin{pmatrix} -C_1 \\ \vdots \\ -C_r \end{pmatrix} S^J(W_1, \dots, W_r).$$

In particular, if $|C_p| \le r - p$ for $p = 1, \dots, r$, then

$$S^{J}\begin{pmatrix} W_1 - C_1 & \dots & W_r - C_1 \\ \vdots & & \vdots \\ W_1 - C_r & \dots & W_r - C_r \end{pmatrix} = S^J(W_1, \dots, W_r).$$

Proof. To prove the first assertion, assume that $|C_p| + i_p < r$ for all p. Apply the Additivity Formula (2.3) with $W'_p := -C_p$. The Schur functions $S^{I/K}$ in the sum on the right hand side of the Additivity Formula are the maximal minors of the matrix whose p'th row is $S(-C_p)[i_p]$ for $p = 1, \dots, r$. It follows from the hypothesis on the degree of C_p that only the first r entries in the p'th row can be nonzero. Hence the sum on the right hand side of (2.3) reduces to its single term corresponding to $K = (0, 1, \dots, r-1)$. Clearly, the latter term is the product on the right hand side of the asserted formula. Thus the first assertion holds.

Assume in particular that $I = (0, \dots, r-1)$. Then the first factor on the right side of the first formula is the determinant of an upper triangular matrix with 1 in the diagonal. Hence the first assertion implies the second. ∎

(2.7) Example. *The Jacobi–Trudi Formula.* When $I = (0, \dots, 0)$, the condition for Lemma (2.6) is that $|C_p| < r$ for all p. For example, let $A = \{a_1, \dots, a_n\}$ be an alphabet with n letters, and let $r = n$. Take $W_p := A$ and $C_p := A - a_p$ in Jacobi's Lemma. Then we obtain the formula,

$$S^{0,\dots,0/J}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = S^{0,\dots,0/0,1,\dots,n-1}\begin{pmatrix} a_1 - A \\ \vdots \\ a_n - A \end{pmatrix} S^J(A, \dots, A).$$

The Schur function on the left side is the determinant $\Delta^J(a_1, \dots, a_n)$. On the right side, the second Schur function is the simple Schur function $S^J(A)$, and the first Schur function is the determinant $\det D$ from Jacobi–Trudi's Formula. Hence we recover Jacobi–Trudi's Formula from (2.6) (and we recover the proof from the proof of (2.6)).

(2.8) Factorization Formula. *Let $(a_1, \dots, a_r)$ be a sequence of r letters, and set $A_{\geq p} := a_p + \dots + a_r$ for $p = 1, \dots, r$. Moreover, let J be a sequence of r non-negative integers, and set $\lambda_p := j_p - (p-1)$ for $p = 1, \dots, r$. Finally, let $B_1, \dots, B_r$ be a sequence of r positive words. Assume that $|B_p| \leq \lambda_p$ for $p = 1, \dots, r$. Then*

$$S^J(A_{\geq 1} - B_1, \dots, A_{\geq r} - B_r) = \prod_{p=1}^{r} S_{\lambda_p}(a_p - B_p). \tag{2.8.1}$$

Proof. Apply the particular case $I = (0, 1, \dots, r-1)$ of Jacobi's Lemma (2.6) with $W_p := A_{\geq p} - B_p$ and $C_p := A_{>p}$. It follows that the Schur function on the left hand side of (2.8.1) is equal to the following Schur function:

$$S^J\begin{pmatrix} A_{\geq 1} - A_{>1} - B_1 & \dots & A_{\geq r} - A_{>1} - B_r \\ \vdots & & \vdots \\ A_{\geq 1} - A_{>r} - B_1 & \dots & A_{\geq r} - A_{>r} - B_r \end{pmatrix}. \tag{1}$$

The latter Schur function is the determinant of a matrix whose (p, q)'th entry is equal to

$$S_{j_q - (p-1)}(A_{\geq q} - A_{>p} - B_q). \tag{2}$$

Consider an entry above the diagonal, that is, for $p < q$. Clearly, the word $A_{\geq q}-A_{>p}-B_q$ is negative. It is equal to $-B_{pq}$, where B_{pq} the sum of the $q-p-1$ letters a_k for $p < k < q$ and the word B_q. In particular, since $|B_q| \leq \lambda_q$, the degree of B_{pq} is strictly less than $\lambda_q + (q-p) = j_q - (p-1)$. Hence the entry (2) vanishes above the diagonal.

Therefore the determinant (1) the product of its diagonal entries. Clearly the p'th diagonal word is a_p-B_p, and so the diagonal entry is $S_{j_p-(p-1)}(a_p-B_p) = S_{\lambda_p}(a_p-B_p)$. Thus the asserted formula has been proved. ∎

(2.9) Remark. In the setup of Lemma (2.8), assume that B_p is the word of an alphabet with β_p letters. Then $\beta_p \leq \lambda_p$ by assumption. Therefore, by Remark (SCHUR.1.4), the p'th factor on the right hand side of (2.8.1) is the product,

$$a_p^{\lambda_p-\beta_p} \prod_{b \in B_p} (a_p - b).$$

In particular, if all the alphabets B_p are empty, then the p'th factor on the right hand side of (2.8.1) is the product, $a^\lambda = a_1^{\lambda_1} \cdots a_r^{\lambda_r}$, and we obtain the formula,

$$S^J(A_{\geq 1}, \dots, A_{\geq r}) = a_1^{j_1} a_2^{j_2-1} \cdots a_r^{j_r-(r-1)}.$$

(2.10) Example. Let $A = \{a_1, \dots, a_n\}$ be an alphabet. Then the determinant $\Delta^J(a_1, \dots, a_n)$ defined in (SYM.6.3) is the multi Schur function,

$$\Delta^J(a_1, \dots, a_n) = S^{0,\dots,0/J} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{1}$$

Assume that $J = (0, 1, \dots, n-1)$. Then the determinant Δ^J is the Vandermonde determinant $\Delta(a_1, \dots, a_n)$. It follows by the symmetry of (2.2), or directly, that the Schur function in (1) is equal to $S^{n-1,\dots,n-1}(a_n, \dots, a_1)$, and hence equal to

$$S^{n-1,\dots,n-1}(A_{\leq n}-A_{<n}, \dots, A_{\leq 1}-A_{<1}) \tag{2}$$

It follows from the Factorization Formula (2.8), applied to the reversed sequence $(a_n, \dots, a_1)$ and $B_i := A_{<n-i+1}$, that the Schur function (2) is equal to the product,

$$S_{n-1}(a_n-A_{<n}) \cdots S_1(a_2 - A_{<2}) S_0(a_1).$$

Hence, the Factorization Formula together with Remark (SCHUR.1.4) implies the equation of (SYM.6.7),

$$\Delta(a_1, \dots, a_n) = \prod_{p>q} (a_p - a_q).$$

(2.11) Lemma. *Let $(W_1, \dots, W_r)$ be a sequence of r words, and let J be a sequence of r non-negative integers. Assume for some k and some $d < k$ that the following holds: $W_{k-i} = W_k$ and $j_{k-i} = j_k - i$ for $i = 1, \dots, d$. Then the Schur function $S^{I/J}(W_1, \dots, W_r)$ is unchanged if W_k is replaced by $W_k - B$, where B is positive word of degree at most d.*

Proof. The proof, entirely similar to the proof of Jacobi's Lemma (2.6), is left as an exercise. ∎

(2.12) Example. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be alphabets with n and m letters. As an application of Lemma (2.11) we will prove the formula,

$$S^{m,m+1,\dots,m+n-1}(A - B) = \prod_{p,q}(a_p - b_q). \tag{2.12.1}$$

Set $A_i := a_1 + \dots + a_i$, and let W be any word. Consider for $r = n$ and $J = (m, m+1, \dots, m+n-1)$ the Schur function $S^J(W) = S^J(W, \dots, W)$. Then, by repeated application of Lemma (2.11), it follows that we can replace in the Schur function $S^J(W, \dots, W)$, for $k = n, \dots, 2$, the k'th word W by $W - A_{k-1}$. Hence we obtain the equation,

$$S^{m,m+1,\dots,m+n-1}(W) = S^{m,m+1,\dots,m+n-1}(W-A_0, W-A_1, \dots, W-A_{n-1}).$$

In particular, for $W := A - B$ we obtain the equation,

$$S^{m,m+1,\dots,m+n-1}(A - B) = S^{m,m+1,\dots,m+n-1}(A_{\geq 1} - B, \dots, A_{\geq n} - B).$$

Finally, by applying the Factorization Formula to the Schur function on the right hand side, cf. Remark (2.9), we obtain the formula (2.12.1).

3. Differenciation of Schur functions.

(3.1) Setup. Fix an alphabet A with n letters $a_1, \dots, a_n$. Consider the simple operators ∂^p, π^p, ψ^p, and the general operators $\partial^A = \partial^\omega$, $\pi^A = \pi^\omega$ of Section (DIFF.3). For convenience, define $\partial_A := \partial_\omega$ and $\pi_A := \pi_\omega$.

(3.2) Note. We proved in (DIFF.3.4) that the simple operators satisfy the Coxeter-Moore relations, and in (DIFF.3.5) that the general operators can be defined by any reduced presentation of ω. In this section we will only need the inductive definitions,

$$\partial^A = \partial^1 \cdots \partial^{n-1} \partial^{\bar{A}}, \quad \pi^A = \pi^1 \cdots \pi^{n-1} \pi^{\bar{A}},$$

where $\bar{A} := \{a_1, \dots, a_{n-1}\}$.

(3.3) Lemma. *Fix an integer p such that $1 \leq p < n$. Let W be a word which is symmetric with respect to the letters a_p and a_{p+1}. Then we have the following two identities of power series:*

$$\partial^p S(W - a_p) = S(W),$$
$$\pi^p S(W - a_p) = S(W).$$

Proof. Clearly, $S(W - a_p) = S(W)(1 - a_p)$, and the power series $S(W)$ is symmetric in the letters a_p and a_{p+1}. The operators are linear with respect to polynomials symmetric in a_p and a_{p+1}. Therefore the two first equations of the Lemma result from the following equations,

$$\partial^p (1 - a_p) = 1, \quad \pi^p (1 - a_p) = 1.$$

The latter equations result immediately from the definitions of ∂^p and π^p. $\square$

(3.4) Lemma. *Fix a positive integers $k \leq r$. Consider a Schur function $S^J(W_1, \dots, W_r)$, where J is a sequence of r non-negative integers. Assume that all the words W_q are symmetric with respect to the letters a_p and a_{p+1}. Then,*

$$\partial^p S^J(W_1, \dots, W_k - a_p, \dots, W_r) = S^{j_1, \dots, j_k - 1, \dots, j_r}(W_1, \dots, W_k, \dots, W_r),$$
$$\pi^p S^J(W_1, \dots, W_k - a_p, \dots, W_r) = S^{j_1, \dots, j_k, \dots, j_r}(W_1, \dots, W_k, \dots, W_r).$$

Proof. The Schur function on the left hand sides of the equations is the determinant of a matrix whose k'th column is the sequence of polynomials $S_{j_k - i}(W_k - a_p)$ for $i = 0, \dots, r - 1$ and where the entries of the remaining columns are symmetric in a_p and a_{p+1}. Since the operators ∂^p and π^p are linear with respect to polynomials symmetric in a_p and a_{p+1}, the left sides are therefore equal to the determinants obtained by applying the operators to the entries of the k'th column.

Consider the first equation. Since ∂^p lowers the degree of polynomials by 1, it follows from Lemma (3.3) that $\partial^p S_l(W_k - a_p) = S_{l-1}(W_k)$. Clearly, the first equation of the Lemma is a consequence.

Similarly, it follows from Lemma (3.3) that $\pi^p S_l(W_k - a_p) = S_l(W_k)$, and consequently the second equation of the Lemma holds. $\square$

(3.5) Differenciation Lemma. *Consider for $n \le r$ the Schur function $S^J(W_1, \dots, W_r)$, where J is a sequence of r non-negative integers. Assume that all the words W_q are symmetric in the letters of A. Then the following two formulas hold:*

$$\partial^\omega S^J(W_1 - A_{<1}, \dots, W_n - A_{<n}, W_{n+1}, \dots, W_r) = S^{J-E}(W_1, \dots, W_r),$$
$$\pi^\omega S^J(W_1 - A_{<1}, \dots, W_n - A_{<n}, W_{n+1}, \dots, W_r) = S^J(W_1, \dots, W_r),$$

where $J - E$ is the sequence obtained from J by subtracting $k - 1$ from j_k for $k = 1, \dots, n$.

Proof. The assertion will be proved by induction on n. The formulas have no content when $n = 1$. So we may assume that $n > 1$ and that the assertion holds for the alphabet $\bar{A} := \{a_1, \dots, a_{n-1}\}$.

In the Schur functions on the left sides of the equations, the n'th word $W_n - A_{<n}$ is symmetric with respect to the letters of $\bar{A}$. Therefore, by the inductive hypothesis and the inductive definition of ∂^ω in (3.1), the left hand side of the first equation is equal to the expression,

$$\partial^1 \cdots \partial^{n-1} S^{J-\bar{E}}(W_1, \dots, W_{n-1}, W_n - \bar{A}, W_{n+1}, \dots, W_r),$$

where $J - \bar{E}$ is the sequence obtained from J by subtracting $k-1$ from j_k for $k = 1, \dots, n-1$. Clearly, by applying $n - 1$ times the first equation of Lemma (3.4) for $p = n - 1, \dots, 1$ and $k = n$, it follows that the latter expression is equal to the right hand side of the first formula. Hence the first formula holds.

The proof of the second formula is entirely similar. ∎

(3.6) Remark. By conjugation by ω we obtain from (3.5) the following formulas for ∂_ω and π_ω:

$$\partial_\omega S^J(W_1 - A_{>n}, \dots, W_n - A_{>1}, W_{n+1}, \dots, W_r) = S^{J-E}(W_1, \dots, W_r),$$
$$\pi_\omega S^J(W_1 - A_{>n}, \dots, W_n - A_{>1}, W_{n+1}, \dots, W_r) = S^J(W_1, \dots, W_r).$$

Indeed, in (DIFF.3.6) we proved that $\omega \partial^\omega \omega^{-1} = \partial_\omega$. Consider the first equation of (3.5). Apply the permutation ω to the two sides. The right side is unchanged, because it is symmetric in the letters of A. On the left side, the result is the operator $\omega \partial^\omega = \partial_\omega \omega$ applied to the Schur function $S^J(W_1 - A_{<1}, \dots, W_n - A_{<n}, W_{n+1}, \dots, W_r)$. As ω changes the word $W_k - A_{<k}$ into the word $W_k - A_{>n-k+1}$, we obtain the first formula asserted above. The verification of second formula is completely analogous.

Using the fact that the Schur function $S^J(W_1, \dots, W_r)$ is alternating in J and the words W_q we get analogous formulas when the n words $A_{<k}$ are subtracted from any n different words in the given sequence $(W_1, \dots, W_r)$. For example, from the second formula above we obtain the formula,

$$\pi_\omega S^J(W_1, \dots, W_{r-n}, W_{r-n+1} - A_{>n}, \dots, W_r - A_{>1}) = S^J(W_1, \dots, W_r). \qquad (3.6.1)$$

(3.7) Note. Let J be a sequence of n non-negative integers and set $E := (0, 1, \dots, n-1)$. Then the following two formulas hold:

$$\partial^\omega(a^J) = S^J(A), \tag{3.7.1}$$

$$\pi^\omega(a^J) = S^{J+E}(A), \tag{3.7.2}$$

Indeed, we have that $A_{\geq q} = A - A_{<q}$. Therefore, it follows from the Factorization Formula (SCHUR.2.8), applied to the sequence $J + E$ and the words $B_q := 0$, that

$$S^{J+E}(A - A_{<1}, \dots, A - A_{<n}) = a^J.$$

Hence the two asserted formulas follow from the Differentiation Lemma (3.5).

Note that, by the formulas of Theorem (DIFF.3.3), the left side of (3.7.1) is equal to Δ^J/Δ. Hence the formula (3.7.1) also follows from Theorem (DIFF.3.3) and Jacobi–Trudi's Lemma (SCHUR.1.11). The formula (3.7.2) also follows from the (3.7.1) and the second operator equation of (DIFF.3.3.3).

(3.8) Sergeev-Pragacz's Formula. *Let $B = \{b_1, \dots, b_m\}$ be a second alphabet, disjoint from A. Let J be a strictly increasing sequence of r non-negative integers. Set $\lambda_k := j_k - (k-1)$ for $k = 1, \dots, r$. Assume that $r \geq n$, where n is the number of letters of A, and set $s := r - n$. Then the following formula holds,*

$$S^J(A - B) = \pi^A \pi_B \prod_{k=1}^{s} S_{\lambda_k}(-B_{\leq \lambda_k}) \prod_{k=s+1}^{s+n} S_{\lambda_k}(a_{k-s} - B_{\leq \lambda_k}).$$

Proof. Form the dual sequence J' with respect to an integer N, see (SCHUR.1.9). By choosing N large we may assume that the dual sequence J' has at least m elements. Consider the Schur function $S^J(A - B)$ on the left side of the asserted formula. By Duality (SCHUR.1.9),

$$S^J(A - B) = \varepsilon S^{J'}(B - A) = \varepsilon S^{J'}(B - A, \dots, B - A), \tag{1}$$

where ε is equal to the parity of $\|J\| - r(r-1)/2$. Now apply the Differentiation Formula, in the form of Equation (3.6.1), to the alphabet B and the sequence of equal words $W_q := B - A$. We obtain the formula,

$$S^{J'}(B - A, \dots, B - A) = \pi_B S^{J'}(B - A, \dots, B - A, B_{\leq m} - A, \dots, B_{\leq 1} - A). \tag{2}$$

Again by Duality, see Corollary (SCHUR.2.5), applied to the letters of B and $W := A$,

$$S^{J'}(B - A, \dots, B - A, B_{\leq m} - A, \dots, B_{\leq 1} - A) = \varepsilon S^J(A - B_{\leq \lambda_1}, \dots, A - B_{\leq \lambda_r}), \tag{3}$$

with ε as before. Apply again the Differentiation Lemma to the alphabet A and the words $W_q := A - B_{\le\lambda_q}$. Subtract $A_{<k}$ from the last n words. We obtain the equation,

$$\begin{aligned} S^J(A - B_{\le\lambda_1}, \dots, A - B_{\le\lambda_r}) \\ = \pi^A S^J(A - B_{\le\lambda_1}, \dots, A - B_{\le\lambda_t}, A_{\ge 1} - B_{\le\lambda_{t+1}}, \dots, A_{\ge n} - B_{\le\lambda_r}). \end{aligned} \quad (4)$$

Finally, the Schur function on the right hand side of (4) is equal to the product on the right hand side of Sergeev's Formula, as it follows by applying the Factorization Formula, see Remark (SCHUR.2.9), to a sequence of letters $(a'_1, \dots, a'_s, a_1, \dots, a_n)$ and then specializing the additional letters a'_q to zero. Therefore, the asserted formula follows from Equations (1)–(4). ∎

(3.9) Note. The condition for the Schur function $S^J(A - B)$ that the sequence J has at least n elements can always be obtained by *extending* the sequence J, cf. (SCHUR.1.7).

Clearly, the factor $S_{\le\lambda_k}(-B_{\lambda_k})$ in the first product on the right hand side is only non-zero if the word B_{λ_k} has λ_k letters, that is, if λ_k is less than or equal to the number m of letters of B. Hence the Schur function $S^J(A - B)$ is only non-zero if $\lambda_s \le m$, that is, if $j_s \le m + s - 1$. Sequences J satisfying the latter condition are said to be contained in the (n, m)-*hook*.

Assume that $\lambda_{s+1} \ge m$, that is, $j_{s+1} \ge s + m$. Then the last product in Sergeev-Pragacz's formula is equal to the following product,

$$\prod_{p=1}^{n} a_p^{\lambda_{s+p} - m} \prod_{a \in A,\, b \in B} (a - b).$$

In particular, when $J = (m, m+1, \dots, m+n-1)$, we recover the formula of Example (SCHUR.2.12),

$$S^{m, m+1, \dots, m+n-1}(A - B) = \prod_{a \in A,\, b \in B} (a - b),$$

since $\pi^A(1) = 1$ and $\pi_B(1) = 1$.

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