# Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity

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**Abstract**

This paper is mainly a survey of published results.

We recall the definition of positive definite and (conditionally) negative definite functions on abelian semigroups with involution, and we consider three main examples: $\mathbb{R}^k, [0,\infty[^k, \mathbb{N}_0$–the first with the inverse involution and the two others with the identical involution.

Schoenberg's theorem explains the possibility of constructing rotation invariant positive definite and conditionally negative definite functions on euclidean spaces via completely monotonic functions and Bernstein functions.

It is therefore important to be able to decide complete monotonicity of a given function.

We combine complete monotonicity with complex analysis via the relation to Stieltjes functions and Pick functions and we give a survey of the many interesting relations between these classes of functions and completely monotonic functions, logarithmically completely monotonic functions and Bernstein functions. In Section 6 it is proved that $\log x - \Psi(x)$ and $\Psi'(x)$ are logarithmically completely monotonic (where $\Psi(x) = \Gamma'(x)/\Gamma(x)$), and these results are new as far as we know. We end with a list of completely monotonic functions related to the Gamma function.

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## 1 Positive definite and completely monotonic functions

Positive definite functions on groups or semigroups have a long history and have many applications in probability theory, operator theory, potential theory, moments problems and several other areas. They enter as an important chapter in all treatments of harmonic analysis. They can be traced back to papers by Carathéodory, Herglotz, Bernstein, Matthias, culminating in Bochner's theorem from 1932-1933. See [37] for details.

Shortly after Bochner's characterization of continuous positive definite functions on $\mathbb{R}$ or $\mathbb{R}^k$, his theorem was extended to the framework of locally compact abelian groups: *The continuous positive definite functions on a LCA-group are the Fourier transforms of positive finite Radon measures on the dual group.*

Completely monotonic functions were characterized by Bernstein in 1928 as Laplace transforms of positive measures on $[0, \infty[$, see [17].

Stieltjes moment sequences, i.e. moment sequences of positive measures on $[0, \infty[$ were introduced and characterized by Stieltjes in 1894 in the pioneering work [45], where he introduced the Stieltjes integral with respect to a distribution of mass. His results were extended to a characterization of moment sequences of measures on the real line by Hamburger in 1920-21. However all the results mentioned so far are strongly interrelated and they all depend on measure theory based on Stieltjes' and Lebesgue's ideas.

Positive definite functions, completely monotonic functions and moment sequences share many properties. For example, each of these functions or sequences form a set which is stable under sums, products and multiplication with non-negative constants, and it is also closed in a suitable topology. This is explained in harmonic analysis: They are the transforms of positive measures on certain sets with a commutative operation leading to a convolution of measures. The convolution of two measures is transformed into the product of the transforms of each of the measures. This classical point of view is elaborated in [13].

Let us consider an abelian semigroup $(S, +)$, i.e. the composition $+$ is commutative and associative, and let us assume that $S$ has a neutral element 0 and an involution $*$, i.e. $* : S \to S$ is a bijection satisfying $(s+t)^* = s^* + t^*$ and $(s^*)^* = s$. We recall that a function $f : S \to \mathbb{C}$ is called *positive definite* if for all $n \in \mathbb{N}$ and for all $s_1, \ldots, s_n \in S$ the $n \times n$-matrix $(f(s_i + s_j^*))$ is positive hermitian, i.e.

$$\sum_{i,j=1}^{n} f(s_i + s_j^*) c_i \overline{c_j} \geq 0, \quad \forall\, (c_1, \ldots, c_n) \in \mathbb{C}^n. \tag{1}$$

In the case of $S = \mathbb{R}^k$ considered as an abelian group with addition, we choose the involution $x^* = -x$ and can formulate the famous theorem of Bochner:

**Theorem 1.1** *A function $f : \mathbb{R}^k \to \mathbb{C}$ is positive definite and continuous if and only if it is the Fourier transform of a positive finite measure $\mu$ on $\mathbb{R}^k$, i.e. if and only if*

$$f(x) = \int_{\mathbb{R}^k} e^{-i\langle x, \xi \rangle}\, d\mu(\xi).$$

In the case of $S = [0,\infty[^k$ considered as an abelian semigroup with the identical involution $x^* = x$, we have the following result of Bernstein and Widder (cf. [47, Ch. VI, Th. 21] and [13, Th. 6.5.8].):

**Theorem 1.2** *A function $f : [0,\infty[^k \to \mathbb{R}$ is positive definite and continuous if and only if it is the Laplace transform of a positive finite measure $\mu$ on $\mathbb{R}^k$, i.e. if and only if*

$$f(x) = \int_{\mathbb{R}^k} e^{-\langle x,\xi\rangle}\, d\mu(\xi). \tag{2}$$

Note that in the group case (Theorem 1.1) a positive definite function is automatically bounded, whereas in the semigroup case (Theorem 1.2), this is certainly not true, since $\exp(-\langle x,\xi\rangle)$ is unbounded if $\xi \notin [0,\infty[^k$.

There is another important remark concerning Theorem 1.2. A function $f$ satisfying the conditions of the theorem has a holomorphic extension to the tube

$$\mathbb{C}^k_+ = ]0,\infty[^k + i\mathbb{R}^k$$

given by

$$f(x+iy) = \int_{[0,\infty[^k} e^{-\langle x+iy,\xi\rangle}\, d\mu(\xi).$$

For bounded functions Theorem 1.2 takes the form:

**Corollary 1.3** *A function $f : [0,\infty[^k \to \mathbb{R}$ is positive definite, bounded and continuous if and only if it is the Laplace transform of a positive finite measure $\mu$ on $[0,\infty[^k$, i.e. if and only if*

$$f(x) = \int_{[0,\infty[^k} e^{-\langle x,\xi\rangle}\, d\mu(\xi). \tag{3}$$

**Proof** A function given by (3) is clearly bounded. Conversely, if $f$ is a bounded, continuous and positive definite function, it has a representation (2), and we have to show that $\mathrm{supp}(\mu) \subseteq [0,\infty[^k$. In fact, if there exists $\xi_0 \in \mathrm{supp}(\mu)$ with a negative $j$'th coordinate, we can choose $\varepsilon > 0$ such that $U := \{\xi \in \mathbb{R}^k \mid \xi_j < -\varepsilon\}$ is an open neighbourhood of $\xi_0$. Therefore, with $e_j$ denoting the usual $j$'th standard basis vector in $\mathbb{R}^k$ and for $x_j \geq 0$ we get

$$f(x_j e_j) = \int_{\mathbb{R}^k} e^{-x_j\xi_j}\, d\mu(\xi) \geq \int_U e^{-x_j\xi_j}\, d\mu(\xi) \geq e^{\varepsilon x_j}\mu(U),$$

which contradicts the boundedness since $\mu(U) > 0$. $\square$

Finally, if we consider $S = \mathbb{N}_0 = \{0,1,2,\ldots\}$ as an abelian semigroup under addition and the identical involution $n^* = n$, we can formulate Hamburger's theorem:

**Theorem 1.4** *A sequence $s : \mathbb{N}_0 \to \mathbb{R}$ is positive definite if and only if it is a moment sequence of a positive finite measure $\mu$ on $\mathbb{R}$, i.e.*

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \in \mathbb{N}_0. \tag{4}$$

Note that a moment sequence $(s_n)$ is bounded if and only if the corresponding measure $\mu$ is supported by $[-1,1]$. While the measure $\mu$ in Theorem 1.1 and 1.2 is uniquely determined by $f$, this need not be the case in Theorem 1.4: There exist so-called *indeterminate* moment sequences $(s_n)$, where (4) has more than one and hence infinitely many solutions $\mu$. However, if there is one solution $\mu$ with compact support, then (4) has no other solutions.

There is another way of characterizing the functions of Corollary 1.3, namely as completely monotonic functions.

For simplicity we shall assume $k = 1$.

A function $f : ]0,\infty[ \to \mathbb{R}$ is called *completely monotonic* if it is $C^\infty$ and

$$(-1)^n f^{(n)}(x) \geq 0, \quad n \geq 0, \ x > 0.$$

The set of completely monotonic functions is denoted $\mathcal{C}$. It is a convex cone and stable under multiplication as is seen directly from the definition. It is less obvious that $\mathcal{C}$ is closed under pointwise convergence, but follows from the famous theorem of Bernstein proved in [17]:

**Theorem 1.5** *A function $f : ]0,\infty[ \to \mathbb{R}$ is completely monotonic if and only if it is the Laplace transform of a positive measure $\mu$ on $[0,\infty[$, i.e.*

$$f(x) = \mathbb{L}\mu(x) = \int_0^{\infty} e^{-x\xi} \, d\mu(\xi), \quad x > 0. \tag{5}$$

The measure $\mu$ in Theorem 1.5 is not necessarily finite. Since $f \in \mathcal{C}$ is decreasing, $\lim_{x\to 0^+} f(x) = f(0+)$ exists and equals $\mu([0,\infty[)$.

For rotation invariant functions $f : \mathbb{R}^k \to \mathbb{R}$, i.e. functions of the form $f(x) = \varphi(||x||^2)$ for some function $\varphi : [0,\infty[\to \mathbb{R}$ and $||\cdot||$ denoting the euclidean norm, Schoenberg proved the following important result, cf. [41] (I.J. Schoenberg (1903-1990), Romanian/American):

**Theorem 1.6** *Let $\varphi : [0,\infty[\to \mathbb{R}$ be given. Then $\varphi(||x||^2)$ is continuous and positive definite on $\mathbb{R}^k$ for all $k \geq 1$ if and only if $\varphi$ is the Laplace transform of a positive finite measure on $[0,\infty[$.*

Schoenberg's Theorem is an important tool for constructing rotation invariant positive definite functions on euclidean spaces. It leads to the question of deciding if a function on the half-line is completely monotonic.

There have appeared many proofs of Schoenberg's Theorem. We mention the abstract approach via Schoenberg triples in the monograph [13], which also works for non-continuous positive definite functions. In [43] there is a beautiful analytical proof based on Scheffé's Lemma.

## 2 Conditionally negative definite functions

Conditionally negative definite kernels and functions play an important role in probability theory, Dirichlet spaces, potential theory, and in relation with isometric embeddings of metric spaces into Hilbert spaces, see Schoenberg [42] and the treatment in [13].

Let $(S, +, *)$ be an abelian semigroup with neutral element and involution $*$ like in Section 1.

**Definition 2.1** A function $\psi : S \to \mathbb{C}$ is called *(conditionally) negative definite* if $\psi$ has the following properties:

(i) $\psi$ is *hermitian*, i.e. $\psi(s^*) = \overline{\psi(s)}$, $s \in S$,

(ii) For all $n \geq 2$ and for all $s_1, \ldots, s_n \in S$

$$\sum_{i,j=1}^{n} \psi(s_i + s_j^*) c_i \overline{c_j} \leq 0, \quad \forall\, (c_1, \ldots, c_n) \in \mathbb{C}^n \text{ verifying } \sum_{j=1}^{n} c_j = 0. \tag{6}$$

Inspired by the work of Beurling and Deny about Dirichlet spaces, we called these functions *negative definite* in [14] and [13]. It is reasonable to include "conditionally" in the terminology because of the classical different concept of a negative definite matrix, but it is longer to write.

We note the following easy result going back to Schoenberg, cf. [13, Th. 3.2.2]:

**Theorem 2.2** *A function $\psi : S \to \mathbb{C}$ is conditionally negative definite if and only if* $\exp(-t\psi)$ *is positive definite for all $t > 0$.*

In the important case of $S = \mathbb{R}^k$ with the involution $x^* = -x$, the counterpart of Bochner's theorem is the Lévy-Khinchin formula. (Note that in the group case conditionally negative definite functions automatically satisfy $\Re\psi(s) \geq \psi(0)$ by Theorem 2.2 and the fact that positive definite functions are bounded):

**Theorem 2.3** *A function $\psi : \mathbb{R}^k \to \mathbb{C}$ is conditionally negative definite and continuous if and only if it has an integral representation of the form*

$$\psi(x) = a + i\langle b, x\rangle + Q(x) + \int_{\mathbb{R}^k \setminus \{0\}} \left(1 - e^{-i\langle x, \xi\rangle} - \frac{i\langle x, \xi\rangle}{1 + ||\xi||^2}\right) d\nu(\xi), \tag{7}$$

*where $a \in \mathbb{R}, b \in \mathbb{R}^k$, $Q$ is a positive semidefinite quadratic form on $\mathbb{R}^k$ and $\nu$ (the Lévy-measure) is a positive measure on $\mathbb{R}^k \setminus \{0\}$ satisfying $\int ||\xi||^2/(1 + ||\xi||^2)\, d\nu(\xi) < \infty$.*

**Remark 2.4** For a continuous positive definite function $f : \mathbb{R}^k \to \mathbb{C}$ and $c \in \mathbb{R}$ we see that $\psi(x) = c - f(x)$ is a bounded, continuous, conditionally negative definite function. Conversely, any bounded, continuous, conditionally negative definite function on $\mathbb{R}^k$ has this form. This was proved (even for LCA-groups) by Harzallah [27], cf. [14, Prop. 7.13].

In the case of $S = [0, \infty[^k$ considered as an abelian semigroup with the identical involution $x^* = x$, we have the following result, which is the counterpart of Theorem 1.2 (noting that hermitian functions are real-valued):

**Theorem 2.5** *A function $\psi : [0, \infty[^k \to \mathbb{R}$ is conditionally negative definite and continuous if and only if it has a representation of the form*

$$\psi(x) = a + \langle b, x\rangle - Q(x) + \int_{\mathbb{R}^k \setminus \{0\}} \left(1 - e^{\langle x, \xi\rangle} + \frac{\langle x, \xi\rangle}{1 + ||\xi||^2}\right) d\nu(\xi), \tag{8}$$

*where $a \in \mathbb{R}, b \in \mathbb{R}^k$, $Q$ is a positive semidefinite quadratic form on $\mathbb{R}^k$ and $\nu$ (the Lévy-measure) is a positive measure on $\mathbb{R}^k \setminus \{0\}$ satisfying*

$$\int_{0 < ||\xi|| \le 1} ||\xi||^2 \, d\nu(\xi) < \infty, \quad \int_{||\xi|| > 1} e^{\langle x, \xi\rangle} \, d\nu(\xi) < \infty, \quad \forall x \in [0, \infty[^k.$$

A complete proof of the above theorem can be found in [8].

In the case of $k = 1$ and nonnegative functions, Theorem 2.5 reduces to the following result, see [13, Th. 4.4.3]:

**Theorem 2.6** *A function $\psi : [0, \infty[ \to [0, \infty[$ is conditionally negative definite and continuous if and only if it has a representation of the form*

$$\psi(x) = a + bx + \int_0^\infty \left(1 - e^{-x\xi}\right) d\nu(\xi), \tag{9}$$

*where $a, b \ge 0$ and $\nu$ (the Lévy measure) is a positive measure on $]0, \infty[$ satisfying $\int \xi/(\xi + 1) \, d\nu(\xi) < \infty$.*

These functions, which are the counterparts of completely monotonic functions, are called Bernstein functions. They are considered in detail in Section 5.

Finally, if we consider $S = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ as an abelian semigroup under addition and the identical involution $n^* = n$, we can formulate the counterpart of Hamburger's theorem:

**Theorem 2.7** *A function $\psi : \mathbb{N}_0 \to \mathbb{R}$ is conditionally negative definite if and only if it has an integral representation of the form*

$$\psi(n) = a + bn - cn^2 + \int_{\mathbb{R} \setminus \{1\}} (1 - x^n - n(1 - x)) \, d\nu(x), \tag{10}$$

*where $a, b \in \mathbb{R}, c \geq 0$ and $\nu$ (the Lévy measure) is a positive measure on $\mathbb{R} \setminus \{1\}$ satisfying*

$$\int_{0<|x-1|<1} (1-x)^2 \, d\nu(x) < \infty, \quad \int_{|x-1|\geq 1} |x|^n \, d\nu(x)\infty, \quad \forall n \in \mathbb{N}_0.$$

For the proof of Theorem 2.7 we refer to [13].

Theorem 2.7 is related to infinitely divisible moment sequences in the sense of Tyan [46]. He defined a moment sequence $(s_n)$ to be *infinitely divisible* if

(i) $s_n \geq 0$ for all $n \geq 0$,

(ii) $(s_n^c)$ is a moment sequence for all $c > 0$.

The main result of Tyan can be formulated:

**Theorem 2.8** *A moment sequence $(s_n)$ such that $s_n > 0$ for all $n$ is infinitely divisible if and only if $(-\log s_n)$ is conditionally negative definite on $(\mathbb{N}_0, +)$.*

Tyan's result was reexamined in the case of Stieltjes moment sequences in the recent paper [11], which contains several examples of infinitely divisible Stieltjes moment sequences.

Coming back to rotation invariant functions on $\mathbb{R}^k$ we have the following counterpart of Theorem 1.6. It follows by combination of the previous results.

**Theorem 2.9** *Let $\psi : [0, \infty[ \to [0, \infty[$ be given. Then $\psi(||x||^2)$ is continuous and conditionally negative definite on $\mathbb{R}^k$ for all $k \geq 1$ if and only if $\psi$ is a Bernstein function, i.e. has the form (9).*

# 3 Stieltjes functions

It can be quite difficult to prove that a certain function is completely monotonic, simply because the expression for the $n$'th derivative can be too complicated to handle.

We shall now consider an important subclass of completely monotonic functions:

**Definition 3.1** A function $f : ]0, \infty[ \to \mathbb{R}$ is called a *Stieltjes function,* if it is of the form

$$f(x) = a + \int_0^\infty \frac{d\mu(\xi)}{x+\xi} \tag{11}$$

where $a \geq 0$, and $\mu$ is a positive measure on $[0, \infty[$. (The condition on $\mu$ imposed by (11) can be summarized in $\int 1/(1+\xi) \, d\mu(\xi) < \infty$.)

The set $\mathcal{S}$ of Stieltjes functions is a convex cone such that $\mathcal{S} \subset \mathcal{C}$. The inclusion holds simply because

$$f^{(n)}(x) = (-1)^n n! \int_0^\infty \frac{d\mu(\xi)}{(x+\xi)^{n+1}}, \quad x > 0,\, n \geq 1.$$

(T. J. Stieltjes (1856-1894), Dutch/French).

It is, however, also important to realize that the function (11) can be written

$$f(x) = \mathbb{L}[a\delta_0 + \mathbb{L}\mu(\xi)1_{]0,\infty[}(\xi)\, d\xi](x), \tag{12}$$

(where $\delta_a$ here and in the following denotes the Dirac measure with mass 1 concentrated in the point $a$), i.e. $f$ is grosso modo an iterated Laplace transform. This also shows that $f \in \mathcal{S}$ is completely monotonic.

The cone $\mathcal{S}$ is closed under pointwise convergence. For the proof of this and applications of Stieltjes functions see [28].

Formula (11) also shows that a Stieltjes function has a holomorphic extension to the cut plane $\mathbb{A} := \mathbb{C} \setminus ]-\infty, 0]$.

The formula

$$\frac{1}{x(1+x^2)} = \int_0^\infty e^{-x\xi}(1-\cos(\xi))\, d\xi$$

shows that $1/x(1+x^2)$ is completely monotonic, but it cannot be a Stieltjes function, since it has poles at $\pm i$.

One can also remark that $x^{-\alpha} \in \mathcal{C}$ for all $\alpha \geq 0$, but $x^{-\alpha} \in \mathcal{S}$ if and only if $0 \leq \alpha \leq 1$.

(For $0 < \alpha < 1$ we have

$$\int_0^\infty \frac{d\xi}{(x+\xi)\xi^\alpha} = x^{-\alpha} \int_0^\infty \frac{dt}{(1+t)t^\alpha},$$

showing that $x^{-\alpha} \in \mathcal{S}$. That $x^{-\alpha} \notin \mathcal{S}$ for $\alpha > 1$ follows from Theorem 3.2 below since $\sin(\alpha \operatorname{Arg} z) < 0$ for some $z$ in the upper half-plane. See Example (c) in Section 4.)

The holomorphic extension of $f \in \mathcal{S}$ to $z = x + iy \in \mathbb{A}$ is given as

$$\Im f(x+iy) = \int_0^\infty \frac{-iy\, d\mu(\xi)}{(x+\xi)^2 + y^2}, \tag{13}$$

which shows the "only-if" part of the following characterization of Stieltjes functions:

**Theorem 3.2** *A function $f : ]0,\infty[ \to \mathbb{R}$ is a Stieltjes function if and only if $f(x) \geq 0$ for $x > 0$ and it has a holomorphic extension (also denoted $f$) to the cut plane $\mathbb{A}$ satisfying $\Im f(x+iy) \leq 0$ for $y > 0$.*

The holomorphic extension of a Stieltjes function has of course a non-negative imaginary part in the lower half-plane. On the other hand, a holomorphic function $f : \mathbb{A} \to \mathbb{C}$, which is real on the positive axis, must satisfy $f(\overline{z}) = \overline{f(z)}$ by the reflection principle.

Theorem 3.2 is proved in [1, p. 127] and there attributed to Krein. It is very much connected to the theory of Pick functions, which will be discussed in the next section.

We shall now list some useful stability properties of the cone of Stieltjes functions:

(i) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f(1/x)} \in \mathcal{S}$

(ii) $f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{xf(x)} \in \mathcal{S}$

(iii) $f \in \mathcal{S},\ \lambda > 0 \Rightarrow \frac{f}{\lambda f + 1} \in \mathcal{S}$

(iv) $f, g \in \mathcal{S} \setminus \{0\} \Rightarrow f \circ \frac{1}{g}, \frac{1}{f \circ g} \in \mathcal{S}$

(v) $f, g \in \mathcal{S},\ 0 < \alpha < 1 \Rightarrow f^\alpha g^{1-\alpha} \in \mathcal{S}$.

(vi) $f \in \mathcal{S},\ 0 < \alpha < 1 \Rightarrow f^\alpha \in \mathcal{S}$.

The properties (i),(iii) and (iv) were proved by Hirsch in [28]. Property (ii) was established by Reuter [36] and independently by Itô [31], but already Stieltjes noticed it in a letter to Hermite, see [5, Lettre 426]. The property (v), showing that the cone $\mathcal{S}$ is logarithmically convex, was established in [6]. Property (vi) is a special case of (v) with $g = 1$. All the above properties of $\mathcal{S}$ can easily be deduced from Theorem 3.2 as was pointed out in [7].

**Examples of Stieltjes functions**

(a) $x^{-\alpha}, 0 < \alpha \leq 1$.

(b) $\log((x+b)/(x+a)) = \int_a^b (x+\xi)^{-1}\, d\xi, \quad 0 \leq a < b$.

(c) $1/\log(1+x); \quad (\log(1+x))/x$ (Apply (i) above to example (b) with $a = 0, b = 1$ and apply then (ii)).

(d) $(x \log x)/\log\Gamma(x+1)$, cf. [15].

(e) $(1+1/x)^{-x}$, cf. [2].

(f) See [2]

$$(x+1)\left[e - \left(1 + \frac{1}{x}\right)^x\right] = \frac{e}{2} + \frac{1}{\pi}\int_0^1 \frac{\xi^\xi (1-\xi)^{1-\xi}\sin(\pi\xi)}{x+\xi}\, d\xi.$$

(g) $[\Gamma(1+1/x)]^x$, cf. [3].

(h) See [9]

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1 + \frac{1}{x}\right)^x; \quad \log\Phi.$$

# 4 Pick functions

Pick functions have a long history and a whole book is dedicated to them, see [22].

The upper half-plane is denoted $\mathbb{H}$, i.e. $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$, and the set of holomorphic functions in a domain $G$ is denoted $\mathcal{H}(G)$.

**Definition 4.1** A function $f \in \mathcal{H}(\mathbb{H})$ is called a *Pick function* if $\Im f(z) \geq 0$ for $z \in \mathbb{H}$.

The set of Pick functions is a convex cone $\mathcal{N} \subseteq \mathcal{H}(\mathbb{H})$. (G. Pick (1859-1942), Austrian/Czech).

Pick functions are also called *Nevanlinna functions*. This is reflected in the notation $\mathcal{N}$. (R. Nevanlinna (1895-1980), Finnish).

Pick and Nevanlinna studied interpolation problems for Pick functions, see [35],[33] and Nevanlinna related this class to the indeterminate moment problem, see [33].

Pick functions are closely related to operator monotone functions studied by Loewner, see [22]. We return to this in Theorem 4.10.

**Examples of Pick functions**

(a) $f(z) = \alpha z + \beta\,,\ \alpha \geq 0\,,\ \beta \in \mathbb{R}$.

(b) Any Möbius transformation

$$f(z) = \frac{az+b}{cz+d}\,, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

In fact, $\Im f(x+iy) = y/|cz+d|^2$.

In particular $1/(c-z) \in \mathcal{N}$ for $c \in \mathbb{R}$.

(c) The principal logarithm, i.e. $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$, since the principal argument $\operatorname{Arg} z \in ]0, \pi[$ for $z \in \mathbb{H}$.

(d) $z^\alpha = e^{\alpha \operatorname{Log} z}$ for $0 < \alpha \leq 1$.

(e) $\tan z$, since

$$\Im \tan(x+iy) = \frac{\sinh y \cosh y}{\cos^2 x + \sinh^2 y}.$$

(f) See [16]

$$\frac{\log \Gamma(x+1)}{x \log x}; \quad \frac{\log \Gamma(x+1)}{x}.$$

(g) See [34] for Pick functions related to canonical products with negative zeros.

In 1911 G. Herglotz gave an integral representation of the holomorphic functions defined in the unit disc and having nonnegative real part. Via a conformal mapping of the unit disc onto the upper half-plane, Herglotz' representation is equivalent with the following integral representation of Pick functions.

**Theorem 4.2** *The formula*

$$f(z) = \alpha z + \beta + \int_{-\infty}^{\infty} \frac{tz+1}{t-z} d\tau(t)\ , \quad z \in \mathbb{H} \tag{14}$$

*establishes a one-to-one correspondence between Pick functions $f$ and triples $(\alpha, \beta, \tau)$, where $\alpha \geq 0$, $\beta \in \mathbb{R}$ and $\tau$ is a positive finite measure on $\mathbb{R}$.*

**Remark 4.3** Using the identity

$$(1+t^2)\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) = \frac{tz+1}{t-z},$$

the representation (14) can be written

$$f(z) = \alpha z + \beta + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t), \tag{15}$$

where the positive measure $\sigma$ on $\mathbb{R}$ satisfies $\int (1+t^2)^{-1} d\sigma(t) < \infty$.

The connection between the measures $\tau$ and $\sigma$ is $\sigma = (1+t^2)d\tau(t)$.

It is important to notice that integration term by term in (15) is not permitted. The difference is $\sigma$-integrable but $t(1+t^2)^{-1}$ need not be $\sigma$-integrable as is shown by Example 4.5 below.

We shall next express the triple $(\alpha, \beta, \sigma)$ from (15) in terms of the Pick function $f$.

**Theorem 4.4** *For a Pick function $f$ with representation (15) we have*

$$\alpha = \lim_{y\to\infty} \frac{f(iy)}{iy}\ , \beta = \Re f(i),\ \sigma = \lim_{y\to 0^+} \frac{1}{\pi} \Im f(x+iy) dx \ \textit{vaguely},$$

*i.e.*

$$\lim_{y\to 0^+} \frac{1}{\pi} \int \Im f(x+iy) \varphi(x)\, dx = \int \varphi(x)\, d\sigma(x)$$

*for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support.*

**Example 4.5** For the Pick function $f(z) = \operatorname{Log} z$ we have

$$\lim_{y\to 0^+} \frac{1}{\pi} \operatorname{Arg}(x+iy) = \begin{cases} 0 & \text{for } \ x > 0 \\ \frac{1}{2} & \text{for } \ x = 0 \\ 1 & \text{for } \ x < 0 \end{cases},$$

and the convergence is uniform on compact subsets of $]0,\infty[$ and of $]-\infty,0[$. By Theorem 4.4 we see that $\sigma$ is 0 on $[0,\infty[$ and equal to Lebesgue measure on $]-\infty,0[$. In fact, for any continuous function $\varphi$ with compact support we have

$$\lim_{y\to 0^+}\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi(x)\,\mathrm{Arg}(x+iy)dx=\int_{-\infty}^{0}\varphi(x)dx$$

by the dominated convergence theorem. It follows that the representation (15) is

$$\mathrm{Log}\,z=\int_{-\infty}^{0}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)dt. \tag{16}$$

The imaginary part of a Pick function $f$ is a non-negative harmonic function in the upper half-plane. It therefore either vanishes identically and $f$ is a real constant, or it never vanishes.

**Corollary 4.6** *Let $f\in\mathcal{N}$. Then either $f$ is a real constant or $f(\mathbb{H})\subseteq\mathbb{H}$. In particular $f\in\mathcal{N}\setminus\{0\}$ has no zeros in $\mathbb{H}$ and $-1/f\in\mathcal{N}$.*

The Pick functions $f$ such that $f(\mathbb{H})\subseteq\mathbb{H}$ are called *non-degenerate*, and we notice the following result.

**Proposition 4.7** *Let $f_1,f_2$ be non-degenerate Pick functions. Then $f_1\circ f_2$ is a non-degenerate Pick function.*

For $f\in\mathcal{H}(\mathbb{H})$ the function $z\mapsto\overline{f(\overline{z})}$ is holomorphic in the lower half-plane $\Im z<0$, and it is called the *reflection* of $f$ in the real axis. The reflection of a Pick function $f$ is given in the lower half-plane by the expression (14) or (15). Whenever it is convenient, we shall therefore consider Pick functions as holomorphic functions in $\mathbb{C}\setminus\mathbb{R}$. Given $f\in\mathcal{N}$ it is natural to ask, if there is an analytic continuation of $f$ across some part of the real axis. This is formalized in the following:

**Definition 4.8** For a closed set $A\subseteq\mathbb{R}$ we denote by $\mathcal{N}_A$ the set of $f\in\mathcal{N}$ for which there exists $F\in\mathcal{H}(\mathbb{C}\setminus A)$ such that

$$F(z)=\begin{cases} f(z), & \Im z>0\\ \\ \overline{f(\overline{z})}, & \Im z<0.\end{cases}$$

For $f\in\mathcal{N}_A$ the extension $F$ is necessarily unique and is given for $x\in\mathbb{R}\setminus A$ by

$$F(x)=\lim_{y\to 0^+}f(x+iy)=\lim_{y\to 0^+}\overline{f(x+iy)},$$

and the convergence is uniform on compact subsets of $\mathbb{R}\setminus A$. It follows that $F(x)$ is real for $x\in\mathbb{R}\setminus A$, so the representing measure $\sigma$ from (15) is zero on $\mathbb{R}\setminus A$. This shows the non-trivial part of the following characterization of $\mathcal{N}_A$.

**Theorem 4.9** *A Pick function $f$ with representation (15) belongs to $\mathcal{N}_A$ if and only if* $\mathrm{supp}(\sigma) \subseteq A$, *and in the affirmative case the right-hand side of (15) defines the holomorphic extension to* $\mathbb{C} \setminus A$.

Note that $\mathrm{Log} \in \mathcal{N}_{]-\infty,0]}$.

Let $f$ be a Stieltjes function and assume that $f \neq 0$. Then clearly $-f$, $f(-x)$ and $1/f$ are Pick functions.

Given an open interval $I = ]a,b[$ a function $f : I \to \mathbb{R}$ is called *operator monotone* if $f(A) \leq f(B)$ for all self-adjoint operators $A, B$ in a Hilbert space satisfying $A \leq B$ and having spectrum contained in $I$. Loewner proved that for a function to be operator monotone it is enough to consider $n \times n$-matrices $A, B$ of arbitrary size $n$. In the matrix case $f(A)$ is defined in an elementary way using diagonalization. In the general case $f(A)$ is defined via the functional calculus. Loewner further proved, that the set of operator monotone functions is exactly $\mathcal{N}_{\mathbb{R}\setminus I}$.

We shall now give a characterization of the non-negative operator monotone functions on the positive half-line, thereby bringing the Stieltjes functions in focus.

**Theorem 4.10** *For a function $f : ]0,\infty[ \to [0,\infty[$ the following conditions are equivalent:*

*(i)* $f$ *can be extended to a function in* $\mathcal{N}_{]-\infty,0]}$.

*(ii)* $f$ *has a representation*

$$f(x) = \int_0^\infty \frac{x(1+t)}{x+t}\, d\mu(t),$$

*where $\mu$ is a positive finite measure on* $[0,\infty]$.

*(iii)* $f$ *has the representation*

$$f(x) = \alpha x + x \int_0^\infty \frac{d\sigma(t)}{x+t},$$

*where $\alpha \geq 0$ and $\sigma$ is a positive measure on $[0,\infty[$ satisfying*

$$\int (1+t)^{-1}\, d\sigma(t) < \infty.$$

*(iv)* $f(x)/x$ *is a Stieltjes function.*

**Proof** The equivalence of (ii) and (iii) follows by putting $\alpha = \mu(\{\infty\})$ and $d\sigma(t) = (1+t)1_{[0,\infty[}(t)\, d\mu(t)$.

The equivalence of (iii) and (iv) is easy.

"(iii) $\Rightarrow$ (i)". From (iii) it is clear that $f$ has a holomorphic extension to the cut plane $\mathbb{A}$, and for $z = x + iy, y > 0$ we find

$$\Im f(x+iy) = \alpha y + \int_0^\infty \frac{yt}{(x+t)^2 + y^2}\, d\sigma(t) \geq 0.$$

"(i) $\Rightarrow$ (iii)". We know that

$$f(z) = \alpha z + \beta + \int_{-\infty}^0 \frac{tz+1}{t-z}\, d\tau(t),$$

where $\alpha \geq 0, \beta \in \mathbb{R}$ and $\tau$ is a positive finite measure on $]-\infty, 0]$. Denoting by $\check{\tau}$ the reflection of $\tau$ in the origin, we find

$$f'(z) = \alpha + \int_0^\infty \frac{1+t^2}{(z+t)^2}\, d\check{\tau}(t),$$

so $f'(x) \geq 0$ for $x > 0$, i.e. $f$ is increasing on $]0, \infty[$. For $x > 0$ we have

$$f(x) \leq \alpha x + \beta + x \int_0^\infty \frac{t}{x+t}\, d\check{\tau}(t),$$

and since $f(0+)$ exists and is nonnegative, because $f$ is nonnegative and increasing, we get for $x \to 0^+$:

$$\int_0^\infty \frac{d\check{\tau}(t)}{t} + f(0+) = \beta.$$

This shows that $\tau$ has no mass at 0 and that $\int_0^\infty (1/t)\, d\check{\tau}(t) < \infty$. We can therefore write

$$f(x) = \beta - \int_0^\infty \frac{d\check{\tau}(t)}{t} + \alpha x + \int_0^\infty \left(\frac{1}{t} - \frac{1}{x+t}\right) d\check{\tau}(t) + x \int_0^\infty \frac{t}{x+t}\, d\check{\tau}(t),$$

which shows (iii) with

$$\sigma = \left(\beta - \int_0^\infty \frac{d\check{\tau}(t)}{t}\right) \delta_0 + \left(t + \frac{1}{t}\right) d\check{\tau}(t). \square$$

**Remark 4.11** A proof of Theorem 4.10 is also given in Schilling [39] as well as in his Dissertation [38]. The functions of Theorem 4.10 are called *complete Bernstein functions* by Schilling.

# 5 Bernstein functions

The terminology *Bernstein function* is not universally accepted. It has been used in potential theory in the work of Jacques Deny, which was the source of inspiration for [14]. Bernstein functions are called completely monotone mappings in Bochner [19], who used them in connection with subordination of convolution semigroups. They also appear in connection with the functional calculus of contraction semigroups in Banach spaces, see e.g. [39] and references therein.

**Definition 5.1** A function $f : ]0,\infty[ \to [0,\infty[$ is called a *Bernstein function*, if it is $C^\infty$ and $f' \in \mathcal{C}$. The set of Bernstein functions is denoted $\mathcal{B}$.

The set $\mathcal{B}$ is a convex cone closed under pointwise convergence.

Since a Bernstein function in nonnegative and increasing, it has a nonnegative limit $f(0+)$. Integrating the Bernstein representation of the completely monotonic function $f'$ gives the following integral representation

$$f(x) = \alpha x + \beta + \int_0^\infty (1 - e^{-x\xi})\, d\nu(\xi), \tag{17}$$

where $\alpha, \beta \geq 0$ and $\nu$, called the *Lévy measure*, is a positive measure on $]0,\infty[$ satisfying

$$\int_0^\infty \frac{\xi}{1+\xi}\, d\nu(\xi) < \infty.$$

See [14] for details and compare with Theorem 2.6. It follows easily that

$$\alpha = \lim_{x\to\infty} \frac{f(x)}{x}, \quad f'(x) = \mathbb{L}\left[\alpha\delta_0 + \xi 1_{]0,\infty[}(\xi)\, d\nu(\xi)\right](x), \tag{18}$$

and in particular $f(x) = O(x)$ for $x \to \infty$.

In [18] Bernstein functions are called *Laplace exponents* because of the following result, which shall be compared with Theorem 2.2:

**Theorem 5.2** *For a function $f : ]0,\infty[ \to [0,\infty[$ the following are equivalent*

*(i)* $f \in \mathcal{B}$,

*(ii)* $\exp(-tf) \in \mathcal{C}$ *for all* $t > 0$,

*(iii) There exists a convolution semigroup* $(\eta_t)_{t>0}$ *of positive measures on* $[0,\infty[$ *with* $\eta_t([0,\infty[) \leq 1$ *such that for all* $t > 0$ *and* $x \geq 0$

$$\mathbb{L}\eta_t(x) = \int_0^\infty e^{-x\xi}\, d\eta_t(\xi) = e^{-tf(x)}. \tag{19}$$

(In Theorem 5.2 (iii) a convolution semigroup satisfies by definition $\eta_s * \eta_t = \eta_{s+t}$ and $\lim_{t\to 0} \eta_t = \delta_0$ vaguely, cf. [14, p. 48].)

**Corollary 5.3** *The following composition results hold:*

$$f, g \in \mathcal{B} \Rightarrow f \circ g \in \mathcal{B}, \quad f \in \mathcal{C}, g \in \mathcal{B} \Rightarrow f \circ g \in \mathcal{C}.$$

**Proof** For $t > 0$ and $g(x) \in \mathcal{B}$ we have $\exp(-tg) \in \mathcal{C}$ by (ii). Therefore

$$1 - e^{-tg} \in \mathcal{B}$$

so the composition results follow from (17) and (5). $\square$

**Examples of Bernstein functions**
(a) $f(x) = \alpha x + \beta$, where $\alpha, \beta \geq 0$.
(b) $f(x) = 1 - e^{-xt}$, where $t > 0$.
(c) $f(x) = \log(1 + x)$.
(d) $f(x) = x^\alpha$, where $0 \leq \alpha \leq 1$.
(e) $f(x) = \log \Gamma(x + 1)/x \log(x)$, cf. [15].
(f) $f(x) = (1 + 1/x)^x$, $\quad f(x) = (1 + x)^{1+1/x}$, cf. [2].
(g) $f(x) = \sum_{n=0}^{\infty} \log(1 + xq^n)$, where $0 < q < 1$. The corresponding convolution semigroup is studied in [10].
(h) $(\log P_\rho(x))/x^{1/\rho}$, where $\rho > 1$ and $P_\rho$ is the canonical product (cf. [34])

$$P_\rho(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{n^\rho}\right).$$

The paper [32] contains a supplementary list of Bernstein functions.

As an important application of Corollary 5.3 we note that

$$f \in \mathcal{B}, 0 < \alpha < 1 \Rightarrow f(x)^\alpha,\ f(x^\alpha) \in \mathcal{B}, \tag{20}$$

$$f \in \mathcal{C}, 0 < \alpha < 1 \Rightarrow f(x^\alpha) \in \mathcal{C}. \tag{21}$$

We mention the following stability property of $\mathcal{B}$ from [12, p. 265]

$$f \in \mathcal{B}, n \in \mathbb{N} \Rightarrow (f(x^{1/n}))^n \in \mathcal{B}. \tag{22}$$

**Theorem 5.4** *The following relations between $\mathcal{B}, \mathcal{C}, \mathcal{S}$ hold:*

$$f \in \mathcal{B} \setminus \{0\} \Rightarrow \frac{1}{f} \in \mathcal{C}; \quad f \in \mathcal{S} \setminus \{0\} \Rightarrow \frac{1}{f} \in \mathcal{B}; \quad f \in \mathcal{B} \Rightarrow \frac{f(x)}{x} \in \mathcal{C}.$$

**Proof** Integrating the equation (19) with respect to Lebesgue measure on the half-line, we get (for $f \in \mathcal{B} \setminus \{0\}$)

$$\mathbb{L}\left(\int_0^\infty \eta_t\, dt\right)(x) = \frac{1}{f(x)},$$

which shows that $1/f$ is the Laplace transform of the positive measure $\kappa = \int_0^\infty \eta_t\, dt$ called the *potential kernel* of the convolution semigroup $(\eta_t)_{t>0}$.

The reciprocal of a completely monotonic function need not be a Bernstein function ($f(x) = x^2 \notin \mathcal{B}$). However, it holds for $f \in \mathcal{S} \setminus \{0\}$. In fact, by the stability property (ii) for $\mathcal{S}$ we have

$$\frac{1}{xf(x)} = a + \int_0^\infty \frac{d\mu(\xi)}{x+\xi},$$

hence

$$\frac{1}{f(x)} = ax + \int_0^\infty \frac{x}{x+\xi}\, d\mu(\xi)$$

and for each $\xi \geq 0$ the function

$$x \mapsto \frac{x}{x+\xi} = 1 - \frac{\xi}{x+\xi}$$

is a Bernstein function.

The third implication follows from formula (17) by dividing with $x$. $\square$

**Remark 5.5** The following observation is a simple counterpart of a theorem of Harzallah, cf. Remark 2.4:

$$f \in \mathcal{B} \text{ is bounded } \;\Leftrightarrow\; f = c - g \text{ where } c \geq g \in \mathcal{C}. \tag{23}$$

**Proof** Only "$\Rightarrow$" needs a comment. If $f$ is a bounded Bernstein function then (17) shows that $\alpha = 0$ and $f(\infty) = \beta + \nu(]0,\infty[) < \infty$ by the monotonicity theorem of Lebesgue. Since $\nu$ is a finite measure we can integrate term by term in (17), and the result follows. $\square$

# 6 Logarithmically completely monotonic functions

In [24] the authors call a function $f\ :]0,\infty[\to]0,\infty[$ *logarithmically completely monotonic* if it is $C^\infty$ and

$$(-1)^k[\log f(x)]^{(k)} \geq 0, \text{ for } k = 1, 2, \ldots. \tag{24}$$

If we denote the class of logarithmically completely monotonic functions by $\mathcal{L}$, we have $f \in \mathcal{L}$ if and only if $f$ is a positive $C^\infty$-function such that $-(\log f)' \in \mathcal{C}$.

The functions of class $\mathcal{L}$ have been implicitly studied in [3], and Lemma 2.4(ii) in that paper can be stated as the inclusion $\mathcal{L} \subset \mathcal{C}$, a fact also established in [23].

The class $\mathcal{L}$ can be characterized in the following way, established by Horn[29, Theorem 4.4]:

**Theorem 6.1** *For a function $f : ]0,\infty[\; \to ]0,\infty[$ the following are equivalent:*

*(i)* $f \in \mathcal{L}$,

*(ii)* $f^{\alpha} \in \mathcal{C}$ *for all* $\alpha > 0$,

*(iii)* $\sqrt[n]{f} \in \mathcal{C}$ *for all* $n = 1, 2, \ldots$.

Another way of expressing the conditions of Theorem 6.1 is that the functions in $\mathcal{L}$ are those completely monotonic functions for which the representing measure $\mu$ in (5) is infinitely divisible in the convolution sense: For each $n \in \mathbb{N}$ there exists a positive measure $\nu$ on $[0, \infty[$ with $n$'th convolution power equal to $\mu$, viz. $\nu^{*n} = \mu$. By condition (ii) there exists a family $(\mu_\alpha)_{\alpha>0}$ of positive measures such that the Laplace transform of $\mu_\alpha$ is $f^\alpha$. Note that the family $(\mu_\alpha)_{\alpha>0}$ satisfies the convolution equation $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$ and $\mu_\alpha([0, \infty[) = f(0+) \leq \infty$ for all $\alpha > 0$.

In the special case of $f(0+) = 1$ this is very classical: This is the description of infinitely divisible probability distributions on the half-line. Since there are probability measures on $[0, \infty[$ which are not infinitely divisible, we have $\mathcal{C} \setminus \mathcal{L} \neq \emptyset$.

A proof of Theorem 6.1 is included in [9]. From the proof it is clear that condition (iii) can be replaced by the condition

(iv) $f^{\varepsilon_n} \in \mathcal{C}$ for some sequence $(\varepsilon_n)$ of positive numbers tending to zero.

From property (ii) it is clear that $\mathcal{L} \subset \mathcal{C}$, that $\mathcal{L}$ is closed under pointwise convergence and that $\mathcal{L}$ is stable under products and positive powers.

We also note that $\mathcal{S} \subset \mathcal{L}$, because if $f \in \mathcal{S}$ then $f^\alpha \in \mathcal{S} \subset \mathcal{C}$ for any $0 < \alpha \leq 1$ by property (vi) of Section 3, hence $f \in \mathcal{L}$ by (iii) above.

Let us recall the classical notion of a *logarithmically convex function* as a positive function on an interval such that its logarithm is convex. It is a surprising fact that the sum of two such functions is logarithmically convex. A similar thing does not hold for the class $\mathcal{L}$:

**Example 6.2** We have $e^{-tx} \in \mathcal{L}$ for each $t > 0$, but $e^{-x} + e^{-2x} \notin \mathcal{L}$. In fact, we have

$$e^{-x} + e^{-2x} \in \mathcal{L} \Leftrightarrow -\left(-x + \log(1 + e^{-x})\right)' \in \mathcal{C},$$

but

$$1 + \frac{e^{-x}}{1 + e^{-x}} = \mathbb{L}\left(\delta_0 + \sum_{k=1}^{\infty} (-1)^{k-1} \delta_k\right)(x)$$

so this function is not completely monotonic.

For the Digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$ we have the classical formula, see [25]

$$\log x - \Psi(x) = \mathbb{L}[\frac{1}{1 - e^{-\xi}} - \frac{1}{\xi}](x), \tag{25}$$

which shows that this function is completely monotonic. We even have

$$\log x - \Psi(x) \in \mathcal{L}. \tag{26}$$

To see this we note that

$$k(x) := \frac{1}{x}\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) = \mathbb{L}[\rho](x),$$

where $\rho$ is a nonnegative function on $]0,\infty[$ given by $\rho(\xi) = n+1-\xi$ for $\xi \in\ ]n, n+1]\,,\ n = 0,1,2,\ldots$, so $k \in \mathcal{C}$.

For any $\varepsilon > 0$ the expression

$$e^{-\varepsilon x}\left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) = xe^{-\varepsilon x}k(x)$$

is a positive integrable function on $]0,\infty[$. Normalized to a probability density, it is an infinitely divisible distribution by the Steutel-Kristiansen theorem, cf. [44, Theorem 4.5], since $e^{-\varepsilon x}k(x) \in \mathcal{C}$. It now follows from Theorem 6.1 that $\log(x+\varepsilon) - \Psi(x+\varepsilon) \in \mathcal{L}$ for all $\varepsilon > 0$, which proves (26).

It is also well-known that

$$\Psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2} = \mathbb{L}\left[\frac{\xi}{1-e^{-\xi}}\right](x), \tag{27}$$

so $\Psi' \in \mathcal{C}$. As before we also have $\Psi' \in \mathcal{L}$ since $1/(1-e^{-x}) \in \mathcal{C}$. The latter holds because

$$\frac{1}{1-e^{-x}} = \sum_{k=0}^{\infty} e^{-kx} = \mathbb{L}\left[\sum_{k=0}^{\infty} \delta_k\right](x).$$

The following result was given in Grinshpan and Ismail [26]:

**Lemma 6.3** *Let $\alpha_k, \beta_k,\ k = 1,\ldots,n$ be real numbers such that $\sum_{k=1}^{n} \alpha_k = 0$ and $\beta_k \geq 0$ for all $k$. Then*

$$u(x) := \prod_{k=1}^{n} \Gamma^{\alpha_k}(x+\beta_k) \in \mathcal{L}$$

*if and only if*

$$v(\xi) = \sum_{k=1}^{n} \alpha_k \xi^{\beta_k} \geq 0 \textit{ for } \xi \in\ ]0,1]\,.$$

They apply it to prove that certain products and quotients of shifted $\Gamma$-functions belong to $\mathcal{L}$. To formulate one of their main results we need some notation. For $k = 1,2,\ldots,n$ let $P_{n,k}$ denote the set of all vectors $\mathbf{m} = (m_1,\ldots,m_k)$ whose components are natural numbers such that $1 \leq m_\nu < m_\mu \leq n$ for $1 \leq \nu < \mu \leq k$; and let $P_{n,0} = \emptyset$.

**Theorem 6.4** *For any $a_k > 0, k = 1, \ldots, n,$ define*

$$L_n(x) = \frac{\Gamma(x) \prod_{k=1}^{[n/2]} \left[ \prod_{\mathbf{m} \in P_{n,2k}} \Gamma(x + \sum_{j=1}^{2k} a_{m_j}) \right]}{\prod_{k=1}^{[(n+1)/2]} \left[ \prod_{\mathbf{m} \in P_{n,2k-1}} \Gamma(x + \sum_{j=1}^{2k-1} a_{m_j}) \right]}.$$

*Then $L_n \in \mathcal{L}$.*

For $n = 2$ the theorem states that for any $a_1, a_2 > 0$

$$g_{a_1,a_2}(x) = \frac{\Gamma(x)\Gamma(x + a_1 + a_2)}{\Gamma(x + a_1)\Gamma(x + a_2)} \in \mathcal{L}.$$

It was earlier established by Bustoz and Ismail that $g_{a_1,a_2} \in \mathcal{C}$, see [20].

In [30, Theorem 6.1] is given a list of functions in $\mathcal{L}$.

# 7 Some completely monotonic functions related to the Gamma function

There are many completely monotonic functions related to the $\Gamma$-function, see [30].

For the Digamma function $\Psi(x) = \Gamma'(x)/\Gamma(x)$ we have the classical formula, see [25]

$$(-1)^{n+1}\Psi^{(n)}(x) = n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = \mathbb{L}\left[\frac{\xi^n}{1 - e^{-\xi}}\right](x), \quad n = 1, 2, \ldots, \tag{28}$$

which show that these functions are completely monotonic.

In [21] Clark and Ismail introduced the functions

$$G_m(x) = x^m \Psi(x) \quad \text{and} \quad \Phi_m(x) = -x^m \Psi^{(m)}(x).$$

They proved that $G_m^{(m+1)}$ is completely monotonic for $m = 1, 2, \ldots$ and that $\Phi_m^{(m)}$ is completely monotonic for $m = 1, 2, \ldots, 16$. Clark and Ismail conjectured that $\Phi_m^{(m)}$ is completely monotonic for all natural numbers $m$. This was disproved in [4], where it was shown that the sequence stops being completely monotonic when $m$ becomes very large.

For $\alpha, \beta > 0$ the generalized Mittag-Leffler function is defined by

$$F_{\alpha,\beta}(x) = \Gamma(\beta) \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(\alpha k + \beta)}, \quad x > 0. \tag{29}$$

It was established by Schneider [40] that

$$F_{\alpha,\beta} \in \mathcal{C} \Leftrightarrow 0 < \alpha \leq 1,\ \alpha \leq \beta.$$

Note that $F_{1,1}(x) = e^{-x}$. The complete monotonicity of $F_{\alpha,\beta}$ for $0 < \alpha < 1, \beta = 1$ was established by Feller and Pollard.

# References


[1] Akhiezer, N. I. , *The classical moment problem*. Oliver and Boyd, Edinburgh, 1965.

[2] Alzer, H., Berg, C., *Some Classes of Completely Monotonic Functions*, Ann. Acad. Sci. Fenn. Math. **27** (2002), 445–460.

[3] Alzer, H., Berg, C., *Some Classes of Completely Monotonic Functions, II*, Ramanujan J. **11** (2006), 225–248.

[4] Alzer, H., Berg, C., and Koumandos, S., *On a conjecture of Clark and Ismail*, J. Approx. Theory **134** (2005), 102–113.

[5] Baillaud, B., Bourget, H., *Correspondance d'Hermite et de Stieltjes I,II*, Gauthier-Villars, Paris 1905.

[6] Berg, C., *The Stieltjes cone is logarithmically convex.* In: Complex Analysis Joensuu 1978. Lecture Notes in Mathematics **747**, 46–54. Springer 1979.

[7] Berg, C., *Quelques remarques sur le cône de Stieltjes.* In: *Séminaire de Théorie du potentiel, Paris n.5.* Lecture Notes in Mathematics **814**. Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[8] Berg, C., *Fonctions définies négatives et majoration de Schur.* In: Colloque de Théorie du Potentiel Jacques Deny–Orsay 1983. Lecture Notes in Mathematics **1096**. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[9] Berg, C., *Integral representation of some functions related to the Gamma function*, Mediterr. j. math. **1** (2004), 433–439.

[10] Berg, C., *On a generalized gamma convolution related to the q-calculus.* In: Developments of Mathematics Vol. 13, "Theory and Applications of Special Functions." A volume dedicated to Mizan Rahman. Ed. by M.E.H. Ismail and Erik Koelink. Springer Science+ Business Media, Inc., Berlin 2005.

[11] Berg, C., *On powers of Stieltjes moment sequences, II*, J. Comput. Appl. Math. **199** (2007), 23–38.

[12] Berg, C., Boyadzhiev, K. and R. deLaubenfels, *Generation of generators of holomorphic semigroups*, J. Austral. Math. Soc. (Series A) **55** (1993), 246–269.

[13] Berg, C., Christensen, J. P. R. and Ressel, P., *Harmonic analysis on semigroups. Theory of positive definite and related functions.* Graduate Texts in Mathematics vol. **100**. Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[14] Berg, C., Forst, G., *Potential theory on locally compact abelian groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **87**. Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[15] Berg, C., Pedersen, H.L., *A completely monotone function related to the Gamma function*, J. Comput. Appl. Math. **133** (2001), 219–230.

[16] Berg, C., Pedersen, H.L., *Pick functions related to the gamma function*, Rocky Mount. J. Math. **32** (2002), 507–525.

[17] Bernstein, S. N., Sur les fonctions absolument monotones, *Acta Math.* **52** (1928), 1-66.

[18] Bertoin, J., *Lévy processes*. Cambridge University Press, Cambridge 1996.

[19] Bochner, S., *Harmonic analysis and the theory of probability*. University of California Press, Berkeley and Los Angeles, 1955.

[20] Bustoz, J., Ismail, M. E. H., *On gamma function inequalities*, Math. Comp. **47** (1986), 659–667.

[21] Clark, W.E., Ismail, M.E.H., *Inequalities involving gamma and psi functions*, Anal. Appl. **1** (2003), 129–140.

[22] Donoghue Jr., W.F., *Monotone Matrix Functions and Analytic Continuation*. Springer-Verlag, Berlin-Heidelberg-New York, 1974.

[23] Feng Qi, Bai-Ni Guo, *Complete monotonicities of functions involving the Gamma and Digamma functions*, RGMIA Res. Rep. Coll. **7**, No. 1 (2004), Art. 6. Available online at http://rgmia.vu.edu.au/v7n1.html

[24] Feng Qi, Bai-Ni Guo, and Chao-Ping Chen, *Some completely monotonic functions involving the Gamma and polygamma functions* RGMIA Res. Rep. Coll. **7**, No. 1 (2004), Art. 8. Available online at http://rgmia.vu.edu.au/v7n1.html

[25] Gradshteyn, I. S., Ryzhik, I. M., *Table of Integrals, Series and Products*, Sixth Edition, Academic Press, San Diego, 2000.

[26] Grinshpan, A. Z., Ismail, M. E. H., *Completely monotonic functions involving the gamma and q-gamma functions* Proc. Amer. Math. Soc. **134** no. 4 (2006), 1153–1160.

[27] Harzallah, K., *Fonctions opérant sur les fonctions définies-négatives*, Ann. Inst. Fourier **17**,1 (1967), 443–468.

[28] Hirsch, F., *Intégrales de résolventes et calcul symbolique*, Ann. Inst. Fourier **22, 4** (1972), 239–264.

[29] Horn, R.A., *On infinitely divisible matrices, kernels and functions*, Z. Wahrscheinlichkeitstheorie und Verw. Geb. **8** (1967), 219–230.

[30] Ismail, M.E.H., Lorch, L. and Muldoon, M.E., *Completely monotonic functions associated with the gamma function and its q-analogues*, J. Math. Anal. Appl. **116** (1986), 1–9.

[31] Itô, M., *Sur les cônes convexes de Riesz et les noyaux complètement sous-harmoniques*, Nagoya Math. J. **55** (1974), 111–144.

[32] Jacob, N., Schilling, R.L., *Function spaces as Dirichlet spaces (about a paper by Maz'ya and Nagel)*, Z. Anal. Anwendungen **24** (2005), no. 1, 3–28.

[33] Nevanlinna, R., *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem*, Ann. Acad. Sci. Fenn. Serie A **18** No.5 (1922), 1–53.

[34] Pedersen, H. L., *Canonical products of small order and related Pick functions*, J. Comput. Appl. Math. **175** (2005), 245–263.

[35] Pick, G., *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt sind*, Math. Ann. **77** (1916), 7–23.

[36] Reuter, G. E. H., *Über eine Volterrasche Integralgleichung mit totalmonotonem Kern*, Arch. Math. **7** (1956), 59–66.

[37] Sasvári, Z., *Positive Definite and Definitizable Functions*, Akademie Verlag, Berlin 1994.

[38] Schilling, R.L., *Zum Pfadverhalten von Markovschen Prozessen, die mit Lévy-Prozessen vergleichbar sind.* Dissertation, Universität Erlangen, 1994.

[39] Schilling, R.L., *Subordination in the sense of Bochner and a related functional calculus*, J. Austral. Math. Soc. (Seris A) **64** (1998), 368–396.

[40] Schneider, W. R., *Completely monotone generalized Mittag-Leffler functions*, Expo. Math. **14** (1996), 3–16.

[41] Schoenberg, I. J., *Metric spaces and completely monotone functions*, Ann. of Math. **39** (1938), 811–841.

[42] Schoenberg, I. J., *Metric spaces and positive definite functions*, Trans. Amer. Math. Soc. **44** (1938), 522–536.

[43] Steerneman, A.G.M., van Perlo-ten Kleij, F., *Spherical distributions: Schoenberg (1938) revisited*, Expo. Math. **23** (2005), 281–287.

[44] Steutel, F. W., Van Harn, K., *Infinite divisibility of probability distributions on the real line*. Marcel Dekker, Inc., New York, Basel 2004.

[45] Stieltjes, T.J., *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse **8** (1894), 1–122; **9** (1895), 5–47.

[46] Shu-gwei Tyan, *The structure of Bivariate distribution functions and their relation to Markov processes*. Ph. D. Thesis, (Princeton University 1975).

[47] Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton 1941.

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