Monoidal rectification of diagrams

Nathalie Wahl

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We work with strict monoidal categories, i.e. the associativity morphisms are given by the identity morphism. We will just say "monoidal" for simplicity. In most cases, we will denote by $\otimes$ the product in a monoidal category.

1 monoidal functors

Definition 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is* w-monoidal *if there exists a natural transformation*

$$\nu : F \otimes F \to F(\_ \otimes \_)$$

(as functors from $\mathcal{C} \otimes \mathcal{C} \to \mathcal{D}$) which is associative, i.e. the following diagram of natural transformations commutes:

$$\begin{tikzcd} F \otimes F \otimes F \arrow[r, "id\otimes\nu"] \arrow[d, "\nu\otimes id"'] & F \otimes F(\_ \otimes \_) \arrow[d, "\nu_{\_,\_\otimes\_}"] \\ F(\_ \otimes \_) \otimes F \arrow[r, "\nu"] & F(\_ \otimes \_ \otimes \_) \end{tikzcd}$$

Such a functor is called monoidal *if the natural transformation ν is given by identity morphisms. In particular, if F is monoidal, $F(x) \otimes F(y) = F(x \otimes y)$ for all objects x, y in $\mathcal{C}$.*

Remark 2. It follows from the definition that, for a monoidal functor $F : \mathcal{C} \to \mathcal{D}$, the following diagram commutes:

$$\begin{tikzcd} C(x,x') \otimes \mathcal{C}(y,y') \arrow[rr, "F"] \arrow[d] & & \mathcal{D}(Fx,Fx') \otimes \mathcal{D}(Fy,Fy') \arrow[d] \\ \mathcal{C}(x \otimes y, x' \otimes y') \arrow[r, "F"] & \mathcal{D}(F(x \otimes y), F(x' \otimes y')) \arrow[r, "="] & \mathcal{D}(Fx \otimes Fy, Fx' \otimes Fy') \end{tikzcd}$$

Let Top denote the category of topological spaces with usual product and let $\mathsf{Top}^{\mathcal{C}}_m$ denote the category of w-monoidal functors from $\mathcal{C}$ to Top.

2 Rectification of diagrams

In this section, we follow ideas of Dwyer and Kan.

Let $\mathcal{D}$ be a discrete category and let $\tilde{\mathcal{D}}$ be a category enriched over Top with the same objects as $\mathcal{D}$ and such that there is a functor (path components functor)

$$p : \tilde{\mathcal{D}} \to \mathcal{D},$$

which is the identity on objects and induces a homotopy equivalence

$$\tilde{\mathcal{D}}(x, y) \simeq \mathcal{D}(x, y)$$

for each pair of objects x, y. So $\tilde{\mathcal{D}}$ has a contractible space of morphisms over each morphism in $\mathcal{D}$ and p is the projection.

The functor p induces a functor

$$\mathsf{Top}^{\mathcal{D}} \xrightarrow{p^*} \mathsf{Top}^{\tilde{\mathcal{D}}},$$

from $\mathcal{D}$-*diagrams* to $\tilde{\mathcal{D}}$-*diagrams*. There is also a functor in the other direction:

$$\mathsf{Top}^{\mathcal{D}} \xleftarrow{p_*} \mathsf{Top}^{\tilde{\mathcal{D}}},$$

where p_*F is defined on an object x of $\mathcal{D}$ as the realization of a simplicial space whose space of n-simplices is

$$(p_*F)(x)_n = \coprod_{y_0,\ldots,y_n \in \ Ob\tilde{\mathcal{D}}} F(y_0) \times \tilde{\mathcal{D}}(y_0, y_1) \times \ldots \times \tilde{\mathcal{D}}(y_{n-1}, y_n) \times \mathcal{D}(p(y_n), x).$$

Two functors F and G are said to be *equivalent*, denoted $F \simeq G$, if there is a zig-zag of natural transformations $F \longleftarrow F_1 \longrightarrow \ldots \longleftarrow F_k \longrightarrow G$ which induces homotopy equivalences on objects.

In "Infinite loop space structure(s) on the stable mapping class groups", we prove the following:

Proposition 3. *There is an equivalence of functors*

$$p^*p_*F \simeq F$$

for any F in $\mathsf{Top}^{\tilde{\mathcal{D}}}$, which is natural in F.

As $\mathcal{D}$ and $\tilde{\mathcal{D}}$ have the same objects, the equivalence given above means in particular that $p_*F(x) \simeq F(x)$ for any object x. The functor p_*F is the *rectification* of F.

3 Monoidal rectifications

Suppose now that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are moreover monoidal, and that $p : \mathcal{D} \to \tilde{\mathcal{D}}$ is a monoidal functor. We want to show that the functors p^* and p_* defined above are also functors between the categories of w-monoidal functors:

Proposition 4. *The functors p^* and p_* restrict to functors*

$$p^* : \mathsf{Top}_m^{\mathcal{D}} \longleftrightarrow \mathsf{Top}_m^{\tilde{\mathcal{D}}} : p_*$$

Moreover, the equivalence of proposition 3 is monoidal, i.e. if $F : \tilde{\mathcal{D}} \to \mathsf{Top}$ is a w-monoidal functor, and H denotes the chain of natural transformations giving the equivalence $p^*p_*F \simeq F$, then for each x, y objects of $\tilde{\mathcal{D}}$, the following diagram commutes:

$$\begin{array}{ccc} p^*p_*F(x) \times p^*p_*F(y) & \xrightarrow{p^*p_*\nu_{x,y}} & p^*p_*F(x \otimes y) \\ {\scriptstyle H_x \times H_y}\updownarrow & & \updownarrow{\scriptstyle H_{x\otimes y}} \\ F(x) \times F(y) & \xrightarrow{\nu_{x,y}} & F(x \otimes y) \end{array}$$

Proof. If (F, ν) is a w-monoidal functor from $\mathcal{D}$ to Top, then ν induces a natural transformation $p^*\nu : p^*F(x) \times p^*F(y) \longrightarrow p^*F(x \otimes y)$ as $p^*F(x) \times p^*F(y) = F(px) \times F(py)$ and $p^*F(x \otimes y) = F(px \otimes py)$. Clearly, $p^*\nu$ has the same properies as ν. So p^*F is also a w-monoidal functor.

Now let (F, ν) is a w-monoidal functor from $\tilde{\mathcal{D}}$ to Top. We will construct maps

$$p_*\nu_{x,y} : p_*F(x) \times p_*F(y) \longrightarrow p_*F(x \otimes y)$$

inducing a natural transformation which satisfies the associativity condition of definition 1.

Note first that $(p_*F(x) \times p_*F(y))_n$ can be rewritten as
$\coprod_{(y_0,y_0'),\ldots,(y_n,y_n')\in\ Ob(\tilde{\mathcal{D}}\times\tilde{\mathcal{D}})} F(y_0) \times F(y_0') \times \tilde{\mathcal{D}}(y_0, y_1) \times \tilde{\mathcal{D}}(y_0', y_1') \times \ldots$
. $\ldots \times \tilde{\mathcal{D}}(y_{n-1}, y_n) \times \tilde{\mathcal{D}}(y_{n-1}', y_n') \times \mathcal{D}(p(y_n), x) \times \mathcal{D}(p(y_n'), y)$

This space maps to
$\coprod_{(y_0,y_0'),\ldots,(y_n,y_n')\in\ Ob(\tilde{\mathcal{D}}\times\tilde{\mathcal{D}})} F(y_0 \otimes y_0') \times \tilde{\mathcal{D}}(y_0 \otimes y_0', y_1 \otimes y_1') \times \ldots$
. $\ldots \times \tilde{\mathcal{D}}(y_{n-1} \otimes y_{n-1}', y_n \otimes y_n') \times \mathcal{D}(p(y_n) \otimes p(y_n'), x \otimes y)$

using ν and the monoidal structure of $\mathcal{D}$ and $\tilde{\mathcal{D}}$. Finally, this maps to
$\coprod_{z_0,\ldots,z_n\in\ Ob\tilde{\mathcal{D}}} F(z_0) \times \tilde{\mathcal{D}}(z_0, z_1) \times \ldots \times \tilde{\mathcal{D}}(z_{n-1}, z_n) \times \mathcal{D}(p(z_n), x \otimes y)$ which is $(p_*F(x \otimes y))_n$.

The above defines a simplicial map (using the fact that ν is a natural transformation for d_0 and using remark 2 for d_n). It is a natural transformation (using the monoidal structure of $\mathcal{D}$). The associativity follows from the associativity of F and p, and of the monoidal structure of $\mathcal{D}$ and $\tilde{\mathcal{D}}$.

Finally, to show that the diagram in the proposition commutes, one uses the fact that ν is a natural transformation as the equivalence H is obtained by evaluating morphisms of $\tilde{\mathcal{D}}$. $\square$

Example 5. *Any A_∞-algebra is equivalent, as an A_∞-algebra, to a strict monoid.*

Proof. Consider the category $\mathcal{D}$ having objects the positive integers $\mathbb{N}_0$, and morphism sets $\mathcal{D}(n,m) = \{n_1, \ldots, n_m \in \mathbb{N}_0 | \Sigma n_i = n\}$. One can think of the morphisms as the "partial multiplications". For example, the morphisms from 4 to 2 are $(x_1x_2)(x_3x_4)$,$(x_1x_2x_3)x_4$ and $x_1(x_2x_3x_4)$. Note that there is only one morphism from n to 1. This category corresponds to the PROP of the associative non-Σ operad $(Mor(A^n, A^m) = \coprod Mor(A^{n_1}, A) \times \ldots \times Mor(A^{n_m}, A))$.

Consider also the category $\tilde{\mathcal{D}}$ where these morphisms are replaces by appropriate products of Stasheff polytopes, i.e.

$$\tilde{\mathcal{D}}(n,m) = \coprod_{\{n_1,\ldots,n_m \in \mathbb{N}_0 | \Sigma n_i = n\}} K(n_1) \times \ldots \times K(n_m),$$

where $K(1), K(2)$ are points, $K(3)$ is an interval, $K(4)$ is a pentagon, and so on. The composition is induced by inclusions of products of polytopes as faces of higher dimensional ones.

Now an A_∞-algebra structure on a space X is the same as a monoidal functor $F : \tilde{\mathcal{D}} \to \mathsf{Top}$ with $F(1) = X$. In particular, F and its rectification p_*F are w-monoidal functors. We will now show that for any w-monoidal M functor from $\mathcal{D}$ to Top, $M(1)$ is a strict monoid.

Let $M : \mathcal{D} \to \mathsf{Top}$ be a w-monoidal functor, with natural transformation $\nu_{a,b} : M(a) \times M(b) \to M(a+b)$. Define the multiplication m on $M(1)$ by

$$\begin{tikzcd} M(1) \times M(1) \arrow[r, "\nu_{1,1}"] \arrow[dr, dashed, "m"] & M(2) \arrow[d, "M(\mu)"] \\ & M(1) \end{tikzcd}$$

where μ is the unique morphism in $\mathcal{D}$ from 2 to 1.

Consider the following diagram:

$$\begin{tikzcd}
 & & M(1)\times M(1)\times M(1) \arrow[dll, "\nu_{1,1}\times id"'] \arrow[drr, "id\times\nu_{1,1}"] \arrow[ddddll, dotted, bend right=0] \arrow[ddddrr, dotted] & & \\
M(2)\times M(1) \arrow[ddd, "M(\mu)\times id"'] \arrow[drr, "\nu_{2,1}"] & & & & M(1)\times M(2) \arrow[ddd, "id\times M(\mu)"] \arrow[dll, "\nu_{1,2}"'] \\
 & & M(3) \arrow[dd, shift right, "M(\mu\otimes id)"'] \arrow[dd, shift left, "M(id\otimes\mu)"] & & \\
 & & & & \\
M(1)\times M(1) \arrow[rr, "\nu_{1,1}"] \arrow[drr, dotted] & & M(2) \arrow[d, "M(\mu)"] & & M(1)\times M(1) \arrow[ll, "\nu_{1,1}"'] \arrow[dll, dotted] \\
 & & M(1) & &
\end{tikzcd}$$

The top square commutes because M is w-monoidal, and the two bottom squares commute because ν is a natural transformation. In $\mathcal{D}$, we have $\mu\circ(\mu\otimes id)=\mu\circ(id\otimes\mu)$. Hence the dotted square commute and the multiplication is associative.

Note that one can prove similarly that if N is a w-monoidal functor from $\tilde{\mathcal{D}}$ to Top, then $N(1)$ is an A_∞-space.

As p_*F is a w-monoidal functor from $\mathcal{D}$ to Top, $p_*F(1)$ is a monoid. Now $p_*F(1)=p^*p_*F(1)\simeq F(1)$. We want to show that it is an equivalence of A_∞-algebras. Considering $p_*F(1)$ as an A_∞-algebra comes to considering its lift to $\tilde{\mathcal{D}}$ by p^*. The result follows from the commutation of the following diagram:

$$\begin{tikzcd}
K(n)\times F(1)^n \arrow[r, "\nu", "="'] \arrow[d, leftrightarrow, "\simeq"] & K(n)\times F(n) \arrow[r] \arrow[d, leftrightarrow, "\simeq"] & F(1) \arrow[d, leftrightarrow, "\simeq"] \\
K(n)\times p^*p_*F(1)^n \arrow[r, "p^*p_*\nu"] & K(n)\times p^*p_*F(n) \arrow[r] & p^*p_*F(1)
\end{tikzcd}$$

The left square commutes by proposition 4 and the right one because the equivalence is given by natural transformations. □

Remark 6. The above can be done with any non-sigma operad, i.e. an algebra over a "larger" version of the operad can always be strictified.

Similarly, for a non-sigma operad P, one can strictify other sorts of "P-algebra up to homotopy" (for example, one can consider algebras such that the associativity diagrams commute only up to high homotopy). This can be done by constructing the appropriate "larger" version of the PROP category associated to P.

Remark 7. Proposition 4 cannot be extended in a straightforward way to the symmetric monoidal case. The problem comes from the last step in the construction of the maps $p_*\nu_{x,y}$. An object z_i of $\mathcal{D}$ cannot be uniquely seen as a product so we "would not know what permutation to do". It looks like if one should do some form of transfer, summing on all the possibilities, but I guess that wouldn't be possible/make sense in general.