

HOMOLOGICAL STABILITY FOR CLASSICAL GROUPS

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ABSTRACT. We prove a slope 1 stability range for the homology of the symplectic, orthogonal and unitary groups with respect to the hyperbolic form, over any fields other than \mathbb{F}_2 , improving the known range by a factor 2 in the case of finite fields. Our result more generally applies to the automorphism groups of vector spaces equipped with a possibly degenerate form (in the sense of Bak, Tits and Wall). For finite fields of odd characteristic, and more generally fields in which -1 is a sum of two squares, we deduce a stability range for the orthogonal groups with respect to the Euclidean form, and a corresponding result for the unitary groups.

In addition, we include an exposition of Quillen’s unpublished slope 1 stability argument for the general linear groups over fields other than \mathbb{F}_2 , and use it to recover also the improved range of Galatius–Kupers–Randal-Williams in the case of finite fields, at the characteristic.

1. INTRODUCTION

The homology of the classical groups has long been known to stabilize, in quite large generality, see eg., [3, 4, 5, 8, 11, 16, 19, 25, 28, 31]. Our main result is an improvement on the known stability range for the symplectic, unitary and orthogonal groups over finite fields. The proof is based on an unpublished argument of Quillen in the case of GL_n , which we also present here. The proof works for the groups defined over any field, finite or infinite, as long as it is not \mathbb{F}_2 :

Theorem A. Let \mathbb{F} be a field other than \mathbb{F}_2 . The stabilization maps

$$\begin{aligned} H_d(\mathrm{GL}_n(\mathbb{F}); \mathbb{Z}) &\longrightarrow H_d(\mathrm{GL}_{n+1}(\mathbb{F}); \mathbb{Z}) \\ H_d(\mathrm{Sp}_{2n}(\mathbb{F}); \mathbb{Z}) &\longrightarrow H_d(\mathrm{Sp}_{2n+2}(\mathbb{F}); \mathbb{Z}) \\ H_d(\mathrm{U}_{n,n}^\sigma(\mathbb{F}); \mathbb{Z}) &\longrightarrow H_d(\mathrm{U}_{n+1,n+1}^\sigma(\mathbb{F}); \mathbb{Z}) \\ H_d(\mathrm{O}_{n,n}(\mathbb{F}); \mathbb{Z}) &\longrightarrow H_d(\mathrm{O}_{n+1,n+1}(\mathbb{F}); \mathbb{Z}) \end{aligned}$$

are isomorphisms in degrees $d \leq n - 1$ and surjections in degree $d = n$.

Moreover, if \mathbb{F} is the finite field \mathbb{F}_{ℓ^r} for ℓ a prime number with $\ell^r \neq 2$,

$$H_d(\mathrm{GL}_n(\mathbb{F}_{\ell^r}); \mathbb{F}_\ell) \longrightarrow H_d(\mathrm{GL}_{n+1}(\mathbb{F}_{\ell^r}); \mathbb{F}_\ell)$$

is an isomorphism for $d \leq n + r(\ell - 1) - 3$ and a surjection in degrees $d = n + r(\ell - 1) - 2$.

The case of GL_n is the stability range obtained by Quillen in his note books [22, p10] (which though starts with a couple of bleached pages!), with the improvement of Galatius–Kupers–Randal-Williams [12] in the case of finite fields \mathbb{F}_{ℓ^r} at the characteristic ℓ ; the improved range is obtained here by inputting a twisted coefficient computation [12, Lem 5.2] into Quillen’s argument¹. For the other groups in the case \mathbb{F} a finite field, our result improves the best known stability ranges by a factor of two. (See Essert [8, Thm 3.8] who obtains a bound $d \leq \frac{n}{2} - 1$, slightly improving [19, Thm 8.2], and [29] for symplectic group. Mirzaii [18] has a slope 1 range for finite fields but away from the characteristic.) In the case of an infinite field, our ranges match those obtained by

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¹The paper [12] gives in addition a slope $\frac{2}{3}$ range for $\mathrm{GL}_n(\mathbb{F}_2)$ with \mathbb{F}_2 coefficients, which we do not recover with our methods.

Essert [8, Thm 3.8] and Sah [25, App. B] (see also [18]), except that Sah also gets injectivity in degree n for the general linear groups, which does not hold in our generality, see Remark 6.2.

Our result in the case of symplectic, unitary and orthogonal groups, follows from a more general stability result, stabilizing the automorphism group of any *formed space* with the *hyperbolic form*, as we will explain now. We start by recalling the definition of these objects, and explain how they relate to the above groups.

1.1. Automorphisms of formed spaces. To prove our result, we use the general framework of forms originating in the work of Bak, Tits and Wall (see eg. [2, 27, 32, 33]), and later developed also by Magurn–van der Kallen–Vaserstein [17]. Given a field \mathbb{F} , a (possibly trivial) involution $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, an element $\varepsilon \in \mathbb{F}$ such that $\varepsilon\bar{\varepsilon} = 1$, for $\bar{\varepsilon} := \sigma(\varepsilon)$, and a certain additive subgroup $\Lambda \leq \mathbb{F}$, a *form* q on an \mathbb{F} -vector space E is an element of the quotient

$$\text{Form}(E, \sigma, \varepsilon, \Lambda) := \text{Sesq}_\sigma(E) / X(E, \sigma, \varepsilon, \Lambda)$$

of the sesquilinear forms on E by a subgroup $X(E, \sigma, \varepsilon, \Lambda)$ that depends on the chosen σ, ε and Λ (see Definition 2.1). We call the pair (E, q) a *formed space*. Direct sum induces a monoidal structure on the category of formed spaces and form-preserving maps.

To a form $q \in \text{Form}(E, \sigma, \varepsilon, \Lambda)$, one associates two new maps $\omega_q : E \times E \rightarrow \mathbb{F}$ and $Q_q : E \rightarrow \mathbb{F}/\Lambda$ defined by

$$\omega_q(v, w) = q(v, w) + \overline{\varepsilon q(w, v)} \quad \text{and} \quad Q_q(v) = q(v, v) + \Lambda \in \mathbb{F}/\Lambda,$$

where ω_q is an “ ε -skew symmetric sesquilinear form” by construction, and Q_q should be thought of as a “quadratic refinement” of ω_q . We show in our companion paper [26, Prop 2.7] that the form q is always determined by either ω_q or Q_q . As ω_q and Q_q for various choices of σ, ε and Λ recover the classical notions of alternating, Hermitian and quadratic forms, this allows to study the symplectic, unitary and orthogonal groups, all within one framework. More precisely, for (E, q) a formed space, let

$$\mathbf{G}(E) = \mathbf{G}(E, q) := \text{Aut}(E, q)$$

denote its automorphism group, that is the group of bijective linear maps preserving the form, where we drop q from the notation when q is unambiguous. The classical groups appearing in Theorem A are automorphisms of the formed space $\mathcal{H}^{\oplus n}$ for various choices of the parameters σ, ε and Λ , where $\mathcal{H} = (\mathbb{F}^2, q_{\mathcal{H}})$ is the “hyperbolic” formed space, with

$$q_{\mathcal{H}} : \mathbb{F}^2 \times \mathbb{F}^2 \rightarrow \mathbb{F}$$

defined by $q_{\mathcal{H}}(v, w) = \bar{v}_1 w_2$, for $\bar{v}_1 := \sigma(v_1)$ as above:

Definition 1.1. (Symplectic) For $\sigma = \text{id}$, $\varepsilon = -1$, $\Lambda = \mathbb{F}$, we have $\omega_{q_{\mathcal{H}}}(v, w) = v_1 w_2 - v_2 w_1$ is a non-degenerate alternating form and the *symplectic group* is

$$\text{Sp}_{2n}(\mathbb{F}) = \text{Aut}(\mathbb{F}^{2n}, \omega_{q_{\mathcal{H}}}^{\oplus n}) = \mathbf{G}(\mathcal{H}^{\oplus n}).$$

(Unitary) For $\sigma \neq \text{id}$, $\varepsilon = 1$, $\Lambda = \mathbb{F}^\sigma$, we have $\omega_{q_{\mathcal{H}}}(v, w) = \bar{v}_1 w_2 + \bar{v}_2 w_1$ is a non-degenerate Hermitian form, and the *unitary group* is

$$\text{U}_{n,n}(\mathbb{F}) = \text{U}_{n,n}(\mathbb{F}, \sigma) = \text{Aut}(\mathbb{F}^{2n}, \omega_{q_{\mathcal{H}}}^{\oplus n}) = \mathbf{G}(\mathcal{H}^{\oplus n}).$$

(Orthogonal) For $\sigma = \text{id}$, $\varepsilon = 1$, $\Lambda = 0$, we have $Q_{q_{\mathcal{H}}}(v) = v_1 v_2$ is a quadratic form with associated non-degenerate symmetric bilinear form $\omega_{q_{\mathcal{H}}}(v, w) = v_1 w_2 + v_2 w_1$, and the *orthogonal group* is

$$\text{O}_{n,n}(\mathbb{F}) = \text{Aut}(\mathbb{F}^{2n}, Q_{q_{\mathcal{H}}}^{\oplus n}) = \mathbf{G}(\mathcal{H}^{\oplus n}).$$

So, up to the choice of parameters $(\sigma, \varepsilon, \Lambda)$, the symplectic, unitary and orthogonal groups of Theorem A are really the same group $\mathbf{G}(\mathcal{H}^{\oplus n})$. The fact that $\mathbf{G}(\mathcal{H}^{\oplus n})$ is the same subgroup of $\mathrm{GL}(\mathbb{F}^{2n})$ as $\mathrm{Aut}(\mathbb{F}^{2n}, \omega_{q\mathcal{H}}^{\oplus n})$ in the first two cases and $\mathrm{Aut}(\mathbb{F}^{2n}, Q_{q\mathcal{H}}^{\oplus n})$ in the third is given by [26, Thm A and Prop 2.7], and the fact that the form $q_{\mathcal{H}}^{\oplus n}$ is non-degenerate follows from Lemma 2.7. (A form q is called non-degenerate if the associated form ω_q is non-degenerate, see Definition 2.2.)

With this language in place, Theorem A is actually a consequence of the following more general result:

Theorem A' (see Theorem 6.1). Let $E = (E, q)$ be a formed space over a field $\mathbb{F} \neq \mathbb{F}_2$. Then

$$H_d(\mathbf{G}(E \oplus \mathcal{H}), \mathbf{G}(E); \mathbb{Z}) = 0 \quad \text{for } d \leq g(E)$$

where the *genus* $g(E)$ is the maximal dimension of an isotropic subspace in E modulo the radical (see Definition 2.2).

Note that E in the theorem is allowed to be degenerate, and again only the field \mathbb{F}_2 is excluded; this assumption is used in the form $\mathbb{F}_{\mathbb{F}^\times} = 0$, which is true for all fields other than \mathbb{F}_2 . Three out of four cases in Theorem A are obtained by taking $(E, q) = \mathcal{H}^{\oplus n}$ with the three choices for $(\sigma, \varepsilon, \Lambda)$ of Definition 1.1, given that $\mathcal{H}^{\oplus n}$ has genus n (by Lemma 2.7). The remaining case of the general linear groups in Theorem A is not a corollary of the above theorem, since the trivial form on \mathbb{F}^n has genus zero; this case, also treated in Theorem 6.1, thus receives a certain amount of separate treatment. Theorem 6.1 contains in addition a stability result for certain affine groups, and the improved range for linear groups over finite fields at the characteristic, as in Theorem A. The proof of this improved range does not carry over to likewise give an improved range in the case of the symplectic, orthogonal and unitary groups, the main obstacle being the connectivity of the building used (see Remark 6.2).

1.2. Euclidean orthogonal and unitary groups. We consider in addition stability for the Euclidean orthogonal and unitary groups $\mathrm{O}(n, \mathbb{F})$ and $\mathrm{U}(n, \mathbb{F}, \sigma)$ associated to the form which has an orthonormal basis (see Definition 7.4):

Theorem B. (1) Let \mathbb{F} be a field with $\mathrm{char}(\mathbb{F}) \neq 2$ and so that $-1 = a^2 + b^2$ for some $a, b \in \mathbb{F}$. Then the stabilization map

$$H_d(\mathrm{O}(n, \mathbb{F}); \mathbb{Z}) \longrightarrow H_d(\mathrm{O}(n+1, \mathbb{F}); \mathbb{Z})$$

is an isomorphism in degrees $d \leq \frac{n-4}{2}$ and a surjection in degree $d = \frac{n-3}{2}$.

(2) Let $\mathbb{F} \neq \mathbb{F}_2$, and $\sigma = \overline{(\cdot)}$ a nontrivial involution of \mathbb{F} , such that $-1 = \bar{a}a + \bar{b}b$ for some $a, b \in \mathbb{F}$. Then the stabilization map

$$H_d(\mathrm{U}(n, \mathbb{F}, \sigma); \mathbb{Z}) \longrightarrow H_d(\mathrm{U}(n+1, \mathbb{F}, \sigma); \mathbb{Z})$$

is an isomorphism in degrees $d \leq \frac{n-4}{2}$, and a surjection in degree $d = \frac{n-3}{2}$.

If b in (1) or (2) can be chosen to be 0, the corresponding bounds can be improved by 1.

Cathelineau [4] obtained such a stability result for $\mathrm{O}(n, \mathbb{F})$ with range $d \leq n-1$ under the assumption that \mathbb{F} is an infinite Pythagorean field² of characteristic not 2 (see also [30] for a related result), and Sah [25] for $\mathrm{U}(n, \mathbb{F})$ with a range $d \leq n-1$ in the case $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . See also [7].

Part (1) of the theorem applies in particular to finite fields \mathbb{F}_q for q odd and part (2) to \mathbb{F}_{q^2} with the nontrivial involution for any $q \neq 2$ (see Lemmas 7.1 and 7.2). Besides finite fields, Theorem B in the orthogonal case also applies to some infinite fields which are not Pythagorean, such as $\mathbb{Q}(i)$.

²A *Pythagorean field* is a field in which every sum of two squares is a square.

Theorem B is actually a consequence of our main stability theorem, Theorem A'. Just like in the previous case, there is a more general statement behind:

Theorem B'. Let (\mathbb{F}, σ) be a field with involution satisfying that $-1 = \bar{a}a + \bar{b}b$ for some $a, b \in \mathbb{F}$, and let (E, q) be a non-degenerate formed space. Then the stabilization map

$$H_d(\mathbf{G}(E^{\oplus n}); \mathbb{Z}) \longrightarrow H_d(\mathbf{G}(E^{\oplus n+1}); \mathbb{Z})$$

is injective in degrees $d \leq k(\lfloor \frac{n}{2} \rfloor - e) - 1$, and a surjection in degrees $d \leq k(\lfloor \frac{n-1}{2} \rfloor - e)$, where $k = \dim(E)$ and $e = 0$ if b can be chosen to be 0 in the above equation, and $e = 1$ otherwise.

Theorem B is obtained by applying the above result to the Euclidean formed space $\mathcal{E} = (\mathbb{F}, q_{\mathcal{E}})$, with $q_{\mathcal{E}}(a, b) = \bar{a}b$. This formed space is non-degenerate as long as $\varepsilon \neq -1$, which is the reason for the assumptions in Theorem B—see Remark 7.5 more details about this, for why there is no Euclidian symplectic group.

Remark 1.2. One may wonder whether Theorems B and B', having slope 2 ranges, could be proved using a more direct and simpler stability argument. The groups $\mathbf{G}(E^{\oplus n})$ fit eg. into the framework of [24] and stability with slope 2 will follow if one can show that a certain associated building is at least slope 2 connected (with no a priori non-degeneracy assumption, or assumption on the field). The connectivity of this building is however not known to us. In the case of the groups $O(n, \mathbb{F})$, this building is closely related to the poset of orthonormal frames studied in [30], where a connectivity bound dependent on the Pythagorean number of the field³ is given (see Corollary 1.8 in that paper). The connectivity of a similar building is proved by Cathelineau [4, Prop 4.2] under the hypothesis that \mathbb{F} is Pythagorean, that is with Pythagorean number 1.

1.3. Vanishing ranges. The homology of the general linear groups, symplectic, unitary and orthogonal groups over finite fields was completely computed by Quillen [21] and Fiedorowicz-Priddy [9]⁴ away from the characteristic of the field. At the characteristic, the stable homology of these groups was shown to vanish by the same authors (see also [10, 15]), but their unstable homology remains rather mysterious. Our stability range in the case of finite fields thus translate to improved vanishing ranges for the homology of these groups at the characteristic. We make this explicit here in the symplectic, unitary and orthogonal case. The general linear group case is [12, Cor C].

Combining Theorem A with the vanishing of the stable homology [9, Thm III.4.6] gives:

Corollary C. (1) For $\mathbb{F}_{\ell^r} \neq \mathbb{F}_2$,

$$H_d(\mathrm{Sp}_{2n}(\mathbb{F}_{\ell^r}); \mathbb{F}_{\ell}) = 0 \quad \text{for } 0 < d < n$$

(2) For $\mathbb{F}_{\ell^{2r}} \neq \mathbb{F}_2$,

$$H_d(\mathrm{U}_{n,n}(\mathbb{F}_{\ell^{2r}}); \mathbb{F}_{\ell}) = 0 \quad \text{for } 0 < d < n.$$

(3) For ℓ odd

$$H_d(\mathrm{O}_{n,n}(\mathbb{F}_{\ell^r}); \mathbb{F}_{\ell}) = 0 \quad \text{for } 0 < d < n.$$

Using other results in the litterature, one can complete the above corollary as follows:

Remark 1.3. (1) (Symplectic and unitary groups with $\mathbb{F} = \mathbb{F}_2$) The stability result [8, Thm 3.8] applies to the case $\mathbb{F} = \mathbb{F}_2$ and gives

$$H_d(\mathrm{Sp}_{2n}(\mathbb{F}_2); \mathbb{F}_2) = H_d(\mathrm{U}_{n,n}(\mathbb{F}_2); \mathbb{F}_2) = 0 \quad \text{for } 0 < d < \frac{n-1}{2}.$$

(2) (Orthogonal groups in characteristic 2) The first homology group $H_1(\mathrm{O}_{n,n}(\mathbb{F}_{\ell^r}); \mathbb{F}_{\ell}) = \mathbb{Z}/2$ when $\ell = 2$ [9, II.7.19], so there is no vanishing range in characteristic two for orthogonal groups.

³the smallest integer $p = p(\mathbb{F})$ such that every sum of squares in \mathbb{F} can be written as a sum of p squares

⁴See Theorems IV.2.1, IV.5.2, IV.7.2, V.2.1, V.3.1, V.5.1 in [9].

There is however a stable vanishing for the commutator subgroup $O'_{n,n}(\mathbb{F}_{\ell^r})$ of $O_{n,n}(\mathbb{F}_{\ell^r})$ [9, Thm III.4.6]. (This subgroup identifies with the subgroup $DO_{n,n}(\mathbb{F}_{\ell^r})$ of matrices with trivial Dickson invariant, see [9, II.7.19].) Stability for the commutator subgroups follows from a twisted stability theorem for the groups, as in [24, Cor 3.9], and in fact, the groups $O'_{n,n}(\mathbb{F}_{\ell^r})$ occur as a special case of Theorem 5.16 in [24], which, combined with the vanishing result of [9] gives that

$$H_d(O'_{n,n}(\mathbb{F}_{2^r}); \mathbb{F}_2) = 0 \quad \text{for } 0 < d < \frac{n-5}{3}.$$
⁵

In finite characteristic, there is one isomorphism class of non-degenerate symplectic and Hermitian form in each dimension, and only one orthogonal group in odd dimensions, see [9, Sec II.4,6,7]. In even dimensions and odd characteristic, there are two non-degenerate quadratic forms that are not isomorphic, corresponding to the groups usually called $O^+(2n, \mathbb{F}_{\ell^r}) = O(2n, \mathbb{F}_{\ell^r})$ and $O^-(2n, \mathbb{F}_{\ell^r})$, see [9, II.4.5]. (See Remark 7.5(3) for the characteristic 2 case.) The group $O_{n,n}(\mathbb{F}_{\ell^r})$ defined above is isomorphic to the first or the second of these groups depending on whether $(-1)^n$ has a square root in \mathbb{F}_{ℓ^r} or not. There is though only one stable orthogonal group [9, Prop II.4.11]. (See also Theorem 7.6.)

Our stability results, again combined with the vanishing results of [9], also prove:

Corollary D. (1) For $\ell^r \neq 2$,

$$H_d(U(n, \mathbb{F}_{\ell^{2r}}); \mathbb{F}_{\ell}) = 0 \quad \text{for } 0 < d < \frac{n}{2}.$$

(2) For ℓ an odd prime,

$$H_d(O^{\pm}(n, \mathbb{F}_{\ell^r}); \mathbb{F}_{\ell}) = 0 \quad \text{for } 0 < d < \frac{n-1}{2}$$

where O^{\pm} stands for either O^+ or O^- when n is even, and for O when n is odd.

There is some overlap between the two corollaries as $U_{n,n}(\mathbb{F}_{\ell^{2r}}) \cong U(2n, \mathbb{F}_{\ell^{2r}})$ and $O_{n,n}(\mathbb{F}_{\ell^r})$ is isomorphic to either $O^+(2n, \mathbb{F}_{\ell^r})$ or $O^-(2n, \mathbb{F}_{\ell^r})$. For the unitary groups, both results gives the same range, whereas for the orthogonal groups, the first result is better when there is overlap. (See Remark 7.7 for more details.)

1.4. Buildings. The proofs of the homological stability results proceed by studying the Tits building of a vector space, and the isotropic building of a formed space, together with their actions by the relevant automorphism groups. These buildings are regarded as posets: the poset of nontrivial subspaces of a vector space, and the poset of isotropic subspaces of a formed space (containing the radical). In both cases, we need relative variants of the posets; see Definitions 3.2 and 4.1. These posets are highly connected, and indeed satisfy the Cohen-Macaulay property, i.e. all their intervals are homotopy-spherical: this is well-known for the general linear building, and was proven in a special case by Vogtmann in [31] for the isotropic building. We show in Theorems 4.6 and 4.7 that Vogtmann's argument extends to our more general set-up. Note that it is crucial for obtaining a slope 1 stability with our argument that the building is at least slope 1 connected.

1.5. Spectral sequence argument. We prove the stability theorem using spectral sequences associated to the action of the groups on the buildings: Following Quillen, we associate a small chain complex to the Cohen-Macaulay posets (see Appendix B), which gives a double complex using the group action and hence a spectral sequence (see Section 6.2). Stabilizers of the action appear in the spectral sequence and must be treated in parallel; these are affine versions of the groups. These stabilizer subgroups are more complicated than those arising when using simplicial

⁵where we used that the unitary stable rank of a field is 1 [19, Rem 6.4]

complexes of unimodular vectors or split versions of these (as for example in [5, 16, 19, 24]), see Proposition 6.5. Also, the fact that we work with posets rather than simplicial complexes gives more complicated twisted coefficients in the spectral sequence, in the present case twisted by the Steinberg module, the top homology of the general linear building (the same Steinberg module in all cases!). But these two difficulties are in fact, together, the reason why we can improve the slope in the argument: indeed, they combine to give the vanishing of certain groups in the spectral sequence in the form of the vanishing of the Steinberg coinvariants. This vanishing is well-known for the non-relative building, see eg. [1, Thm. 1.1], and we give here a proof for the relative case (see Theorem 3.9), again following ideas outlined by Quillen, using a join decomposition of the building (Theorem 3.6). In the case of finite fields, this vanishing was improved at the characteristic by Galatius–Kupers–Randal-Williams to a vanishing of the twisted homology in a range [12, Lem 5.2], yielding the improved stability range under the same assumption.

It might be possible to apply the above argument to eg. $GL_n(R)$ for R belonging to a certain class of rings. In addition to the coinvariant vanishings mentioned above, the main properties of the buildings that are used in the proof of the stability theorem are summarized in Section 6.1.

One can also ask whether other families of groups would fit into the arguments presented here, and look for a general set-up making such a spectral sequence work, for example for groups arising as automorphism groups of “uncomplemented versions” of the complemented categories of [20] or the homogeneous categories of [24], as suggested by the examples considered here. We do not know whether such a general set-up can be formulated. We did attempt to formulate an abstract list of ingredients making the above spectral sequence argument work, and found a list that was long and unenlightning.

Organization of the paper: In Section 2, we give a brief introduction to the framework of forms. In Section 3, we define and study the general linear buildings with the associated Steinberg modules, while Section 4 is concerned with the buildings of isotropic subspaces of formed spaces. Section 5 defines and studies the groups acting on the buildings. Section 6 gives the proof of Theorems A and A’ through the more general Theorem 6.1, and deduces Corollary C, while Section 7 treats the case of the Euclidean orthogonal and unitary groups, proving Theorem B’ and Corollary D. Finally, Appendix A gives some basic properties of subspaces of formed spaces, while the short Appendix B associates a chain complex to a Cohen-Macaulay poset.

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CONTENTS

1. Introduction	1
1.1. Automorphisms of formed spaces	2
1.2. Euclidean orthogonal and unitary groups	3
1.3. Vanishing ranges	4
1.4. Buildings	5

1.5. Spectral sequence argument	5
2. Forms and classical groups	7
2.1. The hyperbolic formed space \mathcal{H}	9
3. The buildings of a vector space	11
3.1. A join decomposition of the relative building	12
3.2. Vanishing of relative Steinberg coinvariants	14
4. The buildings of a formed space	15
4.1. Cohen-Macaulay property	17
5. Groups acting on the buildings	23
6. Homological stability	27
6.1. Properties of the groups and buildings	28
6.2. Spectral sequences	31
6.3. Proof of Theorem 6.1	33
7. Euclidean form	39
Appendix A. Forms and isotropic subspaces	43
A.1. Properties of orthogonal complements	43
A.2. The formed space Z^\perp/Z for Z isotropic	45
Appendix B. Cohen-Macaulay posets	46
References	47

2. FORMS AND CLASSICAL GROUPS

We give here a brief introduction to the framework of forms of Bak, Tits and Wall [2, 27, 32, 33]) (see also [17]) restricting to the case of fields, referring to [26] for more details. We introduce the notion of kernel, radical and genus for a vector space equipped with a form, and study the hyperbolic form.

Fix a field \mathbb{F} and a field involution $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ (possibly the identity). Throughout, we will write $\bar{c} = \sigma(c)$ in analogy with complex conjugation. Given a scalar $\varepsilon \in \mathbb{F}$ satisfying $\varepsilon\bar{\varepsilon} = 1$, define

$$\begin{aligned}\Lambda_{min} &= \{a - \varepsilon\bar{a} \mid a \in \mathbb{F}\} \\ \Lambda_{max} &= \{a \in \mathbb{F} \mid a + \varepsilon\bar{a} = 0\}.\end{aligned}$$

These are additive subgroups of \mathbb{F} . Let Λ be an abelian groups such that

$$\Lambda_{min} \leq \Lambda \leq \Lambda_{max}.$$

In fact, for most triples $(\mathbb{F}, \sigma, \varepsilon)$, we actually have $\Lambda_{min} = \Lambda_{max}$ so that there is a unique Λ determined by the triple (see [26, Prop A.1]).

For an \mathbb{F} -vector space E , recall that a map $f : E \times E \rightarrow \mathbb{F}$ is called *sesquilinear* if f is biadditive and satisfies that $f(av, bw) = \bar{a}f(v, w)b$ for all $a, b \in \mathbb{F}$ and $v, w \in E$. The set of sesquilinear forms on a vector space E forms a vector space $\text{Sesq}_\sigma(E)$ with addition and scalar multiplication defined pointwise.

Definition 2.1. A $(\sigma, \varepsilon, \Lambda)$ -quadratic form on E (or just a *form* for short) is an element of the quotient

$$\text{Form}(E, \sigma, \varepsilon, \Lambda) = \text{Sesq}_\sigma(E) / X(E, \sigma, \varepsilon, \Lambda),$$

where $\text{Sesq}_\sigma(E)$ denotes the vector space of sesquilinear forms on E and $X(E, \sigma, \varepsilon, \Lambda)$ is the additive subgroup of all those f satisfying

$$f(v, v) \in \Lambda \quad \text{and} \quad f(w, v) = -\varepsilon \overline{f(v, w)}.$$

A *formed space* (E, q) is a finite-dimensional vector space E equipped with a form q .

The set of forms on E is contravariantly functorial in E : from a linear map $f : E \rightarrow F$ and a form q on F , we produce a form on E by setting $f^*q(v, w) = q(f(v), f(w))$ for $v, w \in E$.

As described in the introduction, to a form q we can associate two new objects which capture the features of q in a more accessible way: a sesquilinear map $\omega_q : E \times E \rightarrow \mathbb{F}$ defined by $\omega_q(v, w) = q(v, w) + \varepsilon \overline{q(w, v)}$, which satisfies that $\omega(v, w) = \varepsilon \overline{\omega(w, v)}$, and a set map $Q_q : E \rightarrow \mathbb{F}/\Lambda$ $Q_q(v) = q(v, v) + \Lambda \in \mathbb{F}/\Lambda$, which satisfies that $Q_q(av) = a\bar{a}Q_q(v)$. Theorem 2.5 in [26] shows that the maps ω_q and Q_q are independent of the choice of representative of q , that they satisfy the equation

$$(2.1) \quad Q_q(v + w) - Q_q(v) - Q_q(w) = \omega_q(v, w) \in \mathbb{F}/\Lambda.$$

and that the form q is completely determined by the pair (ω_q, Q_q) . More than that, Proposition 2.4 in that paper shows that in fact one of ω_q and Q_q is always enough to determine q .

Formed spaces form a category $\mathbf{Forms}(\mathbb{F}, \sigma, \varepsilon, \Lambda)$, in which the morphisms

$$(E, q_E) \rightarrow (E', q_{E'})$$

are the linear maps $f : E \rightarrow E'$ preserving the form, i.e. such that the form

$$f^*q_{E'} = q_E(f(-), f(-))$$

coincides with q_E as forms on E . This condition is well-defined, and is equivalent to preserving both ω_q and Q_q by [26, Thm 2.5]. The morphisms in $\mathbf{Forms}(\mathbb{F}, \sigma, \varepsilon, \Lambda)$ are called *isometries*. The category of formed spaces is symmetric monoidal under the direct sum operation

$$(E, q_E) \oplus (E', q_{E'}) = (E \oplus E', q_{E \oplus E'})$$

where

$$q_{E \oplus E'}((v, w), (v', w')) = q_E(v, v') + q_{E'}(w, w').$$

The monoidal category of formed spaces over a finite field \mathbb{F} with standard parameters $(\sigma, \varepsilon, \Lambda)$ is studied in quite a lot of details in [9, Chap II].

Given a vector space E , let ${}^\sigma E^*$ denote the vector space of σ -skew-linear maps $E \rightarrow \mathbb{F}$, that is, additive maps $f : E \rightarrow \mathbb{F}$ such that $f(av) = \bar{a}f(v)$, with vector space structure defined pointwise from that of \mathbb{F} .

Definition 2.2. Let $E = (E, q)$ be a formed space.

(1) The *kernel* of E is defined to be the kernel $\mathfrak{K}(E) = \ker(b_q)$ of the associated linear map

$$b_q : E \rightarrow {}^\sigma E^*, \quad v \mapsto \omega_q(-, v).$$

We say that (E, q) is *non-degenerate* if $\mathfrak{K}(E) = 0$.

(2) The *orthogonal complement* U^\perp of a subspace $U \leq E$ is defined to be the subspace consisting of all $v \in E$ such that $\omega(v, U) = 0$. That is, U^\perp is the kernel of the composition

$$E \xrightarrow{b_q} {}^\sigma E^* \xrightarrow{\text{incl}^*} \sigma U^*.$$

(3) The *radical* of E is defined to be the set

$$\mathfrak{R}(E) = \{v \in \mathfrak{K}(E) \mid Q_q(v) = 0\}.$$

(4) A subspace U of E is called *isotropic* if $q|_U = 0$ (or equivalently, $\omega_q|_U = 0$ and $Q_q|_U = 0$).

(5) The *genus* $g(E)$ is defined to be the maximum dimension of an isotropic subspace minus the dimension of the radical $\mathfrak{R}(E)$.

Remark 2.3. (1) The radical $\mathfrak{R}(E)$ is a subspace of E , because Q_q is additive on $\mathfrak{R}(E)$ and satisfies $Q_q(cv) = \bar{c}cQ_q(v)$ for $c \in \mathbb{F}, v \in E$. The radical is isotropic. If the characteristic of \mathbb{F} is not 2, then $\mathfrak{R}(E) = \mathfrak{K}(E)$, since equation (2.1) in that case gives that $\omega_q|_{\mathfrak{R}(E)} = 0$ implies $Q_q|_{\mathfrak{R}(E)} = 0$.

(2) Note that orthogonality is a symmetric relation: $\omega_q(v, w) = 0$ if and only if $\omega_q(w, v) = 0$, but note also that the orthogonal complement U^\perp defined above will usually not be a complement in the sense of vector spaces. In fact, we will typically be interested in U^\perp when U is isotropic, in which case we actually have $U \subset U^\perp$! Section A.1 in the appendix is concerned with properties of orthogonal complements that are used in the paper.

Lemma 2.4. Let $E = (E, q)$ and $E' = (E', q')$ be two formed spaces. Then

- (1) $\mathfrak{R}(E \oplus E') = \mathfrak{R}(E) \oplus \mathfrak{R}(E')$
- (2) If $\mathfrak{R}(E') = 0$, then $\mathfrak{R}(E \oplus E') = \mathfrak{R}(E)$.
- (3) $g(E \oplus E') \geq g(E) + g(E')$.

Proof. Statement (1) follows from the fact that, for all $e \in E, e' \in E'$,

$$b_{q \oplus q'}(e + e') = b_q(e) + b_{q'}(e') \in {}^\sigma E^* \oplus {}^\sigma E'^*.$$

Statement (2) follows from (1) and (3) for the fact that $U \oplus U'$ is isotropic whenever $U \leq E$ and $U' \leq E'$ both are. \square

Remark 2.5. It is not true in general that $\mathfrak{R}(E \oplus E') = \mathfrak{R}(E) \oplus \mathfrak{R}(E')$, or that $g(E \oplus E') = g(E) + g(E')$. Take for example the case $(\sigma, \varepsilon, \Lambda) = (\text{id}, 1, 0)$ and let $E = (\mathbb{F}, q)$ with $q(a, b) = ab$, and $E' = (\mathbb{F}, q')$ with $q'(a, b) = -ab$. If the characteristic of \mathbb{F} is two, this is a counterexample to the first statement as $\mathfrak{R}(E \oplus E') = \langle (1, 1) \rangle$. If not, it is a counterexample to the second one as $\langle (1, 1) \rangle$ is isotropic.

2.1. The hyperbolic formed space \mathcal{H} . We will be particularly interested in the following basic two-dimensional hyperbolic formed space:

Definition 2.6. Given (\mathbb{F}, σ) , define a formed space $\mathcal{H} = (\mathbb{F}^2, q_{\mathcal{H}})$ by setting

$$q_{\mathcal{H}}(v, w) = \bar{v}_1 w_2.$$

The following result gives properties of the formed space \mathcal{H} , which show that it is a natural ‘‘atomic object’’ to stabilize with. These properties will be used in Section 6, where our main homological stability result is proved.

Lemma 2.7. The formed space $\mathcal{H} = (\mathbb{F}^2, q_{\mathcal{H}})$ of Definition 2.6 has kernel $\mathfrak{R}(\mathcal{H}) = \mathfrak{K}(\mathcal{H}) = 0$ and genus $g(\mathcal{H}) = 1$ and, given any formed space (E, q) , we have that

$$\mathfrak{R}(E \oplus \mathcal{H}) = \mathfrak{R}(E) \quad \text{and} \quad g(E \oplus \mathcal{H}) = g(E) + 1.$$

In particular, $g(\mathcal{H}^{\oplus n}) = n$.

Proof. Let e_1, e_2 be the standard basis for \mathcal{H} , and α_1, α_2 the dual basis for ${}^\sigma \mathcal{H}^*$. We have

$$\omega_q(v, w) = \bar{v}_1 w_2 + \varepsilon \bar{v}_2 w_1,$$

so $b_q(e_1) = \varepsilon \alpha_2$ and $b_q(e_2) = \alpha_1$. Hence b_q is bijective and $\mathfrak{R}(\mathcal{H}) = 0$. That shows that \mathcal{H} itself is not isotropic, so $g(\mathcal{H}) < 2$. But the span of e_1 is isotropic, so $g(\mathcal{H}) = 1$.

Let (E, q) be a formed space. By Lemma 2.4 and the above calculation, $\mathfrak{R}(E \oplus \mathcal{H}) = \mathfrak{R}(E)$ and $g((E \oplus \mathcal{H})) \geq g(E) + 1$. In Lemma 4.3, we will see that all the maximal isotropic subspaces of $E \oplus \mathcal{H}$ have the same dimension, so it is enough to consider the dimension of a maximal isotropic subspace containing $U \oplus L$ for $U \leq E$ and $L \leq \mathcal{H}$ maximal isotropic. Now, using Lemma A.3, we have

$$(U \oplus L)^\perp = U^\perp \cap L^\perp = (U^\perp \cap E) \oplus (L^\perp \cap \mathcal{H}) = (U^\perp \cap E) \oplus L$$

from which it follows that $(U \oplus L)^\perp / U \oplus L = (U^\perp \cap E) / U$. By maximality of U , we have that the right hand side has no isotropic vector. Hence the same holds for the left hand side. \square

Finally, we show how \mathcal{H} relates to other formed spaces. Let $\mathbf{IVect}_{\mathbb{F}}$ denote the category of finite-dimensional \mathbb{F} -vector spaces and isomorphisms, and $\mathbf{IForms}(\mathbb{F}, \sigma, \varepsilon, \Lambda)$ the category of formed spaces and bijective isometries. There are functors

$$F : \mathbf{IForms}(\mathbb{F}, \sigma, \varepsilon, \Lambda) \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathbf{IVect}_{\mathbb{F}} : \Phi$$

with F the forgetful functor and Φ defined on objects by

$$\Phi(V) = (V \oplus {}^\sigma V^*, q) \quad \text{with} \quad q((v, \alpha), (w, \beta)) = \beta(v),$$

and on morphisms by

$$\Phi(g) = g \oplus (g^{-1})^*.$$

The isomorphism $\mathbb{F}^2 \cong \mathbb{F} \oplus {}^\sigma \mathbb{F}^*$ taking (a, b) to $(a, b\sigma)$ defines an isomorphism of formed spaces

$$\mathcal{H} \cong \Phi(\mathbb{F}).$$

More generally, choosing a basis for V determines an isomorphism

$$\Phi(V) \cong \mathcal{H}^{\oplus \dim V}.$$

The following proposition will be used in Section 7:

Proposition 2.8. Let (E, q) be a nondegenerate formed space over a field \mathbb{F} . Then there is an isomorphism

$$(E, q) \oplus (E, -q) \cong \mathcal{H}^{\oplus \dim E},$$

which is natural with respect to bijective isometries of E .

This result is essentially a special case of [32, Thm 3]. We give a translation of Wall's proof to our set-up.

Proof. We show that $(E, q) \oplus (E, -q) \cong \Phi(E)$. Define $\phi_E : E \oplus E \rightarrow E \oplus {}^\sigma E^*$ by

$$\phi_E(u, v) = (u - g_q(u - v), b_q(u - v))$$

for b_q as in Definition 2.2 and $g_q : E \xrightarrow{q(\cdot, \cdot)} {}^\sigma E^* \xrightarrow{b_q^{-1}} E$ taking v to v' if $q(w, v) = \omega_q(w, v')$ for every $w \in E$. One checks that ϕ_E is a vector space isomorphism and, writing $\Phi(E) = (E \oplus {}^\sigma E^*, \bar{q})$, that it is an isometry by checking that the form $\phi_E^* \bar{q} - (q \oplus (-q)) \in X(E \oplus E, \sigma, \varepsilon, \Lambda_{\min}) \leq X(E \oplus E, \sigma, \varepsilon, \Lambda)$.

For naturality, suppose that $\alpha : (E, q) \rightarrow (E', q')$ is a bijective isometry. We need to check that $(\alpha \oplus (\alpha^{-1})^*) \circ \phi_E = \phi_{E'} \circ (\alpha \oplus \alpha)$. This is true because α preserves q and ω_q , and hence also b_q and g_q . \square

For further use, we note that the formed spaces (E, q) and $(E, -q)$ (or their squares) are isomorphic under certain assumptions on the field:

Lemma 2.9. Let (E, q) be a formed space over a field \mathbb{F} .

- (1) If there exists $a \in \mathbb{F}$ such that $\bar{a}a = -1$, then $(E, q) \cong (E, -q)$.
- (2) If there exists $a, b \in \mathbb{F}$ such that $\bar{a}a + \bar{b}b = -1$, then $(E \oplus E, -q \oplus -q) \cong (E \oplus E, q \oplus q)$.

Proof. To prove (1), we use the isomorphism $\alpha : E \rightarrow E$ given by multiplication by a for a such that $\bar{a}a = -1$. Then $q(\alpha(v), \alpha(w)) = -q(v, w)$ by our choice of a .

For (2), we pick $a, b \in \mathbb{F}$ such that $\bar{a}a + \bar{b}b = -1$ and let

$$\beta = \begin{bmatrix} \bar{a} & -\bar{b} \\ b & a \end{bmatrix} : E^{\oplus 2} \rightarrow E^{\oplus 2}.$$

It is bijective since it has inverse

$$\begin{bmatrix} -a & -\bar{b} \\ b & -\bar{a} \end{bmatrix}.$$

It induces an isometry $(E \oplus E, -q \oplus -q) \rightarrow (E \oplus E, q \oplus q)$ since

$$\begin{aligned} & q \oplus q((\bar{a}v - \bar{b}w, bv + aw), (\bar{a}v' - \bar{b}w', bv' + aw')) \\ &= q(\bar{a}v - \bar{b}w, \bar{a}v' - \bar{b}w') + q(bv + aw, bv' + aw') = -q(v, v') - q(w, w'). \end{aligned} \quad \square$$

3. THE BUILDINGS OF A VECTOR SPACE

In this section, we define and study the buildings and relative buildings of subspaces of a vector space, and collect their basic properties. We show that the known vanishing of the coinvariants of the top homology of the standard building also holds in the relative case. Before introducing the building, we recall some basic definitions about posets.

The dimension of a poset is defined to be the maximal length d of a chain $a_0 < a_1 < \dots < a_d$ of elements in the poset. We will here only consider posets in which any maximal chain has the same finite length; these are called *graded posets*. In a graded poset, we define the *rank* of an element a as the length of any maximal chain ending at a .

A poset \mathcal{P} is said to be *n-connected* if its realization is n -connected, and *spherical* if it is $(\dim \mathcal{P} - 1)$ -connected. For $x, y \in \mathcal{P} \cup \{-\infty, \infty\}$, we define the *interval* (x, y) as the subposet

$$(x, y) = \{z \in \mathcal{P} \mid x < z < y\} \quad \text{with in particular} \quad (-\infty, y) = \mathcal{P}_{<y} \quad \text{and} \quad (x, \infty) = \mathcal{P}_{>x}.$$

Definition 3.1. A graded poset \mathcal{P} is *Cohen-Macaulay* if for every $-\infty \leq x \leq y \leq \infty$, the interval (x, y) is spherical, i.e. if $\pi_i(|(x, y)|) = 0$ for $i < \dim(x, y)$.

Being Cohen-Macaulay implies being spherical, which is the case $x = -\infty$ and $y = \infty$.

We recall the Tits building of a vector space, and define its relative analogue that will be relevant to us.

Definition 3.2. Let V be a finite-dimensional vector space over a field \mathbb{F} . Define the *building* of V as

$$\mathcal{T}(V) = \{0 < W < V\},$$

to be the poset of nontrivial proper subspaces of V . For vector spaces $V_0 \leq V$, we define the *relative building* as

$$\mathcal{T}(V, V_0) = \{W < V \mid W + V_0 = V\}.$$

The building and relative building have the following properties:

Theorem 3.3. Let V be a finite-dimensional vector space and V_0 a subspace. Then $\mathcal{T}(V)$ and $\mathcal{T}(V, V_0)$ are Cohen-Macaulay and for $W \in \mathcal{T}(V)$ and $W' \in \mathcal{T}(V, V_0)$,

$$\begin{array}{l|l} \dim \mathcal{T}(V) = \dim V - 2 & \dim \mathcal{T}(V, V_0) = \dim V_0 - 1, \\ \mathcal{T}(V)_{<W} \cong \mathcal{T}(W) & \mathcal{T}(V, V_0)_{<W'} \cong \mathcal{T}(W', W' \cap V_0), \\ \mathcal{T}(V)_{>W} \cong \mathcal{T}(V/W) & \mathcal{T}(V, V_0)_{>W'} \cong \mathcal{T}(V/W'). \\ \text{rank}(W) = \dim W - 1 & \text{rank}(W') = \dim(W' \cap V_0) \end{array}$$

Proof. The Cohen-Macaulay property of $\mathcal{T}(V)$ is the Solomon-Tits theorem; for instance see [23, p. 118]. For $\mathcal{T}(V, V_0)$, Vogtmann [31, Cor. 1.3] shows that it is spherical, which implies the Cohen-Macaulay property by the non-relative case, once we have checked that the intervals are non-relative buildings as stated.

This isomorphism $\mathcal{T}(V, V_0)_{<W'} \cong \mathcal{T}(W', W' \cap V_0)$ takes $X \leq W'$ to itself. This is well-defined because if $X \leq W'$,

$$(X + V_0) \cap W' = X + (V_0 \cap W'),$$

so the latter equals W' if and only if $X + V_0 \geq W'$, which holds if and only if $X + V_0 = V$ since $W' + V_0 = V$. The isomorphism $\mathcal{T}(V, V_0)_{>W'} \cong \mathcal{T}(V/W')$ is given by taking $X \in \mathcal{T}(V, V_0)_{>W'}$ to $X/W' \in \mathcal{T}(V/W')$. The statements for $P(V)$ are checked likewise. \square

In Section 6, we will also need augmented variants of the buildings $\mathcal{T}(V)$ and $\mathcal{T}(V, V_0)$:

Definition 3.4. For $V_0 \leq V$, define

$$\begin{aligned}\bar{\mathcal{T}}(V) &= \{0 < W \leq V\}, \\ \bar{\mathcal{T}}(V, V_0) &= \{W \leq V \mid W + V_0 = V\}.\end{aligned}$$

These differ from the buildings we studied so far by the addition of a maximal element. (In particular they are contractible.) The addition of a maximal element increases the dimensions by one, and does not affect the Cohen-Macaulay property:

Theorem 3.5. For $V_0 \leq V$ finite-dimensional vector spaces, the buildings $\bar{\mathcal{T}}(V)$ and $\bar{\mathcal{T}}(V, V_0)$ are Cohen-Macaulay, and:

$$\begin{array}{l|l} \dim \bar{\mathcal{T}}(V) = \dim V - 1 & \dim \bar{\mathcal{T}}(V, V_0) = \dim V_0, \\ \bar{\mathcal{T}}(V)_{<W} \cong \mathcal{T}(W) & \bar{\mathcal{T}}(V, V_0)_{<W} \cong \mathcal{T}(W, W \cap V_0), \\ \bar{\mathcal{T}}(V)_{>W} \cong \bar{\mathcal{T}}(V/W) & \bar{\mathcal{T}}(V, V_0)_{>W} \cong \bar{\mathcal{T}}(V/W).\end{array}$$

Proof. This follows from Theorem 3.3 and the fact that any interval containing the added maximal element is contractible. \square

3.1. A join decomposition of the relative building. The reduced homology of the building,

$$\mathrm{St}(V) = \tilde{H}_{\dim V - 2}(\mathcal{T}(V)),$$

is known as the *Steinberg module* over $\mathrm{GL}(V)$. It is well-known that the $\mathrm{GL}(V)$ -coinvariants of this module vanish when $\dim V \geq 2$: If $\dim(V) \geq 2$, then

$$\mathrm{St}(V)_{\mathrm{GL}(V)} = 0;$$

this is e.g. a special case of [1, Thm. 1.1]. In Section 3.2, we will prove the analogous result for the relative building $\mathcal{T}(V, V_0)$, using a join decomposition of the building which we now describe.

Given two posets \mathcal{P} and \mathcal{Q} , their *join* $\mathcal{P} * \mathcal{Q}$ is the poset which, as a set, is the disjoint union $\mathcal{P} \sqcup \mathcal{Q}$, with the usual order relations of \mathcal{P} and \mathcal{Q} , plus the additional relation that $a < b$ for all $a \in \mathcal{P}$ and all $b \in \mathcal{Q}$. Note that the realization of a join of two posets is the (topological) join of their realizations.

Theorem 3.6. For any subspace $U \leq V_0$, there is poset map

$$\phi : \mathcal{T}(V, V_0) \rightarrow \mathcal{T}(V/U, V_0/U) * \mathcal{T}(V, U)$$

which is natural with respect to triples $(V \geq V_0 \geq U)$ and induces a homotopy equivalence on realizations.

This result is stated in Quillen's notes [22] with U a line. See also [12, Lem 3.8].

Let

$$\mathrm{St}(V, V_0) := \tilde{H}_{\dim V_0 - 1}(\mathcal{T}(V, V_0))$$

denote the *relative Steinberg module*. The proposition has the following consequence:

Corollary 3.7. There is an isomorphism

$$\mathrm{St}(V, V_0) \cong \mathrm{St}(V/U, V_0/U) \otimes_{\mathbb{Z}} \mathrm{St}(V, U)$$

which is natural with respect to triples $(V \geq V_0 \geq U)$.

Proof. This follows from Theorem 3.6 using the Künneth Theorem for reduced homology of joins [13, eq. 2.3], along with the fact that both $\mathcal{T}(V/U, V_0/U)$ and $\mathcal{T}(V, U)$ are spherical (Theorem 3.3). \square

To prove Theorem 3.6, we will make use of the following (probably well-known) lemma:

Lemma 3.8. Let $f : \mathcal{P} \rightarrow \mathcal{P}$ be a poset map which is weakly monotonic, i.e., $f(x) \geq x$ for all $x \in \mathcal{P}$ or $f(x) \leq x$ for all $x \in \mathcal{P}$. Then f induces a homotopy equivalence

$$|\mathcal{P}| \xrightarrow{\sim} |\text{Im}(f)|.$$

Proof. Let $i : \text{Im}(f) \rightarrow \mathcal{P}$ be the inclusion. Then $|f|$ and $|i|$ are homotopy inverses, because $f \circ i \geq \text{id}_{\text{Im}(f)}$ and $i \circ f \geq \text{id}_{\mathcal{P}}$ in the first case, and the same but reversing the inequalities in the second case. \square

Proof of Theorem 3.6. The map $\phi : \mathcal{T}(V, V_0) \rightarrow \mathcal{T}(V/U, V_0/U) * \mathcal{T}(V, U)$ is defined by

$$\phi(W) = \begin{cases} W/U \in \mathcal{T}(V/U, V_0/U) & \text{if } W + U < V, \\ W \in \mathcal{T}(V, U) & \text{if } W + U = V. \end{cases}$$

The map ϕ is clearly well-defined as $W + V_0 = V$ implies $W/U + V_0/U = V/U$. Let us verify that it is a poset map. Suppose that $W \leq W'$. It is clear that $\phi(W) \leq \phi(W')$ if both $W + U = V$ and $W' + U = V$, or if neither hold. If exactly one holds, it must be that $W' + U = V$ and $W + U < V$: in this case $\phi(W) < \phi(W')$ in the join poset, by definition thereof.

As it does not appear straightforward to describe a homotopy inverse to ϕ , we will show that ϕ is a homotopy equivalence using the Quillen fiber lemma [23, Prop. 1.6]. In our situation, $\mathcal{Q} = \mathcal{T}(V/U, V_0/U) * \mathcal{T}(V, U)$, so there are two cases. If $q = W \in \mathcal{T}(V/U, V_0/U)$, then $\phi^{-1}(\mathcal{Q}_{\leq W})$ is contractible because it has a maximal element: $W + U$. (In this case, $W + U < V$.)

Next suppose that $q = W \in \mathcal{T}(V, U)$. That is, assume $W + U = V$. As a subposet of $\mathcal{T}(V, V_0)$,

$$\phi^{-1}(\mathcal{Q}_{\leq W}) = \{X \in \mathcal{T}(V, V_0) \mid X + U < V \text{ or } X \leq W\}$$

We will show this poset is contractible by replacing it with its image under a monotonic self-map twice, which does not change the homotopy type by Lemma 3.8, after which it will become clearly contractible. Define

$$f : \phi^{-1}(\mathcal{Q}_{\leq W}) \rightarrow \phi^{-1}(\mathcal{Q}_{\leq W})$$

by

$$f(X) = \begin{cases} X & \text{if } X \leq W, \\ X + U & \text{if } X + U < V \text{ but } X \not\leq W, \end{cases}$$

In checking that f is a poset map, the only interesting case is when $X \leq X'$, and $X \leq W$ but $X' \not\leq W$ (the other way around being impossible). In that case

$$f(X) = X \leq X' + U = f(X'),$$

as needed. Now f is clearly monotonic, so $\phi^{-1}(\mathcal{Q}_{\leq W}) \simeq \text{Im}(f)$ by Lemma 3.8.

We have that

$$\text{Im}(f) = \{Y \in \mathcal{T}(V, V_0) \mid U \leq Y \text{ or } Y \leq W\}.$$

Indeed, if $U \leq Y$ or $Y \leq W$ with $Y \in \mathcal{T}(V, V_0)$, then $Y \in \phi^{-1}(\mathcal{Q}_{\leq W})$ and $Y = f(Y)$. Now we define a new map

$$g : \text{Im}(f) \rightarrow \text{Im}(f)$$

by $g(Y) = Y \cap W$. Here it would be clear that g is a monotone poset map. However, this time we must verify that $g(Y) \in \mathcal{T}(V, V_0)$ so that $g(Y)$ is indeed inside $\text{Im}(f)$. So we need to check that

$$(Y \cap W) + V_0 = V.$$

In case $Y \leq W$, this is obvious (as $Y + V_0 = V$), so assume instead that $U \leq Y$. Now recall that $W + U = V$. Then we have

$$\begin{aligned} (Y \cap W) + V_0 &= (Y \cap W) + U + V_0 \\ &= Y \cap (W + U) + V_0 = Y + V_0 = V. \end{aligned}$$

So g is a well-defined monotonic map. The image is $\text{Im}(g) = \{X \in \mathcal{T}(V, V_0) \mid X \leq W\}$ which is contractible because it has a maximal element: W . \square

3.2. Vanishing of relative Steinberg coinvariants. The goal of this section is to prove a relative analogue of the vanishing of the Steinberg invariants. The relative Steinberg module is the module $\text{St}(V, V_0) := \tilde{H}_{\dim V_0 - 1}(\mathcal{T}(V, V_0))$ considered in the previous section. Before stating the relative vanishing result, we define the groups acting on this relative module.

Let $V_0 \leq V$ be a subspace as above, and consider the group

$$A^T(V, V_0) = \{g \in \text{GL}(V) \mid gV_0 = V_0 \text{ and } g \text{ induces } \text{id}_{V/V_0} \text{ on } V/V_0\}.$$

This group can be equivalently described as that subgroup of $\text{GL}(V)$ of elements g such that $g(v) - v \in V_0$ for all $v \in V$. And if we identify V with $V/V_0 \oplus V_0$, in block matrix form it is the group

$$\begin{bmatrix} I_{V/V_0} & 0 \\ * & \text{GL}(V_0) \end{bmatrix}.$$

This group acts on the relative complex $\mathcal{T}(V, V_0)$ as it preserves V_0 . The main result of the section is the following:

Theorem 3.9. If $0 < V_0 \leq V$ are vector spaces over a field $\mathbb{F} \neq \mathbb{F}_2$, then

$$\text{St}(V, V_0)_{A^T(V, V_0)} = 0.$$

In the case of $\mathbb{F} = \mathbb{F}_p$ a finite field, Lemma 5.2 in [12] gives the more general vanishing of $H_d(A^T(V, V_0); \text{St}(V, V_0) \otimes \mathbb{F}_p)$ in the range $d < r(p-1) - 1$.

Proof. Let $n = \dim V$ and assume first that $V_0 = L$ is a line. In that case, $\mathcal{T}(V, L)$ is discrete (the set of complementary hyperplanes), and its reduced homology is given by a short exact sequence

$$0 \rightarrow \text{St}(V, L) \rightarrow \mathbb{Z}\mathcal{T}(V, L) \rightarrow \mathbb{Z} \rightarrow 0.$$

We have

$$A^T(V, L) \cong \begin{bmatrix} I_{n-1} & 0 \\ * & \text{GL}_1 \mathbb{F} \end{bmatrix} = \mathbb{F}^{n-1} \rtimes \mathbb{F}^\times.$$

Now, $\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times$ acts on $\mathbb{Z}\mathcal{T}(V, L)$ via its action on the set $\mathcal{T}(V, L)$ of hyperplanes complementary to L . The latter action is transitive with isotropy subgroup \mathbb{F}^\times . In other words,

$$\mathbb{Z}\mathcal{T}(V, L) \cong \mathbb{Z} \uparrow_{\mathbb{F}^\times}^{\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times}.$$

So by Shapiro's lemma

$$H_i(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; \mathbb{Z}\mathcal{T}(V, L)) = H_i(\mathbb{F}^\times; \mathbb{Z}).$$

Hence, by applying $H_*(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; -)$ to the above short exact sequence we get a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_1(\mathbb{F}^\times; \mathbb{Z}) \xrightarrow{a} H_1(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; \mathbb{Z}) \xrightarrow{b} H_0(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; \text{St}(V, L)) \\ \xrightarrow{c} H_0(\mathbb{F}^\times; \mathbb{Z}) \xrightarrow{d} H_0(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; \mathbb{Z}) \rightarrow 0. \end{aligned}$$

Clearly d is an isomorphism, so $c = 0$ and b is surjective. It will suffice to show that a is surjective; then $b = 0$ and we can conclude $H_0(\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times; \text{St}(V, L)) = 0$ as desired.

To that end, consider the spectral sequence for the group extension

$$1 \rightarrow \mathbb{F}^{n-1} \rightarrow \mathbb{F}^{n-1} \rtimes \mathbb{F}^\times \rightarrow \mathbb{F}^\times \rightarrow 1.$$

It has

$$E_{01}^2 = H_0(\mathbb{F}^\times; H_1(\mathbb{F}^{n-1}; \mathbb{Z})) = (\mathbb{F}^{n-1})_{\mathbb{F}^\times}.$$

Now the action of \mathbb{F}^\times on \mathbb{F}^{n-1} is via its \mathbb{F} -vector space structure, and we claim that for any \mathbb{F} -vector space M , we have that $M_{\mathbb{F}^\times} = 0$. Indeed, since $\mathbb{F} \neq \mathbb{F}_2$, there exists a scalar $x \neq 0, 1$. That is: $x, (1-x) \in \mathbb{F}^\times$. Then for any $y \in M_{\mathbb{F}^\times}$,

$$y = x \cdot y + (1-x) \cdot y = y - y = 0.$$

It now follows that projection $\mathbb{F}^{n-1} \rtimes \mathbb{F}^\times \rightarrow \mathbb{F}^\times$ and hence also inclusion $\mathbb{F}^\times \rightarrow \mathbb{F}^{n-1} \rtimes \mathbb{F}^\times$ induces an isomorphism on $H_1(-; \mathbb{Z})$. So the map a above is indeed surjective as we needed.

We consider now the case where $\dim(V_0) > 1$. Pick a line $L < V_0$ and note that

$$A^T(V, L) \leq A^T(V, V_0).$$

Hence it suffices to show that $\text{St}(V, V_0)_{A^T(V, L)} = 0$. But by Corollary 3.7,

$$\text{St}(V, V_0) \cong \text{St}(V/L, V_0/L) \otimes \text{St}(V, L),$$

equivariantly with respect to $A^T(V, L)$, whose elements preserve both L and V_0 . Since $A^T(V, L)$ acts trivially on $\text{St}(V/L, V_0/L)$,

$$\text{St}(V, V_0)_{A^T(V, L)} \cong \text{St}(V/L, V_0/L) \otimes \text{St}(V, L)_{A^T(V, L)}.$$

However, $\text{St}(V, L)_{A^T(V, L)} = 0$ by the previously checked case. \square

4. THE BUILDINGS OF A FORMED SPACE

Recall that a formed space $E = (E, q)$ is a vector space E equipped with a form q . We will study the automorphism groups of formed spaces through their building of isotropic subspaces. Just as in the case of vector spaces, we will need a relative version of the building. We start by defining and giving the first basic properties of the buildings. In Section 4.1, we will prove that the building and relative building are Cohen-Macaulay.

Recall from Definition 2.2 that $\mathfrak{R}(E)$ denotes the radical of E , and that a subspace $U \leq E$ is isotropic if q vanishes on U .

Definition 4.1. Let $E = (E, q)$ be a formed space as in Definition 2.1. We define its building to be the poset

$$\mathcal{T}(E) = \{W < E \mid W \text{ isotropic, } \mathfrak{R}(E) < W\}$$

ordered by inclusion. For an isotropic subspace $U \in \mathcal{T}(E)$, we define the relative building to be

$$\mathcal{T}(E, U) = \left\{ W < E \mid \begin{array}{l} W \text{ isotropic, } \mathfrak{R}(E) < W, \\ W + U^\perp = E \end{array} \right\}.$$

Note that $\mathcal{T}(E, U)$ is an upper-closed subposet of $\mathcal{T}(E)$, i.e. $\mathcal{T}(E, U) \subset \mathcal{T}(E)$ and for any $W \in \mathcal{T}(E, U)$, the upper interval $\mathcal{T}(E)_{>W} \subset \mathcal{T}(E, U)$.

Note also that the buildings $\mathcal{T}(V)$ and $\mathcal{T}(V, V_0)$ associated to vector spaces are not special cases of this construction. Indeed, if $E = (E, 0)$ is a formed space with trivial form, then $\mathfrak{R}(E) = E$ and $\mathcal{T}(E)$ is empty.

Lemma 4.2. The buildings of isotropic subspaces are insensitive to the radical of E :

$$\mathcal{T}(E) \cong \mathcal{T}(E/\mathfrak{R}(E)) \quad \text{and} \quad \mathcal{T}(E, U) \cong \mathcal{T}(E/\mathfrak{R}(E), U/\mathfrak{R}(E))$$

where $E/\mathfrak{R}(E)$ is equipped with the unique form such that the projection $E \rightarrow E/\mathfrak{R}(E)$ is an isometry.

Proof. The existence of a unique form on $E/\mathfrak{R}(E)$ such that the projection $E \rightarrow E/\mathfrak{R}(E)$ is an isometry is given by Lemma A.7. As the projection is an isometry, it takes isotropic subspaces to isotropic subspaces. And if $U/\mathfrak{R}(E) \subset E/\mathfrak{R}(E)$ is isotropic, then so is $U + \mathfrak{R}(E)$, as $\mathfrak{R}(E)$ is the radical of E . Finally, for $\mathfrak{R}(E) < W \leq E$ isotropic, $W + U^\perp = E$ if and only if $W/\mathfrak{R}(E) + (U/\mathfrak{R}(E))^\perp = E/\mathfrak{R}(E)$. \square

The following result is essentially [27, Prop 1]:

Lemma 4.3. In a formed space (E, q) , all maximal isotropic subspaces have the same dimension.

Proof. Suppose that there are two maximal isotropic subspaces $A, B \subset V$ with $\dim A < \dim B$. Both A and B contain $\mathfrak{R}(E)$, since e.g. $A + \mathfrak{R}(E)$ is isotropic. Since they are also isotropic, we have

$$A \cap \mathfrak{R}(E) = \mathfrak{R}(E) = B \cap \mathfrak{R}(E).$$

Choose an injection $f : A \rightarrow B$ which is the identity on $\mathfrak{R}(E)$. Then we have a bijective map $A \rightarrow \text{Im}(f)$ which is trivially an isometry, and restricts to the identity on $\mathfrak{R}(E)$. So by Witt's Lemma [26, Thm 3.4] it may be extended to a bijective isometry \tilde{f} of E . But then $\tilde{f}^{-1}(B)$ is an isotropic subspace properly containing A , contradicting maximality. \square

We compute the dimensions of the buildings:

Lemma 4.4. Let (E, q) be a formed space, and let $U \in \mathcal{T}(E)$. Then both $\mathcal{T}(E)$ and $\mathcal{T}(E, U)$ are graded posets, with dimensions

$$\dim \mathcal{T}(E) = g(E) - 1, \quad \text{and} \quad \dim \mathcal{T}(E, U) = g(E) - \dim U + \dim \mathfrak{R}(E) = g(U^\perp).$$

Proof. For $\mathcal{T}(E)$, this follows from the definitions and Lemma 4.3. For $\mathcal{T}(E, U)$, observe that maximal elements of $\mathcal{T}(E, U)$ are in particular maximal isotropic subspaces of E , having dimension $g(E) + \dim \mathfrak{R}(E)$. By Lemma A.5, the minimal elements have the form $L \oplus \mathfrak{R}(E)$ where $E = L \oplus U^\perp$. By Lemma A.1 and using that $\mathfrak{R}(E) \leq U$, we have

$$\dim(U^\perp) = \dim(E) - \dim(U) + \dim(\mathfrak{R}(E))$$

So the minimal elements have dimension $\dim U$, and

$$\dim \mathcal{T}(E, U) = g(E) + \dim \mathfrak{R}(E) - \dim U.$$

The last equality in the statement is given by Proposition A.8. \square

The following result shows that the lower intervals of the buildings and relative buildings of isotropic subspaces are buildings and relative buildings of vector spaces, and that the upper intervals are isotropic buildings of smaller formed spaces.

Lemma 4.5. Let (E, q) be a formed space, and let $W, U \in \mathcal{T}(E)$, and $W' \in \mathcal{T}(E, U)$. Then

$$\begin{array}{ll} (1) \mathcal{T}(E)_{<W} \cong \mathcal{T}(W/\mathfrak{R}(E)) & | \quad (3) \mathcal{T}(E, U)_{<W'} \cong \mathcal{T}(W'/\mathfrak{R}(E), (U^\perp \cap W')/\mathfrak{R}(E)) \\ (2) \mathcal{T}(E)_{>W} \cong \mathcal{T}(W^\perp) & | \quad (4) \mathcal{T}(E, U)_{>W'} \cong \mathcal{T}((W')^\perp). \end{array}$$

Moreover

$$\text{rank}(W) = \dim W - \dim \mathfrak{R}(E) - 1 \quad \text{and} \quad \text{rank}(W') = \dim(W' \cap U^\perp) - \dim \mathfrak{R}(E) = \dim W' - \dim U.$$

Proof. Statement (1) follows from identifying the right hand side with the set of proper subspaces of W properly containing $\mathfrak{R}(E)$. For (2), $\mathcal{T}(W^\perp)$ is the set of isotropic subspaces of W^\perp properly containing $W + \mathfrak{R}(E)$ (by Lemma A.6), which gives an inclusion of the right hand side into the left hand side in (2). This inclusion is surjective because any isotropic subspace containing W lies in W^\perp , which proves (2). Statement (3) follows if we can check that for $X < W'$,

$$X + U^\perp = E \quad \text{if and only if} \quad X + (U^\perp \cap W') = W'.$$

One direction holds because $(X + U^\perp) \cap W' = X + (U^\perp \cap W')$, and the other holds since $W' + U^\perp = E$. Finally (4) follows from (2), since $\mathcal{T}(E, U) \subset \mathcal{T}(E)$ is upper-closed.

The ranks computations follow from the lower intervals computations and their earlier computation of the dimensions of the lower intervals (Theorem 3.3), using Lemma A.1 to compute $\dim(W' \cap U^\perp)$. \square

4.1. Cohen-Macaulay property. In this section, we prove that the building and relative building of a formed space are Cohen-Macaulay. The arguments presented here, most particularly in the relative case, are adapted from Vogtmann [31, Sec 1]. (Vogtmann acknowledges Igusa for some parts of the argument, so our acknowledgements go to him too, by transitivity.) The goal of the paper [31] is to prove homological stability in the case of non-degenerate orthogonal groups over a field of characteristic 0, using the same buildings as those considered here. However in the process of proving that these buildings are Cohen-Macaulay, Vogtmann also uses buildings associated to certain degenerate formed space. The paper [31] does not use the language of formed spaces, and sometimes uses coordinates that are specific to the example considered. Also a different convention is used for the interaction with the radical in the degenerate case. But, modulo this, the argument presented here is a rather direct generalization of the one given in [31].⁶

First we handle the posets $\mathcal{T}(E)$.

Theorem 4.6. Let $E = (E, q)$ be a formed space. Then $\mathcal{T}(E)$ is Cohen-Macaulay.

Proof. We will prove the theorem by induction on the genus $g(E)$, starting from $g(E) = 0$, in which case the result trivially holds. So let $E = (E, q)$ be a formed space of genus $g(E) \geq 1$. By Lemma 4.2, we may assume that $\mathfrak{R}(E) = 0$.

The lower and upper intervals

$$\mathcal{T}(E)_{>W} \cong \mathcal{T}(W^\perp) \quad \text{and} \quad \mathcal{T}(E)_{<W} \cong \mathcal{T}(W/\mathfrak{R}(W))$$

are already known to be Cohen-Macaulay by Theorem 3.3 and by the induction hypothesis (as $W > 0$, and $g(W^\perp) = g(E) - \dim W$ by Proposition A.8). So it suffices to check that $\mathcal{T}(E)$ is spherical (of dimension $g(E) - 1$). Fix a nonzero isotropic vector $u \in E$, which exists by our assumption on the genus. Consider the subposet

$$\mathcal{P} = \left\{ W \in \mathcal{T}(E) \mid W \cap u^\perp > 0 \right\}.$$

Its realization is contractible, since there is a homotopy between the identity map of \mathcal{P} and the constant map at $\langle u \rangle$, given via monotone maps by

$$W \geq W \cap u^\perp \leq W \cap u^\perp + \langle u \rangle \geq \langle u \rangle$$

(using Lemma 3.8 for each inequality). On the other hand, every $W \in \mathcal{T}(E)$ of dimension at least 2 is contained in \mathcal{P} , since u^\perp is a hyperplane in E . Hence the complement $\mathcal{T}(E) - \mathcal{P}$ is discrete, consisting only of minimal elements. Therefore, by Lemma B.2, there is a homotopy equivalence

$$|\mathcal{T}(E)| / |\mathcal{P}| \simeq \bigvee_{L \in \mathcal{T}(E) - \mathcal{P}} \Sigma |\mathcal{T}(E)_{>L}|$$

Since $\mathcal{T}(E)_{>L} = \mathcal{T}(L^\perp)$ and $g(L^\perp) = g(E) - 1$ (by Proposition A.8), we conclude using the induction hypothesis that the quotient $|\mathcal{T}(E)| / |\mathcal{P}|$ is homotopy-equivalent to a wedge of spheres of dimension $g(E) - 1$. This proves the theorem, since $|\mathcal{P}|$ is contractible. \square

We now turn to the case of the posets $\mathcal{T}(E, U)$.

Theorem 4.7. For all formed spaces E and all $U \in \mathcal{T}(E)$, the poset $\mathcal{T}(E, U)$ is Cohen-Macaulay.

⁶Vogtmann only obtains a slope 3 stability for the groups $O_{n,n}(\mathbb{F})$ in [31]. The reason for this is that she does not use the optimal spectral sequence argument.

We will prove the theorem by induction on $\dim E$ and we make two different arguments depending on whether $\dim U/\mathfrak{R}(E)$ is one or greater than one. We treat the latter case first.

4.1.1. *Case $\mathcal{T}(E, U)$ with $\dim(U/\mathfrak{R}(E)) > 1$.*

Lemma 4.8 (Induction step 1). Suppose that $\mathcal{T}(E, U)$ is Cohen-Macaulay for all formed spaces E with dimension less than d and for all $U \in \mathcal{T}(E)$. Then $\mathcal{T}(E, U)$ is Cohen-Macaulay for all formed spaces E with dimension d and for all $U \in \mathcal{T}(E)$ with $\dim(U/\mathfrak{R}(E)) > 1$.

To prove this first induction step, we will need a variant of the building of isotropic subspaces. For $U \in \mathcal{T}(E)$, let

$$\mathcal{T}'(E, U) = \left\{ W < E \mid W \text{ isotropic, } W \cap \mathfrak{R}(E) = 0, W + U^\perp = E \right\}.$$

So the elements W of $\mathcal{T}'(E, U)$ are required to be disjoint from the radical instead of containing it.

Lemma 4.9. Suppose $\mathfrak{R}(E) = 0$, $U \in \mathcal{T}(E)$ of dimension at least 2 and $L < U$ is codimension 1 in U . Then $\mathcal{T}'(L^\perp, U)$ is Cohen-Macaulay of the same dimension as $\mathcal{T}(L^\perp, U)$.

Note that $L \leq U$ implies $U \leq L^\perp$, and U contains $\mathfrak{R}(L^\perp) = L$, so the above posets make sense.

Proof. Define a map

$$p : \mathcal{T}'(L^\perp, U) \rightarrow \mathcal{T}(L^\perp, U)$$

by setting $p(W) = W \oplus L$. This is a strictly increasing, in fact rank preserving, poset map, the rank in the source being $\dim(W \cap U^\perp)$ and in the target $\dim((W \oplus L) \cap U^\perp) - \dim L$. Using [23, Cor. 9.7], it follows that the domain will be Cohen-Macaulay of the same dimension as the codomain, if for each $Y \in \mathcal{T}(L^\perp, U)$, the poset fiber $p^{-1}(\mathcal{T}(L^\perp, U)_{\leq Y})$ under Y , is Cohen-Macaulay. Now

$$\begin{aligned} p^{-1}(\mathcal{T}(L^\perp, U)_{\leq Y}) &= \left\{ W < L^\perp \mid W \cap L = 0, W + L \leq Y, W + U^\perp = L^\perp \right\} \\ &= \left\{ W < Y \mid W \cap L = 0, W + (U^\perp \cap Y) = Y \right\} \end{aligned}$$

identifies with the general linear building denoted ${}^L T^{U^\perp \cap Y, Y}$ in [31, Sec 1]. We have $\dim L \geq 1$ and $\dim(U^\perp \cap Y) = \dim Y - 1$ because $U^\perp < L^\perp$ has codimension one by Lemma A.1 (since $U \cap \mathfrak{R}(E) = \mathfrak{R}(E) = 0$). Hence we can apply [31, Cor. 1.5], which gives that ${}^L T^{U^\perp \cap Y, Y} = p^{-1}(\mathcal{T}(L^\perp, U)_{\leq Y})$ is Cohen-Macaulay, as required. \square

Proof of Lemma 4.8. Suppose E has dimension d and $U \in \mathcal{T}(E)$ is such that $\dim(U/\mathfrak{R}(E)) > 1$. We may assume without loss of generality that $\mathfrak{R}(E) = 0$, since $\mathcal{T}(E, U) \cong \mathcal{T}(E/\mathfrak{R}(E), U/\mathfrak{R}(E))$ (Lemma 4.2).

As above, choose $L < U$ of codimension one. Because $\dim U > 1$ by assumption, L is nonzero, which implies that $L^\perp < E$ is a proper subspace by Lemma A.1. Hence we are given, by assumption, that $\mathcal{T}(L^\perp, U)$ is Cohen-Macaulay. By Lemma 4.9, this implies that also $\mathcal{T}'(L^\perp, U)$ is Cohen-Macaulay. Consider the poset map

$$j : \mathcal{T}(E, U) \rightarrow \mathcal{T}'(L^\perp, U), \quad W \mapsto W \cap L^\perp.$$

The map makes sense because

$$U^\perp + (W \cap L^\perp) = (U^\perp + W) \cap L^\perp = L^\perp,$$

since $U^\perp < L^\perp$, and $W \cap L = 0$, because $W + L^\perp = E$, so $\mathfrak{R}(E) = W^\perp \cap (L + \mathfrak{R}(E))$ contains $W \cap L$, which is isotropic so lies in $\mathfrak{R}(E) = 0$. Moreover j is strictly increasing, in fact rank preserving, as the rank of $W \in \mathcal{T}'(L^\perp, U)$ is $\dim(W \cap U^\perp)$, just like in $\mathcal{T}(E, U)$.

Let $Y \in \mathcal{T}'(L^\perp, U)$. We claim that the corresponding upper poset fiber of j has the form

$$j^{-1}\left(\mathcal{T}'(L^\perp, U)_{\geq Y}\right) \cong \mathcal{T}(E', U')$$

where $E' < E$ and $U' \leq U$ are certain subspaces, with E' proper. This poset will then by hypothesis be Cohen-Macaulay, showing by [23, Cor. 9.7] that $\mathcal{T}(E, U)$ is indeed Cohen-Macaulay, which will prove the result.

We start by associating an E' and U' to any given $Y \in \mathcal{T}'(L^\perp, U)$. As $Y \oplus L$ is an isotropic subspace of L^\perp , Lemma A.5, shows that there exists an isotropic space Z such that

$$L^\perp = (Y \oplus L)^\perp \oplus Z.$$

Then

$$E = Y^\perp \oplus Z,$$

since $Y^\perp \cap L^\perp = (Y \oplus L)^\perp$ showing that $Y^\perp \cap Z = 0$, and $Y^\perp + L^\perp = E$ as $\dim Y^\perp = \dim E - \dim Y$, $\dim L^\perp = \dim E - \dim L$ and $\dim(Y^\perp \cap L^\perp) = \dim E - \dim Y - \dim L$ by Lemma A.1 and the previous computation, using the fact that $\mathfrak{R}(E) = 0$ so that $Y \cap \mathfrak{R}(E) = 0 = L \cap \mathfrak{R}(E)$. Hence $Y \oplus Z$ is nondegenerate by Lemma A.5, and

$$E = Y \oplus Z \oplus (Y \oplus Z)^\perp.$$

Let

$$E' = (Y \oplus Z)^\perp \quad \text{and} \quad U' = U \cap E'.$$

Note that E' is a proper subspace of E , since $Y > 0$. We have

$$U' \geq L > 0$$

since $Y, Z \leq L^\perp$. Because $\mathfrak{R}(E') = \mathfrak{R}(E) = 0$ by Lemma A.4, it makes sense to speak of $\mathcal{T}(E', U')$.

Now we define poset maps

$$f : j^{-1}\left(\mathcal{T}'(L^\perp, U)_{\geq Y}\right) \rightleftharpoons \mathcal{T}(E', U') : g$$

by setting

$$f(W) = W \cap E' \quad \text{and} \quad g(X) = X \oplus Y.$$

The map f makes sense because $\mathfrak{R}(E') = 0$ and

$$W + U'^\perp \geq W + U^\perp = E \geq E',$$

while $(W \cap E') \cap (U')^\perp = (W \cap U^\perp) \cap E'$ since any element of E' is $U \cap (Y \oplus Z)$. In particular, $W \cap E' < E'$. (This also shows that j is degree preserving.) The map g makes sense because

$$X < E' \leq Y^\perp$$

showing that $X \oplus Y$ is an isotropic strict subspace of E , and

$$(X \oplus Y) + U^\perp = E$$

because $Y + U^\perp = L^\perp \geq Z$, so

$$X + Y + U^\perp = X + Y + Z + U^\perp = X + E'^\perp + U^\perp = X + U'^\perp \geq E',$$

while $Y + Z + E' = E$.

It remains to check that the two maps are inverses of each other. Starting with $X \in \mathcal{T}(E', U')$, we have $f \circ g(X) = (X \oplus Y) \cap E' = X$. And starting with $W \in j^{-1}\left(\mathcal{T}'(L^\perp, U)_{\geq Y}\right)$, we have $W \geq Y$, which implies that $W \leq Y^\perp = Y \oplus E'$ as $E = Z \oplus Y^\perp$. Hence

$$g \circ f(W) = (W \cap E') \oplus Y = W. \quad \square$$

4.1.2. *Case $\mathcal{T}(E, U)$ with $\dim(U/\mathfrak{R}(E)) = 1$.* We assume now that U be an isotropic subspace of E such that $\mathfrak{R}(E) < U$ has codimension one. In this situation, we can pick a $u \in U$ such that

$$U = \mathfrak{R}(E) \oplus \langle u \rangle.$$

Then $U^\perp = u^\perp < E$ is a hyperplane.

If it happens that $U^\perp = U$, then $\mathcal{T}(E, U)$ is zero-dimensional, so Cohen-Macaulay. Hence we may assume $U^\perp > U$, and fix some isotropic vector

$$v \in U^\perp - U.$$

(This is possible by Proposition A.8, which insures that $g(U^\perp) \geq 1$ given that $g(E) \geq 2$; if $g(E) = 1$ there is nothing to prove.) For each constant $\lambda \in \mathbb{F}$, define

$$v_\lambda = v - \lambda u.$$

This v_λ is an isotropic vector not contained in $\mathfrak{R}(E)$, so v_λ^\perp is likewise a hyperplane in E . Also,

$$U \leq v_\lambda^\perp.$$

The hyperplanes u^\perp and v_λ^\perp are distinct, because $v_\lambda \notin U = u^{\perp\perp}$.

Now for U , u , v and λ as above, define

$$\mathcal{P}_\lambda = \mathcal{P}_\lambda(E, u, v) = \left\{ W \in \mathcal{T}(E, U) \mid (W \cap v_\lambda^\perp) + U^\perp = E \right\}.$$

The idea of the proof of the Cohen-Macaulay property of $\mathcal{T}(E, U)$ in the present case, is that $\mathcal{T}(E, U)$ is covered by the subposets \mathcal{P}_λ for various λ , while they and their intersections are already known to be Cohen-Macaulay by induction. We prove these facts in the next three lemmas.

Lemma 4.10. For all $W \in \mathcal{T}(E, U)$, there is some λ such that $W \in \mathcal{P}_\lambda$.

Proof. Since $W + U^\perp = E$, there exists a complement to U^\perp lying in W , generated by some vector w . Set

$$\lambda = \frac{\omega(w, v)}{\omega(w, u)},$$

which is possible since $w \notin U^\perp$. Then $w \in v_\lambda^\perp$, so $W \in \mathcal{P}_\lambda$. □

Lemma 4.11. Under the above assumptions, there is a homotopy equivalence

$$\mathcal{P}_\lambda \simeq \mathcal{T}(v_\lambda^\perp, U + \langle v_\lambda \rangle).$$

Note that the right hand side makes sense because, by Lemma A.6,

$$\mathfrak{R}(v_\lambda^\perp) = \mathfrak{R}(E) + \langle v_\lambda \rangle \leq U + \langle v_\lambda \rangle.$$

Proof. We define maps

$$f : \mathcal{P}_\lambda \longleftrightarrow \mathcal{T}(v_\lambda^\perp, U + \langle v_\lambda \rangle) : g$$

by setting

$$f(W) = W \cap v_\lambda^\perp + \langle v_\lambda \rangle \quad \text{and} \quad g(V) = V.$$

The map f lands in the claimed codomain because $f(W)$ is isotropic, $\mathfrak{R}(v_\lambda^\perp) = \mathfrak{R}(E) + \langle v_\lambda \rangle < f(W) < v_\lambda^\perp$, the last inequality following from the fact that v_λ^\perp is not isotropic because it would otherwise be equal to $\mathfrak{R}(E) + \langle v_\lambda \rangle$, contradicting, through Witt's lemma, our assumption that $U \neq U^\perp$. Finally, $(W \cap v_\lambda^\perp) + U^\perp = E$ implies $(W \cap v_\lambda^\perp) + (U^\perp \cap v_\lambda^\perp) = v_\lambda^\perp$. The map g lands in the claimed codomain because $v_\lambda \in V$ (as $v_\lambda \in \mathfrak{R}(v_\lambda^\perp)$), so $V + U^\perp$ contains v_λ^\perp , and hence contains $v_\lambda^\perp + U^\perp$, which is E since v_λ^\perp and U^\perp are two distinct hyperplanes.

The maps f and g are both poset maps as they both preserve inclusions. The compositions is $f \circ g(V) = V$, because $v_\lambda \in V$. The other composition satisfies

$$g \circ f(W) \geq W \cap v_\lambda^\perp \leq W$$

in \mathcal{P}_λ , since $W \in \mathcal{P}_\lambda$ gives that also $W \cap v_\lambda^\perp \in \mathcal{P}_\lambda$. This gives a homotopy $g \circ f \sim \text{id}_{\mathcal{P}_\lambda}$. Therefore f is a deformation retraction. \square

Lemma 4.12. If $\lambda \neq \mu$, then

$$\mathcal{P}_\lambda \cap \mathcal{P}_\mu = \mathcal{T}(E, U + \langle v \rangle),$$

as subposets of $\mathcal{T}(E, U)$. In particular, the intersection does not depend on λ or μ .

Proof. If $W \in \mathcal{T}(E, U + \langle v \rangle)$, then W contains a complement to $\langle u, v \rangle^\perp$, that is, the map

$$f : W \rightarrow \mathbb{F}^2, \quad w \mapsto (\omega(u, w), \omega(v, w))$$

is surjective. In particular, for any $\alpha \in \mathbb{F}$, there exists $w \in W$ with

$$(\omega(u, w), \omega(v, w)) = (1, \bar{\alpha}).$$

This w lies in $W \cap v_\alpha^\perp$ but not in U^\perp , showing that $W \in \mathcal{P}_\alpha$, for all α .

Conversely, suppose $W \in \mathcal{P}_\lambda \cap \mathcal{P}_\mu$. Then there exist $a, b \in W$ with $a \in v_\lambda^\perp$, $b \in v_\mu^\perp$, and neither a nor b in U^\perp . Scaling appropriately, we may assume $\omega(a, u) = \omega(b, u) = 1$. Then

$$\omega(a, v) = \lambda \quad \text{and} \quad \omega(b, v) = \mu.$$

Since

$$\det \begin{bmatrix} 1 & \lambda \\ 1 & \mu \end{bmatrix} \neq 0,$$

this shows that $W \xrightarrow{b} \sigma E^* \rightarrow \sigma \langle u, v \rangle^*$ is surjective, i.e. that W contains a complement to $\langle u, v \rangle^\perp$, so $W \in \mathcal{T}(E, U + \langle v \rangle)$. \square

For each $\lambda \in \mathbb{F}$, let

$$i_\lambda : \mathcal{T}(E, U + \langle v \rangle) \longrightarrow \mathcal{P}_\lambda$$

be the inclusion map, recalling from Lemma 4.12 that $\mathcal{T}(E, \langle U, v \rangle)$ is the intersection of \mathcal{P}_λ with \mathcal{P}_μ for any $\mu \neq \lambda$.

Lemma 4.13. Let $\lambda, \mu \in \mathbb{F}$ be distinct. Then

- (1) the map $\tilde{H}_*(\mathcal{P}_\lambda \cap \mathcal{P}_\mu) \xrightarrow{(i_{\lambda*}, i_{\mu*})} \tilde{H}_* \mathcal{P}_\lambda \oplus \tilde{H}_* \mathcal{P}_\mu$ is surjective;
- (2) the image of the set map $\pi_1(\mathcal{P}_\lambda \cap \mathcal{P}_\mu) \rightarrow \pi_1 \mathcal{P}_\lambda * \pi_1 \mathcal{P}_\mu$ sending x to $i_{\lambda*}(x) i_{\mu*}(x)^{-1}$ contains $\pi_1 \mathcal{P}_\lambda$.

Proof. First, observe that v_λ^\perp and v_μ^\perp are distinct hyperplanes in E . Indeed, if $v_\lambda^\perp = v_\mu^\perp$, then $v_\lambda \in \langle v_\mu \rangle + \mathfrak{R}(E)$ by Lemma A.2, contradicting that $\langle u, v \rangle \cap \mathfrak{R}(E) = 0$. Hence we can pick an isotropic vector $c \in v_\mu^\perp$ such that $\omega(c, v_\lambda) = 1$. This is possible by applying Lemma A.5 to $\langle v_\lambda \rangle \leq v_\mu^\perp$, since $v_\lambda^\perp \cap v_\mu^\perp$ has codimension one in v_μ^\perp . The above properties imply that $c \notin u^\perp$ since

$$v_\mu^\perp \cap u^\perp = \langle u, v \rangle^\perp \leq v_\lambda^\perp.$$

Consequently, $\langle c \rangle + \mathfrak{R}(E) \in \mathcal{P}_\mu$. Now define $\phi : v_\lambda^\perp \rightarrow v_\lambda^\perp$ by

$$\phi(x) = x - \omega(c, x)v_\lambda.$$

This is an isometry of v_λ^\perp (though not of E). Notice that $\text{Im}(\phi) \leq c^\perp$ and $\phi|_{\mathfrak{R}(E)} = \text{id}_{\mathfrak{R}(E)}$. Also $x \in u^\perp$ if and only if $\phi(x) \in u^\perp$.

We now define a map $j_\lambda : \mathcal{P}_\lambda \longrightarrow \mathcal{T}(E, U + \langle v \rangle)$ by

$$j_\lambda(W) = \phi(W \cap v_\lambda^\perp) + \langle c \rangle.$$

This subspace is indeed isotropic, since c is isotropic and orthogonal to $\text{Im}(\phi)$. Also it contains $\mathfrak{R}(E)$. Lastly, since $W \in \mathcal{P}_\lambda$, $W \cap v_\lambda^\perp$ contains a vector w which is not in u^\perp . Then $\phi(w)$ is in

v_λ^\perp but not in u^\perp . As $\phi(w)$ and c are distinct, given that $c \notin v_\lambda^\perp$, this implies that $\langle \phi(w), c \rangle$ is a complement to $\langle u, v \rangle^\perp$. Hence $j_\lambda(W) \in \mathcal{T}(E, \langle U, v \rangle)$ as claimed.

Now, j_λ is a poset map. We claim that $i_\lambda \circ j_\lambda$ is homotopic to the identity map of \mathcal{P}_λ , and that $i_\mu \circ j_\lambda$ is null-homotopic. For the latter claim, observe that

$$\phi(W \cap v_\lambda^\perp) + \langle c \rangle \geq \langle c \rangle + \mathfrak{R}(E),$$

while $\langle c \rangle + \mathfrak{R}(E) \in \mathcal{P}_\mu$. For the former claim, observe that

$$j_\lambda(W) \geq \phi(W \cap v_\lambda^\perp) \leq W \cap v_\lambda^\perp + \langle v_\lambda \rangle \geq W \cap v_\lambda^\perp \leq W.$$

All of the intermediate terms are contained in \mathcal{P}_λ , because both $W \cap v_\lambda^\perp$ and $\phi(W \cap v_\lambda^\perp)$ contain a vector complementary to u^\perp . This completes the claims.

Exchanging the roles of λ and μ also produces a map $j_\mu : \mathcal{P}_\mu \rightarrow \mathcal{T}(E, U + \langle v \rangle)$ such that $i_\mu \circ j_\mu$ is homotopic to the identity map of \mathcal{P}_μ , and that $i_\lambda \circ j_\mu$ is null-homotopic. These together prove part (1) of the lemma, since

$$\begin{bmatrix} i_{\lambda*} & i_{\mu*} \end{bmatrix} \begin{bmatrix} j_{\lambda*} \\ j_{\mu*} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

For part (2), let $y \in \pi_1 \mathcal{P}_\lambda$ and $x = j_{\lambda*}(y)$. Then, using the homotopy relations established above, we get that $i_{\lambda*}(x)i_{\mu*}(x)^{-1}$ maps to y . \square

Now we can handle the case where $\dim(U/\mathfrak{R}(E)) = 1$:

Lemma 4.14 (Induction step 2). Suppose that $\mathcal{T}(E, U)$ is Cohen-Macaulay for all formed spaces E and all $U \in \mathcal{T}(E)$ such that either $\dim E < d$, or $\dim E = d$ and $\dim(U/\mathfrak{R}(E)) > 1$. Then $\mathcal{T}(E, U)$ is Cohen-Macaulay for all formed spaces E and all $U \in \mathcal{T}(E)$ such that $\dim E = d$ and $\dim(U/\mathfrak{R}(E)) = 1$.

Proof. Once again we may assume that $\mathfrak{R}(E) = 0$. The upper and lower intervals

$$\mathcal{T}(E, U)_{<W} = \mathcal{T}(W/\mathfrak{R}(E), (W \cap U^\perp)/\mathfrak{R}(E)) \quad \text{and} \quad \mathcal{T}(E, U)_{>W} = \mathcal{T}(W^\perp)$$

are already known to be Cohen-Macaulay by Theorem 3.3 and Theorem 4.6. Hence we need only show that $\mathcal{T}(E, U)$ is spherical of dimension

$$\dim \mathcal{T}(E, U) = g(U^\perp) = g(E) - \dim U = g(E) - 1$$

(using Proposition A.8 for the last equality). By our hypothesis and Lemma 4.11, each subposet \mathcal{P}_λ is homotopy equivalent to a wedge of spheres of dimension

$$g(U + \langle v_\lambda \rangle^\perp) = g(E) - 2$$

(which is one less than its actual dimension). The same holds for the intersection $\mathcal{P} = \mathcal{P}_\lambda \cap \mathcal{P}_\mu$, for any $\lambda \neq \mu$ using now Lemma 4.12.

By Lemma 4.10, we have an equality of sets

$$\mathcal{T}(E, U) = \bigcup_{\lambda \in \mathbb{F}} \mathcal{P}_\lambda.$$

Since \mathcal{P}_λ is upper-closed in $\mathcal{T}(E, U)$, this actually implies $|\mathcal{T}(E, U)| = \bigcup_{\lambda \in \mathbb{F}} |\mathcal{P}_\lambda|$ as well.

Assuming $\dim \mathcal{T}(E, U) \geq 1$, it is therefore path connected, being a union of path-connected spaces with nonempty intersection.

When $\dim \mathcal{T}(E, U) \geq 2$, we need to show it is 1-connected. If $\dim \mathcal{T}(E, U) \geq 3$, each \mathcal{P}_λ , and their intersection \mathcal{P} , is 1-connected. By Seifert-van Kampen's theorem [14, Thm. 1.20] (choosing a basepoint within \mathcal{P}), $\mathcal{T}(E, U)$ is then 1-connected. If $\dim \mathcal{T}(E, U) = 2$, the \mathcal{P}_λ 's and \mathcal{P} are still path-connected, so Seifert-van Kampen still applies to compute $\pi_1(\mathcal{T}(E, U))$, and gives zero because of Lemma 4.13(2).

Now it suffices to show that $\mathcal{T}(E, U)$ has its reduced homology concentrated in degree $g(E) - 1$. We begin with $|\mathcal{P}_0| \cup |\mathcal{P}_1|$. Its Mayer-Vietoris sequence establishes that $\tilde{H}_i(|\mathcal{P}_0| \cup |\mathcal{P}_1|) = 0$ for all i except $i = g(E) - 1$ and $g(E) - 2$. In those degrees, we get

$$\begin{aligned} 0 \rightarrow \tilde{H}_{g(E)-1}(|\mathcal{P}_0| \cup |\mathcal{P}_1|) &\rightarrow \tilde{H}_{g(E)-2}(|\mathcal{P}_0| \cap |\mathcal{P}_1|) \xrightarrow{\phi} \tilde{H}_{g(E)-2}|\mathcal{P}_0| \oplus \tilde{H}_{g(E)-2}|\mathcal{P}_1| \\ &\rightarrow \tilde{H}_{g(E)-2}(|\mathcal{P}_0| \cup |\mathcal{P}_1|) \rightarrow 0. \end{aligned}$$

The map ϕ is surjective by Lemma 4.13(1), establishing that the reduced homology of $|\mathcal{P}_0| \cup |\mathcal{P}_1|$ is concentrated in degree $g(E) - 1$.

Now we argue with transfinite induction that

$$P_S = \bigcup_{\lambda \in S} |\mathcal{P}_\lambda|$$

has its reduced homology concentrated in degree $g(E) - 1$ for any $S \subset \mathbb{F}$ of cardinality at least 2. This will complete the proof, since $P_{\mathbb{F}} = |\mathcal{T}(E, U)|$. It is necessary to check that

$$P_{S \cup \lambda} = P_S \cup |\mathcal{P}_\lambda|$$

has the desired property if P_S does, and that the union of a chain of P_S 's has the desired property if all of its terms do. The latter is true because homology commutes with directed colimits of sub-simplicial complexes. The former is true by a Mayer-Vietoris sequence:

$$\begin{aligned} 0 \rightarrow \tilde{H}_{g(E)-1}(P_S) &\rightarrow \tilde{H}_{g(E)-1}(P_S \cup |\mathcal{P}_\lambda|) \rightarrow \tilde{H}_{g(E)-2}(P_S \cap |\mathcal{P}_\lambda|) \xrightarrow{\phi} \tilde{H}_{g(E)-2}|\mathcal{P}_\lambda| \\ &\rightarrow \tilde{H}_{g(E)-2}(P_S \cup |\mathcal{P}_\lambda|) \rightarrow 0. \end{aligned}$$

We have again used that the map ϕ is surjective by Lemma 4.13(1). \square

Proof of Theorem 4.7. We prove the theorem by induction on the dimension of E . The poset $\mathcal{T}(E, U)$ can only exist if $\dim(E) \geq 2$, so that there may exist $U \in \mathcal{T}(E)$. If $\dim(E) = 2$ and $\mathcal{T}(E) \neq \emptyset$, then $\mathfrak{R}(E) = 0$ and any $U \in \mathcal{T}(E)$ necessarily has dimension 1. In this case $\mathcal{T}(E, U)$ has dimension 0 by Lemma 4.4 and the Cohen-Macaulay condition follows trivially. The induction step is given by Lemmas 4.8 and 4.14 combined. \square

5. GROUPS ACTING ON THE BUILDINGS

In this section, we define families of groups acting on the buildings and relative buildings of subspaces and isotropic subspaces. We study the relationships between these groups as well as properties of the actions.

Definition 5.1. (1) Let V be a vector space and $V_0 \leq V$ a subspace. Define

$$\mathbf{A}(V, V_0) = \{g \in \mathrm{GL}(V) \mid g|_{V_0} = \mathrm{id}_{V_0}\}$$

to be the subgroup of the general linear group $\mathrm{GL}(V)$ of automorphisms fixing V_0 pointwise.

(2) Let $E = (E, q)$ be a formed space and $U \in \mathcal{T}(E)$. Define

$$\mathbf{A}(E, U) := \{g \in \mathbf{G}(E) \mid g|_U = \mathrm{id}_U\}$$

to be the subgroup of the bijective isometries $\mathbf{G}(E)$ of automorphisms fixing U pointwise.

The group $\mathbf{A}(V, V_0)$ just defined is closely related to the group

$$\mathbf{A}^T(V, V_0) = \{g \in \mathrm{GL}(V) \mid gV_0 = V_0 \text{ and } g \text{ induces } \mathrm{id}_{V/V_0} \text{ on } V/V_0\}$$

of Section 3.2. Indeed, these are isomorphic via duality, justifying the notation \mathbf{A}^T :

Lemma 5.2. Let $V_0 \leq V$ be vector spaces. Dualizing gives an isomorphism

$$\mathbf{A}^T(V, V_0) \cong \mathbf{A}(V^*, (V/V_0)^*), \quad g \mapsto (g^*)^{-1}.$$

Proof. The map $\mathrm{GL}(V) \rightarrow \mathrm{GL}(V^*)$ sending $g \mapsto (g^*)^{-1}$ is an isomorphism, so we just need to check that $g \in \mathbf{A}^T(V, V_0)$ if and only if $(g^*)^{-1} \in \mathbf{A}(V^*, (V/V_0)^*)$. But, since V_0 is the intersection of the kernels of all elements in $(V/V_0)^*$, g preserves V_0 if and only if g^* preserves $(V/V_0)^*$. Furthermore, g induces the identity map modulo V_0 if and only if g^* equals the identity map on $(V/V_0)^*$. \square

Remark 5.3. By definition, forgetting the form on E allows us to consider $\mathbf{A}(E, U)$ as a subgroup of $\mathbf{A}(E, U)$. But [26, Lem 3.8] shows that $\mathbf{A}(E, U)$ is a subgroup of $\mathbf{A}^T(E, U^\perp)$ as well. So there is a diagram of group inclusions:

$$\begin{array}{ccccc} & & \mathbf{A}(E, U) & & \\ & \hookrightarrow & & \hookrightarrow & \\ \mathbf{A}(E, U) & & & & \mathrm{GL}(E) \\ & \hookrightarrow & & \hookrightarrow & \\ & & \mathbf{A}^T(E, U^\perp) & & \end{array}$$

For vector spaces $V_0 \leq V$, the group $\mathrm{GL}(V)$ acts on $\mathcal{T}(V)$ and the group $\mathbf{A}(V, V_0)$ acts on $\mathcal{T}(V, V_0)$ of Definition 3.2. For a formed space E , $\mathbf{G}(E)$ acts on the poset $\mathcal{T}(E)$, and $\mathbf{A}(E, U)$ acts on the poset $\mathcal{T}(E, U)$ of Definition 4.1. This is because an isometry of E preserves the isotropic subspaces and preserves $\mathfrak{R}(E)$, and also preserves U^\perp if it preserves U .

Proposition 5.4. Let $V_0 \leq V$ be vector spaces, $E = (E, q)$ a formed space and $U \in \mathcal{T}(E)$. The actions of $\mathrm{GL}(V)$ on $\mathcal{T}(V)$, of $\mathbf{A}^T(V, V_0)$ on $\mathcal{T}(V, V_0)$, of $\mathrm{GL}(E)$ on $\mathcal{T}(E)$ and of $\mathbf{A}(E, U)$ on $\mathcal{T}(E, U)$ are all transitive on the elements of any given dimension.

Proof. Transitivity of the action of $\mathrm{GL}(V)$ on the elements of a given rank in $\mathcal{T}(V)$ follows from the fact that any isomorphism of a subspace W can be extended to an isomorphism of the whole space using any choice of complement to W inside V . For the action of $\mathbf{A}^T(V, V_0)$ on $\mathcal{T}(V, V_0)$, recall that the rank of $W \in \mathcal{T}(V, V_0)$ is $\dim(W \cap V_0)$. Given two elements W, W' of the same rank, we pick an isomorphism $W \cap V_0 \rightarrow W' \cap V_0$ and extend it to an isomorphism of V_0 . Now extend this isomorphism to the rest of V by picking a complement $W_0^{(\prime)}$ of $W^{(\prime)} \cap V_0$ inside $W^{(\prime)}$ and identify these complements via their canonical isomorphism to V/V_0 . For the action of $\mathbf{G}(E)$ on $\mathcal{T}(E)$, transitivity is given by Witt's Lemma [26, Thm 3.4] and for the action of $\mathbf{A}(E, U)$ on $\mathcal{T}(E, U)$, it is given by [26, Cor 3.9]. \square

Witt's Lemma [26, Thm 3.4] gives that any desired linear automorphism of an isotropic subspace $W \in \mathcal{T}(E)$ preserving the radical can be achieved by an isometry of a E . But in fact something much stronger is true, as we will see now. We start by fixing some notation.

For X an element in a building \mathcal{P} of subspaces of a vector space, and Γ a subgroup of the general linear group acting on \mathcal{P} , we denote by Γ_X the stabilizer of X , that is the subgroup of elements of Γ preserving X as a set, and Γ_{id_X} the subgroup of elements fixing X elementwise.

Proposition 5.5. Let $E = (E, q)$ be a formed space.

(1) For any $W \in \mathcal{T}(E)$, the map

$$\mathbf{G}(E)_W \rightarrow \mathrm{GL}(W)_{\mathfrak{R}(E)} \times \mathbf{G}(W^\perp/W)$$

sending f to its restriction to W and its induced map on W^\perp/W , is split surjective.

(2) For any $U \in \mathcal{T}(E)$ and $W \in \mathcal{T}(E, U)$, the analogous map

$$\mathbf{A}(E, U)_W \rightarrow \mathbf{A}^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}} \times \mathbf{G}(W^\perp/W)$$

is split surjective.

The map in the second part of the statement makes sense because, by [26, Lem 3.8] (see also Remark 5.3), restriction from E to W defines a homomorphism from $\mathbf{A}(E, U)_W$ to

$$\mathbf{A}(W, W \cap U) \cap \mathbf{A}^T(W, W \cap U^\perp),$$

which is equal to $A^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}}$ given that $W \cap U = \mathfrak{R}(E)$ by [26, Lem 3.10].

The proof of the proposition will make use of the following lemma:

Lemma 5.6. Let $E = (E, q)$ be a formed space and $W \in \mathcal{T}(E)$. For any complement C to W^\perp in E , there exists $E' \leq E$ such that

- (1) $E = W \oplus C \oplus E'$,
- (2) $W^\perp = W \oplus E'$,
- (3) E' is orthogonal to $W \oplus C$.

In particular, E' projects isomorphically and isometrically onto W^\perp/W .

Proof. Using Lemmas A.3 and A.2 and the fact that $C + W^\perp = E$, we get

$$C^\perp \cap (W + \mathfrak{R}(E)) = C^\perp \cap (W^\perp)^\perp = (C + W^\perp)^\perp = \mathfrak{R}(E)$$

and so $C^\perp \cap W = \mathfrak{R}(E)$. Also $C \cap W^\perp = 0$ implies

$$C^\perp + W = C^\perp + (W^\perp)^\perp = (C \cap W^\perp)^\perp = E.$$

Let E' be a complement to $\mathfrak{R}(E)$ in $(W \oplus C)^\perp$. We have

$$W \cap (W \oplus C)^\perp \leq W \cap C^\perp = \mathfrak{R}(E),$$

so E' is disjoint from W . In fact, $W \oplus E' = W^\perp$, since the left is contained in the right and

$$W^\perp = W^\perp \cap (C^\perp + W) = (W + C)^\perp + W.$$

It then follows that $E = (W \oplus C) \oplus E'$, with $W \oplus C$ orthogonal to E' . \square

Proof of Proposition 5.5. We will prove part (2); part (1) is similar (simply set $U = \mathfrak{R}(E)$, and skip the last paragraph). Using that $U \cap W = \mathfrak{R}(E)$ by [26, Lem 3.10], and applying Lemma A.5 to an arbitrary complement Z_0 of $\mathfrak{R}(E)$ in U , we can choose a linear complement C of W^\perp ,

$$C \oplus W^\perp = E$$

which is isotropic, and such that $U \leq C + \mathfrak{R}(E)$.

Let E' be a subspace of E with the properties guaranteed in Lemma 5.6: $E = W \oplus C \oplus E'$, $W \oplus E' = W^\perp$, and E' projects isomorphically and isometrically onto W^\perp/W (where the latter is given the induced form of Proposition A.8). Identifying E' with this quotient then defines a homomorphism

$$\mathbf{G}(W \oplus C)_W \times \mathbf{G}(W^\perp/W) \longrightarrow \mathbf{G}(E)_W.$$

Furthermore, since $U \leq C + \mathfrak{R}(E) \leq C \oplus W$, this homomorphism restricts to a map

$$\mathbf{A}(W \oplus C, U)_W \times \mathbf{G}(W^\perp/W) \longrightarrow \mathbf{A}(E, U)_W.$$

This map and the one in the statement fit in the following commuting triangle:

$$\begin{array}{ccc} \mathbf{A}(E, U)_W & \xrightarrow{\quad} & A^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}} \times \mathbf{G}(W^\perp/W) \\ & \swarrow & \uparrow \downarrow \\ & & \mathbf{A}(W \oplus C, U)_W \times \mathbf{G}(W^\perp/W) \end{array}$$

where the map $\mathbf{A}(W \oplus C, U)_W \rightarrow A^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}}$ is the restriction. We will construct a section of the top horizontal map by constructing one of the vertical map.

Observe that the map

$$\phi : C \rightarrow \sigma(W/\mathfrak{R}(E))^*, \quad v \mapsto \omega_q(-, v)$$

is injective as $C \cap W^\perp = 0$. It is bijective by a dimension count using Lemma A.1:

$$\dim C = \text{codim } W^\perp = \dim W - \dim(\mathfrak{R}(E)).$$

We define a homomorphism

$$\Psi : \mathbf{A}^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}} \rightarrow \mathbf{A}(W \oplus C, U)_W$$

by sending f to $f \oplus \tilde{f}$, where

$$\tilde{f} = \phi^{-1} \circ f^{*-1} \circ \phi.$$

(Here f preserves $\mathfrak{R}(E)$ and hence induces a map on $W/\mathfrak{R}(E)$.) The map Ψ will define a homomorphism if we can check that it is well-defined: that is, that $f \oplus \tilde{f}$ is an isometry of $W \oplus C$ that fixes U —by construction, $\Psi(f)$ preserves W . Since both W and C are isotropic, $\Psi(f)$ is an isometry if and only if for all $w \in W$, $c \in C$,

$$\omega_q(f(w), \tilde{f}(c)) = \omega_q(w, c),$$

and \tilde{f} is precisely defined so that this holds. Lastly we need to check that $\Psi(f)$ is the identity on U if f the identity modulo $W \cap U^\perp$ and the identity on $\mathfrak{R}(E)$. Since $U = (U \cap C) \oplus \mathfrak{R}(E)$, it suffices to show that \tilde{f} is the identity on $U \cap C$. Let $u \in U \cap C$. Then

$$\phi(u)|_{U^\perp} = 0.$$

Since f is the identity modulo $W \cap U^\perp$,

$$f^* : \sigma(W/\mathfrak{R}(E))^* \rightarrow \sigma(W/\mathfrak{R}(E))^*$$

is the identity on the subspace of those elements which vanish on $(W \cap U^\perp)/\mathfrak{R}(E)$. But $\phi(u)$ is such an element. It follows that $\tilde{f}(u) = u$, as needed. \square

Finally, we will need the following result in Section 6:

Lemma 5.7. Let $E = (E, q)$ be a formed space, $U \in \mathcal{T}(E)$, and $W \in \mathcal{T}(E, U)$. Then

$$\mathbf{A}(U^\perp/U, W \cap U^\perp/\mathfrak{R}(E)) \cong \ker \left(\mathbf{A}(E, U)_W \xrightarrow{res} \mathrm{GL}(W) \right),$$

naturally with respect to isometries $E \rightarrow E'$ inducing a bijection on radicals.

Recall here that $U \cap W = \mathfrak{R}(E)$ by [26, Lem 3.10].

Proof. The kernel of the restriction to W in the statement is

$$K = \{g \in \mathbf{G}(E) \mid g|_{W+U} = id_{W+U}\}$$

There is a homomorphism

$$\phi : K \rightarrow \mathbf{A}(U^\perp/U, W \cap U^\perp/\mathfrak{R}(E))$$

because each $g \in K$ preserves U , and hence also U^\perp , and fixes W . We claim that this map is an isomorphism; we will define an inverse.

To describe a map in the other direction, choose a complement C to U^\perp in E , which is contained in W ; this is possible since $W + U^\perp = E$. Apply Lemma 5.6 to get a subspace $E' \leq E$ such that $E = U \oplus C \oplus E'$, with $U^\perp = U \oplus E'$ and E' is orthogonal to $U \oplus C$. As E' projects isomorphically and isometrically onto U^\perp/U , this determines an injection

$$\psi : \mathbf{G}(U^\perp/U) \cong \mathbf{G}(E') \hookrightarrow \mathbf{G}(E),$$

by extending with the identity map of $U \oplus C$. By construction $\psi(f)$ is the identity on U (which contains $\mathfrak{R}(E)$). When f is the identity on $W \cap U^\perp$, $\psi(f)$ is also the identity on W , since

$$W = W \cap (U^\perp \oplus C) = (W \cap U^\perp) \oplus C.$$

Therefore, ψ restricts to a homomorphism

$$\mathbf{A}(U^\perp/U, W \cap U^\perp) \rightarrow K.$$

One then verifies that ψ is an inverse to ϕ . Hence the kernel of restriction to W is as claimed. \square

6. HOMOLOGICAL STABILITY

In this section, we consider the group inclusions

$$(6.1) \quad \mathrm{GL}(V) \longrightarrow \mathrm{GL}(V \oplus \mathbb{F}) \quad \text{and} \quad \mathbf{A}(V, V_0) \longrightarrow \mathbf{A}(V \oplus \mathbb{F}, V_0)$$

as well as

$$(6.2) \quad \mathbf{G}(E) \rightarrow \mathbf{G}(E \oplus \mathcal{H}) \quad \text{and} \quad \mathbf{A}(E, U) \longrightarrow \mathbf{A}(E \oplus \mathcal{H}, U)$$

given by extending an automorphism to be the identity on the added summand \mathbb{F} or \mathcal{H} , where \mathcal{H} is the hyperbolic space of Definition 2.6. These maps are our main *stabilization maps*. We will also need the inclusion

$$(6.3) \quad \mathbf{A}^T(V, V_0) \longrightarrow \mathbf{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}),$$

likewise by extending by the identity of the added summand; it corresponds to the above stabilization map $\mathbf{A}(V, V_0) \rightarrow \mathbf{A}(V \oplus \mathbb{F}, V_0)$ via the identification of Lemma 5.2.

Our main result for this section is the following stability result for these maps, stated in terms of the vanishing in a range of the corresponding relative homology groups. The bounds are given in terms of the dimension of the vector spaces and the genus, and dimension of the radical, of the formed space, as defined in Section 2 (Definition 2.2).

Theorem 6.1. (1) Let V be a finite-dimensional vector space over a field $\mathbb{F} \neq \mathbb{F}_2$, and $V_0 \leq V$ a subspace. Then

$$\begin{aligned} H_d(\mathrm{GL}(V \oplus \mathbb{F}), \mathrm{GL}(V)) &= 0 && \text{for } d \leq \dim V \\ H_d(\mathbf{A}(V \oplus \mathbb{F}, V_0), \mathbf{A}(V, V_0)) &= 0 && \text{for } d \leq \dim V - \dim V_0. \end{aligned}$$

Moreover, if $\mathbb{F} = \mathbb{F}_{\ell^r}$ for ℓ a prime with $\ell^r \neq 2$,

$$\begin{aligned} H_d(\mathrm{GL}(V \oplus \mathbb{F}_{\ell^r}), \mathrm{GL}(V); \mathbb{F}_{\ell}) &= 0 && \text{for } d \leq \dim V + r(\ell - 1) - 2 \\ H_d(\mathbf{A}(V \oplus \mathbb{F}_{\ell^r}, V_0), \mathbf{A}(V, V_0); \mathbb{F}_{\ell}) &= 0 && \text{for } d \leq \dim V - \dim V_0 + r(\ell - 1) - 2. \end{aligned}$$

(2) Let $E = (E, q)$ be a formed space over a field $\mathbb{F} \neq \mathbb{F}_2$, and let $U \in \mathcal{T}(E)$. Then

$$\begin{aligned} H_d(\mathbf{G}(E \oplus \mathcal{H}), \mathbf{G}(E)) &= 0 && \text{for } d \leq \mathfrak{g}(E) \\ H_d(\mathbf{A}(E \oplus \mathcal{H}, U), \mathbf{A}(E, U)) &= 0 && \text{for } d \leq \mathfrak{g}(E) - (\dim(U) - \dim \mathfrak{R}(E)). \end{aligned}$$

The proof of the theorem is given in Section 6.3. To prepare for it, we first assemble in Section 6.1 the properties of the groups and buildings that will be needed for the proof, building on Section 5, and construct in Section 6.2 the relevant spectral sequences, building on the short Appendix B.

Remark 6.2. (1) Note that $r(\ell - 1) - 2 > 0$ whenever $q = \ell^r > 4$, and equals 0 for $q = 3$ or 4. So the second part of (1) in the theorem gives an improvement for $\mathrm{GL}(\mathbb{F}_q)$ for all $q > 4$. This improvement was first found by Galatius–Kupers–Randal-Williams [12]. Our proof here is really different, though we use one lemma from their paper, about the vanishing of the homology $H_*(\mathbf{A}^T(V, V_0); \mathrm{St}(V, V_0) \otimes \mathbb{F}_{\ell})$ in the range $* \leq r(\ell - 1) - 2$ (see Lemma 5.2 in that paper). This lemma is an independent computation, that does not use the rest of the paper.

Note that the map $H_2(\mathrm{GL}_2(\mathbb{F}_4)) \rightarrow H_2(\mathrm{GL}_3(\mathbb{F}_4))$ is not injective, so the stability range cannot be improved in that case. Indeed, the latter group being 0 with \mathbb{F}_2 coefficients by the theorem and Quillen’s vanishing result [21], while one can check that the first is not. Indeed, the 16–dimensional representation ρ induced by the C_2 subgroup generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, permuting the elements of \mathbb{F}_4^2 , splits as a sum of 10 trivial representations and 6 copies of the sign representations σ . Hence the Steiffel-Withney class

$$w(\rho) = w(6\sigma) = w(\sigma)^6 = (1 + t)^6 = 1 + t^2 + t^4 + t^6,$$

with t the generator in $H^1(C_2) = H^1(\mathbb{R}P^\infty)$. As it has a non-trivial component t^2 of degree 2, and is the restriction to a subgroup of a representation of $\mathrm{GL}_2(\mathbb{F}_4)$, this shows that $H^2(\mathrm{GL}_2(\mathbb{F}_4); \mathbb{F}_2)$ must be non-zero. But then the same holds for $H_2(\mathrm{GL}_2(\mathbb{F}_4); \mathbb{F}_2)$ by the universal coefficient theorem⁷.

(2) It is natural to wonder whether an analogous improvement of the offset also holds in the case of the automorphism groups of formed spaces for finite fields at the characteristics. The argument given here in the case of the linear groups does not transfer to these other groups because of one main difference between the buildings used in the two cases: for the linear groups, there is a natural augmented version of the building, $\overline{\mathcal{T}}(V)$ and $\overline{\mathcal{T}}(V, V_0)$, that is contractible, and this is the building we use for the proof. Such an augmented version (with nice properties!) does not exist in the isotropic case. An argument in the set-up of [12] instead could potentially work (if such an improvement is possible!), but requires the study the connectivity of new simplicial complexes.

6.1. Properties of the groups and buildings. Let V be a finite-dimensional vector space and V_0 a subspace. Recall from Definitions 3.2 and 3.4 the building and relative building

$$\begin{aligned} \mathcal{T}(V) &= \{0 < W < V\}, & \mathcal{T}(V, V_0) &= \{W < V \mid W + V_0 = V\}, \\ \overline{\mathcal{T}}(V) &= \{0 < W \leq V\}, & \overline{\mathcal{T}}(V, V_0) &= \{W \leq V \mid W + V_0 = V\}. \end{aligned}$$

Note that, just as we have two choices A and A^T for the definition of the relative groups, there are two choices for the relative buildings, the other choice being

$$\overline{\mathcal{T}}^T(V, V_0) := \{0 \leq W < V \mid W \cap V_0 = 0\}.$$

Sending W to $(V/W)^*$ defines an isomorphism

$$\overline{\mathcal{T}}^T(V, V_0) \xrightarrow{\cong} \overline{\mathcal{T}}(V^*, (V/V_0)^*)$$

which is order preserving if we define the order on $\overline{\mathcal{T}}^T(V, V_0)$ to be given by reversed inclusion. Both $A(V, V_0)$ and $A^T(V, V_0)$ act on both posets $\overline{\mathcal{T}}(V, V_0)$ and $\overline{\mathcal{T}}^T(V, V_0)$. To prove the stability theorem, we will use the action of $A^T(V, V_0)$ on $\overline{\mathcal{T}}(V, V_0)$ and, dually, the action of $A(V, V_0)$ on $\overline{\mathcal{T}}^T(V, V_0)$.

Lemma 6.3. The isomorphism $\overline{\mathcal{T}}^T(V, V_0) \cong \overline{\mathcal{T}}(V^*, (V/V_0)^*)$ given above is equivariant with respect to the action of $A(V, V_0)$ on $\overline{\mathcal{T}}^T(V, V_0)$ and of $A^T(V, V_0)$ on $\overline{\mathcal{T}}(V, V_0)$, where the groups are identified via the isomorphism of Lemma 5.2.

Proof. We need to check that for any $W \in \overline{\mathcal{T}}^T(V, V_0)$ and $g \in A(V, V_0)$, we have that $(V/g(W))^* = (g^{-1})^*(V/W)^*$, which holds because $f \in (g^{-1})^*(V/W)^*$ if and only if $g^*f \in (V/W)^*$, that is if and only if $f \circ g$ vanishes on W , and is the case if and only if f vanishes on $g(W)$. \square

For $E = (E, q)$ a formed spaced and U an isotropic subspace, recall from Definition 4.1 the building and relative building

$$\begin{aligned} \mathcal{T}(E) &= \{\mathfrak{R}(E) < W < E \mid W \text{ isotropic}\}, \\ \mathcal{T}(E, U) &= \{\mathfrak{R}(E) < W < E \mid W \text{ isotropic}, W + U^\perp = E\} \end{aligned}$$

where $\mathfrak{R}(E)$ is the radical of E of Definition 2.2. Note that $\mathcal{T}(E, \mathfrak{R}(E)) = \mathcal{T}(E)$. There are no distinct ‘‘augmented’’ variants of $\mathcal{T}(E)$ and $\mathcal{T}(E, U)$, because E itself is never an isotropic subspace unless $E = \mathfrak{R}(E)$, in which case all variants of the building are empty.

The following result summarizes properties of the buildings that will be needed in the stability argument.

⁷This argument was suggested to us by Oscar Randal-Williams.

Proposition 6.4. Let V be a finite dimensional vector space and $V_0 \leq V$ be a subspace. Let $E = (E, q)$ be a formed space and $U \in \mathcal{T}(E)$.

(1) The posets $\overline{\mathcal{T}}(V), \overline{\mathcal{T}}(V, V_0), \mathcal{T}(E)$ and $\mathcal{T}(E, U)$ are Cohen-Macaulay with:

Poset P	$\dim(P)$	$\text{rank}(W)$ for $W \in P$	lower intervals
$\overline{\mathcal{T}}(V)$	$\dim V - 1$	$\dim W - 1$	$\overline{\mathcal{T}}(V)_{<W} \cong \mathcal{T}(W)$
$\overline{\mathcal{T}}(V, V_0)$	$\dim V_0$	$\dim(W \cap V_0)$	$\overline{\mathcal{T}}(V, V_0)_{<W} \cong \mathcal{T}(W, W \cap V_0)$
$\mathcal{T}(E)$	$g(E) - 1$	$\dim W - \dim \mathfrak{R}(E) - 1$	$\mathcal{T}(E)_{<W} \cong \mathcal{T}(W/\mathfrak{R}(E))$
$\mathcal{T}(E, U)$	$g(U^\perp)$	$\dim W - \dim U$	$\mathcal{T}(E, U)_{<W} \cong \mathcal{T}(W/\mathfrak{R}(E), (W \cap U^\perp)/\mathfrak{R}(E))$.

(Note that $g(U^\perp) = g(E) - \dim U + \dim \mathfrak{R}(E)$ by Lemma A.8.)

(2) The natural inclusions of vector spaces (resp. formed spaces) induce rank-preserving maps of posets

$$\begin{array}{ccc} \overline{\mathcal{T}}(V) & \longrightarrow & \overline{\mathcal{T}}(V \oplus \mathbb{F}) \\ \overline{\mathcal{T}}(V, V_0) & \longrightarrow & \overline{\mathcal{T}}(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{T}(E) & \longrightarrow & \mathcal{T}(E \oplus \mathcal{H}) \\ \mathcal{T}(E, U) & \longrightarrow & \mathcal{T}(E \oplus \mathcal{H}, U), \end{array}$$

In each case, the dimension of the codomain is exactly one more than that of the domain.

Moreover, the maps are equivariant with respect to the stabilization maps

$$\begin{array}{ccc} \text{GL}(V) & \longrightarrow & \text{GL}(V \oplus \mathbb{F}) \\ \text{A}^T(V, V_0) & \longrightarrow & \text{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{G}(E) & \longrightarrow & \mathbf{G}(E \oplus \mathcal{H}) \\ \mathbf{A}(E, U) & \longrightarrow & \mathbf{A}(E \oplus \mathcal{H}, U). \end{array}$$

Proof. Statement (1) is given by Theorems 3.5, 3.3, 4.6 and 4.7, and Lemmas 4.4 and 4.5.

For (2), we have that the natural inclusion, taking W in the left hand poset to itself as an element of the right hand poset, is a poset map. The map is rank-preserving by (1), where in the case of $\mathcal{T}(E)$, we use that $\mathfrak{R}(E \oplus \mathcal{H}) = \mathfrak{R}(E)$ by Lemmas 2.7, and in the case of $\mathcal{T}(E, U)$, also that

$$W \cap U^\perp = W \cap (U^\perp \cap E),$$

as subspaces of $E \oplus \mathcal{H}$, which is true since $W \leq E$, noting also that $U^\perp \cap E$ identifies with the orthogonal complement of U as a subspace of E .

The claim about dimensions follows from part (1), together with the fact that $g(E \oplus \mathcal{H}) = g(E) + 1$ by Lemma 2.7, and also the observation that, as subspaces of $E \oplus \mathcal{H}$,

$$U^\perp = (U^\perp \cap E) \oplus \mathcal{H},$$

so that $g(U^\perp) = g(U^\perp \cap E) + 1$ by Lemma 2.7. The equivariance of the maps follows from the compatibility of the stabilization maps defined on the groups and the buildings. \square

The following proposition gives a description of the stabilizers of the action of the groups $\text{GL}(V), \mathbf{G}(E), \text{A}^T(V, V_0)$ and $\mathbf{A}(E, U)$ on their associated buildings. In each case, the stabilizer is an extension with kernel another group of one of these types, making an inductive argument possible. In the cases of $\text{GL}(V)$ and $\text{A}^T(V, V_0)$, the kernel is actually a group of type $\mathbf{A}(-, -)$ rather than $\text{A}^T(-, -)$, but these are isomorphic by Lemma 5.2.

Recall that, for a group G acting on a building P , we use the notation G_W for the stabilizer of $W \in P$, and G_{id_W} for the subgroup of elements that restrict to the identity on W .

Proposition 6.5. Let $V_0 \leq V$ be finite-dimensional vector spaces. Let $E = (E, q)$ be a formed space and $U \in \mathcal{T}(E)$.

(1) The groups $\text{GL}(V), \text{A}^T(V, V_0), \mathbf{G}(E), \mathbf{A}(E, U)$ act (respectively) on the posets $\overline{\mathcal{T}}(V), \overline{\mathcal{T}}(V, V_0), \mathcal{T}(E), \mathcal{T}(E, U)$, transitively on the elements of a given rank.

(2) For an element W in each of these posets, there is a split short exact sequence

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbf{A}(V, W) & \longrightarrow & \mathrm{GL}(V)_W & \xrightarrow{res} & \mathrm{GL}(W) & \longrightarrow & 1, \\
1 & \longrightarrow & \mathbf{A}(V_0, W \cap V_0) & \longrightarrow & \mathbf{A}^T(V, V_0)_W & \xrightarrow{res} & \mathbf{A}^T(W, W \cap V_0) & \longrightarrow & 1, \\
1 & \longrightarrow & \mathbf{A}(E, W) & \longrightarrow & \mathbf{G}(E)_W & \xrightarrow{res} & \mathrm{GL}(W)_{\mathfrak{R}(E)} & \longrightarrow & 1, \\
1 & \longrightarrow & \mathbf{A}(U^\perp/U, W \cap U^\perp) & \longrightarrow & \mathbf{A}(E, U)_W & \xrightarrow{res} & \mathbf{A}^T(W, W \cap U^\perp)_{id_{\mathfrak{R}(E)}} & \longrightarrow & 1.
\end{array}$$

Each of these short exact sequences commutes with the stabilization homomorphisms (6.1), (6.2), (6.3) in the left two terms of the sequence, and the identity on the right term. When $W = W_0$ is of rank 0, the second and fourth sequences degenerate to isomorphisms

$$(6.4) \quad \mathrm{GL}(V_0) \xrightarrow{\cong} \mathbf{A}^T(V, V_0)_{W_0} \quad \text{and} \quad \mathbf{G}(U^\perp/U) \xrightarrow{\cong} \mathbf{A}(E, U)_{W_0}.$$

(3) Let X be a rank 0 element in the poset $\mathcal{T}^T(V, W)$, $\mathcal{T}^T(V_0, W \cap V_0)$, $\mathcal{T}(E, W)$ or $\mathcal{T}(U^\perp/U, W \cap U^\perp)$ associated to the kernel groups in each of the four exact sequences in (2). The splitting in (2) may be chosen so that it commutes with the stabilizer of X in the kernel. (By (6.4), this stabilizer identifies with $\mathrm{GL}(X)$ in the general linear cases and with $\mathbf{G}(W^\perp/W)$ in the isotropic cases.)

Proof. Statement (1) follows from Proposition 5.4.

The identifications of the kernels in the four sequences in (2) follows directly from the definitions of the groups in the first three cases, and is given by Lemma 5.7 in the last case. Surjectivity of the right hand map will follow from the existence of a splitting. We will in all cases construct a splitting satisfying the extra condition in (3), proving (2) and (3) at the same time.

In the first case, a rank 0 element X in $\overline{\mathcal{T}}^T(V, W)$ is a complement of W in V . Writing $V = W \oplus X$ defines the required splitting $\mathrm{GL}(W) \times \mathrm{GL}(X) \rightarrow \mathrm{GL}(V)$, noting that $\mathrm{GL}(X)$ identifies indeed canonically with the stabilizer of X for the action of $\mathbf{A}(V, W)$ on $\overline{\mathcal{T}}^T(V, W)$. In the second case, a rank 0 element X in $\overline{\mathcal{T}}^T(V_0, W \cap V_0)$ is a complement to $W \cap V_0$ in V_0 . Picking a complement W' of $W \cap V_0$ inside W , we have $V = W' \oplus (W \cap V_0) \oplus X$. This gives an inclusion

$$\mathbf{A}^T(W, W \cap V_0) \times \mathrm{GL}(X) \longrightarrow \mathbf{A}^T(W \oplus X, V_0),$$

likewise giving the required splitting.

A rank 0 element X in $\mathcal{T}(E, W)$ can be written as $X = C \oplus \mathfrak{R}(E)$ with C a complement of W^\perp in E . As in the proof of Proposition 5.5, we write $E = W \oplus C \oplus E'$ with $E \cong W^\perp/W$. The lemma shows that this gives a splitting

$$\mathrm{GL}(W)_{\mathfrak{R}(E)} \times \mathbf{G}(W^\perp/W) \longrightarrow \mathbf{G}(E)$$

of the restriction map. As elements of $\mathbf{A}(E, W)$ are the identity modulo W^\perp (by [26, Lem 3.8]), we have that $\mathbf{G}(W^\perp/W)$ identifies with the stabilizer subgroup $\mathbf{A}(E, W)_X$, which finishes the proof in this case. The result is similarly given by Proposition 5.5 in the last case too.

The compatibility with the stabilization maps is a direct check. \square

Finally we will need the following retract information:

Lemma 6.6. Let V be a finite-dimensional vector space and $E = (E, q)$ a formed space. The natural inclusions

$$\mathrm{GL}(V) \longrightarrow \mathrm{GL}(\mathbb{F} \oplus V)_{\mathbb{F}} \quad \text{and} \quad \mathbf{G}(E) \longrightarrow \mathbf{G}(\mathcal{H} \oplus E)_L$$

are split injective, compatibly with the stabilization maps. Here $L \leq \mathcal{H}$ is any choice of isotropic line in the formed space \mathcal{H} of Definition 2.6.

Proof. The former splitting arises from viewing V as $(\mathbb{F} \oplus V)/\mathbb{F}$, and the latter from $E \cong L^\perp/L$. For this last equality, we need the check that $L \oplus E = L^\perp$ as subspaces of $\mathcal{H} \oplus E$. The left side is contained in the right; the reverse inclusion holds since $L^\perp \cap \mathcal{H} = L$ as \mathcal{H} is 2-dimensional and nondegenerate by Lemma 2.7. \square

Lemma 6.7. Let $V_0 \leq V$ be vector spaces, E a formed space, and $U \in \mathcal{T}(E)$. Let W_0 be a rank 0 element in $\overline{\mathcal{T}}(V, V_0)$ (resp. $\mathcal{T}(E, U)$). The inclusions

$$\mathrm{GL}(V_0) \cong \mathbf{A}^T(V, V_0)_{W_0} \longrightarrow \mathbf{A}^T(V, V_0) \quad \text{and} \quad \mathbf{G}(U^\perp/U) \cong \mathbf{A}(E, U)_{W_0} \longrightarrow \mathbf{A}(E, U)$$

associated to the isomorphisms (6.4) are split, compatibly with the stabilization maps.

Proof. In the first case, W_0 is a complement to V_0 in V and the splitting is given by restriction to V_0 . In the second case, W_0 is a complement to U^\perp in E and the splitting sends $f \in \mathbf{A}(E, U)$ to its induced map on U^\perp/U . \square

6.2. Spectral sequences. For $V_0 \leq V$ finite-dimensional vector spaces, recall from Section 3.1 the Steinberg module

$$\mathrm{St}(V) = \tilde{H}_{\dim \mathcal{T}(V)}(\mathcal{T}(V)) \quad \text{and} \quad \mathrm{St}(V, V_0) = \tilde{H}_{\dim \mathcal{T}(V, V_0)}(\mathcal{T}(V, V_0)),$$

the top homology groups of the posets $\mathcal{T}(V)$ and $\mathcal{T}(V, V_0)$.

Remark 6.8. We use the convention that $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$, so that $\mathrm{St}(\mathbb{F}^1) = \mathbb{Z}$.

Theorem 6.9. Consider a tuple $(\Gamma, \Gamma', \mathcal{P}, \mathcal{P}')$ of one of the following four types:

- | | | | |
|-------------------------------------|--|--|---|
| (1) $\Gamma = \mathrm{GL}(V)$ | $\Gamma' = \mathrm{GL}(V \oplus \mathbb{F})$ | $\mathcal{P} = \overline{\mathcal{T}}(V)$ | $\mathcal{P}' = \overline{\mathcal{T}}(V \oplus \mathbb{F})$, |
| (2) $\Gamma = \mathbf{G}(E)$ | $\Gamma' = \mathbf{G}(E \oplus \mathcal{H})$ | $\mathcal{P} = \mathcal{T}(E)$ | $\mathcal{P}' = \mathcal{T}(E \oplus \mathcal{H})$, |
| (3) $\Gamma = \mathbf{A}^T(V, V_0)$ | $\Gamma' = \mathbf{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F})$ | $\mathcal{P} = \overline{\mathcal{T}}(V, V_0)$ | $\mathcal{P}' = \overline{\mathcal{T}}(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F})$, |
| (4) $\Gamma = \mathbf{A}(E, U)$ | $\Gamma' = \mathbf{A}(E \oplus \mathcal{H}, U)$ | $\mathcal{P} = \mathcal{T}(E, U)$ | $\mathcal{P}' = \mathcal{T}(E \oplus \mathcal{H}, U)$. |

where V is an \mathbb{F} -vector space of dimension at least 1, $0 < V_0 < V$ a subspace, $E = (E, q)$ a formed space, and $U \in \mathcal{T}(E)$. Then there is a spectral sequence converging to zero in all degrees in cases (1) and (3), and in degrees $p + q \leq \dim \mathcal{P}$ in cases (2) and (4), with

$$E_{pq}^1 = \begin{cases} H_q(\Gamma', \Gamma) & p = -1, \\ H_q(\Gamma'_{W_p}, \Gamma_{W_p}; \mathrm{St}_p) & 0 \leq p \leq \dim \mathcal{P} \\ H_q(\Gamma'_{W_p}; \mathrm{St}_p) & p = \dim \mathcal{P} + 1 \\ 0 & p > \dim \mathcal{P} + 1. \end{cases}$$

Here $W_p \in \mathcal{P}'$ is any element of rank p , with $W_p \in \mathcal{P}$ if $p \leq \dim \mathcal{P}$. Also

$$\mathrm{St}_p = \begin{cases} \mathrm{St}(W_p) & \text{in case (1),} \\ \mathrm{St}(W_p/\mathfrak{R}(E)) & \text{in case (2),} \\ \mathrm{St}(W_p, W_p \cap V_0) & \text{in case (3),} \\ \mathrm{St}(W_p/\mathfrak{R}(E), (W_p \cap U^\perp)/\mathfrak{R}(E)) & \text{in case (4).} \end{cases}$$

Moreover, if $\mathbb{F} \neq \mathbb{F}_2$, we have

$$E_{\dim P+1,0}^1 = 0$$

in all cases. In case (1) and (3), if $\mathbb{F} = \mathbb{F}_{\ell^r}$ with $\ell^r \neq 2$ and we take homology with coefficients in \mathbb{F}_ℓ , then we additionally have

$$E_{\dim P+1,q}^1 = 0 \quad \begin{array}{l} \text{for all } q \text{ in case (1),} \\ \text{for all } q \leq r(\ell - 1) - 2 \text{ in case (3).} \end{array}$$

These spectral sequences come from the action of the groups Γ and Γ' on chain complexes of modules associated to their corresponding posets \mathcal{P} and \mathcal{P}' , as given by the following theorem:

Theorem 6.10. Let (Γ, \mathcal{P}) be as in case (1), (2), (3) or (4) in Theorem 6.9. Then there is a chain complex $C_\bullet = C_\bullet(\Gamma, \mathcal{P})$ of Γ -modules with

$$C_p = \begin{cases} \text{St}_p \uparrow_{\Gamma_{W_p}}^\Gamma & 0 \leq p \leq \dim(\mathcal{P}), \\ \mathbb{Z} & p = -1 \end{cases}$$

where $W_p \in \mathcal{P}$ has rank p and St_p is as in Theorem 6.9. This chain complex is exact in all cases and degrees, except for

$$H_{\dim \mathcal{P}}(C_\bullet) = \tilde{H}_{\dim \mathcal{P}}(\mathcal{P})$$

in cases (2) and (4).

Proof of Theorem 6.10. From Theorems 3.5, 4.6 and 4.7 we have that $\overline{\mathcal{T}}(V)$, $\overline{\mathcal{T}}(V, V_0)$, $\mathcal{T}(E)$, and $\mathcal{T}(E, U)$ are all Cohen-Macaulay. Let \mathcal{P} be one of the posets $\overline{\mathcal{T}}(V)$, $\mathcal{T}(E)$, $\overline{\mathcal{T}}(V, V_0)$ or $\mathcal{T}(E, U)$. Applying Theorem B.1 to \mathcal{P} in each case gives the above associated chain complexes, using the identification of the lower intervals $\mathcal{P}_{<W_p}$ given in Proposition 6.4. The homology of the chain complex is the reduced homology of \mathcal{P} , which is trivial in the cases of $\overline{\mathcal{T}}(V)$ and $\overline{\mathcal{T}}(V, V_0)$ since they are contractible, and non-trivial only in its top degree in the cases of $\mathcal{T}(E)$ and $\mathcal{T}(E, U)$ since they are Cohen-Macaulay. \square

Proof of Theorem 6.9. Let $\Gamma, \Gamma', \mathcal{P}, \mathcal{P}'$ be as in the theorem. We have $\dim \mathcal{P}' = \dim \mathcal{P} + 1$ by Proposition 6.4(2). Consider the complexes $C_\bullet = C_\bullet(\Gamma, \mathcal{P})$ and $C'_\bullet = C_\bullet(\Gamma', \mathcal{P}')$ be the complexes of Γ' and Γ -modules given in Theorem 6.10. Choose projective resolutions Q_\bullet and Q'_\bullet of \mathbb{Z} over Γ and Γ' . Then for every $p \geq -1$, stabilization by \mathbb{F} (resp. \mathcal{H}) induces a map of complexes

$$Q_\bullet \otimes_G C_p \longrightarrow Q'_\bullet \otimes_{G'} C'_p$$

and

$$H_i(\Gamma', \Gamma; C', C) = H_i(\text{Cone}(Q_\bullet \otimes_G C_p \rightarrow Q'_\bullet \otimes_{G'} C'_p)).$$

Assembling these cones for every p , we get a double complex

$$\text{Cone}_{p,q} = Q_{q-1} \otimes_G C_p \oplus Q'_q \otimes_{G'} C'_p$$

with horizontal differential that of C_\bullet and C'_\bullet , and vertical differential that of the cone. There are two spectral sequences associated to this double complex. The first one has

$${}^1E: E_{pq}^1 = Q_{q-1} \otimes_G H_p(C_\bullet) \oplus Q'_q \otimes_{G'} H_p(C'_\bullet).$$

Now by Theorem 6.10, $H_*(C'_\bullet) = 0 = H_*(C_\bullet)$ in cases (1) and (3). In cases (2) and (4), $H_p(C_\bullet) = 0$ when $p < \dim \mathcal{P}$, and $H_p(C'_\bullet) = 0$ when $p \leq \dim \mathcal{P}$. Since there are no nonzero terms in the range $p + q \leq \dim \mathcal{P}$, the spectral sequence converges to 0 in those degrees.

Hence the other spectral sequence associated to the same double complex converges to 0 in the same range of degrees. It has E^1 -term

$${}^2E: E_{pq}^1 = \begin{cases} H_q(\Gamma', \Gamma) & p = -1, \\ H_i(\Gamma', \Gamma; C'_p, C_p) = H_i(\Gamma', \Gamma; \text{St}_p \uparrow_{\Gamma'_{W_p}}^\Gamma, \text{St}_p \uparrow_{\Gamma_{W_p}}^\Gamma) & 0 \leq p \leq \dim \mathcal{P} \\ H_q(\Gamma'; C'_p) = H_i(\Gamma'; \text{St}_p \uparrow_{\Gamma'_{W_p}}^\Gamma) & p = \dim \mathcal{P} + 1 \end{cases}$$

This can be rewritten as in the statement of the theorem using Shapiro's lemma and its relative version.

Finally we consider the terms

$$E_{p,q}^1 = H_0(\Gamma'_{W_p}; \text{St}_p) \quad \text{when } p = \dim \mathcal{P} + 1.$$

In case (1) and (3), $W_p = V \oplus \mathbb{F}$ as it is a maximal subspace, so Γ'_{W_p} is the whole group $\mathrm{GL}(V \oplus \mathbb{F})$ or $\mathrm{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F})$, and $E_{p,0}^1$ is the corresponding Steinberg coinvariants when $q = 0$. These vanish by Theorem 3.9 and its absolute version (see eg., [1, Thm. 1.1]).

When $\mathbb{F} = \mathbb{F}_{\ell^r}$, taking homology with coefficients in \mathbb{F}_{ℓ} , we moreover have in case (1) $E_{p,q}^1 = H_q(\mathrm{GL}(V \oplus \mathbb{F}); \mathrm{St}(V) \otimes \mathbb{F}_{\ell}) = 0$ for all q (see eg., [12, Sec 5.1]), giving the additional vanishing statement in that case, while in case (2), $E_{p,q}^1 = H_q(\mathrm{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}); \mathrm{St}(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}) \otimes \mathbb{F}_{\ell}) = 0$ for all $q \leq r(\ell - 1) - 2$ by [12, Lem 5.2]).

For cases (2) and (4), the action of Γ'_{W_p} on

$$\mathrm{St}_p = \begin{cases} \mathrm{St}(W_p / \mathfrak{R}(E)) & \text{in case (2),} \\ \mathrm{St}(W_p / \mathfrak{R}(E), (W_p \cap U^{\perp}) / \mathfrak{R}(E)) & \text{in case (4).} \end{cases}$$

is through the quotients

$$\begin{cases} \mathrm{GL}(W_p)_{\mathfrak{R}(E)} & \text{in case (2),} \\ \mathrm{A}^T(W_p, W_p \cap U^{\perp})_{id_{\mathfrak{R}(E)}} & \text{in case (4).} \end{cases}$$

of Proposition 6.5, which surject onto

$$\begin{cases} \mathrm{GL}(W_p / \mathfrak{R}(E)) & \text{in case (2),} \\ \mathrm{A}^T(W_p / \mathfrak{R}(E), (W_p \cap U^{\perp}) / \mathfrak{R}(E)) & \text{in case (4).} \end{cases}$$

The corresponding coinvariants therefore reduce to coinvariants of the same form as in the linear cases, and vanish by the same theorems. \square

6.3. Proof of Theorem 6.1. Recall from equation (6.4) and Lemma 6.7 that there are split inclusions of groups

$$\mathrm{GL}(V_0) \cong \mathrm{A}^T(V, V_0)_{W_0} \rightarrow \mathrm{A}^T(V, V_0) \quad \text{and} \quad \mathbf{G}(U^{\perp}/U) \cong \mathbf{A}(E, U)_{W_0} \rightarrow \mathbf{A}(E, U)$$

for W_0 a rank 0 element of $\overline{\mathcal{T}}(V, V_0)$ (resp. $\mathcal{T}(E, U)$), which are compatible with the stabilization maps. Hence we get maps of pairs

$$(6.5) \quad \begin{aligned} (\mathrm{GL}(V_0 \oplus \mathbb{F}), \mathrm{GL}(V_0)) &\longrightarrow (\mathrm{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F}), \mathrm{A}^T(V, V_0)) \\ (\mathbf{G}(U^{\perp}/U \oplus \mathcal{H}), \mathbf{G}(U^{\perp}/U)) &\longrightarrow (\mathbf{A}(E \oplus \mathcal{H}, U), \mathbf{A}(E, U)) \end{aligned}$$

inducing injective maps in homology.

For the rest of the section, set $\alpha = 0$ unless we are in the linear case, with $\mathbb{F} = \mathbb{F}_{\ell^r}$ and we take coefficients in \mathbb{F}_{ℓ} , in which case we set

$$\alpha = r(\ell - 1) - 2.$$

We will prove the result by showing that the following two statements hold for every $d \geq 0$:

(I_d) Let (G, G', A, A') be either of the following

- (1) $G = \mathrm{GL}(V_0) \quad G' = \mathrm{GL}(V_0 \oplus \mathbb{F}) \quad A = \mathrm{A}^T(V, V_0) \quad A' = \mathrm{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F})$
- (2) $G = \mathbf{G}(U^{\perp}/U) \quad G' = \mathbf{G}(U^{\perp}/U \oplus \mathcal{H}) \quad A = \mathbf{A}(E, U) \quad A' = \mathbf{A}(E \oplus \mathcal{H}, U),$

with $\mathcal{P} = \overline{\mathcal{T}}(V, V_0)$ in the first case and $\mathcal{P} = \mathcal{T}(E, U)$ in the second case. Then the homomorphism (6.5) induces an isomorphism

$$H_d(G', G) \xrightarrow{\cong} H_d(A', A) \quad \text{if } d \leq \dim \mathcal{P} + 1 + \alpha;$$

(II_d) Let (G, G') be either of the following

- (1) $G = \mathrm{GL}(V) \quad G' = \mathrm{GL}(V \oplus \mathbb{F})$
- (2) $G = \mathbf{G}(E) \quad G' = \mathbf{G}(E \oplus \mathcal{H})$

with $\mathcal{P} = \overline{\mathcal{T}}(V)$ in the first case and $\mathcal{P} = \mathcal{T}(E)$ in the second case. Then

$$H_d(G', G) = 0 \quad \text{if } d \leq \dim \mathcal{P} + 1 + \alpha.$$

We start by showing that (I_d) and (II_d) together imply

(I_d') Let (A, A') be either of the following

- (1) $A = \mathbf{A}^T(V, V_0) \quad A' = \mathbf{A}^T(V \oplus \mathbb{F}, V_0 \oplus \mathbb{F})$
- (2) $A = \mathbf{A}(E, U) \quad A' = \mathbf{A}(E \oplus \mathcal{H}, U),$

with $\mathcal{P} = \overline{\mathcal{T}}(V, V_0)$ in the first case and $\mathcal{P} = \mathcal{T}(E, U)$ in the second case. Then

$$H_d(A', A) = 0 \quad \text{if } d \leq \dim \mathcal{P} + \alpha.$$

Proof that (I_d) and (II_d) imply (I_d') . Indeed, as $d \leq \dim \mathcal{P} + \alpha < \dim \mathcal{P} + 1 + \alpha$, we can apply (I_d) in each case to get an isomorphism $H_d(G', G) \cong H_d(A', A)$ for (G, G') as in (I_d) . Now to apply (II_d) in case (1) we need $\dim \overline{\mathcal{T}}(V, V_0) \leq \dim \overline{\mathcal{T}}(V_0) + 1$, which holds as both sides of the inequality equal $\dim V_0$. In case (2), we need $\dim \mathcal{T}(E, U) \leq \dim \mathcal{T}(U^\perp/U) + 1$, which holds as both sides equal $g(U^\perp)$ by Propositions 6.4 and A.8. \square

Before proving that (I) and (II) hold, we start by checking that they do together prove Theorem 6.1.

Proof that (I_d) and (II_d) imply Theorem 6.1 in degree d . (II_d) gives the non-relative part of the statement in the theorem, as $\dim \mathcal{P} + 1 = \dim V$ in case (1) and $\dim \mathcal{P} + 1 = g(E)$ in case (2). The relative part of the statement is given by (I_d') using also that $A(V, V_0) \cong A^T(V^*, (V/V_0)^*)$ in the general linear case, together with the computation $\dim \overline{\mathcal{T}}(V^*, (V/V_0)^*) = \dim(V/V_0)^* = \dim V - \dim V_0$. In the isotropic case, it follows using the computation $\dim \mathcal{T}(E, U) = g(E) - \dim U + \dim \mathfrak{R}(E)$ of Proposition 6.4. \square

We will prove the statements (I_d) and (II_d) together, by induction on the degree d . The two statements trivially hold for $d = 0$. So we assume for induction that both statements hold for all G, G', A, A' for all $d < q$, and we will prove that (I_q) and (II_q) then also hold. We start by proving (I_q) .

Proof that (I_d) and (II_d) for all $d < q$ imply (I_q) . Let G, G', A, A' be the groups and $\mathcal{P}, \mathcal{P}'$ the associated posets, as in (I_q) and Theorem 6.9. Theorem 6.9(2) associates a spectral sequence to the tuple $(A, A', \mathcal{P}, \mathcal{P}')$, which converges to zero in degrees $p + q \leq \dim \mathcal{P}$, in fact in all degrees in case (1), and has

$$E_{pq}^1 = \begin{cases} H_q(A', A) & \text{if } p = -1, \\ H_q(A'_{W_p}, A_{W_p}; \text{St}_p) & \text{if } 0 \leq p \leq \dim \mathcal{P} \\ H_q(A'_{W_p}; \text{St}_p) & \text{if } p = \dim \mathcal{P} + 1 \\ 0 & \text{if } p > \dim \mathcal{P} + 1 \end{cases}.$$

where $W_p \in \mathcal{P}$ (or \mathcal{P}') has rank p , and

$$\text{St}_p = \text{St}(W_p, W_p \cap V_0) \quad \text{or} \quad \text{St}_p = \text{St}(W_p / \mathfrak{R}(E), W_p \cap U^\perp / \mathfrak{R}(E))$$

in the first and second cases respectively.

Recall that the map $(G', G) \rightarrow (A', A)$ in the statement is induced by the inclusion of the stabilizer of an element W_0 of rank 0 in P , using the identification $G' \cong A'_{W_0}$ from (6.4). Hence the differential $d^1 : E_{0,q}^1 \rightarrow E_{-1,q}^1$ coincides with the map we are interested in, and (I_q) can be rephrased as saying that this differential is an isomorphism as long as

$$q \leq \dim \mathcal{P} + 1 + \alpha.$$

By Lemma 6.7, the pair of inclusions

$$(G', G) \rightarrow (A', A)$$

is split. Hence the induced map in homology is automatically injective, so we only need to check its surjectivity. Since $E_{-1,q}^\infty = 0$ (as $q - 1 \leq \dim \mathcal{P}$ in case (2) as $\alpha = 0$ in that case), and no differentials can leave this position, it suffices to check that no differentials except for d^1 can hit this position. This will follow if we can show that

$$E_{p,q-p}^1 = 0 \text{ when } 1 \leq p \leq \dim \mathcal{P} + 1,$$

recalling that $E_{p,q-p}^1 = 0$ if $p > \dim \mathcal{P} + 1$ for any q . The case $p = \dim \mathcal{P} + 1$ is given by Theorem 6.9 as $E_{p,q-p}^1 = E_{\dim \mathcal{P} + 1, q-p}^1$ with $q - p = 0$ in all cases with $\alpha = 0$, and $q - p \leq \alpha$ in the finite field case at the characteristic with $\alpha = r(\ell - 1) - 2$.

So we consider the case $p \leq \dim \mathcal{P}$, in which case

$$E_{p,q-p}^1 = H_{q-p}(A'_{W_p}, A_{W_p}; \text{St}_p).$$

Write $W_p^0 = W_p \cap V_0$ in the general linear case and $W_p^0 = W_p \cap U^\perp$ in the isotropic case. Recall from Proposition 6.5 that there are group extensions for A_{W_p} and A'_{W_p}

$$\begin{aligned} 1 &\longrightarrow A_0 := \mathbf{A}(V_0, W_p^0) \longrightarrow A_{W_p} \longrightarrow \mathbf{A}^T(W_p, W_p^0) \longrightarrow 1 \\ 1 &\longrightarrow A'_0 := \mathbf{A}(V_0 \oplus \mathbb{F}, W_p^0) \longrightarrow A'_{W_p} \longrightarrow \mathbf{A}^T(W_p, W_p^0) \longrightarrow 1 \end{aligned}$$

in the general linear case and

$$\begin{aligned} 1 &\longrightarrow A_0 := \mathbf{A}(U^\perp/U, W_p^0) \longrightarrow A_{W_p} \longrightarrow \mathbf{A}^T(W_p, W_p^0)_{id_{\mathfrak{R}(E)}} \longrightarrow 1 \\ 1 &\longrightarrow A'_0 := \mathbf{A}(U^\perp/U \oplus \mathcal{H}, W_p^0) \longrightarrow A'_{W_p} \longrightarrow \mathbf{A}^T(W_p, W_p^0)_{id_{\mathfrak{R}(E)}} \longrightarrow 1 \end{aligned}$$

in the isotropic case. As these are compatible with the stabilization maps, we get an associated Leray-Hochschild-Serre spectral sequence of the form

$$\begin{aligned} E_{a,b}^{r2} &= H_a(\mathbf{A}^T(W_p, W_p^0)_{(id_{\mathfrak{R}(E)})}; H_b(A'_0, A_0; \text{St}_p)) \\ &= H_a(\mathbf{A}^T(W_p, W_p^0)_{(id_{\mathfrak{R}(E)})}; H_b(A'_0, A_0) \otimes \text{St}_p), \end{aligned}$$

converging to $H_{q-p}(A'_{W_p}, A_{W_p}; \text{St}_p)$, where we have used that St_p is free abelian and that the action of A'_0 on it is trivial in both cases because the action is through the cokernel of the above exact sequence, where A'_0 is the kernel. We want to show that this E^2 page is zero in total degree

$$a + b = q - p \leq \dim \mathcal{P} + 1 + \alpha - p \quad \text{with } 1 \leq p \leq \dim \mathcal{P}.$$

We may here apply the induction hypothesis to the groups $H_b(A'_0, A_0)$, because the degree

$$b \leq q - p < q$$

is less than q .

In the general linear case, $A_0 = \mathbf{A}(V_0, W_p^0) \cong \mathbf{A}^T(V_0^*, (V_0/W_p^0)^*)$ and by (Γ_b) , $H_b(A'_0, A_0) = 0$ as long as $b \leq \dim V_0 - \dim W_p^0 + \alpha = \dim \mathcal{P} + \alpha - p$. In the isotropic case (where $\alpha = 0$ always), $A_0 = \mathbf{A}(U^\perp/U, W_p^0)$ so $H_b(A'_0, A_0) = 0$ by (Γ_b) as long as $b \leq \dim \mathcal{T}(U^\perp/U, W_p^0) = g(U^\perp/U) - \dim W_p^0 = \dim \mathcal{P} - p$ in this case too, where we used that $g(U^\perp/U) = g(U^\perp)$ (Proposition A.8).

So in both cases $E_{ab}^{r2} = 0$ whenever $b \leq \dim \mathcal{P} + \alpha - p$.

The only remaining case is that $b = \dim \mathcal{P} + 1 + \alpha - p$, in which case $a = 0$. Writing

$$\mathcal{P}(A_0) = \overline{\mathcal{T}}(V_0^*, (V_0/W_p^0)^*) \cong \overline{\mathcal{T}}^T(V_0, W_p^0) = \mathcal{P}^T(A_0) \quad \text{or} \quad \mathcal{P}(A_0) = \mathcal{T}(U^\perp/U, W_p^0)$$

in each respective case, where the isomorphism $\mathcal{P}(A_0) \cong \mathcal{P}^T(A_0)$ is that of Lemma 6.3, the same computation as the one we just did gives that $b \leq \dim \mathcal{P}(A_0) + 1 + \alpha$ in that case, so we can apply (I_b). Consider first the linear case. Let $X \in \mathcal{P}^T(A_0)$ be a rank 0 element, that is a complement of W_p^0 inside V_0 . Then its image $(V_0/X)^*$ in $\overline{\mathcal{T}}(V_0^*, (V_0/W_p^0)^*)$ is also rank 0 and (I_b) shows that there is an isomorphism

$$H_b(G', G) \longrightarrow H_b(A^T(V_0^* \oplus F^*, (V_0/W_p^0)^* \oplus F^*), A^T(V_0^*, (V_0/W_p^0)^*))$$

for

$$G = A^T(V_0^*, (V_0/W_p^0)^*)_{(V_0/X)^*} \quad \text{and} \quad G' = A^T(V_0^* \oplus \mathbb{F}^*, (V_0/W_p^0)^* \oplus \mathbb{F}^*)_{(V_0/X)^*}.$$

Recall from Lemma 6.3 that the isomorphism $\mathcal{P}(A_0) \cong \mathcal{P}^T(A_0)$ is equivariant with respect to the isomorphism $A^T(V_0^*, (V_0/W_p^0)^*) \cong A(V_0, W_p^0) = A_0$, so the stabilizers G of $(V_0/X)^* \in \mathcal{P}(A_0)$ identifies under this latter isomorphism with the stabilizer of $X \in \mathcal{P}^T(A_0)$, and likewise for G' . Hence (I_b) gives an isomorphism

$$H_b((A'_0)_X, (A_0)_X) \xrightarrow{\cong} H_b(A'_0, A_0)$$

By Proposition 6.5(3), conjugation by $A^T(W_p, W_p^0)$ is the identity on the stabilizer subgroups $(A'_0)_X$ and $(A_0)_X$. Hence we get in this case

$$E_{0, n-p}^{\prime 2} \cong H_0(A^T(W_p, W_p^0); \text{St}(W_p, W_p^0)) \otimes H_{n-p}((A'_0)_X, (A_0)_X)$$

which is zero because the first factor vanishes by Theorem 3.9.

In the isotropic case, let X be a rank 0 element in $\mathcal{T}(U^\perp/U, W_p^0)$. Then (I_b) gives an isomorphism

$$H_b((A'_0)_X, (A_0)_X) \xrightarrow{\cong} H_b(A'_0, A_0)$$

while Proposition 6.5(3) gives that $A^T(W_p, W_p^0)_{id_{\mathfrak{R}(E)}}$ acts trivially on $(A'_0)_X, (A_0)_X$. Hence we get that

$$E_{0, n-p}^{\prime 2} \cong H_0(A^T(W_p, W_p^0)_{id_{\mathfrak{R}(E)}}; \text{St}(W_p/\mathfrak{R}(E), W_p^0/\mathfrak{R}(E))) \otimes H_{n-p}((A'_0)_X, (A_0)_X)$$

which is also zero by Theorem 3.9, as

$$A^T(W_p, W_p^0)_{id_{\mathfrak{R}(E)}} \rightarrow A^T(W_p/\mathfrak{R}(E), W_p^0/\mathfrak{R}(E))$$

is surjective and the action on $\text{St}(W_p/\mathfrak{R}(E), W_p^0/\mathfrak{R}(E))$ is through this surjection. Hence all the necessary entries in the previous spectral sequence are also zero, which finishes the proof of (I_q). \square

Proof that (I_d) for all $d \leq q$ and (II_d) for all $d < q$ imply (II_q). Fix now G, G' as in (II_q), with associated posets \mathcal{P} and \mathcal{P}' . Let $\Sigma G = \text{GL}(\mathbb{F} \oplus V)$, $\Sigma G' = \text{GL}(\mathbb{F} \oplus V \oplus \mathbb{F})$ in the general linear case, and likewise $\Sigma G = \mathbf{G}(\mathcal{H} \oplus E)$ and $\Sigma G' = \mathbf{G}(\mathcal{H} \oplus E \oplus \mathcal{H})$ in the isotropic case, and let $\Sigma \mathcal{P} = \overline{\mathcal{T}}(\mathbb{F} \oplus V)$ or $\mathcal{T}(\mathcal{H} \oplus E)$ be the poset associated to ΣG in each case, and $\Sigma \mathcal{P}'$ defined similarly. Theorem 6.9(1) associates a spectral sequence to the data $(\Sigma G, \Sigma G', \Sigma \mathcal{P}, \Sigma \mathcal{P}')$ in each case, which converges to zero in degrees $p + q \leq \dim \Sigma \mathcal{P} = \dim \mathcal{P} + 1$, and in all degrees in the general linear case. It has

$$E_{pq}^1 = \begin{cases} H_q(\Sigma G', \Sigma G) & \text{if } p = -1, \\ H_q(\Sigma G'_{W_p}, \Sigma G_{W_p}; \text{St}_p) & \text{if } 0 \leq p \leq \dim \mathcal{P} + 1 \\ H_q(\Sigma G'_{W_p}; \text{St}_p) & \text{if } p = \dim \mathcal{P} + 2 \\ 0 & \text{if } p > \dim \mathcal{P} + 2 \end{cases}$$

where $W_p \in \Sigma \mathcal{P}$ (or $\Sigma \mathcal{P}'$) has rank p with

$$\text{St}_p = \text{St}(W_p) \quad \text{or} \quad \text{St}(W_p/\mathfrak{R}(E))$$

in the general linear case and isotropic case respectively, where we recall from Lemma 2.7 that $\mathfrak{R}(\mathcal{H} \oplus E) = \mathfrak{R}(E)$. We have also used the fact given by Proposition 6.4(2) that $\dim \mathcal{P}' = \dim \mathcal{P} + 1$.

Note that W_p has dimension $p + 1$ in the general linear case and dimension $p + 1 + \dim \mathfrak{R}(E)$ in the isotropic case (see Proposition 6.4). In both cases, $\text{St}_0 = \text{St}(\mathbb{F}^1)$ is the constant module \mathbb{Z} .

We assume $q \leq \dim \mathcal{P} + 1 + \alpha$ and need to show that $H_q(G', G) = 0$. This group does not as such appear in the spectral sequence, but taking $W_0 = \mathbb{F} \leq \mathbb{F} \oplus V$ in the general linear case and $W_0 = L \leq \mathcal{H} \leq \mathcal{H} \oplus E$ in the isotropic case, Lemma 6.6 shows that there is an injective map

$$H_q(G', G) \rightarrow H_q(\Sigma G'_{W_0}, \Sigma G_{W_0}) = E_{0,q}^1.$$

We will show that the differential

$$d^1 : E_{0,q}^1 = H_q(\Sigma G'_{W_0}, \Sigma G_{W_0}) \rightarrow E_{-1,q}^1 = H_q(\Sigma G', \Sigma G)$$

is injective when $q \leq \dim \mathcal{P} + 1$. It will follow that the composition

$$s_1 : H_q(G', G) \rightarrow H_q(\Sigma G', \Sigma G)$$

is also injective in this range of degrees. This composition is known as the ‘‘lower suspension’’ and the same argument applied to $(\Sigma G', \Sigma G)$ will show that also the map $s_2 : H_q(\Sigma G', \Sigma G) \rightarrow H_q(\Sigma^2 G', \Sigma^2 G)$ is injective in this range. Statement (II_q) will then follow if we show that the composition

$$H_q(G', G) \xrightarrow{s_1} H_q(\Sigma G', \Sigma G) \xrightarrow{s_2} H_q(\Sigma^2 G', \Sigma^2 G)$$

is the zero map. This is a by now standard argument, which can be found e.g. in [24, Prop. 4.22]. We sketch it here for completeness.

Consider the commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_q(G') & \xrightarrow{f_1} & H_q(G', G) & \xrightarrow{f_2} & H_{q-1}(G) & \xrightarrow{l'_2} & H_{q-1}(G') & \cdots \\ & & \downarrow & & \downarrow s_1 & & \downarrow l_2 & \swarrow \phi_* & & \\ \cdots & \longrightarrow & H_q(\Sigma G') & \xrightarrow{g_1} & H_q(\Sigma G', \Sigma G) & \xrightarrow{g_2} & H_{q-1}(\Sigma G) & \longrightarrow & \cdots & \\ & & \downarrow l_1 & & \downarrow s_2 & & \downarrow & & & \\ \cdots & \xrightarrow{l'_1} & H_q(\Sigma^2 G') & \xrightarrow{h_1} & H_q(\Sigma^2 G', \Sigma^2 G) & \xrightarrow{h_2} & H_{q-1}(\Sigma^2 G) & \longrightarrow & \cdots & \end{array}$$

where the vertical maps are all induced by the lower suspensions and the horizontal maps come from the long exact sequence associated to the ‘‘upper suspension’’, which is our stabilization map.

For convenience, write $(X, Z) = (E, \mathcal{H})$ in the isotropic case, or $(X, Z) = (V, \mathbb{F})$ in the general linear case, so that G (resp. $G', \Sigma G$) is the group of automorphisms of X (resp. $X \oplus Z, Z \oplus X$). The natural isomorphism $X \oplus Z \rightarrow Z \oplus X$ induces an isomorphism $\phi : G' \rightarrow \Sigma G$ which commutes with the respective inclusions of G into these groups. This shows that l_2 and l'_2 have the same kernel, so the composition

$$g_2 \circ s_1 = l_2 \circ f_2 = 0$$

in the diagram. Hence the image of s_1 lies in the image of g_1 , and thus the image of $s_2 \circ s_1$ lies in the image of $s_2 \circ g_1 = h_1 \circ l_1$.

Now, let $\psi : \Sigma G' \rightarrow \Sigma^2 G$ be the isomorphism induced by the isometry $Z \oplus X \oplus Z \rightarrow Z \oplus Z \oplus X$ sending $(z, x, z') \mapsto (z', z, x)$. This time the triangle

$$\begin{array}{ccc} & \Sigma G' & \\ \psi \swarrow & \downarrow \Sigma & \\ \Sigma^2 G & \xrightarrow{\text{incl}} & \Sigma^2 G' \end{array}$$

does not quite commute. But it does commute up to conjugation in the target: specifically, conjugation by the automorphism of $Z \oplus Z \oplus X \oplus Z$ switching the first and last factors. Therefore it induces a commutative triangle in homology, showing that the induced maps l_1 and l'_1 have the same image. It follows that $h_1 \circ l_1 = 0$. Combining this with the previous paragraph, we can conclude that $s_2 \circ s_1 = 0$ as claimed.

It remains only to check injectivity of the differential $d^1 : E_{0,q}^1 \rightarrow E_{-1,q}^1$ in the above spectral sequence. We are assuming

$$q \leq \dim \mathcal{P} + 1 + \alpha,$$

so $E_{0,q}^\infty = 0$. The differential d^1 is the only one that can leave that position. It will necessarily be injective once we have shown that no other differential hits that position. We will do this by showing that

$$E_{p,q-p+1}^1 = 0 \quad \text{for all } 1 \leq p \leq \dim \mathcal{P} + 2.$$

When $p = \dim \mathcal{P} + 2 = \dim \Sigma \mathcal{P} + 1$, we have $q - p + 1 \leq \alpha$ and $E_{p,q-p+1}^1 = 0$ by Theorem 6.9 as either $q - p + 1 = \alpha = 0$ or we are in a case where this homology group vanishes for any q . So we are left to consider the case $p \leq \dim \mathcal{P} + 1$. We have

$$E_{p,q-p+1}^1 = H_{q-p+1}(\Sigma G'_{W_p}, \Sigma G_{W_p}; \text{St}_p).$$

We use the Leray-Hochschild-Serre spectral sequence associated to the short exact sequences of Proposition 6.5. The E^2 -term of that spectral sequence has the form

$$E_{a,b}^2 = H_a(\text{GL}(W_p)_{\mathfrak{R}(E)}; H_b(A', A; \text{St}_p))$$

converging to $H_{a+b}(\Sigma G'_{W_p}, \Sigma G_{W_p}; \text{St}_p)$, with

$$A = \mathbf{A}(\mathbb{F} \oplus V, W_p), \quad A' = \mathbf{A}(\mathbb{F} \oplus V \oplus \mathbb{F}, W_p)$$

in the general linear case, and

$$A = \mathbf{A}(\mathcal{H} \oplus E, W_p), \quad A' = \mathbf{A}(\mathcal{H} \oplus E \oplus \mathcal{H}, W_p)$$

in the isotropic case. We want to show that $E_{a,b}^2 = 0$ for all $a+b = q-p+1$ with $1 \leq p \leq \dim \mathcal{P} + 1$ and $q \leq \dim \mathcal{P} + 1 + \alpha$.

We start by analysing the isotropic case (where $\alpha = 0$). When $a = 0$, we have

$$\begin{aligned} E_{0,q-p+1}^2 &= H_0(\text{GL}(W_p)_{\mathfrak{R}(E)}; H_{q-p+1}(A', A; \text{St}(W_p/\mathfrak{R}(E)))) \\ &= H_0(\text{GL}(W_p)_{\mathfrak{R}(E)}; H_{q-p+1}(A', A) \otimes \text{St}(W_p/\mathfrak{R}(E))). \end{aligned}$$

By (I_{q+1-p}) , which we may use as $q+1-p \leq q$, there is an isomorphism

$$H_{q-p+1}(\mathbf{G}(W_p^\perp/W_p \oplus \mathcal{H}), \mathbf{G}(W_p^\perp/W_p)) \rightarrow H_{q-p+1}(A', A)$$

as, using Lemma 2.7 and the dimension and rank calculations of Proposition 6.4,

$$\begin{aligned} q - p + 1 &\leq \dim \mathcal{P} - p + 2 = g(E) - p + 1 = g(\mathcal{H} \oplus E) - p \\ &= g(\mathcal{H} \oplus E) - \dim(W_p) + \dim \mathfrak{R}(\mathcal{H} \oplus E) + 1 = \dim \mathcal{T}(\mathcal{H} \oplus E, W_p) + 1. \end{aligned}$$

Now $\mathbf{G}(W_p^\perp/W_p)$ and $\mathbf{G}(W_p^\perp/W_p \oplus \mathcal{H})$ are the stabilizers of rank 0 elements in the buildings associated to A and A' , and Proposition 6.5(3) shows that the action of $\text{GL}(W_p)_{\mathfrak{R}(E)}$ on these stabilizer subgroups is trivial. Hence the E^2 -term can be rewritten as

$$E_{0,q-p+1}^2 = H_0(\text{GL}(W_p)_{\mathfrak{R}(E)}; \text{St}(W_p/\mathfrak{R}(E))) \otimes H_{q-p+1}(A', A),$$

which is zero because $\text{GL}(W_p)_{\mathfrak{R}(E)}$ surjects onto $\text{GL}(W_p/\mathfrak{R}(E))$, whose coinvariants in the module $\text{St}(W_p/\mathfrak{R}(E))$ vanish by [1, Thm. 1.1], after checking that

$$\dim(W_p/\mathfrak{R}(E)) = p + 1 \geq 2.$$

When $a \geq 1$, we have $b \leq q - p < q$ and we may use (I'_b) . We also have $b \leq q - p \leq \dim \mathcal{T}(\mathcal{H} \oplus E, W_p)$ just as above. Hence $H_b(A', A) = 0$ and hence $E'_{a,b}{}^2 = 0$ in that case. This shows that $E'_{a,b}{}^2 = 0$ for all $a + b = q - p + 1$, which finishes the proof of (II_q) in the isotropic case.

In the general linear case, we have

$$A \cong A^T(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*),$$

and similarly for A' , fitting together into an isomorphism of pairs. Also,

$$\dim \overline{\mathcal{T}}(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*) + \alpha = \dim V + 1 - (p + 1) + \alpha = \dim \mathcal{P} - p + 1 + \alpha \geq q - p.$$

This shows just as in the previous case that $E'_{a,b}{}^2 = 0$ for all $a + b = q - p + 1$ by (I'_b) , as long as $a \geq 1$.

When $a = 0$, we have

$$\begin{aligned} E'_{0,q-p+1}{}^2 &= H_0(\mathrm{GL}(W_p); H_{q-p+1}(A', A; \mathrm{St}(W_p))) \\ &= H_0(\mathrm{GL}(W_p); H_{q-p+1}(A', A) \otimes \mathrm{St}(W_p)). \end{aligned}$$

By Lemma 6.3, there is an isomorphism $\overline{\mathcal{T}}^T(\mathbb{F} \oplus V, W_p) \cong \overline{\mathcal{T}}(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*)$, which is equivariant with respect to the isomorphism $A \cong A^T(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*) = A^T$, and likewise for the suspended versions. Pick a rank 0 element X in $\overline{\mathcal{T}}^T(\mathbb{F} \oplus V, W_p)$, that is a complement of W_p in V , with $(\mathbb{F} \oplus V/X)^*$ its image in $\overline{\mathcal{T}}(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*)$. By (I_{q-p+1}) , which applies since

$$\dim \overline{\mathcal{T}}(\mathbb{F}^* \oplus V^*, (\mathbb{F} \oplus V/W_p)^*) + 1 + \alpha \geq q - p + 1$$

as checked above, there is an isomorphism

$$H_{q-p+1} \left(A_{(\mathbb{F} \oplus V/X)^*}^T, A_{(\mathbb{F} \oplus V/X)^*}^T \right) \longrightarrow H_{q-p+1} \left((A^T)', A^T \right).$$

Now under the isomorphism $A^T \cong A$, the group $A_{(\mathbb{F} \oplus V/X)^*}^T$ identifies with the stabilizer of X , which is rank 0 in the building associated to A , and likewise for the stabilized version. The action of $\mathrm{GL}(W_p)$ is trivial on this stabilizer (as can be check directly, or via Proposition 6.5(3)). Hence we get a decomposition

$$E'_{0,q-p+1}{}^2 = H_0(\mathrm{GL}(W_p); \mathrm{St}(W_p)) \otimes H_{q-p+1}(A', A).$$

And this is zero because the coinvariants vanish by [1, Thm. 1.1] as W_p as dimension $p + 1 \geq 2$. This finishes the proof of (II_q) , and of the theorem. \square

Proof of Corollary C. By [9, Thm III.4.6], the stable homology of the groups $\mathrm{Sp}_{2n}(\mathbb{F}_{p^r})$ and $\mathrm{U}_{n,n}(\mathbb{F}_{p^{2r}})$ vanishes at the characteristic, and that of $\mathrm{O}_{n,n}(\mathbb{F}_{p^r})$ vanishes at the characteristic when p is odd. The result then follows from Theorem 6.1. \square

7. EUCLIDEAN FORM

In this section, we will use our homological stability result stabilizing by adding hyperbolic planes, to deduce in some cases, a stability result for the unitary and orthogonal groups of the standard Euclidean form on \mathbb{F}^n , proving Theorems B and B' as well as Corollary D. The idea is to relate the stabilizations maps $\oplus E$, for E a non-degenerate formed space, to the stabilization maps $\oplus \mathcal{H}$, under specific assumptions on the field.

Let \mathbb{F} be a field with involution σ . We will assume that the pair (\mathbb{F}, σ) satisfies one of the following properties:

- (A) There exists $a \in \mathbb{F}$ such that $\bar{a}a = -1$.
- (B) There exist $a, b \in \mathbb{F}$ such that $\bar{a}a + \bar{b}b = -1$.

Clearly (A) implies (B). Also (A) always holds in characteristic 2 since in that case $-1 = 1 = \bar{1} \cdot 1$. When $\mathbb{F} = \mathbb{C}$ is the complex numbers, (A) is satisfied when σ is the identity but *not* when σ is complex conjugation.

When $\sigma = \text{id}$, Property (A) (resp. (B)) says that -1 is a square (resp. a sum of two squares) in \mathbb{F} . For finite fields of odd characteristic, we have the following:

Lemma 7.1. Let \mathbb{F}_{p^r} be a finite field of odd characteristic. Any nonsquare element in \mathbb{F}_{p^r} can be written as a sum of two squares.

In particular, for any finite field \mathbb{F}_{p^r} , the pair $(\mathbb{F}_{p^r}, \sigma = \text{id})$ satisfies Property (B).

Proof. Let $q = p^r$. Exactly half of the nonzero elements of \mathbb{F}_q are squares because the multiplicative group $\mathbb{F}_q - \{0\}$ is cyclic of even order. Zero is also a square, so the total number of squares is $(q+1)/2$. If the set of squares were closed under addition, these elements would form a subgroup of the additive group \mathbb{F}_q , but that cannot be the case because q is not divisible by $(q+1)/2$. Hence there exist elements $a, b \in \mathbb{F}_q$ such that $a^2 + b^2$ is not a square. But we can then get any other nonsquare by scaling appropriately, as the quotient of any two nonsquare elements is a square (since $|\mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}| = 2$). \square

Lemma 7.2. Let $(\mathbb{F}_{p^{2r}}, \sigma)$ be a finite field with the non-trivial involution $\sigma = (-)^{p^r}$. Then the norm map

$$N : \mathbb{F}^\times \rightarrow (\mathbb{F}^\sigma)^\times, \quad a \mapsto \bar{a}a$$

is surjective. In particular $(\mathbb{F}_{p^{2r}}, \sigma \neq \text{id})$ always satisfies Property (A).

Note that the norm map is not surjective in general, as exemplified for instance by $\mathbb{F} = \mathbb{C}$ and σ complex conjugation.

Proof. Set $q = p^r$. The map N is a group homomorphism of multiplicative groups given explicitly by the formula $N(a) = a^{q+1}$. Its domain is a cyclic group of order $q^2 - 1$. So the image is cyclic of order $\frac{q^2-1}{q+1} = q - 1$, the same as the order of the codomain. \square

The following result allows us to compare stabilization by E , for E non-degenerate, with stabilization by \mathcal{H} :

Proposition 7.3. Let $E = (E, q)$ be a nondegenerate formed space over (\mathbb{F}, σ) .

- (1) If (\mathbb{F}, σ) satisfies Property (A), then $E^{\oplus 2} \cong \mathcal{H}^{\oplus \dim E}$.
- (2) If (\mathbb{F}, σ) satisfies Property (B), then $E^{\oplus 4} \cong \mathcal{H}^{\oplus 2 \dim E}$.

Proof. Let $E' = (E, -q)$. By Proposition 2.8,

$$E \oplus E' \cong \mathcal{H}^{\oplus \dim E}.$$

If there exists $a \in \mathbb{F}$ such that $a\bar{a} = -1$, then multiplying by a gives an isomorphism $E' \cong E$, proving the first case. In the second case, Lemma 2.9 says that $E' \oplus E' \cong E \oplus E$. Applying this to a sum of two copies of the above equation proves the result. \square

Just as we did with \mathcal{H} earlier, we can consider stabilization by E , that is the map

$$\mathbf{G}(F) \rightarrow \mathbf{G}(F \oplus E)$$

extending an isometry by defining it to be the identity on the added E . We are now ready to prove Theorem B' from the introduction, which says that, for any non-degenerate formed space $E = (E, q)$ and taking $F = E^{\oplus n}$, the induced map

$$H_d(\mathbf{G}(E^{\oplus n})) \longrightarrow H_d(\mathbf{G}(E^{\oplus n+1}))$$

is injective in degrees $d \leq k(\lfloor n/2 \rfloor - e) - 1$, and surjective in degrees $d \leq k(\lfloor (n-1)/2 \rfloor - e)$, where $e = 0$ in the case of Property (A) and $e = 1$ in the case of Property (B), with $k = \dim E$.

Proof of Theorem B'. Let $E' = (E, -q)$. Lemma 2.9 and Proposition 2.8 give that

$$E' \oplus E' \cong E \oplus E \quad \text{and} \quad E \oplus E' \cong \mathcal{H}^{\oplus k}.$$

Consequently, the stability map $\mathbf{G}(E^{\oplus n}) \rightarrow \mathbf{G}(E^{\oplus n} \oplus \mathcal{H}^{\oplus k})$ factors as

$$\mathbf{G}(E^{\oplus n}) \rightarrow \mathbf{G}(E^{\oplus n} \oplus E) \rightarrow \mathbf{G}(E^{\oplus n} \oplus E \oplus E') \cong \mathbf{G}(E^{\oplus n} \oplus \mathcal{H}^{\oplus k}).$$

So by Theorem 6.1 (applied k times), the first map, which is the stabilization by E , induces an injection on homology in degrees $d < g(E^{\oplus n})$. Meanwhile, so long as $n \geq 2$,

$$E^{\oplus n} \cong E^{\oplus n-2} \oplus E' \oplus E',$$

so, using now that we also have $\mathcal{H} \cong E' \oplus E$, the stability map

$$\mathbf{G}(E^{\oplus n-2} \oplus E') \rightarrow \mathbf{G}(E^{\oplus n-2} \oplus E' \oplus \mathcal{H}^{\oplus k}) = \mathbf{G}(E^{\oplus n-2} \oplus E' \oplus E' \oplus E)$$

factors as

$$\begin{array}{ccccc} \mathbf{G}(E^{\oplus n-2} \oplus E') & \longrightarrow & \mathbf{G}(E^{\oplus n-2} \oplus E' \oplus E') & \longrightarrow & \mathbf{G}(E^{\oplus n-2} \oplus E' \oplus E' \oplus E) \\ & & \cong \downarrow & & \downarrow \cong \\ & & \mathbf{G}(E^{\oplus n}) & \longrightarrow & \mathbf{G}(E^{\oplus n+1}). \end{array}$$

Hence by Theorem 6.1, the stabilization map $\mathbf{G}(E^{\oplus n}) \rightarrow \mathbf{G}(E^{\oplus n+1})$ induces a surjection on homology in degrees $d \leq g(E^{\oplus n-2} \oplus E')$.

Now we estimate the two genera mentioned above. By Proposition 7.3,

$$E^{\oplus n} \cong \mathcal{H}^{\oplus k \lfloor n/2 \rfloor} \oplus E^{\oplus e} \quad \text{or} \quad E^{\oplus n} \cong \mathcal{H}^{\oplus 2k \lfloor n/4 \rfloor} \oplus E^{\oplus e},$$

respectively in cases (A) or (B). Using Lemma 2.4 and the fact that any $x \in \mathbb{R}$ satisfies $2 \lfloor x/2 \rfloor \geq \lfloor x \rfloor - 1$, we get

$$g(E^{\oplus n}) \geq k(\lfloor n/2 \rfloor - e).$$

Similarly, for $n \geq 3$,

$$E^{\oplus n-2} \oplus E' \cong E^{\oplus n-3} \oplus \mathcal{H}^{\oplus k} \cong \mathcal{H}^{\oplus k(\lfloor (n-3)/2 \rfloor + 1 - e)} \oplus E^{\oplus e}$$

So

$$g(E^{\oplus n-2} \oplus E') \geq k(\lfloor (n-1)/2 \rfloor - e).$$

If $n = 2$, then this last conclusion is vacuously true. If $n < 2$, then all of the claims in the proposition statement are vacuous. \square

We explain now how Theorem B' gives a stability result (Theorem B) for the the unitary groups $U(n, \mathbb{F}, \sigma)$ and orthogonal groups $O(n, \mathbb{F})$ defined using the standard Euclidean form.

Definition 7.4. Set $\Lambda = \Lambda_{\min}$. The *Euclidean unitary group* is the group

$$U(n, \mathbb{F}, \sigma) = \mathbf{G}(\mathcal{E}^{\oplus n})$$

where $\mathcal{E} = (\mathbb{F}^1, q_{\mathcal{E}})$ with

$$q_{\mathcal{E}}(a, b) = \bar{a}b,$$

and the *Euclidean orthogonal group* is the special case

$$O(n, \mathbb{F}) = U(n, \mathbb{F}, \text{id}).$$

- Remark 7.5.** (1) Note that $\omega_{q_\varepsilon}(a, b) = (1 + \varepsilon)\bar{a}b$, so that q is nondegenerate as long as $\varepsilon \neq -1$. In particular this one-dimensional formed space \mathcal{E} is always nondegenerate in the orthogonal case ($\sigma = id$, $\varepsilon = 1$, $\Lambda = 0$) and in the unitary case ($\sigma \neq id$, $\varepsilon = 1$, $\Lambda = \mathbb{F}^\sigma$), provided that $\text{char}(\mathbb{F}) \neq 2$.
- (2) (Unitary groups, $\text{char}(\mathbb{F}) = 2$) In the proof of Theorem B below, we will show that the group $U(n, \mathbb{F}, \sigma)$ in characteristic 2 actually always identifies, as long as $\mathbb{F} \neq \mathbb{F}_2$, with the automorphism group of a non-degenerate formed space, with parameter $\varepsilon \neq 1$, allowing to apply our results to these groups too.
- (3) (Orthogonal groups, $\text{char}(\mathbb{F}) = 2$) The formed space \mathcal{E} in characteristic 2 is degenerate in our sense, with kernel $\mathfrak{K}(\mathcal{E}) = \mathbb{F}$. However the radical $\mathfrak{R}(\mathcal{E}) = 0$ and the formed space \mathcal{E} is defined as “non-degenerate” in for example [9]. For finite fields of even characteristic, [9, Sec 7.8] states that, in the orthogonal case, there is exactly one formed space with trivial radical in each odd dimension, namely the formed space $\mathcal{H}^{\oplus n} \oplus \mathcal{E}$. This formed space is degenerate in our sense, so does not satisfy the hypothesis of Theorem B’, but it does however fit in our first stability result, namely Theorem A’, being a genus n formed space.
- (4) (Symplectic groups) There is no “Euclidian symplectic group” since $\omega_{\mathcal{E}}$ is trivial when $\varepsilon = -1$ and $\sigma = id$, but also $Q_{\mathcal{E}}$ is trivial when $\Lambda = \mathbb{F}$.

Proof of Theorem B. If \mathbb{F} has characteristic not 2, we have seen in Remark 7.5 that the Euclidian formed space \mathcal{E} is non-degenerate in both the orthogonal and unitary case, and the result follows directly from Theorem B’ with $k = 1$.

If $\mathbb{F} \neq \mathbb{F}_2$ has characteristic 2 and $\sigma \neq id$, choose any $\alpha \in \mathbb{F} - \mathbb{F}^\sigma$ and set $\varepsilon = \bar{\alpha}/\alpha$. Then $\varepsilon\bar{\varepsilon} = 1$ and $\varepsilon \neq 1 = -1$. From [26, Lem 2.10], we have that multiplication by $\frac{1}{\alpha}$ defines a natural isomorphism $\text{Form}(E, \sigma, 1, \Lambda) \longrightarrow \text{Form}(E, \sigma, \varepsilon, \frac{1}{\alpha}\Lambda)$. Let $\mathcal{E}' = (\mathbb{F}^1, \frac{1}{\alpha}q_{\mathcal{E}})$ be the image of \mathcal{E} under this map for $E = \mathbb{F}^1$, and note that $(\mathcal{E}')^{\oplus n}$ is likewise the image of $\mathcal{E}^{\oplus n}$ with $E = \mathbb{F}^n$. By naturality, the automorphism groups $\mathbf{G}(\mathcal{E}'^{\oplus n})$ and $\mathbf{G}(\mathcal{E}^{\oplus n})$ are the same subgroup of $\text{GL}(E)$. The result then follows by applying Theorem B’ to the formed space \mathcal{E}' . \square

Another consequence of Property (B) is that the stable isometry groups do not depend on the choice of nondegenerate form used to stabilize by.

Theorem 7.6. Assume (\mathbb{F}, σ) satisfies Property (B). Let $E = (E, q)$ be a nondegenerate formed space. The stable group

$$\text{colim}_n \mathbf{G}(E^{\oplus n})$$

does not depend on the choice of E , up to group isomorphism.

Proof. By Proposition 7.3, $E^{\oplus 4} \cong \mathcal{H}^{\oplus 2 \dim E}$. Hence $\text{colim}_n \mathbf{G}(E^{\oplus n}) \cong \text{colim}_m \mathbf{G}(\mathcal{H}^{\oplus m})$, as the two colimits have cofinal subsystems which coincide. \square

Finally, we prove Corollary D. The proof does not actually directly use Theorem B’, but rather deduces the result from Theorem 6.1 using the same ideas.

Proof of Corollary D. For (1) with $n = 2m$ even, we have that $U(2m, \mathbb{F}_{p^{2r}}) \cong U_{m,m}(\mathbb{F}_{p^{2r}})$. By Corollary C, the d th homology of this group thus vanishes for $0 < d < m = \frac{n}{2}$, giving the result. For $n = 2m + 1$, we have that $U(2m + 1, \mathbb{F}_{p^{2r}}) \cong \mathbf{G}(\mathcal{E} \oplus \mathcal{H}^{\oplus m})$ as there is only one isomorphic class of non-degenerate Hermitian form in each dimension [9, II.6.7]. The result then follows from combining the vanishing theorem [9, Thm III.4.6] with Theorem 6.1, as $g(\mathcal{E} \oplus \mathcal{H}^{\oplus m}) = m = \frac{n-1}{2}$ (Lemma 2.7), given that $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$ when n is odd.

For (2), there is also only one isomorphism class of non-degenerate quadratic form over finite fields in odd dimensions [9, II.4.5] and the same argument as above gives the vanishing of $H_d(O(n, \mathbb{F}_{p^r}), \mathbb{F}_p)$ when p is odd, using the vanishing theorem [9, Thm III.4.6] (which only works in odd characteristic for orthogonal groups) and Theorem 6.1.

In even dimension $n = 2m$ there are two non-isomorphic quadratic forms [9, II.4.5], with the corresponding orthogonal groups denoted $O^+(2m, \mathbb{F}_{p^r})$ and $O^-(2m, \mathbb{F}_{p^r})$, though with only one stable group [9, Prop II.4.11]. The adding rules for isomorphism classes of quadratic forms [9, II.4.7] imply that we can always write such a quadratic form as $\mathcal{H}^{\oplus m-1} \oplus \mathcal{E} \oplus \mathcal{E}^\pm$, where the last summand is either the Euclidean form, or the form with automorphism group $O^-(1, \mathbb{F}_{p^r})$. Combining [9, Thm III.4.6] and Theorem 6.1 gives a vanishing for $0 < d < m - 1 = \frac{n-2}{2} = \lfloor \frac{n-1}{2} \rfloor$ when n is even. \square

Remark 7.7. The group $O_{m,m}(\mathbb{F}_{p^r})$ is isomorphic to $O^+(2m, \mathbb{F}_{p^r})$ or $O^-(2m, \mathbb{F}_{p^r})$ depending on whether its discriminant $(-1)^m$ is a square or not in \mathbb{F}_{p^r} (see [9, II.4.5]). Note that $H_d(O_{m,m}(\mathbb{F}_{p^r})) = 0$ when $0 < d < m$, giving an improvement by one degree on the vanishing of the corresponding group $O^+(2m, \mathbb{F}_{p^r})$ or $O^-(2m, \mathbb{F}_{p^r})$ accordingly.

APPENDIX A. FORMS AND ISOTROPIC SUBSPACES

In this appendix, we study basic properties of subspaces of formed spaces. Section A.1 is concerned with the study of orthogonal complements and Section A.2 with the construction of a canonical form on the quotient Z^\perp/Z given any isotropic subspace Z of a formed space.

A.1. Properties of orthogonal complements. In Section 2, we have associated to a formed space $E = (E, q)$ the maps

$$\omega_q : E \times E \rightarrow \mathbb{F}, \quad Q_q : E \rightarrow \mathbb{F}, \quad \text{and} \quad \flat_q = \omega_q(-, -) : E \rightarrow E^*.$$

Recall from Definition 2.2 that ‘‘orthogonality’’ is defined using ω_q , where one has to remember that ω_q is typically here *not* an inner product, so that orthogonality does not behave in the usual manner. In particular, orthogonal complements are not actual vector space complements! Recall also from Definition 2.2 the kernel $\mathfrak{K}(E) = \ker \flat_q$ and the radical $\mathfrak{R}(E) \leq \mathfrak{K}(E)$, the subspace where also Q_q vanishes.

Lemma A.1. Let (E, q) be a finite dimensional formed space, and $A \leq E$ a subspace. Then

$$\dim(A^\perp) = \dim(E) - \dim(A) + \dim(A \cap \mathfrak{K}(E)).$$

Proof. The orthogonal complement A^\perp is the kernel of the composition $E \xrightarrow{\flat_q} \sigma E^* \xrightarrow{\text{incl}^*} A^*$, which is the same as the kernel of the composition

$$E \rightarrow E/\mathfrak{K}(E) \rightarrow \sigma(E/\mathfrak{K}(E))^* \xrightarrow{\text{incl}^*} \sigma(A/A \cap \mathfrak{K}(E))^*,$$

in which all three maps are surjective. So $\dim(A^\perp) = \dim(E) - \dim(A/A \cap \mathfrak{K}(E))$. \square

Lemma A.2. Let (E, q) be a finite dimensional formed space, and $A \leq E$ a subspace. Then $(A^\perp)^\perp = A + \mathfrak{K}(E)$.

Proof. Clearly $A + \mathfrak{K}(E)$ is always contained in $(A^\perp)^\perp$. We will check that they have the same dimension. Applying Lemma A.1 to both A^\perp and A gives

$$\begin{aligned} \dim(A^{\perp\perp}) &= \dim(E) - \dim(A^\perp) + \dim(\mathfrak{K}(E)) \\ &= \dim(E) - [\dim(E) - \dim(A) + \dim(A \cap \mathfrak{K}(E))] + \dim(\mathfrak{K}(E)) = \dim(A + \mathfrak{K}(E)). \end{aligned} \quad \square$$

Lemma A.3. Let $A, B \leq E$ be two subspaces of a formed space (E, q) . Then

$$(A + B)^\perp = A^\perp \cap B^\perp.$$

If in addition $\mathfrak{K}(E) \leq A$, then

$$(A \cap B)^\perp = A^\perp + B^\perp.$$

Proof. The first part follows directly from the definitions. For the second statement, we have from Lemma A.2

$$A^\perp + B^\perp = (A^\perp + B^\perp) + \mathfrak{K}(E) = ((A^\perp + B^\perp)^\perp)^\perp.$$

On the other hand

$$(A \cap B)^\perp = ((A \cap B) + \mathfrak{K}(E))^\perp = ((A + \mathfrak{K}(E)) \cap (B + \mathfrak{K}(E)))^\perp = (A^{\perp\perp} \cap B^{\perp\perp})^\perp.$$

The result then follows from applying the first statement to the subspaces A^\perp and B^\perp . \square

Lemma A.4. Let (E, q) be a formed space and $A \leq E$ a nondegenerate subspace. Then

$$E = A \oplus A^\perp \quad \text{with} \quad \mathfrak{K}(E) = \mathfrak{K}(A^\perp) \quad \text{and} \quad \mathfrak{R}(E) = \mathfrak{R}(A^\perp).$$

Proof. If $x \in A \cap A^\perp$, then $x \in \mathfrak{K}(A)$, which is zero by assumption, so A and A^\perp intersect trivially. As also $A \cap \mathfrak{K}(E) = 0$, by Lemma A.1 we get that $\dim A^\perp = \text{codim } A$ and hence $E = A \oplus A^\perp$. Now $\mathfrak{K}(E) \leq \mathfrak{K}(A^\perp)$ since $\mathfrak{K}(E) \leq A^\perp$, but also $\mathfrak{K}(E) \geq \mathfrak{K}(A^\perp)$ since $E = A + A^\perp$. Finally, the radicals are both the kernel of Q_q on $\mathfrak{K}(E)$. \square

Finally, the following lemma is crucial in Section 3: it gives a description of the minimal elements in $\mathcal{T}(E, U)$, it is used in computing the connectivity of $\mathcal{T}(E, U)$, and finally it is used in defining splittings of stabilizer groups.

Lemma A.5. Let (E, q) be a formed space, and let $U \leq E$ be isotropic. Then there exists an isotropic subspace Z such that

$$E = U^\perp \oplus Z.$$

The space Z can be chosen to contain any isotropic subspace Z_0 with $W^\perp \cap Z_0 = 0$, and has the property that $Z \cap \mathfrak{R}(E) = 0$. Furthermore, if also $U \cap \mathfrak{R}(E) = 0$, then

- (1) $U \oplus Z$ is non-degenerate;
- (2) $E = U \oplus Z^\perp = (U \oplus Z) \oplus (U \oplus Z)^\perp$;

Proof. For the first statement, we assume without loss of generality that $\mathfrak{R}(E) \cap U = 0$: if not, replace U with a complement in U to $\mathfrak{R}(E) \cap U$, which does not change U^\perp . Let C be any linear complement to U^\perp , containing Z_0 . Consider the map

$$f : U \rightarrow {}^\sigma C^*$$

sending w to $\omega_q(-, w)$. It is injective since an element of the kernel is orthogonal to both C and U^\perp , hence lies in $\mathfrak{K}(E) \cap U = \mathfrak{R}(E) \cap U = 0$. It is surjective since, if not, there would be a nonzero element of C orthogonal to U , which does not exist. Hence f is bijective.

Let x_1, \dots, x_n be a basis for C , such that x_1, \dots, x_ℓ is a basis for Z_0 . Consider the dual basis for ${}^\sigma C^*$; via f , it induces a basis w_1, \dots, w_n for U with the property that

$$\omega_q(x_i, w_j) = \delta_{ij}.$$

For each j , let $c_j \in \mathbb{F}$ be such that $Q_q(x_j) = [c_j] \pmod{\Lambda}$ and set

$$y_j = x_j - c_j w_j - \sum_{k=1}^{j-1} \omega_q(x_k, x_j) w_k.$$

Set $Z = \langle y_1, \dots, y_n \rangle$. Since all $w_j \in U \leq U^\perp$, Z is another linear complement to U^\perp . Since Z_0 is isotropic, $y_j = x_j$ for $j \leq \ell$, so $Z_0 \leq Z$ still. Using that $\omega_q(x_i, w_j) = \delta_{ij}$ and $\omega_q(w_i, w_j) = 0$, one gets

$$\omega_q(y_i, y_j) = \omega_q(x_i, x_j) - \omega_q(x_i, \omega_q(x_i, x_j) w_i) = 0,$$

for all $i < j$, and

$$Q_q(y_j) = Q_q(x_j) + 0 - \omega_q(x_j, c_j w_j) = 0 \pmod{\Lambda}$$

for all j , where we now also use that $Q_q(w_j) = 0$. As the vanishing of $q_q(y_j)$ implies that of $\omega_q(y_j, y_j)$, it follows that Z is isotropic, as desired, proving the first part of the statement.

For the second part of the statement, we start off by counting dimensions. By Lemma A.1,

$$\dim(U^\perp) = \text{codim } U \quad \text{and} \quad \dim(Z^\perp) = \text{codim } Z$$

since neither of them intersects $\mathfrak{K}(E)$. It follows that

$$\text{codim } Z = \dim U^\perp = \text{codim } U$$

since $E = U^\perp \oplus Z$.

Now, we claim that U is also a complement to Z^\perp . If $x \in U \cap Z^\perp$, then x is orthogonal to both U^\perp and Z , hence to all of E , so $x \in \mathfrak{K}(E)$, forcing $x = 0$ as $U \cap \mathfrak{K}(E) = 0$. Then the above dimension count shows that

$$U \oplus Z^\perp = E,$$

proving the first part of statement (2). As $\mathfrak{K}(U \oplus Z)$ is contained in $Z^\perp \leq E = U \oplus Z^\perp$, and hence in Z . Similarly, $\mathfrak{K}(U \oplus Z)$ is contained in U , so the kernel is zero as $U \cap Z = 0$, proving statement (1). The second part of (1) then follows from applying Lemma A.4. \square

A.2. The formed space Z^\perp/Z for Z isotropic. Given an isotropic subspace $Z \leq E$, we will now construct a canonical form on Z^\perp/Z . We start by proving a few additional properties of orthogonal complements of isotropic subspaces.

Lemma A.6. Let (E, q) be a formed space and $Z \leq E$ an isotropic subspace. Then

$$\mathfrak{K}(Z^\perp) = Z + \mathfrak{K}(E)$$

Proof. Clearly Z and $\mathfrak{K}(E)$ are contained in $\mathfrak{K}(Z^\perp)$, so we will check the reverse inclusion. First,

$$\mathfrak{K}(Z^\perp) = Z^\perp \cap Z^{\perp\perp} = Z + \mathfrak{K}(E),$$

using Lemma A.2. Now, since ω_q vanishes on $Z + \mathfrak{K}(E)$, Q_q is additive there. But Q_q vanishes on Z . So if $z \in Z$ and $k \in \mathfrak{K}(E)$ such that $0 = Q_q(z+k) = Q_q(k)$, then $k \in \mathfrak{K}(E)$, so $z+k \in Z + \mathfrak{K}(E)$ as claimed. \square

Lemma A.7. Let $Z < E$ be a subspace. Pulling back along the quotient map $E \rightarrow E/Z$ induces a bijection between forms q on E/Z with radical R and forms \hat{q} on E with radical $R + Z$. In particular, if (E, \hat{q}) is a formed space and $Z \leq \mathfrak{K}(E)$, there is a uniquely defined form on E/Z such that the projection $E \rightarrow E/Z$ is an isometry; its radical is $\mathfrak{K}(E)/Z$.

It follows that pulling back via the quotient map $E \rightarrow E/Z$ determines a bijection between the isotropic subspaces of E/Z and the isotropic subspaces of E containing Z .

Proof. The projection map $p : E \rightarrow E/Z$ induces a map p^* from forms on E/Z to forms on E by pull-back. It is injective: if $p^*q = p^*q'$, they have the same associated ω and Q , which implies that $\omega_q = \omega_{q'}$ and $Q_q = Q_{q'}$ because p is surjective, which in turn implies $q' = q''$ by [26, Thm 2.5].

If $(E/Z, q)$ has kernel K and radical R , we have that (E, p^*q) has kernel $K + Z$ as \mathfrak{K}_{p^*q} factors as $E \xrightarrow{p} E/Z \xrightarrow{b_q} (E/Z)^* \hookrightarrow E^*$. The radical of (E, p^*q) is $R + Z$, computed similarly. Suppose that (E, \hat{q}) is a formed space with $\mathfrak{K}(E) = Z + R$ for some $R \leq E$. We need to check that there is a form q on E/Z with radical $(R + Z)/Z$ with $p^*q = \hat{q}$. Pick an arbitrary linear complement

$$Z \oplus L = E.$$

The quotient map p induces an isomorphism $L \cong E/Z$. Define q to be the form $q|_L$, seen as a form on E/Z via this isomorphism. Using that $Z \leq \mathfrak{K}(E)$, one checks that $\omega_{p^*q} = \omega_{\hat{q}}$ and $Q_{p^*q} = Q_{\hat{q}}$, which shows that $p^*q = \hat{q}$ by [26, Thm 2.5]. \square

Recall from Definition 2.2 that the genus g of a formed space is the dimension of a maximal isotropic space minus the dimension of the radical.

Proposition A.8. Let (E, q) be a formed space and $Z \leq E$ an isotropic subspace. There is a unique form on Z^\perp/Z such that the quotient map

$$\pi : Z^\perp \rightarrow Z^\perp/Z$$

is an isometry. Furthermore, its radical is the image under the quotient map of $Z + \mathfrak{R}(E)$ and its genus is

$$g(Z^\perp/Z) = g(Z^\perp) = g(E) + \dim \mathfrak{R}(E) - \dim(Z + \mathfrak{R}(E)).$$

Proof. As $Z \leq \mathfrak{R}(Z^\perp)$, we can apply Lemma A.7 to get a unique form on Z^\perp/Z induced by $(Z^\perp, q|_{Z^\perp})$, with the quotient map inducing an isometry. Using in addition Lemma A.6, we get that $\mathfrak{R}(Z^\perp) = Z + \mathfrak{R}(E)$, showing that its radical is as stated.

For the genus computation, the first equality follows as Z^\perp and Z^\perp/Z agree after modding out their radicals. Now, a maximal isotropic subspace of Z^\perp must contain $\mathfrak{R}(Z^\perp) = Z + \mathfrak{R}(E)$. It must then be a maximal isotropic subspace of E as well as it contains Z , so any larger one is necessarily contained in Z^\perp . So its dimension is $g(E) + \dim \mathfrak{R}(E)$. The second equality follows. \square

APPENDIX B. COHEN-MACAULAY POSETS

Recall that a poset is Cohen-Macaulay if every interval in it is spherical (see Definition 3.1). In this section, we show how, for a Cohen-Macaulay poset \mathcal{P} , one can use the homology of lower intervals to construct a chain complex computing the reduced homology of \mathcal{P} . Such a construction appears in Quillen's notebooks [22], and is used in for example [6, Sec 2] and [31, Sec 2], but we do not know of a reference for it including proofs.

Theorem B.1. Let \mathcal{P} be a Cohen-Macaulay poset and denote by $\mathcal{P}_r \subset \mathcal{P}$ the subset of elements of rank r . Then there is a chain complex $\tilde{C}_\bullet(\mathcal{P})$ with terms

$$\tilde{C}_r = \bigoplus_{X \in \mathcal{P}_r} \tilde{H}_{r-1}(\mathcal{P}_{<X})$$

and $\tilde{C}_{-1} = \mathbb{Z}$, and whose homology is the reduced homology $\tilde{H}_*(\mathcal{P})$.

In particular, this chain complex is exact except in degree $\dim \mathcal{P}$. Note that, as $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$, we have

$$\tilde{C}_0(\bar{\mathcal{P}}) = \mathbb{Z} \mathcal{P}_0.$$

Lemma B.2. Let $\mathcal{Q} \leq \mathcal{P}$ be posets such that $\mathcal{P} - \mathcal{Q}$ is discrete, i.e. has no nontrivial order relations. There is a homeomorphism

$$|\mathcal{P}|/|\mathcal{Q}| \simeq \bigvee_{X \in \mathcal{P} - \mathcal{Q}} \Sigma |\mathcal{P}_{<X} * \mathcal{P}_{>X}|$$

which is natural with respect to maps of such pairs, where Σ denotes the unreduced suspension.

Proof. The assumption that $\mathcal{P} - \mathcal{Q}$ is discrete gives that the simplices of $|\mathcal{P}|$ that are not contained in $|\mathcal{Q}|$ are precisely those that contain precisely one element of $\mathcal{P} - \mathcal{Q}$. Hence we have that

$$|\mathcal{P}|/|\mathcal{Q}| \simeq \left(|\mathcal{Q}| \bigcup_{X \in \mathcal{P} - \mathcal{Q}} |\mathcal{P}_{<X} * \{p\} * \mathcal{P}_{>X}| \right) / |\mathcal{Q}|.$$

Noting that, for each $X \in \mathcal{P} - \mathcal{Q}$, we have that $|\mathcal{P}_{<X} * \{X\} * \mathcal{P}_{>X}| \cap |\mathcal{Q}| = |\mathcal{P}_{<X} * \mathcal{P}_{>X}|$, we get the formula given in the statement. Naturality follows as the above decomposition is natural. \square

For all $r \geq 0$, define

$$\mathcal{P}_{\leq r} = \{X \in \mathcal{P} \mid \text{rank } X \leq r\}.$$

In particular, $\mathcal{P}_r = \mathcal{P}_{\leq r} \setminus \mathcal{P}_{\leq r-1}$. Since \mathcal{P}_r is discrete (two distinct comparable elements cannot have the same rank), we get:

Corollary B.3. Let \mathcal{P} be graded poset. Then there is a natural (with respect to rank-preserving poset maps) homotopy equivalence

$$|\mathcal{P}_{\leq r}| / |\mathcal{P}_{\leq r-1}| \rightarrow \bigvee_{X \in \mathcal{P}_r} \Sigma |\mathcal{P}_{< X}|.$$

Proof of Theorem B.1. The rank defines a finite filtration of the realization of \mathcal{P} by closed CW-inclusions. There is an associated spectral sequence beginning with

$$E_{pq}^1 = \tilde{H}_{p+q}(|\mathcal{P}_{\leq p}| / |\mathcal{P}_{\leq p-1}|)$$

and converging to $H_{p+q}(|\mathcal{P}|) = H_{p+q}(\mathcal{P})$. By the previous result,

$$\tilde{H}_j(|\bar{\mathcal{P}}_{\leq p}| / |\bar{\mathcal{P}}_{\leq p-1}|) = \bigoplus_{X \in \mathcal{P}_p} \tilde{H}_{j-1}(|\mathcal{P}_{< X}|)$$

which, by the Cohen-Macaulay property of \mathcal{P} , is zero except when $j-1 = p-1$. Hence the spectral sequence is zero except along the row $q = 0$, and so we have $E_2 = E_\infty$, and also E_∞ canonically identifies with the abutment. In other words, it is merely a chain complex. \square

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