Abstract

This thesis studies stability issues of ruin probabilities in various models occurring in risk theory. The question of stability arises naturally in risk theory since the governing parameters in these models can only be estimated with uncertainty. Moreover, in most cases there are no explicit expressions known for the ruin probabilities.

The main contribution of the thesis is to provide explicit stability bounds for ruin probabilities in various risk models. Since asymptotic behaviour of ruin probability is very important, we consider weighted distance between the ruin probabilities. The proper choice of the weight functions is part of the problem. It is related to the tail behaviour of the ruin probabilities, if the latter is known. Distance between the governing parameters is dictated by the techniques involved. We are following mainly two approaches to obtain our stability bounds: one is based on the stability bounds for stationary distributions of Markov chains developed by Kartashov. A second approach exploits the regenerative property of so-called reversed process.

Contents

Chapter 1

Introduction

1.1 Actuarial risk models and the stability issue

Actuarial risk theory goes back to the works of Lundberg and is used to model the portfolio of an insurance company. The risk process R represents the capital value of a company at each time t. Typically the risk process R takes care of the premiums paid to the insurance company and the claims paid out by the insurance company, thus considers the *insurance risk.* Such a situation is present in the Cramér-Lundberg (also called classical), the Sparre Andersen and many others risk models. Some of the more recent risk models involve also investment possibilities and consider the situation where capital is exposed to *financial* as well as insurance risk. Such risk models were investigated by Paulsen, Norberg, Kalashnikov and many others.

One of the most important characteristics of the risk process is a *ruin probability* $\psi(x)$ which is a probability that the capital falls below the level 0 and is considered as a function of an *initial capital* $x := R(0)$:

$$\psi(x) = \mathbf{P}\left(\inf_{t\geq 0} R(t) < 0 \,\middle|\, R(0) = x\right).$$

The level 0 here is chosen by tradition and convenience, one can consider any other level defining the event ruin instead, see for example Embrechts & Schmidli [13]. However, the initial capital x can be always adopted so that ruin is associated with the level 0. Note also that ruin hardly means bankruptcy of the company. It should be considered as a technical term which expresses riskiness of the business.

Ruin probabilities can be found explicitly only in rare cases. This is so for the Cramér-Lundberg risk model where claims are following the exponential distribution (or a mixture of exponentials), or the Markov modulated risk model with phase-type distributed claims, see Asmussen [3]. However, in many other interesting cases the ruin probability cannot be found explicitly, and thus the main attention is devoted to the asymptotic behaviour of $\psi(u)$ when u tends to ∞. In the Cramer-Lundberg model under the Cramér assumption the ruin probability decays exponentially fast, i.e.,

$$\psi(x) \sim C\exp(-\varepsilon x), \qquad \text{as} x \to \infty,$$

where constants $C \geq 0$, $\varepsilon > 0$. The situation alters if one assumes 'heavy-tailed' claims (all positive exponential moments are infinite) or investments. For example, if the claim size distribution function (d.f.)

is such that the corresponding integrated tail distribution (say, F_Z^I) is subexponential, then the asymptotic behaviour of the ruin probability is given by the function $C(1 - F_Z^I)$, where $C > 0$, see Embrechts *et al.* [12]. Risk models with investments driven by a Lévy process were considered by Kalashnikov and Norberg [31], among others. They investigated the tail behaviour of the ruin probability and showed that with the possibility of risky investments, i.e. investments which may possibly lead to a loss, the ruin probability is in general no longer an exponentially decaying function even in the case of small claims. In some cases, the latter probability decays not faster than a power function. The behaviour of the ruin probability in risk models with capital exposed to financial as well as insurance risks was considered by various authors. Paulsen [41] sets up quite a general model with stochastic investments and shows that the financial risk has a significant influence on the ruin probability. Norberg [39] considers risk processes where the cash flow and the log accumulation factor are independent Brownian motions with drift. By an application of martingale techniques it is shown that the ruin probability exhibits power function behaviour. Those results indicate that even very small random perturbations may have a significant influence on the ruin probability.

The listed examples show that the asymptotic behaviour of ruin probabilities is quite different. This leads to the following considerations.

Stability issues. Let us collect all the parameters of a risk model to a vector valued *governing parameter* or *input characteristic* a. For example, the Cramér-Lundberg model is governed by the premium rate c, the claims occurrence intensity λ and the claim size d.f., say F_Z. In the Sparre Andersen model the intensity λ would be replaced by the claims inter-occurrence times d.f., say F_θ. Depending on the model, the governing parameter a may take more complicated values including parameters of (random) interest, etc. The ruin probability function

$$\psi_a := \{\psi_a(x)\}_{x \geq 0}$$

(we will refer to ψ_a simply as ruin probability) is fully determined by the input a, and thus it represents *an output* characteristic of the model. The examples above show that the ruin probability may have different asymptotic behaviour for different inputs a. Thus, a natural and also crucial question for applications is whether the model under consideration is *stable*, i.e., whether small perturbations in the input characteristic a yield also to the small deviations of the output ψ_a. This question is not trivial, since in general, the explicit expression for the ruin probability is not known, and hence the dependence of ψ_a on the governing parameter a cannot be investigated *directly.* Furthermore, the importance of the stability analysis is motivated by the fact that the governing parameter a is *not known* in practice. One typically would estimate it from the data and replace it by the statistical estimates and empirical distributions or use some other parameter which is believed to be close to the 'true' (or optimal) parameter a. Thus, we are in the situation where the input parameters are not known and for any given input, the output characteristic is also not known (in certain ceases even the asymptotic behaviour of the ruin probability is also not known). One major problem we are studying is whether convergence of the governing parameters (in some sense) implies convergence of the corresponding ruin probabilities (in some sense). The goal of the thesis is to obtain *explicit* bounds for the (unknown) ruin probabilities in terms of the governing parameters.

Stability questions do not originate from actuarial risk theory. On the contrary, this question is well known and has been investigated in the context of stochastic processes and their applications: stationary distributions of Markov chains, finite-time and uniform-in-time stability of regenerative processes, waiting times in queuing theory, reliability, storage theory, etc. The creator of the stability theory is considered to be Lyapunov who considered stability in the theory of differential equations. The general stability problem for stochastic models was proposed by Zolotarev, see [52] and references therein. Its basic statements with a view towards actuarial risk theory are reproduced in Kalashnikov [30, 32]. Stability of the ruin probability is a particular case of the general setting. The above discussed stability in these terms is a 'direct-stability', whereas an 'inverse-stability refers to the deviations of input, in our case a, with respect of the perturbations of output, ψ_a. This question also makes sense in our setting of ruin probabilities. In this thesis stability refers to the 'direct-stability', unless stated otherwise. The inverse problem will be only touched briefly in the discussion of the main result.

Now we formulate the stability problem for the ruin probability.

1.2 Stability problem for the ruin probability

Stability problem. Let $\mathbb{A}$ be the space of governing parameters describing the risk model. The ruin probability ψ is considered as a function of $a \in \mathbb{A}$ and is a mapping from the space $\mathbb{A}$ of governing parameters into a functional space Ψ, i.e. $\psi : \mathbb{A} \to \Psi$, where $\psi_a \in \Psi$ is a function of the initial capital x. Assume that the spaces $\mathbb{A}$ and Ψ are metric spaces $(\mathbb{A}, d_\mathbb{A})$ and (Ψ, d_Ψ). This setup allows for the following notion of stability: The ruin probability ψ is called *stable at the point a* if for any sequence $\{a'\} \subset \mathbb{A}$ converging to a we have the convergence of the corresponding ruin probabilities, i.e.,

$$d_\mathbb{A}(a, a') \to 0 \Rightarrow d_\Psi(\psi_a, \psi_{a'}) \to 0.$$

We will not deal with the whole space of governing parameters but only with a certain subset meeting conditions to be specified later. Throughout the thesis this set will be denoted by $\mathbb{A}$ and will be called the set of *admissible parameters*. If we can find a function f, continuous at 0 with $f(0) = 0$, and a set $\mathbb{A}$ such that

$$d_\Psi(\psi_a, \psi_{a'}) \le f(d_\mathbb{A}(a, a')), \qquad a, a' \in \mathbb{A} \tag{1.1}$$

then the inequality (1.1) is called *a stability bound* for the ruin probability. Often the function f depends on some characteristics of the parameters a and a' (moments, etc.). Our aim is to obtain stability bounds of the form (1.1).

In the sequel we assume that the *original* process is governed by a given parameter a. Together with the original process we consider a *perturbed* process. All quantities (parameters, random variables, processes) referring to a perturbed process will be marked by a prime.

Metrics $d_\mathbb{A}$ and d_Ψ. The choice of metrics $d_\mathbb{A}$ and d_Ψ is of crucial importance for the successful stability analysis and represents part of the problem. Since the most important characteristic of the ruin probability is its asymptotic behaviour, we seek for such stability bounds which allow for a tail comparison of the ruin probabilities. This implies that metric d_Ψ has to be *weighted metric*. We will use the following metrics d_Ψ:

The weighted total variation metric $\|\cdot\|_w$,

$$\|\psi_a - \psi_{a'}\|_w := \int_0^\infty w(u)|\psi_a(du) - \psi_{a'}(du)|; \tag{1.2}$$

The weighted uniform metric $|\cdot|_w$,

$$|\psi_a - \psi_{a'}|_w := \sup_u w(u)|\psi_a(u) - \psi_{a'}(u)|.$$

Here w is a *weight function*, i.e., it is bounded away from 0 and increasing (typically to ∞). The weight function w restricts a set of admissible deviations in terms of the tail behaviour of the ruin probability from the original model. Typically, the choice of w is related to the asymptotic behaviour of the ruin probability, if the latter is known. We will use $w(x) = \exp(\varepsilon x)$ if the ruin probability has an exponential decay, and $w(x) = (1 + x)^\varepsilon$ in the case of power law decay of the ruin probability. However, stability bounds are more interesting when the asymptotics of the ruin probability is not known. Thus, the choice of the metric d_Ψ constitutes a part of the stability problem.

However, sometimes it is easier to obtain the stability bound in some other metric. Then one need to use relations between the two metrics in order for the final result to be stated in a desired metric $d_{\mathbb{A}}$. This task is not always easy since in general, different metrics induce different topologies. This problem has been discussed in Zolotarev [52], Kalashnikov & Rachev [33], Kalashnikov [30]. Coming back to the metrics $\|\cdot\|_w$ and $|\cdot|_w$, we consider that weighted uniform metric may be more useful for practical applications. An inequality $|\cdot|_w \leq \|\cdot\|_w$ allows to rewrite the stability bound (1.1) in terms of the weighted uniform metric $|\cdot|_w$.

The choice of the metric $d_{\mathbb{A}}$ in the parameter space $\mathbb{A}$ is mostly dictated by the techniques to be used. For the better result, one should seek for the weaker metric $d_{\mathbb{A}}$. For example, the weighted total variation metric is too strong: we cannot approximate absolutely continuous density functions by empirical (pure jump) density functions.

1.3 Approaches to the solution

Since stability has been intensively investigated for different stochastic models, before solving the stability of ruin probability problem, we list some important (for our problem) features of already developed results and techniques.

First, stability analysis deals mainly with *stationary* stochastic systems and their stationary characteristics: stability in the limit theorems was investigated by Zolotarev with the help of probability metrics, see [52]; stability bounds for the stationary distributions of MC's were investigated by Borovkov [10], Kartashov [35] and others. The works of Kartashov are based on perturbation theory of linear operators in Banach spaces. Further, the stability of regenerative processes (such process consists of i.i.d. cycles) was extensively studies by Kalashnikov using coupling and crossing arguments, see [28] and the references therein. The finite-time stability bounds for different systems (regenerative process, queuing systems, etc.) may be obtained using the technique of minimal probability metrics, see Zolotarev [52], Kalashnikov [26], Rachev [44]. The uniform-in-time bounds use regenerative arguments.

The other feature of the above mentioned results is that they deal with *bounded* probability metrics (i.e., probability metrics ν is bounded if $\nu \leq C$ for some $C < \infty$), such as uniform, total variation, Prohorov, Fortet-Mourier and other probability metrics. Stability bounds in terms of unbounded metrics can be obtained with the help of existing relations between the probability metrics, and possibly under some additional assumptions on the risk model. This problem is discussed in Kalashnikov [28]. However weighted probability metrics had very limited attention. The only exception, to our knowledge, are stability bounds for the stationary distributions of MC's developed by Kartashov. These results hold in the weighted total variation distance (1.2).

Coming back to the stability of the ruin probability problem let us note the following. Because of the so-called *positive safety loading* condition which ensures that ruin does not occur with probability 1, the risk process typically has drift to ∞, and thus, it does not possess stationary characteristics. This prevents the application of developed results to the risk process directly. However, this does not represent a major problem either. It is well known that the risk process can be related to a dual process $\{V_n\}$ like in queuing and storage. In risk theory this process is known as a *the reversed process* since it is constructing by a certain time-reversing procedure, see Amussen [3], Enikeeva *et. al.* for such constructions. The reversed process satisfies the relation

$$\psi(x) = \lim_{n\to\infty} \mathbf{P}(V_n > x).$$

The construction of the reversed process is typically algebraic (uses path-wise arguments) and does not rely on the probabilistic properties of $\{R_n\}$. Therefore, in some cases $\{V_n\}$ does not represent a MC. Typically this can be solved by supplementing $\{V_n\}$ with the additional coordinates form a certain stationary sequence $\{\varphi_n\}$, which makes the process $\{V_n, \varphi_n\}$ a MC and possibly also a regenerative process. This allows one to employ the well developed results for the stability investigation.

The second problem concerning the unbounded probability metric is more difficult to overcome and may require more sophisticated techniques.

Some of the results for the stability of the ruin probability are based on bounds for general MC's obtained by Kartashov, see Enikeeva *et. al.* [14], Rusaityte [46]. This technique uses a decomposition (also called splitting) of the transition kernel P of a MC $\{V_n, \tau_n\}$,

$$P(v, \Gamma) = K(v, \Gamma) + h(v)\nu(\Gamma), \tag{1.3}$$

where h is a non-negative function, ν is a probability measure and a kernel $K \geq 0$. The shift operator $\mathfrak{K}$ corresponding to kernel K has to meet a certain norm condition, namely, for some weight function w,

$$\|\mathfrak{K}\|_w := \sup_{v\geq 0} \frac{\mathfrak{K}w(v)}{w(v)} < 1. \tag{1.4}$$

For further details about this technical condition we refer to Section 3.2.2. Now we would like to emphasize that this yields the stability bound for the stationary distribution π of a MC $\{V_n, \tau_n\}$ in the weighted total variation metric with *the latter weight* w. Thus, it is very important to use the 'right' (with respect to the risk model) weight function w. In simple cases, like Sparre Andersen model, this does not represent a big problem, and, actually, has been carried out by Kartashov in terms of queuing applications, see [35]. However, in general, it represents the main technical difficulty. In many other risk models it is not possible to prove that the required norm condition $\|\mathfrak{K}\|_w < 1$ holds with the desired weight function w. One needs to construct another weight function which has the same tail behaviour and is related to w. This yields the stability bound for the ruin probability in metric $\|\cdot\|_w$. However, this is not a final bound since it is given in terms of the shift operators of the reversed processes. The bound directly in terms of the governing parameters is obtained by solving the stability problem for the shift operator. It can be solved with the help of minimal metrics. Typically the application of Kartashov's bounds rely on the assumptions which are related to the Cramér condition of the corresponding risk model.

Another approach proposed in Kalashnikov [30] uses the regenerative property of the reversed process. Thus, we refer to his method as *regenerative approach.* This approach deals with the weighted uniform metric $|\cdot|_w$. The two cases, so-called Cramér and non-Cramér, are treated differently. In a *Cramér case*, the original and perturbed reversed processes $\{V_n\}$ and $\{V_n'\}$, respectively, with $V_n \sim G_n$ and $V_n' \sim G_n'$, satisfy the following contraction property:

$$|G_{n+1} - G_{n+1}'|_w \leq \kappa |G_n - G_n'|_w + d_{\mathbb{A}}(a, a'), \tag{1.5}$$

where the constant $\kappa < 1$. Property (1.5) is typically related to the Cramér condition, see examples in Kalashnikov [30]. In order that (1.5) holds one need to prove that $\mathbb{E}w(V_n)$ is uniformly bounded for all $n > 0$. The latter proof uses the regenerative property of the reversed process and test functions technique. The construction of the test function problem is related to the construction of the weight function in the MC approach discussed above. The stability bound follows immediately from the relation (1.5):

$$|\psi - \psi'|_w \le \sup_{n\ge 0} |G_n - G'_n|_w \le \frac{d_{\mathbb{A}}(a, a')}{1-\kappa}. \tag{1.6}$$

Since G_n corresponds to the finite-time ruin probability $\psi^{(n)}$ caused by the first n claims, the above bound also holds for $|\psi^{(n)} - \psi'^{(n)}|_w$. In Kalashnikov [30] this approach was applied to the Cramér-Lundberg risk model and to the model with Lévy processes driven investments. In Rusaityte [47] it was generalized to the Markov modulated risk model with investments driven by a Lévy process.

The *non-Cramér case* refers to the case when the relation (1.5) holds with constant $\kappa \ge 1$. Typically this is the case when Cramér condition of the risk model is violated, for a example the risk model with heavy tailed claims, see [30]. In this case the approach is built upon the ideas from the uniform-in-time comparison of regenerative processes developed by Kalashnikov, see [28, 29] and further references therein. The first step is to prove the finite-time stability bound $\sup_{n\le N} |G_n - G'_n|_w \le d_{\mathbb{A}}(a, a')N$. Then, by the regenerative property of process $\{V_n\}$, to bound the remaining term δ_N, so that $\delta_N \to 0$ when $N \to \infty$. The final uniform-in-time stability bound in terms of $\sup_{n\ge 0} |G_n - G'_n|_w$ follows from the successful choice of $N = N(d_{\mathbb{A}}(a, a'))$. For the further details we refer to [30].

Contribution of the thesis

This thesis is based on the following papers:

1. Continuity estimates for ruin probabilities, [14] (together with F. Enikeeva and V. Kalashnikov, *Scan. Act. J.*, **2001).** This paper proposes a general approach allowing to obtain explicit stability (continuity) bounds for ruin probabilities. The approach is based on the three steps: (i) reverse the risk process; (ii) if necessary, make it a MC by supplementing with additional coordinates, and (iii) apply known stability results. Each of these three steps are well-known in the literature. The paper also proposes another construction of the reversed process. In particular it is useful in the risk models where income depends on the current reserve, for example, the risk models with investments. This approach is illustrated by two examples, the Sparre Andersen and Markov modulated risk models. Application of Kartashov's results for general MC's yields stability bounds in weighted total variation distance with exponential weight.

2. Continuity of the ruin probability in a model with borrowing and investments, [46] (conditionally accepted). In this paper the above approach based on the stability bounds for general MC's is applied to the risk model where capital in excess of a given (large) level is invested to a portfolio driven by a Lévy process, and when capital drops below certain given level, the company is obliged to borrow the missing amount at a constant interest rate. This model is technically more involved and uses different constructions concerning the decomposition of the transition kernel and the explicit expressions in terms of the governing parameters. Distance between the ruin probabilities is weighted with exponential weight when investments take place at constant interest rate, and with the power-law weight in general model. This corresponds to exponential and power-law upper bounds for the ruin probability in the respective models. Explicit stability bounds are obtained for the investments at constant interest rate, the interest rate is perturbed by jumps at Poisson times, and the Black-Scholes price process.

3. Stability of the ruin probability in a Markov modulated model with investments, [47] (submitted). This paper deals with the stability of the ruin probability in a Markov modulated risk model with Lévy process driven investments. We apply the regenerative approach from Kalashnikov [30]. Stability bounds are based on a certain 'contraction' property for the reversed process $\{V_n\}$, and also finiteness of the moments $\mathbb{E}w(V_n)$, where w is a weight function in a weighted distance between ruin probabilities. The latter moment condition is proved with the help of test functions. The final stability bounds are given in weighted uniform metrics.

Some general remarks. 1. The results show that in our case ruin probability *is stable*, i.e. in each model we are able to find a set of admissible governing parameters $\mathbb{A}$ where stability bound holds. The main result of the thesis are the *explicit* stability bounds in the *weighted* metric. Weights play very important role in the thesis. In some cases, when asymptotic behaviour of ruin probability is known, the asymptotically faster weight functions can lead to instability. Such examples are given in Kalashnikov [30].

2. The weighted total variation metric is too strong: there is no convergence between absolutely continuous and pure jump (for example, empirical) distributions. This is a disadvantage of the MC approach since it provides stability bounds in weighted total variation metric between the distributions involved in the governing parameter.

3. The regenerative approach yields stability bounds in a weighted uniform metric, which for practical applications are better.

Outline of the thesis

In Chapter 2 we introduce the risk models which will be considered in the thesis. We start with the general concepts of the risk process $\{R_n\}$ and the corresponding ruin probability ψ. In Section 2.1.1 we present the construction of the so-called reversed process $\{V_n\}$, the stationary distribution of which is identified with the ruin probability. This construction is taken from Enikeeva *et. al.* [14]. It uses a path-wise argument, similar to Asmussen & Petersen [6], and is useful particularly in such cases where income of the insurance company depends on the capital level, i.e., $R_{n+1} - R_n$ depends on R_n (among other processes and r.v.'s). Such a situation occurs, for example, in models with investments. All the technical details in the stability analysis of the ruin probability will be carried out in terms of the reversed process.

Next, we introduce the particular risk models for which stability bounds of ruin probability will be obtained. We consider four models: the Sparre Andersen risk model, well known in actuarial literature; the Markov modulated risk model representing the classical (or Cramér-Lundberg) risk model in a Markovian environment; the model with Lévy processes driven investments and borrowing which is a generalization of risk models considered by Kalashnikov & Norberg [31] (the Lévy process driven investments) and Embrechts & Schmidli [13] (borrowing and investments at constant interest rates). The last model encounters both, random interest driven by a Lévy process and Markov modulation. The last section is devoted to some facts about the asymptotic behaviour of the ruin probabilities in the listed models under assumptions similar to those under which the stability bounds will be obtained.

In Chapter 3 we give the mathematical background on which thesis relies. MC's are used throughout all thesis. In Section 3.1 we introduce some basic notions, like generating operator of a MC, etc. Dynkin's formula is an important tool used in Chapter 7 in a regenerative approach. Further we give some facts about the ergodicity of denumerable (with finite or countable state space) and general (with state space in $\mathbb{R}^k$) MC's. Ergodicity here means existence of a unique stationary distribution of a MC.

Part of the results in thesis rely on the stability bounds for stationary distributions of MC's developed by Kartashov [35]. These bounds together with involved facts about kernel operators and their norms are given in Section 3.2. At the end of the section we discuss the applicability of these results to the stability problem of ruin probability. Kartashov's bounds serve as a main source for the results in Chapters 4 to 6.

Next we introduce *regenerative processes* and the notion of *crossing* of regenerative processes. The stability bounds (uniform-in-time and finite-time) for the regenerative processes using crossing arguments were developed by Kalashnikov, see for example [28]. They rely on finiteness of the moments of crossing times. Such a uniform-in-time stability bound for discrete time regenerative process is given in Proposition 3.18. We use this result in order to compare finite-state MC's, which can be treated as regenerative processes. Also, the regenerative structure of the reversed process is exploited in Chapter 7.

Ergodicity theorems for general MC's and also stability bounds for regenerative processes involve certain moment requirements on the first passage time of the process to a given set. Test function technique is very helpful in proving such conditions. Some results in this direction are given in Section 3.4.

Some facts about probability metrics are given in Section 3.5. In particular, technique of minimal metrics play very important role in this thesis. In Chapter 6 it is used to solve the stability of the shift operator of a MC problem with respect to the governing parameter.

In Chapter 4 we illustrate the applicability of Kartashov's bounds for stationary distributions of MC's in the Sparre Andersen model. The imposed assumptions are related to the Cramér condition and yield the exponential upper bound for the ruin probability $\psi(x) \leq e^{-\varepsilon x}$. The final result is given in a weighted total variation metric with weight function $w(x) = e^{\varepsilon x}$. In this model stability bounds for the ruin probability follow straightforwardly. This example is taken from Enikeeva *et. al.* [14].

In Chapter 5 we apply the same MC approach based on the results of Kartashov to the Markov modulated risk model. We work under assumptions related to the Cramér condition. The final result is given in a weighted total variation metric with exponential weight $e^{\varepsilon^* v}$. Decomposition (1.4) is based on the fact that for every n, V_n attains value $\{0\}$ with positive probability. The main difficulty concerns the norm condition (1.4) of the decomposed kernel K, see (1.3): we have to construct another weight function $W(i,v)$ s.t. $W(i,v) \sim C_i e^{\varepsilon^* v}$ when $v \to \infty$ for some constants $C_i > 0$ and $\|\mathfrak{K}\|_W < 1$. This allows us to obtain the final stability bound for the ruin probability in a weighted total variation metric with the desired weight $e^{\varepsilon^* v}$. This model was treated in Enikeeva *et. al.* [14].

In Chapter 6 we consider a risk model with borrowing at constant interest rate when capital is low and Lévy process driven investments taking place for the capital in excess of some (large) level. Similar to the previous two chapters, we apply MC approach based on the results of Kartashov in order to obtain the stability bounds for ruin probability. The decomposition (1.3) in two previous models was based on the fact that for all n, V_n takes 0 value with positive probability. In current situation, due to borrowing condition, the reversed process $\{V_n\}$ for $n > 0$ stays positive. Now we decompose kernel P using that the reversed process $\{V_n\}$ has a positive density in some interval $(0, d_*)$ and also the 'memoryless' property of exponential claim inter-arrival times. This construction is given in Section 6.2. In order to prove the norm condition (1.4), we consider two cases: *the general model* when investments are driven by a genuine Lévy process U, and *the deterministic model* when process U is deterministic, i.e., $U_t = \alpha t$ for some constant $\alpha > 0$. The reason is, that in two cases, under certain conditions, ruin probability has different asymptotic behaviour and we would like to adjust the weight functions corresponding to it. (Note, that assumption about borrowing for small capital values has no influence on the asymptotics of ruin probability, see Emrechts & Schmidli [13] and Kalashnikov & Norberg [31].) Constructions of the weights are given in the Appendix I of this chapter. In deterministic case we prove the condition (1.4) with weight function w_0, where $w_0(x) \leq \exp(\varepsilon x) \leq c_{w_0} w_0(x)$ $(c_{w_0} > 1)$. This yields the stability bound for ruin probabilities in metric weighted with function $w(x) = \exp(\varepsilon x)$. Similarly, in the general case we construct weight function w_1 satisfying conditions $w_1(x) \leq (1+x)^\varepsilon \leq c_{w_1} w_1(x)$ $(c_{w_1} > 1)$. This yields the stability bound in metric $d_\Psi = \| \cdot \|_w$, where $w(x) = (1+x)^\varepsilon$.

Application of Kartashov's results now yield the stability bounds for the ruin probability in desired metric with respect to the perturbations of the shift operator $\mathfrak{P}$. To express these stability bounds directly in terms of the governing parameters in this case is technically more involved. The corresponding constructions are given in the Appendix II. We use the technique of minimal metrics. In deterministic model the final stability bound with respect to the claims occurrence intensity λ and the claim size d.f. F_Z is given in Section 6.3. The in general model is illustrated by two examples corresponding to the particular choice of the Lévy process U. One is given by the deterministic jumps at Poisson times:

$$U_t = \alpha t + \sum_1^m \alpha_i P_t(\lambda_i), \qquad \alpha > 0, \quad \alpha_i \in \mathbb{R}, \quad \min_i \alpha_i < 0,$$

where $P_t(\lambda_i)$ are independent Poisson processes with parameter λ_i. In this case the final stability bound is given w.r.t. the governing parameter $a = (\lambda, \lambda_1, \ldots, \lambda_m, F_Z, \beta)$, where β is the borrowing interest rate, see Theorem 6.9 and Example 6.10. In the other example we choose

$$U_t = \alpha t + \sigma W_t, \qquad \alpha,\ \sigma > 0,$$

where W is a standard Brownian motion. This corresponds to the Black-Scholes price model. The final bound w.r.t. the governing parameter $a = (\lambda, c, F_Z, \beta, \alpha, \sigma)$ is given in Theorem 6.9 and Example 6.11.

This chapter is based on Rusaityte [46].

In Chapter 7 we find stability bounds for the ruin probability in the Markov modulated risk model with Lévy process driven investments. We apply the regenerative approach developed by Kalashnikov which uses similar 'contraction' property to (1.5) with $\kappa < 1$. Stability bounds follow immediately from the later 'contraction' property and the uniformly for all $n \geq 0$ bounded w-moments of the reversed process $\{V_n\}$. These conditions are given in (C1) and (C2) in Section 7.2.

Similarly to the previous chapter, we treat separately three cases with respect to the Lévy processes driving investments. Depending on the state i of the modulating MC $\{I_n\}$, investments are driven by m independent Lévy processes U^i. The first model *without investments* corresponds to $U_i \equiv 0$ for all i. This model has been already considered in Chapter 5 using MC approach. Stability bounds in this chapter are derived under the same assumption which is related to the Cramér condition. The second case is model *with deterministic investments*, which is given by $U_t^i = \alpha_i t$ with positive constants α_i. Now we require that claims Z^i, where i refers to the state of the modulating chain, possess finite exponential moments. This implies an exponential upper bound for the ruin probability. In both cases we use the exponential weight functions w (with different exponents) in weighted uniform metric $d_\Psi = | \cdot |_w$. The final stability bounds for these two models are given in Subsections 7.2.1 and 7.2.2. The third, *general model*, is given by genuine Lévy processes U^i. In this case we use a power-law weight function to compare ruin probabilities. To our knowledge, asymptotic behaviour of the ruin probability in this models has not been investigated. The stability results suggest power-law upper bounds, but this question is beyond the scope of this thesis and has not been investigated.

To prove these bounds we have to show that conditions (C1) and (C2) holds for each model. These proofs are given in the Appendix. We use the ideas from Kalashnikov [30]. The main technical difficulty in the proof of (C1) is to construct a non-negative test function $\varphi(v, i)$ which is related to the weight function w (asymptotically) and s.t. $\mathcal{A}\varphi(v, i) \leq -\chi\varphi(v, i)$, where $\mathcal{A}$ is a generating operator of MC $\{V_n, I_n\}$, and χ is a positive constant. Then, (C1) follows from the regenerative property of $\{V_n, I_n\}$ and by Dynkin's formula. The proof of (C2) provides the explicit expression for the distance $d(a, a')$. For simplicity, we work under assumption that Lévy processes U'^i in the perturbed model are given by the time transformation $\{U'^i_t\} \stackrel{d}{=} \{U^i_{\beta_i t}\}$ (here $\stackrel{d}{=}$ stands for the identity of the finite-dimensional distributions).

Chapter 2

Risk models and ruin probability

In this chapter we introduce various risk models of interest and the corresponding notation. The first section is devoted to the description of a general risk process and the construction of the so-called reversed process; all the technical details in the following chapters will be carried out in terms of the latter process. In the second section we define those risk processes which will be investigated in the context of stability of ruin probabilities later in the thesis. Some facts about ruin probabilities corresponding to these risk processes are given in the third section.

2.1 General risk models and the concept of ruin probability

A *risk process* $\{R(t)\}_{t\geq 0}$ describes the capital or reserve of an insurance company at each instant of time t. We always assume it to have right-continuous paths with limits from the left for every $t > 0$. The value $R(0) = x$ is the *initial capital*. We assume that the main objects governing the risk process R are the following:

- The cash-flow process of premiums minus claims,

$$P_t = ct - \sum_{i=1}^{N_t} Z_i, \tag{2.1}$$

  where $c > 0$ is a constant premium rate; $\{Z_i\}_{i\geq 1}$ are the positive claims which occur at the random time-points generated by a point process N;

- Investments in a portfolio where a monetary unit at time 0 accumulates to $\exp(U_t)$ units at time t. Here U is a Lévy process called *the interest process*.

The cash-flow process P describes an insurance business in each time interval $[0, t]$. In the case of no investments, P coincides with the risk process R. In the literature one can find more general models, where P is defined as a Lévy process, see Kalashnikov & Norberg [31], or the premium rate may depend on the current reserve at each time (this can also be interpreted as interest), see Asmussen [3] for a discussion and further references. We confine ourselves to the model (2.1) which contains many important examples (the Cramér-Lundberg model, the renewal and Cox models, etc.) and suffices for our purposes. If one tries to capture more features of the real-life insurance business a risk model should also encounter investment possibilities and financial risks. It is often assumed that the cash-flow process P and the interest process U are independent. However, we do not impose such an assumption; in a Markov modulated case with investments these processes are in general dependent, see the corresponding example in Section 2.2.4.

Thus, given the risk process R, *ruin* is the event that the process R ever falls below 0,

$$\{R_t < 0 \text{ for some } t \geq 0\}. \tag{2.2}$$

The probability of ruin with initial capital x will always be denoted by ψ and is given by the quantity

$$\psi(x) = \mathbf{P}\left(\inf_{t\geq 0} R(t) < 0 \,\middle|\, R(0) = x\right), \qquad x > 0. \tag{2.3}$$

Note that there is no particular reason to associate the event of ruin with the level 0. It is for mathematical convenience only in order to simplify certain constructions such as the reversed process, see Section 2.1.1. But it presents no restriction either, since the initial capital x can always be adjusted in such a way that ruin occurs when the risk process falls below the level 0. Such a modification will be carried out in Section 2.2.3.

Aiming at calculating the ruin probability, it suffices to consider the risk process R at the sequence of random times

$$\mathcal{T} = \{T_i\}_{i\geq 0}, \qquad T_0 := 0,$$

where $\mathcal{T}$ comprises the claim occurrence times, i.e. the times when the jumps of the process N happen. This allows one to reduce the continuous time process $R(t)$ to the (discrete time) skeleton process $\{R_n\}$, where

$$R_n = R(T_n). \tag{2.4}$$

Then, the (ultimate) ruin probability in terms of the process $\{R_n\}$ is

$$\psi(x) = \mathbf{P}\left(\inf_{n\geq 0} R_n < 0 \mid R_0 = x\right). \tag{2.5}$$

The probability of ruin up to time T_n is denoted by

$$\psi^{(n)}(x) = \mathbf{P}\left(\min_{0\leq k\leq n} R_k < 0 \,\middle|\, R_0 = x\right).$$

In the sequel we assume that the following holds.

Assumption 2.1. Assume that

$$R_{n+1} = F(R_n, \sigma_{n+1}), \tag{2.6}$$

where $\{\sigma_n\}_{n\geq 1}$ is a (strictly) stationary sequence taking values in an appropriate Polish space Σ, and the function F is right-continuous with respect to its first variable and also strictly increasing, i.e.,

$$\{r < r'\} \Rightarrow \{F(r,\sigma) < F(r',\sigma)\}, \quad \forall \sigma \in \Sigma. \tag{2.7}$$

The condition (2.7) is natural. It infers whatever the given conditions (represented by σ), that if an insurance company has a surplus r which is less than or equal to r' then, at the next step, its reserve $F(r,\sigma)$ will still be less than or equal to $F(r',\sigma)$.

2.1.1 The reversed process and the ruin probability

In a sense, the risk process can be considered as the dual process of certain processes in queuing and storage theory. In particular one process can be constructed from the other one by an appropriate reversing procedure. The connection between the different areas was first discovered by Prabhu [42] in the queuing context. The duality was studied by Siegmund [48], Asmussen & Sigman [4] and many others, for a survey and further references see Asmussen [3].

The construction presented here is taken from Enikeeva *et al.* [14] and uses arguments similar to those in Asmussen & Kella [5] and Asmussen & Petersen [6].

For a fixed $N \geq 1$, define the process $\{V_n^{(N)}\}$ by the following recursive equation:

$$V_{n+1}^{(N)} = (V_n^{(N)} + \eta_{n+1}^{(N)})_+, \quad 0 \leq n \leq N-1, \quad V_0^{(N)} = 0, \tag{2.8}$$

where the random variables (r.v.'s) $\eta_n^{(N)}$ will be defined later. Then

$$V_n^{(N)} = \max(0, \eta_n^{(N)}, \eta_n^{(N)} + \eta_{n-1}^{(N)}, \dots, \eta_n^{(N)} + \dots + \eta_1^{(N)}), \quad 1 \leq n \leq N. \tag{2.9}$$

Let us assume that both $\{R_n\}$ and $\{\eta_n^{(N)}\}$ are defined on the same probability space and thus, $\{V_n^{(N)}\}$ is also defined on the same probability space. Now, let us require that the sequence $\{\eta_n^{(N)}\}_{1\leq n\leq N}$ satisfies the conditions

$$\left\{V_n^{(N)} \leq R_{N-n}\right\} \quad \Rightarrow \quad \left\{V_n^{(N)} + \eta_{n+1}^{(N)} \leq R_{N-n-1}\right\}, \tag{2.10}$$

$$\left\{V_n^{(N)} > R_{N-n}\right\} \quad \Rightarrow \quad \left\{V_n^{(N)} + \eta_{n+1}^{(N)} > R_{N-n-1}\right\}. \tag{2.11}$$

It is possible in these arguments to put $\eta_{n+1}^{(N)} = -(R_{N-n} - R_{N-n-1})$, but later on we need the more general conditions (2.10) and (2.11).

Lemma 2.2. *Given relations (2.10) and (2.11),*

$$\psi(x) = \lim_{N\to\infty} \mathbf{P}\left(V_N^{(N)} > x\right).$$

Proof. Define the stopping time

$$\tau = \min\{i \geq 1 : R_i < 0\},$$

where $\tau = \infty$ if $R_i \geq 0$ for all i. Let us fix $N > 0$ and assume that ruin occurs within $[1, N]$ that is, $\min_{1\leq i\leq N} R_i < 0$. Then

$$\psi^{(N)}(x) = \mathbf{P}(\tau \leq N \mid R_0 = x).$$

By (2.11),

$$\{\tau \leq N\} \subset \{R_\tau < V_{N-\tau}^{(N)}\} \subset \{R_{\tau-1} < V_{N-\tau+1}^{(N)}\} \subset \dots \subset \{x = R_0 < V_N^{(N)}\}$$

It follows that

$$\psi^{(N)}(x) \leq \mathbf{P}\left(V_N^{(N)} > x\right).$$

In the complementary case, there is no ruin within $[1, N]$, and hence, all $R_n \geq 0$, $0 \leq n \leq N$. But $V_0^{(N)} = 0 \leq R_N$. Therefore

$$\begin{aligned}
\{\tau > N\} &= \{V_0^{(N)} \leq R_N\} \cap \{\tau > N\} \\
&\subset \{V_0^{(N)} + \eta_1^{(N)} \leq R_{N-1}\} \cap \{\tau > N\} \\
&= \{(V_0^{(N)} + \eta_1^{(N)})_+ \leq R_{N-1}\} \cap \{\tau > N\} \\
&= \{V_1^{(N)} \leq R_{N-1}\} \cap \{\tau > N\} \subset \cdots \subset \{V_N^{(N)} \leq R_0 = x\} \cap \{\tau > N\} \\
&\subset \{V_N^{(N)} \leq R_0 = x\}
\end{aligned}$$

which yields that

$$1 - \psi^{(N)}(x) \leq \mathbf{P}\left(V_N^{(N)} \leq x\right) = 1 - \mathbf{P}\left(V_N^{(N)} > x\right),$$

or

$$\mathbf{P}\left(V_N^{(N)} > x\right) \leq \psi^{(N)}(x).$$

Therefore

$$\psi^{(N)}(x) = \mathbf{P}\left(V_N^{(N)} > x\right),$$

which completes the proof. □

The proof of Lemma 2.2 is mostly algebraic than probabilistic. It does not use any of the distributional assumptions on the process R. Using Assumption 2.1, we will give an explicit construction of the process $V^{(N)}$. For fixed $\sigma \in \Sigma$ denote

$$F^{-1}(v, \sigma) = \inf\{r : F(r, \sigma) \geq v\}. \tag{2.12}$$

Under Assumption 2.1 and for fixed $N > 0$, let

$$\eta_n^{(N)} = F^{-1}(V_{n-1}^{(N)}, \sigma_{N-n+1}) - V_{n-1}^{(N)}. \tag{2.13}$$

We shall verify conditions (2.10) and (2.11) for the corresponding process $\{V_n^{(N)}\}$ given by (2.8). If $V_n^{(N)} \leq R_{N-n}$, then, by monotonicity of F^{-1},

$$V_n^{(N)} + \eta_{n+1}^{(N)} = F^{-1}(V_n^{(N)}, \sigma_{N-n}) \leq F^{-1}(R_{N-n}, \sigma_{N-n}) = R_{N-n-1}.$$

If $V_n^{(N)} \geq R_{N-n}$, then

$$F^{-1}(V_n^{(N)}, \sigma_{N-n}) \geq F^{-1}(R_{N-n}, \sigma_{N-n}) = R_{N-n-1}.$$

Thus, (2.10) and (2.11) hold and, hence,

$$\psi(x) = \lim_{N\to\infty} \mathbf{P}\left(V_N^{(N)} > x\right).$$

Without loss of generality the sequence $\{\sigma_n\}_{n\geq 1}$ can be embedded in the stationary sequence $\{\sigma_n\}_{-\infty<n<\infty}$. Thus, we can define the sequence $\{V_n\}_{n\geq 0}$ by the relation

$$V_{n+1} = (V_n + \eta_{n+1})_+ , \quad V_0 = 0, \tag{2.14}$$

where

$$\eta_{n+1} = F^{-1}(V_n, \sigma_{-n}) - V_n. \tag{2.15}$$

Evidently, the following equality in distribution holds

$$\{V_n\}_{0\le n\le N} \overset{d}{=} \{V_n^{(N)}\}_{0\le n\le N},$$

and therefore, by Lemma 2.2,

$$\psi(x) = \lim_{n\to\infty} \mathbf{P}(V_n > x). \tag{2.16}$$

The process $\{V_n\}$ will be referred to as the *reversed process*.

The sequence $\{V_n\}$ can be rather general. In order to use standard techniques it is convenient to embed this sequence into a Markov chain (cf. Cox [11], Gnedenko and Kovalenko [18]).

Assumption 2.3. Assume that the sequence $\{V_n\}$ defined by (2.14) can be embedded into a Markov chain

$$W_n = (V_n, \tau_n)\,, \tag{2.17}$$

where $\{\tau_n\}$ is a sequence taking values in a Polish space $\mathbb{T}$.

For any Borel set $B \in \mathbb{R}_+ \times \mathbb{T}$, denote by

$$P(w, B) = \mathbf{P}\left(W_{n+1} = (V_{n+1}, \tau_n) \in B \mid W_n = w\right)$$

the transition probability of the chain (2.17) and by

$$\mathfrak{P}f(w) = \mathbb{E}\left(f(W_{n+1}) \mid W_n = w\right) \tag{2.18}$$

the corresponding *shift operator*. The operator $\mathfrak{P}$ is defined for all measurable functions $f : \mathbb{W} \to \mathbb{R}$ such that the right-hand side of (2.18) is finite.

2.2 Specific models

In this section we introduce more specific risk models to be investigated in the thesis.

2.2.1 The Sparre Andersen risk model

The *Sparre Andersen (S.A.)* risk model assumes continuously paid premiums with intensity $c > 0$ and independent identically distributed (i.i.d.) positive claims Z_n generated by an ordinary renewal process $\{N_t\}_{t\ge 0}$. The skeleton risk process $R = \{R_n\}_{n\ge 0}$, see (2.4), is given by the recursion

$$R_{n+1} = R_n + c\theta_n - Z_n, \qquad R_0 = x, \tag{2.19}$$

where θ_n are the i.i.d. inter-arrival times with distribution function (d.f.) F_θ. This model was introduced as a generalization of a classical risk model by Andersen [1]. In the literature the S.A. model is also called *an ordinary renewal* risk process, to distinguish it from the case when θ_1 has a different distribution, see Grandell [21].

To describe the S.A. risk process it suffices to determine the premium rate c and the distribution functions F_Z and F_θ. Thus we will refer to a S.A. risk model as to a triple (c, F_θ, F_Z).

If $\{N_t\}$ is given by a homogeneous Poisson process (with parameter λ, say) then the S.A. model reduces to the well-known *classical* or *Cramér-Lundberg* risk model. The triple (c, λ, F_Z) will be used to describe a classical risk process.

2.2.2 Markov modulated model

In this section we assume that the underlying point process N is a certain Cox process. Such risk models were extensively treated in Grandell [21], see also Björk & Grandell [8] .

Assume there are m independent classical risk processes R^i governed by the corresponding parameters $a_i = (c_i, \lambda_i^{cl}, F_{Z^i})$. Here λ_i^{cl} stands for the *claims* occurrence intensity of the $i$ht process, c_i is the corresponding premium rte and F_{Z^i} is the distribution of the i.i.d. positive claims Z_j^i. The corresponding Markov modulated risk process with underlying modulating Markov chain is defined as follows.

Markov modulation. Let $J = \{J_t\}_{t\geq 0}$ be a Markov process with state space $\mathbb{E} = \{1, 2, \ldots, m\}$. It leaves state i with intensity λ_i^J and, upon leaving i, it jumps to state j $(j \neq i)$ with probability p_{ij}^J. The risk process R behaves like R^i while the underlying process $J = \{J_t\}_{t\geq 0}$ is in state $i \in \mathbb{E}$.

Let $\{T_n\}_{n\geq 0}$ be the increasing sequence of both, claims occurrence times and jump times of J. The corresponding sequence of inter-occurrence times is $\{\theta_n\}_{n\geq 0}$, where $\theta_n = T_n - T_{n-1}$, $n \geq 1$. Set $\mathcal{I}_n = J_{T_n}$. Then the r.v.'s $\{\theta_k, k \in \{l : \mathcal{I}_l = i\}\}$ are i.i.d. exponentially distributed with parameter

$$\lambda_i = \lambda_i^{cl} + \lambda_i^J. \tag{2.20}$$

The above defined process $\mathcal{I} = \{\mathcal{I}_n\}_{n\geq 0}$ is a Markov chain with transition probabilities

$$p_{ij} = \begin{cases} \lambda_i^J p_{ij}^J / \lambda_i, & j \neq i, \\ \lambda_i^{cl} / \lambda_i, & j = i. \end{cases} \tag{2.21}$$

We assume that the chain $\mathcal{I}$ has the unique stationary distribution $\{\pi_i\}_{i\in\mathbb{E}}$ and that at time 0 it is in steady state, i.e. $\mathbf{P}(\mathcal{I}_0 = i) = \pi_i$. The existence of $\{\pi_i\}_{i\in\mathbb{E}}$ follows, for example, from the existence of the probabilities $\{\pi_i^J\}_{i\in\mathbb{E}}$ such that

$$\pi_j^J = \sum_{i\in\mathbb{E}} \pi_i^J p_{ij}^J. \tag{2.22}$$

Then the probabilities π_i are given by

$$\pi_i = \pi_i^J \left(1 + \frac{\lambda_i}{\lambda_i^J}\right) \left(\sum_{i\in\mathbb{E}} \pi_i^J \left(1 + \frac{\lambda_i}{\lambda_i^J}\right)\right)^{-1}. \tag{2.23}$$

Since ruin can only occur at the claim arrival times, it suffices to consider the risk process R at the times T_n, where it is given by the recursion

$$R_{n+1} = R_n + c_{I_n}\theta^{I_n} - \delta_{I_n I_{n+1}} Z_n^{I_{n+1}}, \qquad R_0 = x, \tag{2.24}$$

where δ_{ij} is the Kronecker symbol and θ^i is exponentially distributed with parameter λ_i, $i \in \mathbb{E}$. Notice that the process R satisfies Assumption 2.1 with $\sigma_n = c_{I_n}\theta^{I_n} - \delta_{I_n I_{n+1}} Z_n^{I_{n+1}}$.

2.2.3 Interest and borrowing

Now we consider the situation when an insurance company with reserve process R^0 also faces a financial risk due to random interest. The cash-flow process P is given by (2.1), where N is a homogeneous Poisson process with parameter λ and the claims Z_n are i.i.d with distribution F_Z. Thus, P is described by the parameters (c, λ, F_Z). The risk process $R^0 = \{R^0(t)\}_{t\geq 0}$ with initial capital $R^0(0) = x^0$ is defined by the following conditions:

(C1) The capital in excess of the level $D^0 > 0$ is invested in a portfolio with price process $\exp(U_t)$, where $U = \{U_t\}$ is a Lévy process on $(-\infty, \infty)$, i.e., if $R^0(u) \geq D^0$ for $u \in [s, s+t)$, then

$$R^0(s+t) = e^{U_{s+t}-U_s}\left(R^0(s) + \int_s^{s+t} e^{U_s - U_u}\, dP_u - D^0\right) + D^0. \tag{2.25}$$

We assume that U is independent of all other processes and r.v.'s of the model.

(C2) There is a minimal requirement of liquid reserve of size d^0, $d^0 < D^0$, and if at some $t \geq 0$ the reserve $R^0(t)$ falls below the level d^0 then the amount $d^0 - R^0(t)$ is borrowed at the constant interest rate $\beta > 0$.

It follows from the assumption (C2) that once the level $d^0 - c/\beta$ has been reached ruin is unavoidable since the risk process will decay thereafter. Therefore, the event

$$\{R^0(t) \leq d^0 - c/\beta \quad \text{for some } t \geq 0\}$$

is called *ruin*.

This risk model can be considered as a generalization of the model in Embrechts & Schmidli [13] where an insurance company invests its money at a constant interest rate (thus, U is a deterministic function). However, this modification is essential, since randomness in the investments leads to a different asymptotic behavior of the ruin probability, see Section 2.3. Ruin probabilities in a model with investments driven by a Lévy process like in (2.25) were investigated by Kalashnikov & Norberg [31]. The above model can be considered as a special case in their set up.

Risk process. For technical reasons it is more convenient if ruin is associated with hitting the level 0; see the discussion following formula (2.3). Therefore, we shall transform our initial risk process R^0 by defining the process

$$R(t) = R^0(t) + \frac{c}{\beta} - d^0, \qquad R(0) = x^0 + \frac{c}{\beta} - d^0 =: x. \tag{2.26}$$

Now, ruin in terms of the process R is the event

$$\{R(t) < 0 \text{ for some } t \geq 0\}.$$

Further on we only deal with the process R which has the same cash-flow process P as R^0 and it satisfies the conditions (C1)-(C2) with D^0 and d^0 replaced by $D = D^0 + c/\beta - d^0$ and $d = c/\beta$, respectively.

Since ruin can only occur at the claim occurrence times T_i, $i \geq 1$, it suffices to study ruin for the skeleton risk process $\{R_n\}_{n\geq 0}$,

$$R_n := R(T_n), \quad n \geq 1, \qquad R_0 := R(0). \tag{2.27}$$

2.2.4 Markov modulation and interest

In this section we consider both random interest and Markov modulation. We start with the basic process R^* given by

$$R^*(s+t) = e^{U_{s+t}-U_s}\left(R^*(s) + \int_s^{s+t} e^{U_s - U_u}\, dP_u\right), \tag{2.28}$$

where P is of the form (2.1) governed by the triple (c, λ^{cl}, F_Z) and U is a Lévy process. Ruin probabilities for such risk processes were investigated by Kalashnikov & Norberg [31], see Section 2.3.3 for their main result.

Assume there are m independent processes R^i of the type (2.28). The notation corresponding to the ith process is equipped with an index i; upper-i for random processes and lower-i for constants (i.e., R^i, R^i, P^i, N^i and c_i, λ_i^{cl}). The underlying modulating Markov process J with state space $\mathbb{E} = \{1, \ldots, m\}$ is as in Section 2.2.2. The corresponding Markov modulated risk process R is defined in the same manner, see Section 2.2.2.

Since ruin can only occur at the claim arrival times, it suffices to consider the risk process R at the times T_n, denoted by $R_n = R(T_n)$, where $\{T_n\}_{n\geq 0}$ is the increasing sequence of claim occurrence times and jump times of J, see Section 2.2.2). The process $\{R_n\}$ is given by the following recursion

$$R_{n+1} = \zeta_{n+1}^{\mathcal{I}_{n+1}}\left(R_n + \chi_{n+1}^{\mathcal{I}_n, \mathcal{I}_{n+1}}\right), \qquad R_0 = x, \tag{2.29}$$

where

$$\begin{aligned}
\zeta_{n+1}^i &= \exp\left(U_{T_{n+1}}^i - U_{T_n}^i\right), \\
\chi_{n+1}^{ji} &= -\delta_{ji} Z_n^i \exp\left(U_{T_{n+1}}^i - U_{T_n}^i\right) + c_i \int_{T_n}^{T_{n+1}} \exp\left(U_{T_{n+1}}^i - U_u^i\right) du,
\end{aligned}$$

and δ_{ij} is the Kronecker symbol.

2.3 About ruin probabilities

In this section we give some well known facts about the asymptotic behavior and bounds of the ruin probabilities for the previously introduced risk models. Such results can give an idea about the quality of the stability bounds. We do not aim at giving a complete survey about the results in the literature; we are only interested in those which hold under similar conditions as the corresponding stability bounds to be derived in this thesis. Note also, that the Markov modulated model with investments has not been investigated so far (except for the degenerate case when the Lévy process U is identically equal to 0).

2.3.1 The Sparre Andersen model

Assume that the S.A. model (c, F_θ, F_Z) satisfies the *net profit* condition $c\mathbb{E}\theta_n > \mathbb{E}Z_n$ (thus we also assume finite mean values); otherwise ruin occurs with probability (w.p.) 1.

Assumption 2.4. There exists a positive constant r^* such that

$$\mathbb{E}\exp(r^*(Z_1 - c\theta_1)) = 1. \tag{2.30}$$

The value r^* is the *Lundberg exponent* (see Grandell [21]) and Assumption 2.4 is called the *Cramér condition* (for S.A. risk model). In the literature, r^* is also called *adjustment coefficient* (see Gerber [19], Asmussen [3]).

Under the Cramér condition the ruin probability ψ satisfies the *Lundberg inequality*

$$\psi(x) \le e^{-r^* x}. \tag{2.31}$$

If, in addition, $\int_0^\infty z e^{r^* z}\, dF_Z(z) < \infty$, then the *Cramér-Lundberg approximation*

$$\lim_{x\to\infty} e^{r^* x}\psi(x) = C \tag{2.32}$$

holds, where $C \in (0,1)$ is an explicit constant, see Grandell [21], Asmussen [3], Rolski *et al.* [45] and others.

Assumption 2.4 implies that the claims Z_i are 'light-tailed' in the sense that the moment generating function $M_Z(r) := \mathbb{E}\exp(rZ_i)$ exists for some positive r. The case $M_Z(r) = \infty$ for all $r > 0$ refers to 'heavy-tailed' claims.

2.3.2 The Markov modulated model

Ruin probabilities in the Markov modulated risk model (or risk model in Markovian environment) were treated by a number of authors: the upper 'Lundberg-type' bounds were investigated by Björk & Grandell [8] and Grandell [21] using martingale techniques; the Cramér-Lundberg approximation based on the Wiener-Hopf factorization technique is considered in Asmussen [2]. The Lundberg bound in this section is taken from Grandell [21] (it also holds for a more general risk model with 'Markov renewal intensity', see Grandell [21] p.105).

We adopt the notation from Section 2.2.2. Assume that $c_i = c$, $F_{Z^i} = F_Z$ for all $i \in \mathbb{E}$, and for some $\varepsilon > 0$,

$$h(\varepsilon) := \int_0^\infty e^{\varepsilon z}\, dF_Z(z) - 1 < \infty.$$

Let P^J be the matrix (p^J_{ij}) and $\operatorname{diag}(\phi(\varepsilon))$ be the diagonal matrix $(\delta_{ij}\phi_i(\varepsilon))$, where

$$\phi_i(\varepsilon) := \mathbb{E}\exp\left(-\varepsilon c\bar\theta^i + h(\varepsilon)\lambda_i^{cl}\bar\theta^i\right),$$

and $\bar\theta^i \sim Exp(\lambda_i^J)$.

Proposition 2.5. *Let* $\varepsilon^* = \sup\left\{\varepsilon \ge 0 : \|\operatorname{diag}(\phi(\varepsilon))P^J\|_{SP} < 1\right\}$*, where* $\|\cdot\|_{SP}$ *is the spectral radius. Then for any* $\varepsilon \in (0, \varepsilon^*)$ *we have*

$$\psi(x) \le C(\varepsilon^* - \varepsilon)e^{-(\varepsilon^*-\varepsilon)x},$$

for some finite constant $C(\varepsilon^* - \varepsilon)$*.*

2.3.3 Random interest

The ruin probability for the risk process R^* in (2.28) was investigated by Kalashnikov & Norberg [31]. In particular, they considered the case when assets may yield negative interest and found upper and lower bounds for the ruin probability. They showed under certain Cramér type conditions that the ruin probability decays like a power function, and they determined upper and lower bounds for the probability of ruin of Cramér-Lundberg type.

Let R^* be the risk process defined by (2.28) and introduce a r.v. θ which is exponentially distributed with parameter λ^{cl} (it can be considered as a generic claim inter-arrival time r.v.). In what follows we assume that

$$\mathbf{P}\left(e^{U_\theta} < 1\right) > 0, \qquad \mathbf{P}\left(e^{U_\theta} \leq 1,\ \int_0^\theta e^{U_\theta - U_u}\, du - Z_1 < 0\right) > 0 \tag{2.33}$$

This requires that the interest process U takes negative values with positive probability, and the later condition holds if, for example, the claims Z_i have infinite support.

Assumption 2.6. Assume that for some $\varepsilon^* > 0$,

$$\mathbb{E}\exp(-\varepsilon^* U_\theta) = 1 \tag{2.34}$$

Condition (2.34) is called *Cramér condition* in Kalashnikov & Norberg [31] (for the risk process R^*). They proved the following result.

Proposition 2.7. *Under the Assumption 2.6, condition (2.33), if the claims Z_i have finite ε^*th-moment and for some $\varepsilon > 0$, $\mathbb{E}\exp(-(\varepsilon^* + \varepsilon)R_\theta) < \infty$, then for any $\delta > 0$ there exist finite constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 x^{-\varepsilon^* - \delta} \lesssim \psi(x) \leq C_2 x^{-\varepsilon^* + \delta}, \qquad x \to \infty,$$

where $f(x) \lesssim g(x)$ means $\limsup_{x\to\infty} \frac{f(x)}{g(x)} \leq 1$.

The upper bound can be considered as an analogue of the Lundberg inequality. The lower bound indicates that ε^* is the 'correct' exponent.

Remark 2.8. Condition (2.34) is needed for the lower bound only. If (2.34) it is not satisfied, one still has the upper bound, although, one cannot say that ε^* is the "correct' exponent (and not even that the ruin probability has power law decay).

Remark 2.9. The result also holds if only capital in excess of a certain level x^* is invested, and below this level the ruin probability satisfies $\inf_{x \leq x^*} \psi(x) > 0$ (see Section 2.3 in Kalashnikov & Norberg [31]). This is obviously the case in a risk model with borrowing.

Borrowing and interest. As it was mentioned in Remark 2.9, the result in Proposition 2.7 also holds in the current situation with borrowing. However, if the interest is deterministic (or absent), this bound is not satisfactory. The case $U = 0$ was treated in the previous section. The case $U_t = \alpha t$ for some $\alpha > 0$ was investigated by Embrechts & Schmidli [13]. They proved that, if

$$\varepsilon^* = \sup\{r : \mathbb{E}\exp(rZ_1) < \infty\} < \infty, \tag{2.35}$$

then for any $\delta > 0$,

$$\lim_{u\to\infty} \psi(x)e^{(\varepsilon^* - \delta)x} = 0, \quad \lim_{u\to\infty} \psi(x)e^{(\varepsilon^* + \delta)x} = \infty.$$

Markov modulation and random interest. The decay of the ruin probability has not been investigated so far.

Chapter 3

Mathematical background

In this chapter we give the necessary mathematical background on which the results of the thesis rely.

Throughout the chapter $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ denotes a measurable space on which all r.v.'s and processes are defined. The collection of all measurable finite-valued functions on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is denoted by $\mathcal{F}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ is the space of finite signed measures. The positive elements in the corresponding spaces are denoted by $\mathcal{F}(\mathcal{X})^+$ and $\mathcal{M}(\mathcal{X})^+$, respectively.

3.1 Markov processes

Markov process play a very important role throughout the thesis. In this section we give the necessary theoretical background from Markov process theory. While the terminology and results for denumerable Markov chains (MC's) (i.e. the state space $\mathcal{X}$ is countable or finite) can be found in many books, like Feller [15], Kalashnikov [29] and others, the terminology referring to general MC's (which means that $\mathcal{X} \subseteq \mathbb{R}^k$) is not unique. For general MC's we refer to Borovkov [10], Meyn & Tweedie [38], Nummelin [40]. The most important results for our purpose are the stability results for the stationary distribution of a MC given in Section 3.2.

Let $X = \{X_n\}$ be a $\mathbf{P}$-measurable homogeneous Markov chain on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with transition probability kernel P, i.e.,

$$\mathbf{P}(X_{k+1} \in \cdot \,|X_k = x) = P(x, \cdot).$$

The n-step transition probabilities $\mathbf{P}(X_n \in \cdot \,|X_0 = x)$ are denoted by $P^n(x, \cdot)$, i.e.,

$$P^n(x, \cdot) = \int P^{n-1}(x, dy)P(y, \cdot).$$

Many properties (concerning ergodicity, finiteness of moments, etc.) of MCs can be proved by using 'local' characteristics. Often such characteristics are formulated in terms of the generating operator.

Definition 3.1. The operator $\mathcal{A} : \mathcal{F}(\mathcal{X}) \to \mathbb{R}$ defined by $\mathcal{A}f(x) = \mathbb{E}_x [f(X_1)] - f(x)$ is called the *generating operator* of the MC X. In the literature, $\mathcal{A}$ is also called the *drift operator*, cf., for example Meyn & Tweedie [38]. If $\mathcal{A}f(x)$ is finite for all $x \in \mathcal{X}$ then f is said to belong to the domain of the definition of $\mathcal{A}$; we write $f \in \mathcal{D}_{\mathcal{A}}$.

An important tool throughout this thesis is *Dynkin's formula* which we give next; cf. Kalashnikov [29] for the proof.

Proposition 3.2 (Dynkin's formula). *If τ is a stopping time w.r.t. the filtration generated by the MC X such that $\mathbb{E}_x\tau < \infty$, $f \in \mathcal{D}_\mathcal{A}$, and for some constant $v < \infty$*

$$\sup_{x\in\mathcal{X}} \mathbb{E}_x|f(X_1) - f(x)| \le v,$$

then

$$\mathbb{E}_x f(X_\tau) = f(x) + \mathbb{E}_x\left[\sum_{k<\tau} \mathcal{A}f(X_k)\right].$$

Definition 3.3. The measure $\pi \in \mathcal{M}(\mathcal{X})^+$ is called *invariant* w.r.t. the Markov chain X if

$$\pi(B) = \int_\mathcal{X} \pi(dx)P(x, B) \qquad \text{for all } B \in \mathcal{B}(\mathcal{X}).$$

It is an *invariant probability measure* if it is invariant w.r.t. to X and $\pi(\mathcal{X}) = 1$.

In what follows, we give some results about the existence of the invariant probability measure π and the convergence of P^n to π. Define

$$\tau_B := \min\{k \ge 1 : X_k \in B\} \quad \text{and} \quad \tau_x := \tau_{\{x\}}. \tag{3.1}$$

We will use the notation $\mathbf{P}_x(\cdot) = \mathbf{P}(\cdot|X_0 = x)$ and $\mathbb{E}_x(\cdot) = \mathbb{E}_x(\cdot|X_0 = x)$.

Denumerable Markov chains

In this case the state space $\mathcal{X}$ is finite or countable. By p^n_{xy} we denote the n-step transition probability from state x to y, $p^1_{xy} =: p_{xy}$.

The MC X is called *irreducible* if for any $x, y \in \mathcal{X}$ there exists $n \ge 1$ such that $p^n_{xy} > 0$. The irreducible MC X is called *recurrent* if for at least one state $x \in \mathcal{X}$,

$$\mathbf{P}_x(\tau_x < \infty) = 1. \tag{3.2}$$

If in addition

$$\mathbb{E}_x(\tau_x) < \infty, \tag{3.3}$$

then X is *positive recurrent.*

The state $x \in \mathcal{X}$ is called *periodic* with *period d* if

$$d = \text{g.c.d.}\{n : p^n_{xx} > 0\},$$

(g.c.d. stands for the greatest common divisor). If a state has period 1 then it is called *aperiodic.*

It is well-known that all states of an irreducible MC share various properties, for example, they all have the same period and relation (3.2) holds for each $x \in \mathcal{X}$. If X is also positive recurrent then (3.3) holds for each $x \in \mathcal{X}$.

Proposition 3.4. *If X is an irreducible positive recurrent MC then there exists a unique invariant probability measure π. If X is also aperiodic, then $p^n_{xy} \to \pi_y$ as $n \to \infty$.*

General Markov chains

Now we consider MCs in the space $\mathcal{X} \subseteq \mathbb{R}^k$.

Definition 3.5. The MC X is *φ-recurrent* for $\varphi \in \mathcal{M}(\mathcal{X})^+$ if

$$\mathbf{P}_x(\tau_B < \infty) = 1 \qquad \forall x \in \mathcal{X} \tag{3.4}$$

for each $B \in \mathcal{B}(\mathcal{X})$ such that $\varphi(B) > 0$. If $\mathbf{P}_x(\tau_B < \infty) > 0$ then the MC X is *φ-irreducible.*

The set $A \in \mathcal{B}(\mathcal{X})$ is called *an atom* for X if there exists a measure $\nu \in \mathcal{M}(\mathcal{X})^+$ such that $P(x, B) = \nu(B)$ for all $B \in \mathcal{B}(\mathcal{X})$ and $x \in A$.

If X is φ-irreducible and $\varphi(A) > 0$ then A is an *accessible atom.*

Proposition 3.6 (Meyn & Tweedie [38], p. 234). *If X is φ-irreducible and contains an accessible atom A such that*

$$\mathbb{E}_A \tau_A < \infty \tag{3.5}$$

then a unique stationary distribution π exists.

Definition 3.7. A φ-irreducible MC is called *periodic* if there exists a finite collection of disjoint sets $\mathcal{X}_i \subset \mathcal{X}$, $i = 1, \ldots, d$ such that

(i) for $x \in \mathcal{X}_i$, $\mathbf{P}(x, \mathcal{X}_{i+1}) = 1, i = 0, \ldots, d-1 (\mathrm{mod}\ d)$

(ii) $\varphi\left(\mathcal{X} \setminus \cup_{i=1}^d \mathcal{X}_i\right) = 0.$

The maximal number d satisfying the above conditions is called *the period* of the MC X. If $d = 1$ then X is *aperiodic.*

The following proposition is taken from Borovkov [10], p. 17 .

Proposition 3.8. *If the MC X is aperiodic and there exist a set $V \in \mathcal{B}(\mathcal{X})$, a probability measure φ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, a number $p \in (0, 1)$ and $n \in \mathbb{N}$ such that*

(I) $\mathbf{P}_x(\tau_V < \infty) = 1$*, for all $x \in \mathcal{X}$,*

(II) $\sup_{x \in V} \mathbb{E}_x \tau_V < \infty$*,*

(III) *$P^n(x, B) > p\varphi(B)$ for all $x \in V$ and $B \in \mathcal{B}(\mathcal{X})$,*

then there exists a unique invariant probability measure π, and

$$\|P^n(x, \cdot) - \pi(\cdot)\| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Here $\|P^n(x, \cdot) - \pi(\cdot)\| := \int_{\mathcal{X}} |P^n(x, dx) - \pi(dx)|$ is the total variation distance, see Section 3.5 for more details.

Remark 3.9. A set $V \in \mathcal{B}(\mathcal{X})$ satisfying condition (I) is called *strongly recurrent* (see Kalashnikov [29], p. 153). In specific problems, this condition usually follows from condition (II), since one often obtains bounds for $\mathbb{E}_x \tau_V$ for all $x \in \mathcal{X}$. In case that $\mathbb{E}_x \tau_V < \infty$ for all $x \in \mathcal{X}$, the set V is called *positive* (see Kalashnikov [29], p. 155).

Remark 3.10. In order to prove conditions like (II) or (3.5), it is convenient to use test functions. The corresponding results are given in Section 3.4.

Example 3.11. Let X be a reflected random walk given by the recursive equations

$$X_0 = 0, \qquad X_{n+1} = (X_n + \eta_n)_+,$$

where $\{\eta_n\}_{n\geq 1}$ is a sequence of i.i.d. r.v.'s with d.f. F_η. If $\mathbb{E}\eta_1 < 0$ then the MC X has a unique stationary distribution π. This example is standard; we include a proof for the sake of illustration.

Proof. Take the probability measure φ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ which is concentrated at 0, i.e., $\varphi(\{0\}) = 1$. Notice that X is φ-irreducible since $\mathbf{P}(\eta_1 < 0) > 0$ (see also Example 2.1 in Nummelin [40]) and that the set $\{0\}$ is an accessible atom.

In order to prove the condition (3.5) for $A = \{0\}$, notice that $\tau_0 = \min\{n > 0 : S_n \leq 0\} =: \tau^S_{(-\infty,0]}$ a.s., where $S_n := \eta_1 + \cdots + \eta_n$ and $F_\eta(x) = \mathbf{P}(\eta_1 \leq x)$, $x \in \mathbb{R}$. Denote by $\mathcal{A}^S$ the generating operator of the MC $\{S_n\}$ and apply Proposition 3.24 from Section 3.4 with the test function

$$g(x) = \begin{cases} (x+\varepsilon)_+, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where the constant $\varepsilon > 0$ is such that $\int_{-\varepsilon}^{\infty} y\, dF_\eta(y) < 0$ (this is possible since $\mathbb{E}\eta_1 < 0$). For $x > 0$,

$$\begin{aligned} \mathcal{A}^S g(x) &= \int_{-x-\varepsilon}^{\infty} (x+\varepsilon+y)\, dF_\eta(y) - (x+\varepsilon) \\ &= -(x+\varepsilon)\mathbf{P}(\eta_1 \leq -(x+\varepsilon)) + \int_{-x-\varepsilon}^{\infty} y\, dF_\eta(y) \\ &\leq \int_{-\varepsilon}^{\infty} y\, dF_\eta(y) < 0. \end{aligned}$$

When $x \leq 0$, $\mathcal{A}^S g(x) \leq \mathbb{E}(\varepsilon + \eta_1)_+ \leq \varepsilon$. Thus, $\mathbb{E}\tau^S_{(-\infty,0]} = \mathbb{E}\tau_0 < \infty$.

Proposition 3.8 yields that $P^n \to \pi$ in the total variation metric (take the set $V = \{0\}$, then conditions (I) and (II) are already proved and (III) holds trivially). ☐

3.2 Stability of Markov processes

This section is devoted to stability bounds for the stationary distribution of a Markov chain obtained by Kartashov, see [35] and [34]. These results serve as the main source for the stability bounds of ruin probabilities obtained by the MC approach.

First we need some results on the norms of the kernel operators.

3.2.1 Kernel operators and their norms

Let K be a kernel on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, i.e., for each fixed $\Gamma \in \mathcal{B}(\mathcal{X})$, $K(\cdot, \Gamma)$ is a measurable function and for each fixed $x \in \mathcal{X}$, $K(x, \cdot) \in \mathcal{M}(\mathcal{X})$. A non-negative kernel with $K(x, \mathcal{X}) = 1$ can be considered as a transition probability kernel P of a certain Markov chain.

One can associate the following two linear operators with the kernel $K(x,\Gamma)$. The first one is the *shift operator* $\mathfrak{K}$ which operates on the functions $f \in \mathcal{F}(\mathcal{X})$ according to the rule

$$\mathfrak{K}f(x) := \int_{\mathcal{X}} f(y)\, K(x,dy), \quad x \in \mathcal{X}. \tag{3.7}$$

The other operator (denoted by $\mathfrak{K}^*$) acts on the finite measures $\mu \in \mathcal{M}(\mathcal{X})$ according to the rule

$$\mathfrak{K}^*\mu(\Gamma) := \int_{\mathcal{X}} K(x,\Gamma)\, \mu(dx), \quad \Gamma \in \mathcal{B}(\mathcal{X}). \tag{3.8}$$

Assume that $\mathcal{M} \subseteq \mathcal{M}(\mathcal{X})$ is a Banach space with a norm $\|\cdot\|$ satisfying the following consistency with the order structure in $\mathcal{M}(\mathcal{X})$ conditions,

$$\begin{aligned}
\|\mu_1\| &\leq \|\mu_1+\mu_2\|, \quad \mu_1,\, \mu_2 \in \mathcal{M}^+, & (3.9)\\
\|\mu_1-\mu_2\| &= \|\mu_1+\mu_2\|, \quad \text{for } \mu_1 \perp \mu_2,\ \mu_1,\mu_2 \in \mathcal{M}^+ & (3.10)\\
|\mu(\mathcal{X})| &\leq \kappa\, \|\mu\|, \quad \mu \in \mathcal{M} & (3.11)
\end{aligned}$$

where $\mathcal{M}^+$ is the positive cone in the spaces $\mathcal{M}$, and κ is a finite constant.

Introduce the dual space $\mathcal{F}$ to the space $\mathcal{M}$ where each element $f^* \in \mathcal{F}$ corresponds to a real-valued measurable function $f : \mathcal{X} \to \mathbb{R}$ according to the rule

$$f^*(\mu) := \int_{\mathcal{X}} f(x)\, \mu(dx) \tag{3.12}$$

with finite norm

$$\|f^*\| = \sup\left\{\left|\int_{\mathcal{X}} f(x)\mu(dx)\right|,\ \|\mu\| \leq 1,\ \mu \in \mathcal{M}\right\}.$$

Slightly abusing notation, the same symbol is used to denote norms in different spaces; the space we refer to will be clear from the context. Let $\mathcal{F}^+$ be the positive cone in $\mathcal{F}$.

The operator $\mathfrak{K}^*$ corresponding to the kernel K such that $\mathfrak{K}^*\mathcal{M} \subseteq \mathcal{M}$, has a finite norm

$$\|\mathfrak{K}^*\| = \sup\{\|\mathfrak{K}^*\mu\|,\ \|\mu\| \leq 1,\ \mu \in \mathcal{M}\},$$

and by the duality of the operators $\mathfrak{K}$ and $\mathfrak{K}^*$ (see Kolmogorov & Fomin [36]),

$$\|\mathfrak{K}^*\| = \|\mathfrak{K}\|.$$

In order to derive stbaility bounds for ruin probabilities it will be natural to use *the weighted total variation norm* $\|\cdot\|_w$ defined by

$$\|\mu\|_w = \int_0^\infty w(x)\, |\mu|(dx), \tag{3.13}$$

where $|\mu|$ is the variation of a measure μ and w is a positive measurable function bounded away from 0. This norm induces the following norm of the operators (see Kartashov [35, 34] or Meyn & Tweedie [38, Section 16.1]),

$$\|\mathfrak{K}\|_w = \|\mathfrak{K}^*\|_w = \sup_{x\in\mathcal{X}} \frac{\mathfrak{K}w(x)}{w(x)}, \tag{3.14}$$

and the norm of the functional $f^* : \mathcal{M} \to \mathbb{R}$, $f^* \in \mathcal{F}$ is

$$\|f^*\|_w = \sup_{x \in \mathcal{X}} \frac{|f(x)|}{w(x)}. \tag{3.15}$$

The norm $\|\cdot\|_w$ satisfies conditions (3.9)–(3.11) with the constant κ given by

$$0 < \kappa = \sup_{x \in \mathcal{X}} \frac{1}{w(x)} < \infty, \tag{3.16}$$

see Kartashov [35, 34].

We are now in the position to state the stability bounds for the stationary distribution of a MC.

3.2.2 Kartashov's results

Here we will give Kartashov's results which are important for the thesis. We will also briefly discuss how these results can be used in order to obtain the stability bounds for ruin probabilities.

As before, we assume that P is the transition probability kernel of a MC with a unique stationary distribution π.

We introduce the following conditions on P and π:
There exist a probability measure $\nu \in \mathcal{M}^+$ and a non-negative measurable function $h \in \mathcal{F}(\mathcal{X})$ such that:

(D1) $\int_0^\infty h(v)\,\pi(dv) > 0$ and $\int_0^\infty h(v)\,\nu(dv) > 0$;

(D2) the kernel

$$K(v, \Gamma) := P(v, \Gamma) - h(v)\nu(\Gamma) \tag{3.17}$$

is non-negative;

(D3) $\|\mathfrak{K}\|_w \le \rho < 1$, where $\mathfrak{K}$ is the shift operator associated with the kernel K and w is a given weight function.

We call (3.17) a *decomposition* of P into the components K, h, ν.

The following result is proved in Kartashov [35, Theorem 8].

Theorem 3.12. *Assume that a MC with transition probability kernel P and the corresponding shift operator $\mathfrak{P}$ satisfy the conditions* **(D1)**–**(D3)** *with the norm $\|\cdot\|_w$ and let $\|\mathfrak{P}\|_w < \infty$. Then each MC with the transition probability kernel P' and the shift operator $\mathfrak{P}'$ such that*

$$\Delta \equiv \|\mathfrak{P}' - \mathfrak{P}\|_w < \frac{1-\rho}{1 + \|\pi\|_w \kappa \rho} \equiv \Delta_0, \tag{3.18}$$

where ρ is from **(D3)** *and κ is given in (3.16), has a unique invariant probability measure π',*

$$\|\pi' - \pi\|_w \le \frac{\Delta \|\pi\|_w}{\Delta_0 - \Delta}. \tag{3.19}$$

and the norm $\|\pi\|_w$ can be bounded by

$$\|\pi\|_w \le \frac{\|\nu\|_w}{1-\rho}. \tag{3.20}$$

Theorem 3.12 serves as a main source for the stability results. We will use its assertion in the following simplified form.

Corollary 3.13. *Under the conditions and with the notation of Theorem 3.12,*

$$\|\pi' - \pi\|_w \le \frac{\Delta\|\nu\|_w}{(1-\rho)(\Delta_1 - \Delta)}, \tag{3.21}$$

if

$$\Delta < \frac{(1-\rho)^2}{1 + (\kappa\|\nu\|_w - 1)\rho} \equiv \Delta_1. \tag{3.22}$$

The following example was considered in Kartashov [35, 34]. The stability bound for the ruin probability in the classical risk model immediately follows from this example, see Chapter 4.

Example 3.14 (Continuation of Example 3.11). Let $X = \{X_n\}$ be a reflected random walk satisfying the conditions in Example 3.11. Its transition kernel P has the form

$$P(v, \Gamma) = \mathbf{P}(v + \eta_1 \in \Gamma, v + \eta_1 > 0) + \mathbf{P}(v + \eta_1 \le 0)\delta_0(\Gamma), \tag{3.23}$$

where $\delta_0(\cdot)$ is the probability measure concentrated at 0. Thus, decomposition **(D2)** is given by the function $h(x) = \mathbf{P}(x + \eta_1 \le 0)$ and the measure $\nu = \delta_0$.

If there exists $\varepsilon > 0$ such that

$$\mathbb{E}\exp(\varepsilon\eta) = \rho < 1, \tag{3.24}$$

then condition **(D3)** holds in the norm $\|\cdot\|_w$, where $w(v) = \exp(\varepsilon v)$, and with the same constant ρ, see Kartashov [34], Theorem 8.1.

Application to the stability of the ruin probability problem

Inequality (3.22) can be regarded as a stability bound of the ruin probability in a risk model governed by a parameter a: quantities with primes correspond to the perturbed risk model governed by the parameter a', and Δ measures the distance between the parameters a and a'. Then the inequality (3.22) provides the stability bound in the weighted total variation distance (see Section 3.5) $\nu(\psi, \psi') = \|\psi' - \psi\|_w$. The relation

$$|\psi' - \psi|_w \le \|\psi' - \psi\|_w \tag{3.25}$$

yields bounds in the weighted uniform metric $|\psi' - \psi|_w$ which is more suitable for applications.

There are two technical problems associated with the application of Theorem 3.12 in ruin theory.

The first one is to prove that the Markov chain X obtained from the reversed process V (see Section 2.1.1) satisfies conditions **(D1)**–**(D3)**. The construction of a non-negative kernel K as in **(D2)** is not very complicated if the MC X has an atom. In this case one can employ splitting techniques as in Nummelin [40].

A more difficult problem is to prove that such a 'decomposed kernel' K satisfies condition **(D3)**. One typically has to impose assumptions related to the Cramér condition of the corresponding risk model. This typically implies an asymptotic upper bound for the ruin probability $\psi(u)$ as $u \to \infty$ (for example, exponential bounds in the case of the Sparre Andersen model or power law bounds in a risk model with risky investments, see Sections 2.2.1 and 2.2.4). Thus, one can obtain 'sharper' stability bounds (w.r.t.

the comparison of the tail behaviors of ruin probabilities) by adjusting the weight function w to this upper bound. This usually causes additional difficulties in proving **(D3)**. Of course, the most interesting case is when the asymptotic decay of the ruin probability is not known (as is the case in a Markov modulated model with random investments, see Section 2.2.4). But even in this case one should seek for the "optimal" weight w (optimality is not well-defined here, but the imposed conditions might still provide some intuition, see the discussion in Section 5).

Another problem concerns the right-hand side of the stability bound (3.19). In particular, the distance between the governing parameters a and a' is measured by the quantity $\Delta = \|\mathfrak{P}' - \mathfrak{P}\|_w$ which is very inconvenient since it is not expressed directly in terms of the input. Note that Theorem 3.12 and also Corollary 3.13 remain valid if one replaces Δ by any upper bound. This reduces the problem to the following: we have to find an upper bound of $\Delta = \|\mathfrak{P}' - \mathfrak{P}\|_w$ which is expressed directly in terms of the inputs a and a'. This can easily be achieved for "simple" risk models, but in more complicated cases various technical problems occur.

3.3 Regenerative processes. Stability bounds

Regenerative processes find their applications in many fields, like queuing, storage, reliability theory, etc. In actuarial risk theory regenerative techniques were used by Kalashnikov [30] in order to obtain stability bounds for the ruin probability. Parts of the thesis (Chapter 7) use ideas of Kalashnikov [30].

In this section we give the necessary definitions together with the stability bounds for the discrete-time regenerative processes. Such bounds for discrete and continuous-time regenerative processes were developed by Kalashnikov, see [25, 26, 27, 28, 29], and the references therein.

Consider a random process $Y = \{Y_t\}_{t\in T}$, where either $T = \mathbb{R}_+$ or $T = \mathbb{N} \cup \{0\}$, in a measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ together with an increasing sequence of finite random times $S = \{S_n\}_{n\geq 0} \subset T$. Define the shifts of (Y, S) by

$$\mathfrak{S}_{S_i}(Y,S) = \Big((Y(S_i + t))_{t\in T}\, , (S_k - S_i)_{k\geq i} \Big) . \tag{3.26}$$

Definition 3.15. The process (Y, S) is said to be *regenerative* if

(i) $\mathfrak{S}_{S_i}(Y, S)$, $i = 0, 1, 2, \ldots$ are identically distributed;

(ii) $\mathfrak{S}_{S_i}(Y, S)$, $i = 0, 1, 2, \ldots$ do not depend on

$$((Y_t)_{t\leq S_i}, S_0, \ldots, S_i) . \tag{3.27}$$

The sequence S is called *a renewal process* and the random times S_i are *renewal or regeneration* epochs. If $T = \mathbb{R}_+$ then (Y, S) is a *continuous time* regenerative process; otherwise, if $T = \mathbb{N}_+ \cup \{0\}$, then (Y, S) is a *discrete time* regenerative process. If $S_0 = 0$ then the process (Y, S) is said to be *without delay* or *zero-delayed.*

If instead of condition (ii) the weaker condition

(ii') $\mathfrak{S}_{S_i}(Y, S)$, $i = 0, 1, 2, \ldots$ do not depend on

$$(S_0, \ldots, S_i) , \tag{3.28}$$

is fulfilled, then the process (Y, S) is called *wide sense regenerative.*

Example 3.16. A φ-recurrent MC X with an accessible atom A can be considered as a regenerative process w.r.t. a renewal process $S = \{S_n\}$ defined by

$$S_n = \min\{k : k > S_{n-1}, X_{S_n} \in A\},$$

(see Example 3.14, p. 27). Indeed, $P(x, B) =: \nu(B)$ for $x \in A$ and by the recurrence property the times S_n are finite, thus (X, S) is a regenerative process.

□

Until the end of this section, (Y, S) and (Y', S') are discrete time zero-delayed (wide-sense) regenerative processes; $Y = \{Y_n\}_{n\geq 0}$ and $Y' = \{Y'_n\}_{n\geq 0}$ with $P_n(\cdot) := \mathbf{P}(Y_n \in \cdot)$ and $P'_n(\cdot) := \mathbf{P}(Y'_n \in \cdot)$. The symbol $\|\cdot\|$ will denote the total variation probability metric. Define

$$\varepsilon_n := \sup_{k\leq n} \|P_k - P'_k\|, \tag{3.29}$$

$$\varepsilon := \lim_{n\to\infty} \varepsilon_n. \tag{3.30}$$

Upper bounds of ε_n provide a *finite-time comparison* of the processes Y and Y' and ε refers to the *uniform-in-time comparison*, see [25, 28, 29] and the references therein. We are interested in the uniform-in-time bounds which are somewhat delicate and use the notion of crossing of a renewal (or regenerative) process.

Let $\overline{S}$ and $\overline{S}'$ be copies of the processes S and S' defined on the same probability space i.e.

$$\overline{S} \overset{d}{=} S, \qquad \overline{S}' \overset{d}{=} S'. \tag{3.31}$$

Then the corresponding inter-renewal times have the same distribution function as those of S and S', say F and F'. Set

$$\sigma := \inf\left\{\overline{S}_k > 0 : \bigcup_{j\geq 1} (\overline{S}_k = \overline{S}'_j)\right\}, \tag{3.32}$$

where $\sigma = \infty$ if $\overline{S}_k \neq \overline{S}'_j$ for all $j, k \geq 1$.

Definition 3.17. A process $(\overline{S}, \overline{S}')$ satisfying (3.31) is called *the crossing* of the renewal processes S and S' and the r.v. σ is *the crossing time.*

The crossing $(\overline{S}, \overline{S}')$ is *successful* if $\mathbf{P}(\sigma < \infty) = 1$.

Proposition 3.18 ([29], p. 222). *If there exists a crossing time σ such that*

$$\mathbb{E} \exp(\lambda\sigma) \leq c_\lambda \tag{3.33}$$

for some constant $\lambda > 0$, then

$$\varepsilon \leq \inf_t \left\{\varepsilon_t + \frac{2c_\lambda}{\lambda \exp(\lambda t)}\right\}.$$

Remark 3.19. In Kalashnikov [29] the above lemma is proved for

$$\varepsilon_t = \|Y(\cdot + t) - Y'(\cdot + t)\|, \qquad \text{and} \quad \varepsilon = \sup_{t\geq 0} \varepsilon_t,$$

i.e. it deals with the total variation distance between the "shifted" processes $Y(\cdot+t)$ and $Y'(\cdot+t)$ instead of the r.v.'s $Y(t)$ and $Y'(t)$. However, all the arguments in the proof remain the same if one substitutes the r.v.'s by the corresponding processes. See also Theorem 1 in Kalashnikov [25] for a similar result.

Condition (3.33) holds if the inter-renewal time distributions F_{S_1} and $F_{S_1'}$ satisfy certain arithmetic properties which we are going to explain now.

Definition 3.20 ([29], p. 181). The distribution $p = \{p_k\}_{k\geq 1}$ of a r.v. in $\mathbb{N}$ belongs to the class of *uniformly aperiodic distributions* $\mathbb{U}(N, \alpha)$, where $N \in \mathbb{N}$ and $0 < \alpha < 1$, if

$$\text{g.c.d.}\{n : \ p_n \geq \alpha, \ 1 \leq n \leq N\} = 1. \tag{3.34}$$

Proposition 3.21 ([29], p. 186). *If $F_{S_1}, F_{S_1'} \in \mathbb{U}(N, \alpha)$ for some $N \in \mathbb{N}$ and $\alpha \in (0, 1)$, and if*

$$\mathbb{E}\exp(\bar{\lambda}S_1) \leq \bar{b}, \qquad \mathbb{E}\exp(\bar{\lambda}S_1')\} \leq \bar{b}, \tag{3.35}$$

for some constants $\bar{\lambda} > 0$ and $\bar{b} < \infty$, then there exist constants $\lambda = \lambda(\bar{\lambda}, N, \alpha) \leq \bar{\lambda}$ and $c = c(\bar{\lambda}, \bar{b}, N, \alpha) < \infty$ (they can be written in a closed form) such that

$$\mathbb{E}\exp(\lambda\sigma) \leq c. \tag{3.36}$$

The following result is a straightforward consequence of Propositions 3.18 and 3.21.

Corollary 3.22 (cf.[29], p. 223). *Under the conditions of Proposition 3.21 and if*

$$\varepsilon_n \leq \alpha n \tag{3.37}$$

for some $\alpha \in (0, 1)$, then

$$\varepsilon \leq h\, \alpha \log\left(\frac{e}{\alpha}\right), \tag{3.38}$$

where $h = \max(c, 1/\lambda)$.

Proof. From Propositions 3.18 and 3.21, it follows that for $\alpha/(\lambda c) \leq 1$,

$$\varepsilon \leq \frac{\alpha}{\lambda} \log\left(\frac{\lambda c e}{\alpha}\right). \tag{3.39}$$

If $\lambda c \geq 1$, then the result follows because $(\alpha/\lambda c)\log(\lambda c e/\alpha)$ is decreasing in λc when $\alpha/(\lambda c) \leq 1$.

Otherwise, if $\lambda c < 1$, (3.36) also holds with $1/\lambda$ instead of c. Applying the same arguments as above we obtain (3.38). □

The following example will be used later in the thesis.

Example 3.23. Consider two MCs $I = \{I_n\}_{n\geq 0}$, $I' = \{I'_n\}_{n\geq 0}$ with values in $\{1, 2, \ldots, m\}$, corresponding transition probabilities $q_i = (q_{ij})$ and $q'_i = (q'_{ij})$ and the same initial state s. The n-step transition probabilities are denoted by $q_i^{(n)} = (q_{ij}^{(n)})$ and ${q'}_i^{(n)} = ({q'}_{ij}^{(n)})$, respectively. We assume that

$$\min\{q_{ij}, q'_{ij}\} \geq \underline{q} > 0.$$

Then, for some constant h (it can be written in explicit form)

$$\sup_{n\geq 0} \|q_s^{(n)} - {q'}_s^{(n)}\| \leq h \max_i \|q_i - q'_i\| \log\left(\frac{e}{\|q_i - q'_i\|}\right), \tag{3.40}$$

when $\max_i \|q_i - q'_i\| \leq 1$.

Proof. The MCs I and I' are also regenerative processes with regenerative state s and the corresponding renewal sequences are the successive visits to the state s. The distributions of the inter-renewal times S_1, S'_1 belong to $\in \mathbb{U}\left(1, \underline{q}\right)$. We will prove the following:

a) $\sup_{k\leq n} \|q_s^{(k)} - {q'}_s^{(k)}\| \leq n \max_i \|q_i - q'_i\|$;

b) $\mathbb{E}\exp(r\sigma_1) < \infty$ for $r < -\log(1 - \underline{q})$.

Relation a) follows by induction using the inequalities

$$\begin{aligned}\sum_j |q_{ij}^{(n+1)} - {q'}_{ij}^{(n+1)}| &\leq \sum_k q_{ik}^{(n)} \sum_j |q_{kj} - q'_{kj}| + \sum_k |q_{ik}^{(n)} - {q'}_{ik}^{(n)}| \sum_j q'_{kj} \\ &\leq \max_i \|q_i - q'_i\| + \sum_k |q_{ik}^{(n)} - {q'}_{ik}^{(n)}|.\end{aligned}$$

Now we will prove b). Since $\min_{ij} q_{ij} \geq \underline{q}$, we have that for any $n \in \mathbb{N}$,

$$\mathbf{P}_s(\sigma > n) \;=\; \mathbf{P}_s(I_1 \neq s, \ldots, I_n \neq s) \;\leq\; \left(1 - \underline{q}\right)^n.$$

This yields b) for any $r < -\log(1 - \underline{q})$. Analogous results hold for the Markov chain I'. An application of Corollary 3.22 with $\alpha = \max_i \|q_i - q'_i\|$ yields (3.40). ☐

3.4 Test functions

The method of *test functions* (also called *Lyapunov or trial functions*, see Borovkov [10], Kalashnikov [23]) originates from the direct Lyapunov method and is widely used in the analysis of stochastic processes (MCs, renewal, regenerative processes) and their applications (queuing systems, etc). Test functions are used in different drift criteria for the positivity, recurrence, transience, regularity and other properties of a MC; see Meyn & Tweedie [38], Borovkov [10] for these results and further references; accessibility and non-accessibility criteria in terms of test functions can be found in Kalashnikov [29]. These results have further applications in regenerative process theory, see Kalashnikov [28]. The stability of random processes in queuing problems was investigated by Kalashnikov [23]. Thus, it is not surprising that test functions also have applications in actuarial risk theory: a risk process can be recognized as a MC or a regenerative process, see Kalashnikov [30].

In this thesis, following the work by Kalashnikov [30], test functions in combination with Dynkin's formula will be used to bound the moments of the reversed process. But first, as promised, we give some results on accessibility, which, together with Propositions 3.6 and 3.8, provide the ergodicity conditions of a MC.

Recall from (3.1) that for any $V \in \mathcal{B}(\mathcal{X})$, $\tau_V = \min\{n > 0 : X_n \in V\}$. The following proposition is taken from Kalashnikov [29], Section 5.2. See also Borovkov [10], Meyn & Tweedie [38] for similar results.

Proposition 3.24. *The relation $\mathbb{E}_x \tau_V < \infty$ holds for all $x \in \mathcal{X}$ if and only if there exist a non-negative test function g and a constant $\varepsilon > 0$ such that*

(i) $\mathcal{A}g(x) \le -\varepsilon$ *for* $x \in V^c$,

(ii) $\mathcal{A}g(x) < \infty$ *for* $x \in V$.

Then,

$$\mathbb{E}_x \tau_V \le \begin{cases} \dfrac{g(x)}{\varepsilon}, & x \in V^c, \\ 1 + \dfrac{1}{\varepsilon}(g(x) + \mathcal{A}g(x)), & x \in V. \end{cases}$$

The renewal sequence is often generated by the random times when a MC visits a certain fixed set V. In such cases the following proposition helps to prove conditions like (3.35), see Kalashnikov [29], Section 5.2.

Proposition 3.25 ([29], p. 115.). *The expectation $\mathbb{E}_x \exp \lambda \tau_V$, $\lambda > 0$ is finite for any $x \in \mathcal{X}$, if and only if there exists a test function $g(x) \ge 1$, $x \in \mathcal{X}$ such that*

(i) $\mathcal{A}g(x) \le -(1 - e^{-\lambda})g(x)$, $x \in V^c$;

(ii) $\mathcal{A}g(x) < \infty$, $x \in V$.

Then,

$$\mathbb{E}_x \exp(\lambda \tau_V) \le \begin{cases} g(x), & x \in V^c, \\ e^{\lambda}(g(x) + \mathcal{A}g(x)), & x \in V. \end{cases}$$

3.5 Probability metrics

In this section we present some parts of the theory of probability metrics. This notion was introduced by Zolotarev [52] in order to measure the distance between the probability distributions of random elements.

Let $\mathbb{X}$ be a set of random elements with values in a complete separable metric space $(\mathcal{X}, d_{\mathcal{X}})$. Suppose that if $X \in \mathbb{X}$ and $X = Y$ a.s. then also $Y \in \mathbb{X}$. For any $X, Y \in \mathbb{X}$ we denote their joint probability distribution by P_{XY} and the marginals by P_X and P_Y, respectively. By $\mathcal{P}_{XY}$ we denote the set of all possible joint distributions P_{XY} of random vectors (X, Y) with fixed marginals P_X and P_Y. Introduce $\mathcal{P}^2 = \{P_{XY}, X, Y \in \mathbb{X}\}$.

Definition 3.26. A non-negative functional $d : \mathcal{P}^2 \to \mathbb{R}_+ \cup \{0\}$ is called a *probability metric* if the following conditions are fulfilled:

1. If $\mathbf{P}(X = Y) = 1$ then $d(P_{XY}) = 0$;
2. $d(P_{XY}) = d(P_{YX})$, for any $X, Y \in \mathbb{X}$;
3. $d(P_{XY}) \leq d(P_{XZ}) + d(P_{ZY})$ for any $X, Y, Z \in \mathbb{X}$.

Probability metrics can be divided into the following two classes.

Definition 3.27. A probability metric d is *simple* if $d(P_{XY})$ is completely defined by the marginals P_X and P_Y. Otherwise, d is called *compound*.

In the case of a simple probability metric d we use the notation $d(P_{XY}) =: d(P_X, P_Y)$. Also, if $\mathcal{X} \subseteq \mathbb{R}^k$ (as it is in our case) instead of the probability distribution P_X we sometimes write the distribution function F_X.

Remark 3.28. We will sometimes write r.v.'s instead of their distribution as the arguments in the probability metric (i.e., $d(X, Y) := d(P_{XY})$). This is done for convenience quite often in the literature (see Zolotarev [52], Kalashnikov [26], Rachev [44] and others), but one has to keep in mind that we are dealing with the corresponding distributions.

We will use the following probability metrics later.

1. *The total variation metric*

$$\begin{aligned} \|P_X - P_Y\| &:= \sup_{B \in \mathcal{B}(\mathcal{X})} |P_X(B) - P_Y(B)| \\ &= \frac{1}{2} \int |P_X(dx) - P_Y(dx)| \,. \end{aligned} \tag{3.41}$$

2. *The weighted total variation metric*

$$\|P_X - P_Y\|_w = \frac{1}{2} \int w(x) \, |P_X(dx) - P_Y(dx)| \,, \tag{3.42}$$

   where a so-called *weight function* w is measurable, positive and bounded away from 0. Further we reserve the symbol w for the weight function and assume that it always satisfies these properties. Sometimes it is required that $w \geq 1$ but we do not assume this unless stated explicitly. It will also be convenient to write $Var_w(X, Y) := \|P_X - P_Y\|_w$ for the weighted total variation distance between the distribution functions P_X and P_Y (and similar, by $Var(X, Y) := \|P_X - P_Y\|$ for $w \equiv 1$), see Remark 3.28.

3. *The supremum metric*, for $X, Y \in \mathbb{R}$,

$$|F_X - F_Y| = \sup_{x \in \mathbb{R}} |F_X(x) - F_Y(x)|. \tag{3.43}$$

4. *The weighted supremum metric* is given by

$$|F_X - F_Y|_w = \sup_{x \in \mathbb{R}} w(x) |F_X(x) - F_Y(x)|, \tag{3.44}$$

   where w is a weight function.

5. *The uniform metric* is defined by

$$i(P_{XY}) := \mathbf{P}(X \neq Y). \tag{3.45}$$

6. The probability metric

$$i_w(P_{XY}) := \frac{1}{2}\mathbb{E}\left[(w(X) + w(y))\mathbf{1}(X \neq Y)\right]. \tag{3.46}$$

   is called *the weighted uniform metric*. If $w \equiv 1$ then i_w coincides with the uniform metric i.

7. *The ζ-metric* is defined by a class of functions $\mathcal{F} \subseteq \mathcal{F}(\mathcal{X})$,

$$\zeta(P_X, P_Y; \mathcal{F}) := \sup\left\{|\mathbb{E}(f(X) - f(Y))| : \ f \in \mathcal{F}\right\}. \tag{3.47}$$

   If one takes $\mathcal{F} = \mathcal{F}_1 := \{f \in \mathcal{F}(\mathcal{X}) : \ |f(x)| \leq 1\}$, then it is well-known (see Zolotarev [52]) that $\zeta(\cdot; \mathcal{F}_1)$ is the total variation metric, i.e.

$$\zeta(P_X, P_Y; \mathcal{F}_1) = \|P_X - P_Y\|.$$

The metrics 1., 3., 5., 7. are well known in the literature and can be found in Zolotarev [52], Kalashnikov [28, 26], Kalashnikon & Rachev [33], Rachev [44] and others. The weighted total variation metric was used by Kartashov [34, 35]; for more weighted metrics see Rachev [44].

Definition 3.29. Let d be a compound metric. The functional

$$\hat{d}(P_X, P_Y) = \inf\{d(P_{XY}) : P_{XY} \in \mathcal{P}_{XY}\} \tag{3.48}$$

is called the *minimal metric* with respect to d.

It is well-known that $\hat{d}$ is a simple probability metric, see Zolotarev [52] for the proof.

The following result can be found in many books and papers about probability metrics, for example see Zolotarev [52], Rachev [44] and others.

Proposition 3.30. *The total variation metric $\|\cdot\|$ is minimal w.r.t. the uniform metric i.*

Recall that $d_{\mathcal{X}}$ is a metric in $\mathcal{X}$ and define

$$\tau(P_{XY}; d_{\mathcal{X}}) := \mathbb{E}d_{\mathcal{X}}(X, Y). \tag{3.49}$$

It is a compound probability metric on the set of r.v.'s for which the quantity in (3.49) is finite. We also introduce the set of functions

$$\mathcal{F}(d_{\mathcal{X}}) := \{f \in \mathcal{F} : \ |f(x) - f(y)| \leq d_{\mathcal{X}}(x, y)\}. \tag{3.50}$$

Proposition 3.31 (Zolotarev [52], Theorem 1.3.3). *The metric $\zeta(\cdot; d_{\mathcal{X}})$ is minimal w.r.t. $\tau(\cdot; d_{\mathcal{X}})$, i.e.,*

$$\hat{\tau}(P_{XY}; d_{\mathcal{X}}) = \zeta(P_X, P_Y; \mathcal{F}(d_{\mathcal{X}})), \qquad X, Y \in \mathbb{X}.$$

This yields the following.

Corollary 3.32. *The weighted total variation metric $\|\cdot\|_w$ is minimal w.r.t. the weighted uniform metric i_w.*

Proof. Let us choose the metric in $\mathcal{X}$,

$$d_w(x, y) := \frac{1}{2}\left((w(x) + w(y))\mathbf{1}(x \neq y))\right).$$

By Proposition 3.31,

$$\hat{\tau}(P_{XY}; d_w) = \zeta(P_X, P_Y; \mathcal{F}(d_w)),$$

where the set of functions $\mathcal{F}(d_w)$ is defined in (3.50) with the metric $d_{\mathcal{X}} = d_w$.

It remains to show that $\zeta(P_X, P_Y; \mathcal{F}(d_w)) = \|\mathbf{P}_X - P_Y\|_w$ for all $X, Y \in \mathbb{X}$. It actually suffices to show that the latter relation holds for all absolutely continuous distributions P_X, P_Y with the corresponding densities p_X, p_Y. The general case follows by approximation.

Let us take the function $f(x) = w(x)\,\mathrm{sign}(p_X(x) - p_Y(x))/2$. Obviously, $f \in \mathcal{F}(d_w)$. Thus,

$$\begin{aligned}
\zeta(P_X, P_Y; \mathcal{F}(d_w)) &\geq \frac{1}{2}\int_0^\infty w(x)\,\mathrm{sign}(p_X(x) - p_Y(x))(p_X(x) - p_Y(x))\,dx \\
&= \frac{1}{2}\int_0^\infty w(x)|p_X(x) - p_Y(x)|\,dx \\
&= \|P_X - P_Y\|_w.
\end{aligned}$$

This completes the proof. □

It often turns out that it is easier to deal with compound probability metrics than with simple ones. The following property of minimal metrics will be used later in the thesis.

Example 3.33 (Zolotarev [51]). Let $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ be two collections of r.v.'s on $\mathcal{X}$. Then,

$$Var(X, Y) \leq \sum_i Var(X_i, Y_i). \tag{3.51}$$

Proof. Using that $Var = \hat{i}$ and

$$\{X \neq Y\} \subset \{X_1 \neq Y_1\} \cup \cdots \cup \{X_m \neq Y_m\},$$

we have

$$Var(X, Y) \leq i(X, Y) = \mathbf{P}(X \neq Y) \leq \sum_{n=1}^m \mathbf{P}(X_n \neq Y_n) = \sum_{n=1}^m i(X_n, Y_n). \tag{3.52}$$

These relation hold for any $P_{XY} \in \mathcal{P}_{XY}$. Let now $P_{X_iY_i} \in \mathcal{P}_{X_iY_i}$, where classes $\mathcal{P}_{X_iY_i}$ are determined by the marginals P_X and P_Y. If we minimize (3.52) over the distributions $\prod_1^m P_{X_iY_i}$, where $P_{X_iY_i} \in \mathcal{P}_{X_iY_i}$, we obtain (3.51). □

Remark 3.34. Note that all the arguments in the example above remain also valid for the weighted total variation metric $\|\cdot\|_w$, i.e.

$$Var_w(X, Y) \leq \sum_i Var_w(X_i, Y_i)$$

for X and Y as in Example 3.33.

Chapter 4

The Sparre Andersen model

This chapter is based on Enikeeva, *et al.* [14]. Using the MC approach based on the results of Kartashov (see Section 3.2) we derive the stability bound for the ruin probability in the S.A. risk model from Section 2.2.1.

The reversed process. Recall the risk process $\{R_n\}_{n\geq 0}$ defined in (2.19). It satisfies Assumption 2.1 with the governing sequence $\{\sigma_n\}$ consisting of the i.i.d. r.v.'s $\sigma_n = c\theta_n - Z_n$. Thus, the construction (2.14)–(2.15) is applicable and yields

$$V_{n+1} = (V_n + Z_n - c\theta_n)_+ \,. \tag{4.1}$$

The process $V = \{V_n\}_{n\geq 0}$ is Markov with shift operator

$$\mathfrak{P}f(v) = \mathbb{E}f\left((v + Z_1 - c\theta_1)_+\right). \tag{4.2}$$

From (2.16) and (4.1) we have

$$\psi(v) = \lim_{n\to\infty} \mathbf{P}\left((V_n + Z_n - c\theta_n)_+ > v\right). \tag{4.3}$$

The MC V possesses a unique stationary distribution by virtue of the net profit condition $c\mathbb{E}\theta_n > \mathbb{E}Z_n$, see Example 3.11.

The stability bound. Now we prove that the MC V satisfies the conditions of Theorem 3.12. This will imply the desired stability bound for the ruin probability, see the discussion in Section 3.2.2. The following assumption is related to the Cramér condition (2.30).

Assumption 4.1. *Assume that there exists constants $\varepsilon > 0$ and $\rho < 1$ such that*

$$\mathbb{E}\exp(\varepsilon(Z_n - c\theta_n)) \leq \rho. \tag{4.4}$$

Note that (4.4) yields the net profit condition.

First we will prove the conditions **(D1)–(D3)** from Section 3.2.2. The decomposition (3.17) of the transition kernel P is based on the following representation:

$$P(x,\Gamma) = \mathbf{P}(v + Z_n - c\theta_n \in \Gamma, v + Z_n - c\theta_n > 0) + \mathbf{P}(v + Z_n - c\theta_n \leq 0)\delta_0(\Gamma), \tag{4.5}$$

where $\delta_0(\Gamma)$ is the probability measure concentrated at 0. Thus, the kernel K in (3.17) is defined by the function $h(v) = \mathbf{P}(v + Z_n - c\theta_n \le 0)$ and the probability measure $\nu = \delta_0$. Take the weight function $w(v) = e^{\varepsilon v}$. Using condition (4.4), we have

$$\begin{aligned}
\|\mathfrak{K}\|_w &= \sup_{v\ge 0} e^{-\varepsilon v} \int_0^\infty e^{\varepsilon y}\, K(v; dy) \\
&= \sup_{v\ge 0} e^{-\varepsilon v}\, \mathbb{E}\left(e^{\varepsilon(v+Z_n-c\theta_n)}\, \mathbf{1}(Z_n - c\theta_n + v > 0)\right) \\
&\le \mathbb{E}e^{\varepsilon(Z_n-c\theta_n)} \le \rho. \qquad (4.6)
\end{aligned}$$

Thus, the conditions of Theorem 3.12 are satisfied. The inequality

$$\begin{aligned}
\|\mathfrak{P} - \mathfrak{P}'\|_w &= \sup_{v\ge 0} e^{-\varepsilon v} \int_0^\infty e^{\varepsilon y}\, |P'(v; dy) - P(v; dy)| \\
&\le \sup_{v\ge 0} e^{-\varepsilon v} \left(\int_0^\infty e^{\varepsilon(v+z)}\, |F_{Z'}(dz) - F_Z(dz)| \right. \\
&\qquad \left. + \int_0^\infty e^{\varepsilon(v+z)}\, F_Z(dz) \int_0^\infty \left| F_{\theta'}\left(d\frac{y}{c}\right) - F_\theta\left(d\frac{y}{c}\right)\right| \right) \\
&= \|F_{Z'} - F_Z\|_w + \mathbb{E}e^{\varepsilon Z}\, \|F_{\theta'} - F_\theta\| =: \mu(a, a') \qquad (4.7)
\end{aligned}$$

yields the following result.

Theorem 4.2. *Let the non-perturbed model governed by the parameter a satisfy Assumption 4.1. Then for any a' s.t. $\mu(a, a') \le (1-\rho)^2$, where $\mu(a, a')$ is defined by (4.7) and ρ is from (4.4), we have*

$$|\psi_a - \psi_{a'}|_w \le \frac{1}{1-\rho}\, \frac{\mu(a, a')}{(1-\rho)^2 - \mu(a, a')}. \qquad (4.8)$$

Proof. We apply Corollary 3.13. Trivially,

$$\kappa = \sup_{x\ge 0} \frac{1}{w(x)} = 1, \qquad \|\delta_0\|_w = 1, \qquad \Delta_1 = (1-\rho)^2.$$

Plugging these expressions in (3.21) and using $|\cdot|_w \le \|\cdot\|_w$ we obtain (4.8). □

Example 4.3. Let θ and θ' be exponentially distributed with parameters λ and λ', respectively, i.e., we consider the classical risk model. Then, using

$$\|F_{\theta'} - F_\theta\| \le \frac{2}{\lambda \vee \lambda'} |\lambda - \lambda'|, \qquad (4.9)$$

we obtain the stability bound (4.8) with

$$\mu(a, a') = \|F_{Z'} - F_Z\|_w + \mathbb{E}e^{\varepsilon Z}\, \frac{2}{\lambda \vee \lambda'} |\lambda - \lambda'|.$$

Chapter 5

The Markov modulated risk model

This chapter is based on Enikeeva *et. al.* [14]. We derive stability bounds for the ruin probabilities in the Markov modulated risk model. Similarly to the previous chapter, we use Markov chain approach.

Recall the Markov modulated risk model from Section 2.2.2; the skeleton risk process $\{R_n\}$ is given by (2.24). First we construct the corresponding reversed process $\{V_n\}$.

The reversed process. In order to define a reversed process we first introduce a MC $I = \{I_n\}_{n\geq 0}$ which may be considered as the 'reversed' chain with respect to $\mathcal{I}$ in the sense that $I_n = \mathcal{I}_{-n}$. Thus, I is a stationary Markov chain with state space $\mathbb{E} = \{1, \ldots, m\}$, transition probabilities

$$q_{ij} = \begin{cases} p_{ji}\pi_j/\pi_i, & j \neq i, \\ p_{ii}, & j = i, \end{cases} \tag{5.1}$$

and initial distribution $\{\pi_i\}$ as in (2.23) which is also the stationary distribution of I. The n-step transition probabilities will be denoted by $q_i^{(n)} = \{q_{ij}^{(n)}\}$. We assume that I is independent of the random elements involved in the definition of R^i for all $i \in \mathbb{E}$.

Now, following the construction (2.14)–(2.15), we define the reversed process as follows,

$$V_{n+1} = \left(V_n + \eta^{I_n, I_{n+1}}\right)_+, \qquad V_0 = 0, \tag{5.2}$$

where

$$\eta^{ji} = \delta_{ji} Z^i - c_i \theta^i, \qquad i, j \in \mathbb{E}. \tag{5.3}$$

The process

$$W_n = (V_n, I_n), \quad n \geq 0, \tag{5.4}$$

is a MC with values in $\mathbb{R}_+ \times \mathbb{E}$, with shift operator

$$\mathfrak{P} f(v, i) = q_{ii}\, \mathbb{E} f\left((v + Z^i - c_i\, \theta^i)_+, i\right) + \sum_{j \neq i} q_{ij} \mathbb{E} f\left((v - c_j\, \theta^j)_+, j\right) \tag{5.5}$$

and stationary distribution Π. Existence of Π will be shown in Chapter 7, see Remark 7.14. Relation (2.16) yields

$$\psi(x) = \lim_{n\to\infty} \mathbf{P}(V_n > x) = \sum_{i\in\mathbb{E}} \int_x^\infty \Pi(dy, i). \tag{5.6}$$

The main assumptions. Denote by $M_{Z^i}(\varepsilon) := \mathbb{E}\exp(\varepsilon Z_n^i)$ the moment generating function (m.g.f.) of the claims Z_n^i and assume that $M_{Z^i}(\varepsilon) < \infty$ for some $\varepsilon > 0$ and all $i \in \mathbb{E}$. We introduce a matrix $P(\varepsilon)$ with entries $p_{ij}(\varepsilon)$ defined by

$$p_{ij}(\varepsilon) := p_{ij}\mathbb{E}e^{\varepsilon\eta_{ij}} = \begin{cases} p_{ii}\,\mathbb{E}e^{\varepsilon(Z^i - c_i\theta^i)} = p_{ii}M_{Z^i}(\varepsilon)\dfrac{\lambda_i}{\lambda_i + c_i\varepsilon}, & j = i, \\ p_{ij}\,\mathbb{E}e^{-\varepsilon c_j\theta^j} = p_{ij}\dfrac{\lambda_j}{\lambda_j + c_j\varepsilon}, & j \neq i, \end{cases}$$

where the probabilities p_{ij} are from (2.21). This matrix is positive and therefore, its spectral radius $\|P(\varepsilon)\|_{SP}$ is equal to the maximal eigenvalue which is positive and denoted by $d(\varepsilon)$. Note that $d(0) = 1$.

Assumption 5.1. Assume that there exists a constant $\varepsilon^* > 0$ such that

$$\|P(\varepsilon^*)\|_{SP} = d(\varepsilon^*) < 1.$$

Assumption 5.1 can also be expressed in terms of the embedded reversed process (V, I). We define another matrix $Q(\varepsilon)$ with elements

$$q_{ij}(\varepsilon) := q_{ij}\mathbb{E}e^{\varepsilon\eta_{ij}} = \begin{cases} q_{ii}\,\mathbb{E}\,e^{\varepsilon(Z^i - c_i\theta^i)}, & j = i, \\ q_{ij}\,\mathbb{E}\,e^{-\varepsilon c_j\theta^j}, & j \neq i, \end{cases}$$

where the probabilities q_{ij} are defined in (5.1). Evidently,

$$Q(\varepsilon) = T^{-1}P^t(\varepsilon)T,$$

where $T = \mathrm{diag}(\pi_1, \ldots, \pi_m)$, T^{-1} is the inverse of T and $P^t(\varepsilon)$ is the transpose of $P(\varepsilon)$. Therefore, the maximal eigenvalue of $Q(\varepsilon)$ is equal to $d(\varepsilon)$. We denote by $\gamma(\varepsilon)$ the eigenvector (column) of the matrix $Q(\varepsilon)$ corresponding to the eigenvalue $d(\varepsilon)$ and by $\gamma^P(\varepsilon)$ the corresponding eigenvector of $P^t(\varepsilon)$. Then

$$\gamma(\varepsilon) = T^{-1}\gamma^P(\varepsilon).$$

Denote by $\gamma_i(\varepsilon)$ the components of the vector $\gamma(\varepsilon)$. All these components are positive by the Perron-Frobenius theory. For definiteness, let us assume that that $\gamma_1(\varepsilon) = 1$. Evidently, $\gamma(\varepsilon)$ is a continuous function of ε.

Decomposition. Now we will construct a decomposition of the transition kernel P of the MC (V, I) and will prove conditions **(D1)**–**(D3)** on p. 26. We decompose P in the following way:

$$P((v,i);(\Gamma,j)) = K((v,i);(\Gamma,j)) + h(v,i)\,\nu(\Gamma,j), \tag{5.7}$$

where

$$\begin{aligned} h(v,i) &= \min_{j\in\mathbb{E}}\frac{q_{ij}}{\pi_j}\min_{i\in\mathbb{E}}\mathbf{P}(v + Z^i - c_i\theta^i \le 0), \\ \nu(\Gamma,j) &= \delta_0(\Gamma)\pi_j, \end{aligned}$$

where δ_0 is a probability measure on $[0,\infty)$ s.t. $\delta_0(\{0\}) = 1$. Conditions **(D1)**, **(D2)** trivially hold. We now focus on **(D3)**.

We would like to compare the ruin probabilities in a weighted metric with weight function

$$w(v) = \exp(\varepsilon^* v).$$

Such bounds would follow from Kartashov's results (see Theorem 3.12 and Corollary 3.13) and relation (5.6) if $\|\mathfrak{K}\|_{W_{\varepsilon^*}} < 1$ for a weight function $W_{\varepsilon^*}(v,i) = C_i \exp(\varepsilon^* v)$, where constants $C_i > 0$. However, one may only conclude from Assumption 5.1 that $\frac{\mathfrak{K}W_{\varepsilon^*}(v)}{W_{\varepsilon^*}(v)} \leq \rho < 1$ for large enough v (see Lemma 5.2). Therefore, it becomes necessary to consider a slight modification of W_{ε^*}. Now we focus on the construction of such function W.

For each $\varepsilon \geq 0$, define a weight function W_ε by

$$W_\varepsilon(v,i) = \gamma_i(\varepsilon)\, e^{\varepsilon v}, \qquad v \geq 0,\ i \in \mathbb{E}. \tag{5.8}$$

Lemma 5.2. *Let Assumption 5.1 hold and take the constant $\varepsilon \in (0, \varepsilon^*]$. Then,*

$$\sup_{v \geq V^*} \max_{i \in \mathbb{E}} \frac{\mathfrak{K}W_\varepsilon(v,i)}{W_\varepsilon(v,i)} \leq \rho^*, \tag{5.9}$$

where the constants $V^* = V^*(\varepsilon) < \infty$ and $\rho^* = \rho^*(\varepsilon) < 1$ are given in (5.11).

Proof. Since $K \leq P$, it suffices to prove that (5.9) holds with $\mathfrak{P}$ instead of $\mathfrak{K}$. From (5.5) we have

$$\mathfrak{P}W_\varepsilon(v,i) = \sum_{j \in \mathbb{E}} q_{ij} \gamma_j(\varepsilon) \mathbb{E} \exp\left(\varepsilon(v + \eta^{ij})_+\right),$$

where the η^{ij}'s are given in (5.3). Using

$$\begin{aligned} q_{ij}\, \mathbb{E} \exp\left(\varepsilon(v+\eta^{ij})_+\right) &\leq q_{ij}\, \left(\mathbb{E}\exp\left(\varepsilon(v + \delta_{ij} Z^j - c_j \theta^j)\right) + 1\right) \\ &= q_{ij}(\varepsilon)\, e^{\varepsilon v} + q_{ij}, \end{aligned} \tag{5.10}$$

and taking into account that $\sum_{j\in\mathbb{E}} q_{ij}(\varepsilon)\gamma_j(\varepsilon) = d(\varepsilon)\gamma_i(\varepsilon)$, we have

$$\frac{\mathfrak{P}W_\varepsilon(v,i)}{W_\varepsilon(v,i)} \leq d(\varepsilon) + \frac{\overline{\gamma}(\varepsilon)}{\underline{\gamma}(\varepsilon)} e^{-\varepsilon v},$$

where $\overline{\gamma}(\varepsilon) := \max_i \gamma_i(\varepsilon)$ and $\underline{\gamma}(\varepsilon) := \min_i \gamma_i(\varepsilon)$. Choosing V^* as solution to the equation

$$\frac{\overline{\gamma}(\varepsilon)}{\underline{\gamma}(\varepsilon)} e^{-\varepsilon V^*} = \frac{1 - d(\varepsilon)}{2}$$

we prove (5.9) with

$$V^*(\varepsilon) = \frac{1}{\varepsilon} \log \frac{2\overline{\gamma}(\varepsilon)}{(1-d(\varepsilon))\underline{\gamma}(\varepsilon)}, \qquad \rho^*(\varepsilon) = \frac{1 + d(\varepsilon)}{2}. \tag{5.11}$$

This completes the proof of the lemma. ∎

Remark 5.3. It follows from the proof of Lemma 5.2 that for any $0 < \varepsilon_0 \le \varepsilon^*$ there exists $V^*(\varepsilon_0, \varepsilon^*)$ such that

$$\sup_{\varepsilon \in [\varepsilon_0, \varepsilon^*]} \rho^*(\varepsilon) < 1. \tag{5.12}$$

Furthermore, for all $\varepsilon \in [\varepsilon_0, \varepsilon^*]$ there exists one $V^* = V^*(\varepsilon_0, \varepsilon^*)$ satisfying Lemma 5.2.

Lemma 5.4. *Let Assumption 5.1 hold. Then there exist constants* $0 < \varepsilon_* \le \varepsilon^*$ *and* $\rho_*(\bar{v}) < 1$ *such that, for any* $\bar{v} \ge 0$,

$$\sup_{v \le \bar{v}} \max_{i \in \mathbb{E}} \frac{\mathfrak{K} W_{\varepsilon_*}(v,i)}{W_{\varepsilon_*}(v,i)} \le \rho_*(\bar{v}). \tag{5.13}$$

Proof. Using that $W_0(v,i) = \gamma_i(0) \equiv 1$, we have

$$\frac{\mathfrak{P} W_0(v,i)}{W_0(v,i)} = 1.$$

This together with (5.7) yields

$$\frac{\mathfrak{K} W_0(v,i)}{W_0(v,i)} = 1 - \left(\min_{j \in \mathbb{E}} \frac{q_{ij}}{\pi_j} \right) s(v),$$

where

$$s(v) = \min_{i \in \mathbb{E}} \mathbf{P}(v + Z^i - c_i \theta^i \le 0).$$

Evidently, $s(v) > 0$ for any $v \ge 0$, and $s(v) \to 0$ as $v \to \infty$.

It follows that (5.13) holds for $\varepsilon_* = 0$ and the left hand side on (5.13) does not exceed

$$1 - s(\bar{v}) \min_{i,j \in \mathbb{E}} \frac{q_{ij}}{\pi_j}.$$

The continuity of $\gamma_i(r)$ with respect to r and Assumption 5.1 infer that relation (5.13) holds for some positive (sufficiently small) ε_* and appropriate $\rho_*(\bar{v}) < 1$, which completes the proof. □

Take ε^* as in Assumption 5.1 and $\varepsilon_* \le \varepsilon^*$ from Lemma 5.4. Put

$$W(v,i) = \gamma_i(r(v)) \exp(r(v)v), \quad v \ge 0, \ i \in \mathbb{E}, \tag{5.14}$$

where

$$r(v) = \varepsilon_* + \frac{(\varepsilon^* - \varepsilon_*)\chi v}{1 + \chi v}, \quad \chi > 0. \tag{5.15}$$

Evidently, $r(0) = \varepsilon_*$ and $r(v) \to \varepsilon^*$ as $v \to \infty$. Note also that the function $r(v)v$ satisfies the Lipschitz condition

$$|r(v_1)v_1 - r(v_2)v_2| \le \varepsilon^* \, |v_1 - v_2|. \tag{5.16}$$

Lemma 5.5. *Let Assumption 5.1 hold. If* W *is defined by (5.14) and (5.15), then* $\|\mathfrak{K}\|_W = \rho < 1$ *for sufficiently small* $\chi > 0$.

Proof. Introduce the function

$$f_v(z,i) = \gamma_i(r(v))\,\exp(r(v)z), \quad i \in \mathbb{E},\ z \geq 0,\ v \geq 0.$$

Then,

$$\begin{aligned}
\|\mathfrak{K}\|_W &= \sup_{v\geq 0, i\in\mathbb{E}} \frac{\mathfrak{K}W(v,i)}{W(v,i)} \\
&= \sup_{v\geq 0, i\in\mathbb{E}} \frac{\mathfrak{K}(f_v + (W - f_v))(v,i)}{W(v,i)} \\
&\leq \sup_{v\geq 0, i\in\mathbb{E}} \frac{\mathfrak{K}(f_v)(v,i)}{W(v,i)} + \sup_{v\geq 0, i\in\mathbb{E}} \frac{\mathfrak{K}(W - f_v)(v,i)}{W(v,i)}. \qquad (5.17)
\end{aligned}$$

We will prove that for appropriately chosen $\chi > 0$, the first supremum in (5.17) is strictly less than 1, and the second can be made arbitrarily small, implying the assertion of the lemma.

By Lemma 5.2 and Remark 5.3, there exist $V^* \geq 0$ and $0 < \rho^* < 1$ such that

$$\sup_{i\in\mathbb{E}, v\geq V^*} \frac{\mathfrak{K}f_v(v,i)}{f_v(v,i)} \leq \rho^*.$$

By Lemma 5.4, there exists $0 < \rho_*(V^*) < 1$ and $\chi_1 > 0$ such that, for all $\chi < \chi_1$,

$$\sup_{i\in\mathbb{E}, v\leq V^*} \frac{\mathfrak{K}f_v(v,i)}{f_v(v,i)} \leq \rho_*(V^*).$$

Thus, the first supremum in (5.17) is bounded by $\max\{\rho^*, \rho_*(V^*)\}$.

Now we turn to the second term in (5.17). Since $K \leq P$, we bound the following quantity.

$$\left|\frac{\mathfrak{P}(W - f_v)(v,i)}{W(v,i)}\right| \leq \frac{e^{-r(v)v}}{\gamma_i(r(v))}\mathbb{E}_{v,i}\left|\gamma_{I_1}(r(V_1))e^{r(V_1)V_1} - \gamma_{I_1}(r(v))e^{r(v)V_1}\right|, \qquad (5.18)$$

where $\mathbb{E}_{v,i}$ is the conditional expectation given $V_0 = v$, $I_0 = i$, and hence, see (5.2), $V_1 = (v + \eta^{iI_1})_+$. It follows from the Lipschitz condition (5.16) and Assumption 5.1, that for all $\chi \geq 0$, the right-hand side of (5.18) is uniformly bounded. Thus, by Lebesgue's dominated convergence theorem, the right-hand side converges to 0 when $\chi \to 0$ (recall that $\gamma_i(r)$ is continuous in r). Thus, for any $\delta > 0$, there exists $\chi_2 > 0$ such that

$$\left|\frac{\mathfrak{P}(W - f_v)(v,i)}{W(v,i)}\right| \leq \delta$$

for all $\chi \leq \chi_2$, $i \in \mathbb{E}$, and $v \geq 0$.

This yields that

$$\|\mathfrak{K}\|_W \leq \max(\rho^*, \rho_*(V^*)) + \delta$$

for all $\chi \leq \min(\chi_1, \chi_2)$. Since δ can be chosen as small as necessary, the lemma is proved. □

The stability bound. We have proved that the decomposition (5.7) of the kernel P satisfies the conditions **(D1)–(D3)** with weight function W defined in (5.14) and (5.15). Thus, Corollary 3.13 is applicable to a MC (V_n, I_n). For the comparisson of ruin probabilities we will use weight function

$$w(x) = \exp(\varepsilon^* x),$$

where ε^* is from Assumption 5.1. Using the representation of the ruin probability (5.6) and the relation

$$\varepsilon^* v \le \frac{\varepsilon^* - \varepsilon_*}{\chi} + r(v)v,$$

we have

$$\begin{aligned}
\|\psi_a - \psi_{a'}\|_w &= \int_0^\infty e^{\varepsilon^* v} |\psi_a(dv) - \psi_{a'}(dv)| \\
&\le e^{(\varepsilon^* - \varepsilon_*)/\chi} \int_0^\infty e^{r(v)v} \, |\psi_a(dv) - \psi_{a'}(dv)| \\
&\le e^{(\varepsilon^* - \varepsilon_*)/\chi} \sum_{i\in\mathbb{E}} \int_0^\infty e^{r(v)v} \, |\Pi'(dv,i) - \Pi(dv,i)| \\
&\le \frac{e^{(\varepsilon^* - \varepsilon_*)/\chi}}{\inf_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \underline{\gamma}(\varepsilon)} \, \|\Pi' - \Pi\|_W. \qquad (5.19)
\end{aligned}$$

The quantity $\|\mathfrak{P}' - \mathfrak{P}\|_W$ can be bounded as follows:

$$\begin{aligned}
\|\mathfrak{P}' - \mathfrak{P}\|_W &= \sup_{v\ge 0} \max_{i\in\mathbb{E}} \sum_{j\in\mathbb{E}} \int_0^\infty \frac{\gamma_j(r(y)) e^{r(y)y}}{\gamma_i(r(v)) e^{r(v)v}} \, |P'((v,i);(dy,j)) - P((v,i);(dy,j))| \\
&\le h_\gamma \sup_{v\ge 0} \max_{i\in\mathbb{E}} \sum_{j\in\mathbb{E}} \int_0^\infty e^{r(y)y - r(v)v} \, |P'((v,i);(dy,j)) - P((v,i);(dy,j))|,
\end{aligned}$$

where

$$h_\gamma = \frac{\sup_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \overline{\gamma}(\varepsilon)}{\inf_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \underline{\gamma}(\varepsilon)}. \qquad (5.20)$$

Now,

$$\int_0^\infty e^{r(y)y - r(v)v} \, |P'((v,i);(dy,j)) - P((v,i);(dy,j))| \;\le\; |q'_{ij} - q_{ij}| \, C_{ij} \,+\, q'_{ij} \, \Delta_{ij},$$

where

$$\begin{aligned}
C_{ij} &:= \sup_{v\ge 0} \mathbb{E} \exp\left(r(v + \delta_{ij} Z^j - c_j \theta^j)_+ (v + \delta_{ij} Z^j - c_j \theta^j)_+ \,-\, r(v)v \right), \\
\Delta_{ij} &= \sup_{v\ge 0} \int_0^\infty e^{r(y)y - r(v)v} \left| P(v + \delta_{ij} Z^j - c_j \theta^j \in dy) - P(v + \delta_{ij} Z'^j - c'_j \theta'^j \in dy) \right|.
\end{aligned}$$

The following bounds use the Lipschitz condition (5.16) and relation (4.9),

$$\begin{aligned} C_{ij} &\leq \sup_{v\geq 0} \mathbb{E}\exp\big(r(v+Z^j)(v+Z^j) - r(v)v\big) \\ &\leq M_{Z^j}(\varepsilon^*) \\ \Delta_{ij} &\leq 2M_{Z^j}(\varepsilon^*)\left(\frac{c_j}{\lambda_j}\wedge\frac{c'_j}{\lambda'_j}\right)\left|\frac{\lambda'_j}{c'_j} - \frac{\lambda_j}{c_j}\right| + \delta_{ij}\|F_{Z^j} - F_{Z'^j}\|_w. \end{aligned}$$

This yields,

$$\begin{aligned} \|\mathfrak{P}' - \mathfrak{P}\|_W &\leq h_\gamma \max_{i\in\mathbb{E}} \sum_{j\in\mathbb{E}} |q'_{ij} - q_{ij}| M_{Z^j}(\varepsilon^*) \\ &\quad + 2h_\gamma \max_{i\in\mathbb{E}} \sum_j M_{Z^j}(\varepsilon^*)\left(\frac{c_j}{\lambda_j}\wedge\frac{c'_j}{\lambda'_j}\right)\left|\frac{\lambda'_j}{c'_j} - \frac{\lambda_j}{c_j}\right| \\ &\quad + h_\gamma \max_{i\in\mathbb{E}} \|F_{Z^i} - F_{Z'^i}\|_w \\ &=: \mu(a,a'), \end{aligned} \tag{5.21}$$

This leads to the following result.

Theorem 5.6. *If the Markov modulated risk process satisfies Assumption 5.1 and $\varepsilon_* > 0$ is taken from Lemma 5.4, then we have the following stability bound:*

$$\|\psi_a - \psi_{a'}\|_w \leq \frac{\mu(a,a')}{\Delta_0 - \mu(a,a')} \frac{\overline{\kappa}\, e^{(\varepsilon^* - \varepsilon_*)/\chi}}{1-\rho} \sum_{j\in\mathbb{E}} \pi_j \gamma_j(\varepsilon_*), \tag{5.22}$$

for any a' s.t. $\mu(a,a') < \Delta_0$, where μ is from (5.21), and Δ_0 is from (5.24).

Proof. We apply Corollary 3.13 to the MC (V_n, I_n) with weight funtion W. Using relations

$$\begin{aligned} \|\nu\|_W &= \sum_{i\in\mathbb{E}} \pi_i\gamma_i(\varepsilon_*) \\ \kappa &:= \sup_{v\geq 0, i\in\mathbb{E}} \frac{1}{W(v,i)} \leq \left(\min_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \underline{\gamma}(\varepsilon)\right)^{-1} =: \overline{\kappa}, \end{aligned}$$

we have the following stability bound for stationary distribution Π of a MC (V, I):

$$\|\Pi' - \Pi\|_W \leq \frac{\|\mathfrak{P}' - \mathfrak{P}\|_W}{\Delta_0 - \|\mathfrak{P}' - \mathfrak{P}\|_W} \frac{1}{1-\rho} \sum_{j\in\mathbb{E}} \pi_j\gamma_j(\varepsilon_*), \tag{5.23}$$

if the perturbed process (V', I') satisfies

$$\|\mathfrak{P}' - \mathfrak{P}\|_W \leq \frac{(1-\rho)^2}{1 + (\overline{\kappa}\sum_i \pi_i\gamma_i(\varepsilon_*) - 1)\rho} =: \Delta_0. \tag{5.24}$$

Bound (5.22) follows from relations (5.19) and (5.21).

□

Chapter 6

Model with borrowing and investments

In this section we obtain stability bounds for the ruin probability in the risk model with borrowing and investments introduced in Section 2.2.3. We use the MC approach from Sections 3.2.

The risk process $\{R(t)\}_{t\geq 0}$ and the skeleton risk process $\{R_n\}_{n\geq 0}$ at the claims occurrence times $\{T_n\}_{n\geq 1}$ ($R_0 := R(0)$), are defined in (2.26) and (2.27), respectively. Let θ be a generic r.v. for the i.i.d. claim inter-occurrence times $\theta_i := T_i - T_{i-1}$ (i.e., it is exponentially distributed with parameter λ) and it is independent of all other r.v. and processes of a risk model. In the rest of this chapter we require the following to hold true.

Assumption 6.1. *There exists $\varepsilon^* > 0$ such that*

$$\mathbb{E}\exp\left(-\varepsilon^* U_\theta\right) < 1. \tag{6.1}$$

The condition (6.1) is related to the analogue of the Cramér condition 2.6 and will be discussed and illustrated by some examples in Section 6.4. At this stage we only use that (6.1) implies

$$\lim_{t\to\infty} U_t = \infty \qquad a.s. \tag{6.2}$$

Indeed, the relation

$$\mathbb{E}\exp\left(-\varepsilon U_\theta\right) = \int_0^\infty \lambda e^{-\lambda t}\mathbb{E}e^{-\varepsilon U_t}\,dt < 1$$

can only hold if $\mathbb{E}e^{-\varepsilon U_t} < 1$ for some $t > 0$. Then, by Jensen's inequality, $e^{-\varepsilon \mathbb{E}U_t} < 1$, which yields $\mathbb{E}U_t > 0$. The strong law of large numbers and the stationary independent increments of U imply that $\lim_{t\to\infty} U_t = \infty$ a.s.

In order to carry out the construction we first introduce another process X, which is closely related to the risk process R.

Process X. In the construction of the reversed process and in the further investigations we will rely on the properties of the process X which is defined by the same assumptions (C1) and (C2) as the risk process R with the exception that the accumulated claim amount process is identically equal to 0 (i.e. we assume zero claims for the process X). It is convenient to define X on the whole real line, i.e. $X = \{X_t\}_{t\in\mathbb{R}}$ (this will be useful when constructing the reversed process, see Section 6.1). For this reason we consider the claim occurrence times sequence $\{T_i\}_{i\geq 1}$ as a part of the increasing sequence of random times in $\mathbb{R}$, $\{T_i\}_{i\in\mathbb{Z}}$, such that

$$\theta_i := T_i - T_{i-1}, \qquad i \in \mathbb{Z}, \tag{6.3}$$

are i.i.d. exponentially distributed with parameter λ, and $T_1 = \min\{T_i, T_i > 0\}$. Then, the Poisson process $\{N_t\}$ in the condition (C1) is given by

$$N_t = \max\{i : 0 < T_i \leq t\}.$$

We first describe the paths of the process X. Conditions (C1)-(C2) imply that X has different path behavior in the regions $[0, d)$, $[d, D)$ and $[D, \infty)$. Therefore, we describe the process X through processes X^1, X^2 and X^3, which are defined as follows. For fixed $v > 0$ we define

$$X_t^1(v) = e^{U_t}\left(v - D + c\int_0^t e^{-U_u}\,du\right)_+ + D; \tag{6.4}$$

$$X_t^2(v) = v + ct; \tag{6.5}$$

$$X_t^3(v) = v\exp(\beta t). \tag{6.6}$$

Here v plays the role of the initial value at time 0. If the process X ever assumes values in the regions $(0, d]$, $(d, D]$ and $(D, \infty]$, then it behaves as X^3, X^2 and X^1 in the corresponding regions. Relations (6.2) and $\{-U_u\}_{u\geq 0} \stackrel{d}{=} \{U_{-u}\}_{u\geq 0}$ ($\stackrel{d}{=}$ stands for the identity of the finite-dimensional distributions) yield

$$\lim_{t\to-\infty}\int_0^t e^{-U_u}\,du = -\infty \quad a.s. \tag{6.7}$$

implying that for any $v \geq D$, $\lim_{t\to\infty} X_t^1(v) = \infty$ a.s. and that the following r.v. is well defined and

$$\tau(v) := \sup\{t : X_t^1(v) \leq D\} > -\infty \qquad a.s. \tag{6.8}$$

Thus, the process X assumes values in the mentioned above regions a.s.

The dynamics of the process X can be seen from the stochastic differential equation,

$$dX_t = (X_t - D)_+ d\tilde{U}_t - (d - X_t)_+\beta dt + cdt,$$

where

$$d\tilde{U}_t = dU_t^c + \frac{1}{2}d[U]_t^c + \left(e^{\Delta U_t} - 1\right) \tag{6.9}$$

and U^c, ΔU and $[U]$ are the continuous part, the jump part and the optional variance of the process U respectively.

Similarly to (6.4)–(6.6), we introduce processes $\left(\mathfrak{S}_s X_t^i\right)(v)$ which depend on the shifted process $(U_{s+t} - U_s)$, where $s \in \mathbb{R}$ is fixed,

$$\left(\mathfrak{S}_s X_t^1\right)(v) \quad := \quad e^{U_{s+t}-U_s}\left(v - D + c\int_0^t e^{-(U_{s+u}-U_s)}\,du\right)_+ + D, \tag{6.10}$$

and $\left(\mathfrak{S}_s X_t^2\right)(v) \equiv X_t^2(v)$ and $\left(\mathfrak{S}_s X_t^3\right)(v) \equiv X_t^3(v)$, because X^2 and X^3 do not depend on U. Obviously, $\left(\mathfrak{S}_0 X_t^1\right)(v) \equiv X_t^1$. These equations define the process $\left(\mathfrak{S}_s X_t\right)(v)$, and because of the stationary increments of U,

$$\{\left(\mathfrak{S}_s X_t\right)(v)\} \stackrel{d}{=} \{X_t(v)\}, \qquad s \in \mathbb{R}.$$

Therefore, we write $X_t(v)$ suppressing the shift $\mathfrak{S}_s$ for identities in distribution.

Lemma 6.2. *Let X' be defined by the same equations as X, with U replaced by an independent copy U'. Then, for any s and t with $s \cdot t \geq 0$, we have*

$$X_{s+t}(v) = (\mathfrak{S}_s X_t)(X_s(v)) \stackrel{d}{=} X_t'(X_s(v)). \tag{6.11}$$

Proof. Relation (6.11) holds for the deterministic processes X^2 and X^3. It only remains to prove that X^1 satisfies (6.11) for $v > D$ and s, t such that $X_u^1 > D$ for all $u \in [0, s+t]$ (or $u \in [s+t, 0]$, if $s, t \leq 0$). For such v, s, t we have

$$\begin{aligned}
X_{s+t}^1(v) &= e^{U_{s+t}-U_t}\, e^{U_t}\left(v - D + c\int_0^t e^{-U_u}\,du + c\int_0^s e^{-U_{t+u}}\,du\right) + D \\
&= e^{U_{s+t}-U_t}\left(e^{U_t}\left(v - D + c\int_0^t e^{-U_u}\,du\right) \pm D + c\int_0^s e^{-(U_{t+u}-U_t)}\,du\right) + D \\
&\stackrel{d}{=} e^{U_s'}\left(X_t^1(v) - D + c\int_0^s e^{-U_u'}\,du\right) + D \equiv {X'}_t^1\left(X_s^1(v)\right),
\end{aligned}$$

where $\{U_s'\} \stackrel{d}{=} \{U_{t+s} - U_t\} \stackrel{d}{=} \{U_t\}$ and U' is independent of U_t and therefore is independent of $X_s^1(v)$. □

Let us return to our risk process $\{R_n\}$. It satisfies the following recursive equation

$$R_0 = u, \qquad R_n = \mathfrak{S}_{T_{n-1}} X_{\theta_n}(R_{n-1}) - Z_n, \quad n \geq 1. \tag{6.12}$$

6.1 Reversed process

The construction (2.14)–(2.15) is applicable to the risk process $\{R_n\}$ defined in (2.4) with

$$\sigma_n = (\theta_n, Z_n, \{U_{T_{n-1}+u} - U_{T_{n-1}}\}_{u \in [0,\theta_n)}),$$

and the function F defined by the right-hand expression in (6.12). Let us consider $\mathfrak{S}_s X_t$ as a function, which for every $v \geq 0$, $\mathfrak{S}_s X_t : v \to \mathfrak{S}_s X_t(v)$. It then follows from the definition (6.10) that $\mathfrak{S}_{T_n} X_{-\theta_n}$ is the inverse function to $\mathfrak{S}_{T_{n-1}} X_{\theta_n}$. Therefore,

$$R_{n-1} = \mathfrak{S}_{T_n} X_{-\theta_n}(R_n + Z_n). \tag{6.13}$$

The process X is non-negative and, therefore, the construction (2.14)–(2.15) together with (6.13) gives us the following reversed process,

$$\begin{aligned} V_0 &= 0, \\ V_n &= \mathfrak{S}_{T_n} X_{-\theta_n}(V_{n-1} + Z_n), \end{aligned} \tag{6.14}$$

which satisfies the relation (2.16).

Since $\{\sigma_n\}$ is i.i.d., the sequence $\{V_n\}$ constitutes a homogeneous Markov chain. Because of the negative drift, the Markov chain $\{V_n\}$ has a stationary distribution given by

$$\pi(x) = \lim_{n\to\infty} \mathbf{P}(V_n \le x) = 1 - \psi(x), \tag{6.15}$$

see (2.16). We denote the transition kernel and the shift operator of the Markov chain $\{V_n\}$ respectively by

$$P(v, \Gamma) = \mathbf{P}\left(V_{n+1} \in \Gamma | V_n = v\right) = \mathbf{P}(X_{-\theta}(v + Z) \in \Gamma), \tag{6.16}$$

$$\mathfrak{P} f(v) = \int_0^\infty f(x)\, P\left(v, dx)\right) = \mathbb{E} f\left(X_{-\theta}(v + Z)\right), \tag{6.17}$$

where f is any measurable function. Given a non-negative kernel K, the corresponding shift operator $\mathfrak{K}$ is defined analogously to (6.17) with P replaced by K.

By virtue of (6.15) the problem of stability of the ruin probability is reduced to the stability of the stationary distribution π of the MC $\{V_n\}$. Similarly to the two previous chapters, we would like to apply Kartashov's result from Section 3.2.2. Thus, our aim is to prove that the MC $\{V_n\}$ satisfies conditions (D1)-(D3). Assumption 6.1 will be crucial in the proof of (D3).

6.2 Decomposition of the kernel P

In this section we construct a decomposition (3.17) of P into K, h, G such that $K \ge 0$. As we have assumed positive claims Z_i, it follows that there exist $q > 0$ and $0 < z_1 < z_2 < \infty$ such that

$$F_Z(z_2) - F_Z(z_1) \ge q. \tag{6.18}$$

Let $d_* = \min\{d, z_1\}$. For every $v \ge d_*$, $\theta^*(v)$ is the unique solution to the equation

$$X_{-\theta_*(v)}(v) = d_*. \tag{6.19}$$

For $v \le D$, $\theta_*(v)$ is a deterministic strictly increasing function of v, which is easy to see from (6.5) and (6.6). For $v > D$, we have $X_{-\theta_*(v)}(v) = X_{-\theta_*(D)}\left(X_{\tau(v)}(x)\right)$, where $\tau(v)$ is defined in (6.8). It follows from (6.11) that $\theta_*(v) = -\tau(v) + \theta_*(D)$. This proves the existence and uniqueness of $\theta_*(v)$. Also, $\theta_*(v)$ is increasing in v as $\tau(v)$ is decreasing (see (6.8) and (6.4)).

The decomposition of the transition kernel P in (6.16) is based on the fact that for any initial value $v \ge 0$ at time $n \ge 0$, the distribution of V_{n+1} has a strictly positive density in the interval $(0, d_*)$.

Lemma 6.3. *Let $v \ge 0$ and $\Gamma \subset (0, d_*)$. Then the transition kernel P satisfies the relation*

$$P(v, \Gamma) \ge h(v) G(\Gamma),$$

where

$$h(v) \quad = \quad q\,\mathbf{P}(\theta > \theta_*(v+z_2)), \tag{6.20}$$

is a positive function, and G is a probability measure on $(0, d_*)$ given by

$$G(\Gamma) \quad = \quad \mathbf{P}(X_{-\theta}(d_*) \in \Gamma). \tag{6.21}$$

The value q is defined in (6.18), and the r.v.'s θ and $\theta_*(v+z_2)$ are independent.

Proof. Let $v \geq 0$ and $\Gamma \subset (0, d_*)$. From the definition (6.16) of the transition kernel P,

$$\begin{aligned}
\mathbf{P}(v, \Gamma) &= \mathbf{P}\big(X_{-\theta}(v+Z) \in \Gamma\big) \\
&= \mathbf{P}\big(X_{-\theta}(v+Z) \in \Gamma, Z \in [z_1, z_2)\big) + \mathbf{P}\big(X_{-\theta}(v+Z) \in \Gamma, Z \notin [z_1, z_2)\big) \\
&\geq q \inf_{z \in [z_1, z_2]} \mathbf{P}\big(X_{-\theta}(v+z) \in \Gamma\big) \\
&= q \inf_{z \in [z_1, z_2]} \mathbf{P}\big(X_{-\theta}(v+z) \in \Gamma \,|\, \theta > \theta_*(v+z)\big)\, \mathbf{P}\big(\theta > \theta_*(v+z)\big),
\end{aligned} \tag{6.22}$$

where the latter equality follows from the relation $\{X_{-\theta}(v+z) \in \Gamma\} \subset \{\theta > \theta_*(v+z)\}$ for any $z \in [z_1, z_2)$. Provided that $\theta > \theta_*(v+z)$ we have from (6.11)

$$\begin{aligned}
X_{-\theta}(v+z) &= X_{-\theta' - \theta_*(v+z)}(v+z) \\
&\overset{d}{=} X'_{-\theta'}\big(X_{-\theta_*(v+z)}(v+z)\big) \\
&= X'_{-\theta'}(d_*),
\end{aligned}$$

where $\theta' := \theta - \theta_*(v+z)$ and by the lack-of-memory of the exponential distribution, θ' and $\theta_*(v+z)$ are conditionally independent and $\mathbf{P}\big(\theta' \leq x \,|\, \theta \geq \theta_*(v+z)\big) = \mathbf{P}(\theta \leq x)$. Hence, the latter together with (6.22) and using that $\theta_*(v)$ is increasing in v, yields

$$\begin{aligned}
\mathbf{P}(v, \Gamma) &\geq q\,\mathbf{P}\big(X'_{-\theta'}(d_*) \in \Gamma\big) \inf_{z \in [z_1, z_2]} \mathbf{P}\big(\theta > \theta_*(v+z)\big) \\
&= q\,\mathbf{P}\big(X'_{-\theta'}(d_*) \in \Gamma\big)\mathbf{P}(\theta > \theta_*(v+z_2)),
\end{aligned}$$

This proves the lemma. □

6.3 Deterministic investments

In this section we obtain stability bounds for ruin probabilities in the special case when the Lévy process degenerates to a drift function, i.e.

$$U_t = \alpha t, \qquad \alpha > 0. \tag{6.23}$$

Assumption 6.1 now trivially holds. In this case the behavior of the ruin probability was treated by Embrechts & Schmidli [13], see Section 2.3.3.

In the following we assume that there exists $\varepsilon^* > 0$ such that

$$\mathbb{E}\exp(\varepsilon^* Z) < \infty. \tag{6.24}$$

Under this condition the ruin probability has exponential decay as $u \to \infty$. From these results it seems reasonable to compare the ruin probabilities in a weighted metric with weight function

$$w(v) = \exp(\varepsilon v)$$

where $\varepsilon < \varepsilon^*$. If $\|\mathfrak{K}\|_w < 1$, then Kartashov's results (see Theorem 3.12 and Corollary 3.13) are applicable and the stability bounds for the ruin probability follow. However, one may only conclude from condition (6.24) that $\frac{\mathfrak{K}w(v)}{w(v)} \leq \rho < 1$ for large enough v (see Lemma 6.13 in the Appendix 6.5.1). Therefore, it becomes necessary to consider a slight modification of w.

Lemma 6.4. *Let $\mathbb{E}\exp(\varepsilon^* Z) < \infty$ and fix a positive $\varepsilon < \varepsilon^*$. Then there exists a function w_0 depending on ε which satisfies*

$$w_0(v) \leq \exp(\varepsilon v) \leq c_{w_0} w_0(v), \qquad \forall v \geq 0, \tag{6.25}$$

for some constant $c_{w_0} > 1$, and

$$\|\mathfrak{K}\|_{w_0} = \rho < 1.$$

The construction of w_0 and the explicit value of c_{w_0} are given in Appendix 6.5.1; see in particular Lemma 6.15.

Now, by virtue of Corollary 3.13, we obtain stability bounds for the ruin probability with weight function w_0. Relation (6.25) provides the desired bounds with weight function w. For the sake of simplicity and illustration we focus on stability bounds with respect to the parameter $a = (\lambda, F_Z)$ keeping α, β and c fixed. Bounds with respect to all parameters can also be obtained. This, however, is technically more involved; see the discussion in Section 6.6.1.

Let

$$\mu(a, a') \quad = \quad \|F_Z - F'_Z\|_{f_0} + h_\lambda |\lambda - \lambda'|, \tag{6.26}$$

where

$$f_0(x) = e^{\varepsilon^* x} \qquad \text{and } h_\lambda = \frac{1}{\lambda \vee \lambda'} \left(\mathbb{E}f_0(Z) + \mathbb{E}f_0(Z')\right).$$

Theorem 6.5. *Let $\mathbb{E}\exp(\varepsilon^* Z) < \infty$, $\mathbb{E}\exp(\varepsilon^* Z') < \infty$ and fix $\varepsilon < \varepsilon^*$. Take ρ and w_0 as in Lemma 6.4. If*

$$\mu(a, a') < \frac{(1-\rho)^2}{1 + (e^{\varepsilon d_*} - 1)\rho} =: \Delta_0,$$

then the following stability bound holds

$$|\psi_a - \psi_{a'}|_w \leq \frac{c_{w_0} e^{\varepsilon d_*}}{(1-\rho)} \frac{\mu(a, a')}{\Delta_0 - \mu(a, a')}. \tag{6.27}$$

Proof. Lemma 6.22 in Appendix II yields the inequality

$$\|\mathfrak{P}_a - \mathfrak{P}_{a'}\|_{w_0} \quad \leq \quad \mu(a, a'). \tag{6.28}$$

The constants κ and $\|G\|_{w_0}$ (see (3.16) and (6.21)) can be bounded as follows

$$\kappa \quad = \quad \sup_{v \geq 0} \frac{1}{w_0(v)} = 1,$$

$$\|G\|_{w_0} \quad = \quad \int_0^{v^*} w_0(v)\, G(dv) \leq \exp(\varepsilon d_*).$$

This together with (6.28) and Corollary 3.13 yields

$$\|\psi' - \psi\|_{w_0} \le \frac{e^{\varepsilon d_*}}{(1-\rho)} \frac{\mu(a,a')}{\Delta_0 - \mu(a,a')}. \tag{6.29}$$

From (3.25) and (6.25),

$$|\psi_a - \psi_{a'}|_w \le c_{w_0} \|\psi_a - \psi_{a'}\|_{w_0}.$$

Now (6.27) follows from (6.29) and the latter inequality. □

6.4 The general model

In this section we consider the general model, where U is a Lévy process satisfying Assumption 6.1. Additionally we assume that

$$\mathbb{E}Z^{\varepsilon^*} < \infty, \tag{6.30}$$

where ε^* is as in (6.1).

To illustrate Assumption 6.1 we consider two examples.

Example 6.6. Let

$$U_t = \alpha t + \sum_1^m \alpha_i P_t(\lambda_i), \qquad \alpha > 0, \quad \alpha_i \in \mathbb{R}, \quad i = 1, \dots, m, \quad \min_i \alpha_i < 0, \tag{6.31}$$

where $P_t(\lambda_i)$ are independent Poisson processes with parameter λ_i. Denote

$$g(\varepsilon) = -\varepsilon\alpha + \sum_{i=1}^m \lambda_i(e^{-\varepsilon\alpha_i} - 1). \tag{6.32}$$

Then

$$\mathbb{E}\exp(-\varepsilon^* U_\theta) = \mathbb{E}\exp(\theta g(\varepsilon^*)) = \frac{\lambda}{\lambda - g(\varepsilon^*)}.$$

From $g(0) = 0$ and $g''(\varepsilon) > 0$ it follows that there exists $\varepsilon^* > 0$ such that relation (6.1) holds if and only if $g'(0) < 0$. Therefore, Assumption 6.1 holds if and only if

$$\alpha + \sum_1^m \alpha_i \lambda_i > 0. \tag{6.33}$$

Example 6.7. Let

$$U_t = \alpha t + \sigma W_t, \tag{6.34}$$

where W is standard Brownian motion and σ is a positive constant. In this case,

$$\mathbb{E}\exp(-\varepsilon^* U_\theta) = \mathbb{E}\exp\left(\theta((\varepsilon^*\sigma)^2/2 - \varepsilon^*\alpha)\right) = \frac{\lambda}{\lambda - (\varepsilon^*\sigma)^2/2 + \varepsilon^*\alpha}.$$

Assumption 6.1 holds if and only if $\alpha > 0$. In this case, $\varepsilon^* < 2\alpha/\sigma^2$.

The weight function

$$w(v) = (1+v)^{\varepsilon}$$

with $\varepsilon < \varepsilon^*$ seems reasonable for the tail-comparison of the ruin probabilities in this case. However, we cannot expect that $\|\mathfrak{K}\|_w < 1$ holds for this function: Assumption 6.1 together with the condition (6.30) yields $\frac{\mathfrak{K}w(v)}{w(v)} \le \rho < 1$ for large enough v (see Lemma 6.19). Similarly to the model with deterministic investments, we consider a slight modification of w.

Lemma 6.8. *Under Assumption 6.1 and condition (6.30), for any fixed $\varepsilon < \varepsilon^*$ there exists a function w_1 which satisfies*

$$w_1(v) \le (1+v)^{\varepsilon} \le c_{w_1} w_1(v), \qquad \text{for all } v \ge 0, \tag{6.35}$$

for some constant $c_{w_1} > 1$, and

$$\|\mathfrak{K}\|_{w_1} = \rho < 1.$$

The construction of w_1 and the explicit expression of c_{w_1} are given Appendix 6.5.2; see in particular Lemma 6.20.

Now Corollary 3.13 together with the inequalities (6.35) provides the following stability bound. The proof is analogous to the one of Theorem 6.5. The non-perturbed model is governed by the parameter $a = (\lambda, c, F_Z, \beta, a_U)$, (where a_U stands for the parameters governing the process U), and all quantities with primes refer to the perturbed model governed by the parameter a'.

Theorem 6.9. *Let $\mathbb{E}Z^{\varepsilon^*} < \infty$, $\mathbb{E}(Z')^{\varepsilon^*} < \infty$ and w_1 be as in Lemma 6.8. Assume that $\mu(a, a')$ satisfies*

$$\|\mathfrak{P}' - \mathfrak{P}\|_{w_1} \quad \le \quad \mu(a, a'). \tag{6.36}$$

and

$$\mu(a, a') < \frac{(1-\rho)^2}{1 + ((1+d_*)^{\varepsilon} - 1)\rho} =: \Delta_0. \tag{6.37}$$

Then the corresponding ruin probabilities satisfy

$$|\psi_a - \psi_{a'}|_w \le \frac{(1+d_*)^{\varepsilon} c_{w_1}}{1-\rho} \frac{\mu(a, a')}{\Delta_0 - \mu(a, a')}. \tag{6.38}$$

We conclude this section with two examples illustrating the use of Theorem 6.9.

Example 6.10. Let U be the process in (6.31) satisfying the condition (6.33). We assume that the parameters c, α and α_i are fixed and, therefore, not included in the governing parameter $a = (\lambda, \lambda_1, \ldots, \lambda_m, F_Z, \beta)$. In this case the inequality (6.36) is fulfilled for

$$\mu(a, a') \quad = \quad h_\lambda |\lambda - \lambda'| + h_\beta \left| \frac{\lambda}{\beta} - \frac{\lambda'}{\beta'} \right| + h_Z \|F_Z - F_Z'\|_{f_1} + \sum_1^m h_{\lambda i} |\lambda_i - \lambda_i'|.$$

Explicit expressions for h_λ, h_β, h_Z, $h_{\lambda i}$ and the function f_1 together with the proof are given in Appendix 6.6; see in particular Example 6.24.

Example 6.11. Let U be defined by (6.34). Define the governing parameter as $a = (\lambda, c, F_Z, \beta, \alpha, \sigma)$. Assume that parameter a' satisfies

$$\frac{\alpha'}{\sigma'^2} > \max\left(\frac{2\alpha}{\sigma^2} - \sqrt{\frac{2\lambda}{\sigma^2} - \left(\frac{\alpha}{\sigma^2}\right)^2}, 0 \right) \qquad \text{and} \qquad \frac{c\sigma'^2}{c'\sigma^2} \ge \frac{1}{2}. \tag{6.39}$$

Then Theorem 6.9 holds with $\varepsilon < 2\min(\alpha/\sigma^2, \alpha'/{\sigma'}^2)$ and

$$\begin{aligned}\mu(a,a') &= h_{\lambda c}|\frac{\lambda}{c}-\frac{\lambda'}{c'}| + h_\beta\left|\frac{\lambda}{\beta}-\frac{\lambda'}{\beta'}\right| + h_Z\|F_Z - F_Z'\|_{f_0} \\ &\quad + h_{\alpha\sigma}\left|\frac{\alpha}{\sigma^2}-\frac{\alpha'}{{\sigma'}^2}\right| + h_{\lambda\sigma}\left|\frac{\lambda}{\sigma^2}-\frac{\lambda'}{{\sigma'}^2}\right| + h_{c\sigma}\left|\frac{c}{\sigma^2}-\frac{c'}{{\sigma'}^2}\right|^\delta, \end{aligned} \tag{6.40}$$

where $0 < \delta < 1/2$, and the constants are given in the proof of Lemma 6.29.

6.5 Appendix I

6.5.1 Construction of the weight function in the deterministic model

As promised in Section 6.3, in this section we construct a weight function w_0 which satisfies relation (6.25) and

$$\|\mathfrak{K}\|_{w_0} = \sup_{v\geq 0}\frac{\mathfrak{K}w_0(v)}{w_0(v)} \leq \rho < 1, \tag{6.41}$$

in the particular case where $U_t = \alpha t$.

First, we prove an inequality which will be used later in the proofs.

Lemma 6.12. *For any $\varepsilon \geq 0$ such that $\mathbb{E}e^{\varepsilon Z} < \infty$,*

$$\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)} \leq \mathbb{E}\left[\exp\left(\varepsilon v(e^{-\alpha\theta}-1)\right)\right]\mathbb{E}\Big[\exp\left(\varepsilon(Z+D)\right)\Big], \qquad v \geq 0, \tag{6.42}$$

where the shift operator $\mathfrak{P}$ is defined in (6.17).

Proof. We first prove two inequalities,

$$X_{-t}(v) \leq v, \tag{6.43}$$
$$X_{-t}(v) \leq D + ve^{-\alpha t} \tag{6.44}$$

for $t, v \geq 0$. For $U_t = \alpha t$ the process X which coincides with X^1, X^2 and X^3 in the corresponding regions (see (6.4)–(6.6) and the discussion afterwards) is deterministic. Relation (6.43) is immediate.

Recall $\tau(v)$ from (6.8). Inequality (6.44) is valid for $v \leq D$, and for $\big(v > D$ and $-t \leq \tau(v)\big)$, because in this case $X_{-t}(v) \leq D$. For $\big(v > D$ and $-t > \tau(v)\big)$,

$$\begin{aligned} X_{-t}(v) &= e^{-\alpha t}\left(v - D + c\int_0^{-t} e^{-\alpha u}\,du\right) + D \\ &\leq e^{-\alpha t}v + D, \end{aligned}$$

and thus, (6.44) holds.

Now, by (6.17),

$$\begin{aligned}\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)} &= \frac{\mathbb{E}\exp\left(\varepsilon X_{-\theta}(v+Z)\right)}{\exp(\varepsilon v)} \\ &= \mathbb{E}\left[\exp\left(\varepsilon v\left(\frac{X_{-\theta}(v+Z)}{v+Z}-1\right)\right)\exp\left(\varepsilon Z\frac{X_{-\theta}(v+Z)}{v+Z}\right)\right]. \end{aligned} \tag{6.45}$$

By (6.44),

$$\begin{aligned}\varepsilon v\left(\frac{X_{-\theta}(v+Z)}{v+Z}-1\right) &\le \varepsilon v\left(\frac{D+e^{-\alpha\theta}(v+Z)}{v+Z}-1\right)\\ &\le \varepsilon D+\varepsilon v\left(e^{-\alpha\theta}-1\right),\end{aligned}$$

and by (6.43),

$$\varepsilon Z\frac{X_{-\theta}(v+Z)}{v+Z}\le \varepsilon Z.$$

This together with (6.45) yields (6.42). □

Inequality (6.42) is the basis for the next result.

Lemma 6.13. *Let* $\mathbb{E}\exp(\varepsilon^* Z)<\infty$ *for* $\varepsilon^*>0$. *Then*
(i) there exists $V<\infty$ *such that*

$$\sup_{\varepsilon\in[0,\varepsilon^*]}\sup_{v\ge V}\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)}\le 1. \tag{6.46}$$

(ii) for any $0<\varepsilon_*\le\varepsilon^*$ *and any* $\rho^*\in(0,1)$ *there exists* $V^*=V^*(\varepsilon_*,\rho^*)<\infty$ *such that*

$$\sup_{\varepsilon\in[\varepsilon_*,\varepsilon^*]}\sup_{v\ge V^*}\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)}\le \rho^*. \tag{6.47}$$

The constants V *and* V^* *are given by (6.52) and (6.53).*

Proof. Relation (6.42) and inequality

$$\begin{aligned}\mathbb{E}\exp\left(\varepsilon v(e^{-\alpha\theta}-1)\right) &= \left(\int_0^t+\int_t^\infty\right)\lambda e^{-\lambda t}\exp\left(\varepsilon v(e^{-\alpha t}-1)\right)dt\\ &\le 1-e^{-\lambda t}+e^{-\lambda t}\exp\left(\varepsilon v(e^{-\alpha t}-1)\right),\end{aligned} \tag{6.48}$$

for any $t>0$, yield

$$\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)}\le\left[1-e^{-\lambda t}\left(1-\exp\left(\varepsilon v(e^{-\alpha t}-1)\right)\right)\right]\mathbb{E}\exp(\varepsilon(Z+D))=:f_\varepsilon(v). \tag{6.49}$$

Set $\eta\le 1$ and consider the conditions under which $f_\varepsilon(v)\le\eta$, i.e.

$$e^{-\lambda t}\exp\left(\varepsilon v\left(e^{-\alpha t}-1\right)\right)\le\frac{\eta}{\mathbb{E}\exp(\varepsilon(Z+D))}+e^{-\lambda t}-1. \tag{6.50}$$

The latter is only possible if the right-hand expression is positive, which is equivalent to

$$t<\frac{1}{\lambda}\log\left(\frac{\mathbb{E}e^{\varepsilon(Z+D)}}{\mathbb{E}e^{\varepsilon(Z+D)}-\eta}\right).$$

Take

$$t=\frac{1}{\lambda}\log\left(\frac{\mathbb{E}e^{\varepsilon(Z+D)}-\gamma}{\mathbb{E}e^{\varepsilon(Z+D)}-\eta}\right), \tag{6.51}$$

where $\gamma \in (0,\eta)$. Inserting this in (6.50) we see that (6.50) holds for any $v \geq V = V(\varepsilon)$, where

$$V(\varepsilon) = \left(1 - \left(\frac{\mathbb{E}e^{\varepsilon(Z+D)} - \eta}{\mathbb{E}e^{\varepsilon(Z+D)} - \gamma}\right)^{\alpha/\lambda}\right)^{-1} \frac{1}{\varepsilon} \log\left(\frac{\mathbb{E}e^{\varepsilon(Z+D)}}{\gamma}\right).$$

We now consider two cases.

(i) Let $\eta = 1$ and take $\gamma = e^{-\varepsilon}$. This yields (6.46) with

$$V := \sup_{\varepsilon\in(0,\varepsilon^*]} V(\varepsilon) = \left(1 - \left(\frac{\mathbb{E}e^{\varepsilon^*(Z+D)} - 1}{\mathbb{E}e^{\varepsilon^*(Z+D)} - e^{-\varepsilon^*}}\right)^{\alpha/\lambda}\right)^{-1} \sup_{\varepsilon\in(0,\varepsilon^*]} \frac{\log\left(\mathbb{E}e^{\varepsilon(Z+D+1)}\right)}{\varepsilon}. \tag{6.52}$$

(ii) Set $\eta = \rho^* < 1$ and take $\gamma = \rho^*/2$. This yields (6.47) with

$$V^* = \left(1 - \left(\frac{\mathbb{E}e^{\varepsilon^*(Z+D)} - \rho^*}{\mathbb{E}e^{\varepsilon^*(Z+D)} - \rho^*/2}\right)^{\alpha/\lambda}\right)^{-1} \frac{1}{\varepsilon_*} \log\left(\frac{2\mathbb{E}e^{\varepsilon^*(Z+D)}}{\rho^*}\right). \tag{6.53}$$

This completes the proof. □

Lemma 6.14. *Under the conditions of Lemma 6.13 there exist* $\varepsilon_0 = \varepsilon_0(V) > 0$ *and* $\rho_0 = \rho_0(V) < 1$ *such that*

$$\sup_{\varepsilon\leq\varepsilon_0} \sup_{v\leq V} \frac{\mathfrak{K}\exp(\varepsilon v)}{\exp(\varepsilon v)} \leq \rho_0. \tag{6.54}$$

Proof. By definition of K (see condition (D2)), for any $\varepsilon \leq \varepsilon^*$ and $v \geq 0$,

$$\mathfrak{K}\exp(\varepsilon v) \quad = \quad \mathfrak{P}\exp(\varepsilon v) - h(v)\int_0^{d_*} \exp(\varepsilon y)\, G(dy). \tag{6.55}$$

We choose the specification of h and G as in (6.20) and (6.21). In particular, G is a probability measure on $[0,d_*]$. Since h is positive and monotone decreasing to 0, it follows that for any $v \leq V$ and $\varepsilon \leq \varepsilon^*$,

$$\frac{\mathfrak{K}\exp(\varepsilon v)}{\exp(\varepsilon v)} \quad \leq \quad \frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)} - \frac{h(V)}{\exp(\varepsilon^* V)}.$$

From (6.42) and since $\mathbb{E}\exp\left(\varepsilon v(e^{-\alpha\theta} - 1)\right) \leq 1$,

$$\frac{\mathfrak{P}\exp(\varepsilon v)}{\exp(\varepsilon v)} \leq \mathbb{E}\exp(\varepsilon(Z+D)).$$

Since the right-hand expression converges to 1 when $\varepsilon \to 0$ and $h(V) > 0$, we can choose $\varepsilon_0 \in (0,\varepsilon^*)$ such that

$$\mathbb{E}\exp(\varepsilon_0(Z+D)) \leq 1 + \frac{1}{2}\frac{h(V)}{\exp(\varepsilon^* V)}.$$

The last three inequalities give that (6.54) holds with

$$\rho_0 = 1 - \frac{1}{2}\frac{h(V)}{\exp(\varepsilon^* V)}.$$

□

The weight function w_0 is given by the following construction. Let ε_0 be from Lemma 6.14. Fix $\varepsilon_* \in (0, \varepsilon_0)$ and $\varepsilon \in (\varepsilon_*, \varepsilon^*)$. For any $\chi \geq 0$ define

$$r(v) \quad = \quad \varepsilon_* + (\varepsilon^* - \varepsilon_*)\frac{(\chi \log(1+v)) \wedge C}{1 + (\chi \log(1+v)) \wedge C}, \quad v \geq 0. \tag{6.56}$$

where

$$C = \frac{\varepsilon - \varepsilon_*}{\varepsilon^* - \varepsilon}. \tag{6.57}$$

We define the desired weight function by

$$w_0(v) = \exp(r(v)v). \tag{6.58}$$

Lemma 6.15. *Under the conditions of Lemma 6.13 there exists $\chi > 0$ such that $\|\mathfrak{K}\|_{w_0} = \rho < 1$.*

Proof. Write

$$f_v(z) = \exp(r(v)z).$$

Then,

$$\begin{aligned} \|\mathfrak{K}\|_{w_0} &= \sup_{v\geq 0} \frac{\mathfrak{K}w_0(v)}{w_0(v)} \\ &= \sup_{v\geq 0} \frac{\mathfrak{K}(f_v + (w_0 - f_v))(v)}{w_0(v)} \\ &\leq \sup_{v\geq 0} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} + \sup_{v\geq 0} \frac{\mathfrak{K}(w_0 - f_v)(v)}{f_v(v)}. \end{aligned} \tag{6.59}$$

We prove that for appropriately chosen χ, the first supremum in (6.59) is strictly less than 1, and the second one can be made arbitrarily small, implying the assertion of the lemma.

We first prove that for some $\rho_1 < 1$ to be specified,

$$\sup_{v\geq 0} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} \leq \rho_1. \tag{6.60}$$

Let V be as in Lemma 6.13 and take $\varepsilon_0(V)$, $\rho_0(V)$ as in Lemma 6.14. For $\varepsilon_* \in (0, \varepsilon_0)$ take $V^* = V^*(\varepsilon_*, \rho_0)$ as in Lemma 6.13. Without loss of generality we assume $V^* \geq V$. Now we require χ to be such that $r(V^*) \leq \varepsilon_0$, which follows from

$$r(V^*) \leq \varepsilon_* + (\varepsilon^* - \varepsilon_*)\frac{\chi \log(1+V^*)}{1 + \chi \log(1+V^*)} \leq \varepsilon_0.$$

The latter inequality is equivalent to

$$\chi \leq \chi_1 := \frac{\varepsilon_0 - \varepsilon_*}{(\varepsilon^* - \varepsilon_0)\log(1+V^*)}.$$

Because $r(v) \in [\varepsilon_*, \varepsilon^*]$ for any $v \geq 0$, we have that

$$\frac{(\mathfrak{K}f_v)(v)}{f_v(v)} = \frac{\int_0^\infty K(v,dy)f_v(y)}{f_v(v)} \leq \sup_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \frac{\int_0^\infty K(v,dy)e^{\varepsilon y}}{e^{\varepsilon v}}.$$

This, together with Lemma 6.14 yields

$$\sup_{v\le V}\frac{(\mathfrak{K}f_v)(v)}{f_v(v)}\le\sup_{\varepsilon\in[0,\varepsilon_0]}\sup_{v\le V}\frac{\mathfrak{K}e^{\varepsilon v}}{e^{\varepsilon v}}\le\rho_0.$$

Similarly, by Lemma 6.13,

$$\sup_{v\ge V^*}\frac{(\mathfrak{K}f_v)(v)}{f_v(v)}\le\sup_{\varepsilon\in[\varepsilon_*,\varepsilon^*]}\sup_{v\ge V^*}\frac{\mathfrak{K}e^{\varepsilon v}}{e^{\varepsilon v}}\le\rho_0.$$

And finally, equality (6.55) yields that for any $v\in[V,V^*]$,

$$\frac{(\mathfrak{K}f_v)(v)}{f_v(v)}\le\sup_{\varepsilon\in[0,\varepsilon^*]}\frac{\mathfrak{K}e^{\varepsilon v}}{e^{\varepsilon v}}\le 1-\frac{h(V^*)}{e^{\varepsilon^*V^*}}.$$

Thus, (6.60) holds with

$$\rho_1=\max\left\{\rho_0,1-\frac{h(V^*)}{\exp(\varepsilon^*V^*)}\right\}$$

This proves (6.60).

Now we turn to the estimate of the second term in (6.59). We will prove that for sufficiently small $\chi>0$, the quantity

$$\sup_{v\ge0}\frac{\mathfrak{K}(w_0-f_v)(v)}{f_v(v)}\tag{6.61}$$

becomes arbitrarily small. Since the kernel $K\le P$, and $r(y)$ is an increasing function, it follows that

$$\begin{aligned}\mathfrak{K}(w_0-f_v)(v)&=\int_0^\infty K(v,dy)\left(e^{r(y)y}-e^{r(v)y}\right)\\&\le\int_v^\infty P(v,dy)\left(e^{r(y)y}-e^{r(v)y}\right)\\&=\mathbb{E}\left[\left(e^{r(V_{n+1})V_{n+1}}-e^{r(v)V_{n+1}}\right)\mathbf{1}\left(V_{n+1}\ge v\right)|V_n=v\right]\end{aligned}$$

Using that $e^{r(y)y}-e^{r(v)y}$ is an increasing function of y, together with the inequality

$$X_{-\theta}(v+Z)\le v+Z+D,$$

we conclude that for any $A\ge0$,

$$\begin{aligned}\frac{\mathfrak{K}(w_0-f_v)(v)}{f_v(v)}&\le\left(\int_0^A+\int_A^\infty\right)\frac{e^{r(v+z+D)(v+z+D)}-e^{r(v)(v+z+D)}}{e^{r(v)v}}dF_Z(z)\\&=:I_1+I_2.\end{aligned}\tag{6.62}$$

Choose χ so small that

$$\chi\le1-\frac{C}{1+C}=:\chi_2,$$

where C is defined by (6.57). Notice that

$$\frac{d}{dv}(r(v)v) \le \chi(\varepsilon^* - \varepsilon_*) + r(v). \tag{6.63}$$

The right-hand side is less or equal than ε^*, and so $r(v)v$ is Lipschitz with constant ε^*. Applying this to I_2, we obtain that for any $\delta_2 > 0$ there exists $A = A(\delta_2)$ such that

$$I_2 \le \int_A^\infty e^{\varepsilon^*(D+z)}\, dF_Z(z) \le \delta_2.$$

Since $\frac{d}{dv}(r(v)v)$ is increasing in v, (6.63) together with a Taylor expansion argument gives

$$r(v+z+D)(v+z+D) \le r(v)v + (\chi(\varepsilon^* - \varepsilon_*) + r(v+z+D))\,(z+D).$$

Inserting the latter inequality in I_1, one obtains

$$\begin{aligned} I_1 &\le Ae^{r(v)(A+D)}\left(e^{\chi(\varepsilon^*-\varepsilon_*)(A+D)}e^{(r(v+A+D)-r(v))(A+D)} - 1\right) \\ &\le Ae^{\varepsilon(A+D)}\left(e^{\chi(\varepsilon^*-\varepsilon_*)(A+D)(1+\log(1+A+D))} - 1\right) \end{aligned}$$

The last expression is less than any fixed $\delta_1 > 0$ for $\chi \le \chi_3$, where

$$\chi_3 := \frac{\log\left(1 + e^{-\varepsilon(A+D)}\delta_1/A\right)}{(\varepsilon - \varepsilon_*)(A+D)(1+\log(1+A+D))}.$$

The theorem is proved if we choose δ_1 and δ_2 such that $\rho_1 + \delta_1 + \delta_2 < 1$, and $\chi = \min\{\chi_1, \chi_2, \chi_3\}$.

Finally, the function w_0 satisfies the desired property

$$e^{\varepsilon v} \le c_{w_0} w_0(v), \qquad \text{for all } v \ge 0,$$

with

$$c_{w_0} = (\varepsilon^* - \varepsilon_*)\frac{C}{1+C}(e^{C/\chi} - 1). \tag{6.64}$$

This concludes the proof of the lemma. □

6.5.2 Construction of the weight function for the general model

In this section we will construct a weight function w_1 satisfying (6.35) and such that

$$\|\mathfrak{K}\|_{w_1} = \sup_{v\ge 0} \frac{\mathfrak{K}w_1(v)}{w_1(v)} \le \rho < 1, \tag{6.65}$$

for the general model, i.e. U is a Lévy process satisfying Assumption 6.1.

We prove an inequality which will be used later in the proofs.

Lemma 6.16. *Let $\varepsilon > 0$ be such that $\mathbb{E}Z^\varepsilon < \infty$. Then, for $v \ge 0$,*

$$\frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le \mathbb{E}\left[\left(\frac{1+D}{1+v} + e^{-U_\theta}\right)^\varepsilon\right]\mathbb{E}\left[\left(1 + \frac{Z}{1+v}\right)^\varepsilon\right]. \tag{6.66}$$

Proof. First, we prove that for every $t \geq 0$ and $v \geq 0$,

$$X_{-t}(v) \leq D + (1+v)e^{U_{-t}}. \tag{6.67}$$

It holds trivially for $v \leq D$ and $\big(v > D$ and $-t \leq \tau(v)\big)$, because $X_{-t}(v) \leq D$ in this case. So we consider the case $\big(v > D$ and $-t > \tau(v)\big)$, where the process X coincides with X^1 and by (6.4),

$$X_{-t}(v) \quad = \quad e^{U_{-t}}\left(v + D + c\int_0^{-t} e^{-U_u}\,du\right) + D,$$

from which (6.67) follows. Notice that

$$\begin{aligned}\frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} &= \mathbb{E}\left(\frac{1+X_{-\theta}(v+Z)}{1+v}\right)^\varepsilon \\ &= \mathbb{E}\left[\left(\frac{1+X_{-\theta}(v+Z)}{1+v+Z}\right)^\varepsilon\left(\frac{1+v+Z}{1+v}\right)^\varepsilon\right].\end{aligned} \tag{6.68}$$

By (6.67),

$$\left(\frac{1+X_{-\theta}(v+Z)}{1+v+Z}\right)^\varepsilon \leq \left(\frac{1+D}{1+v} + e^{U_{-\theta}}\right) \stackrel{d}{=} \left(\frac{1+D}{1+v} + e^{-U_\theta}\right).$$

The latter inequality together with (6.68) leads to (6.66). □

Lemma 6.17. *Let Assumption 6.1 hold with $\varepsilon^* > 0$ and assume that $\mathbb{E}Z^{\varepsilon^*} < \infty$. Then there exists $V < \infty$ such that*

$$\sup_{\varepsilon \leq \varepsilon^*}\sup_{v \geq V}\frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} \leq 1.$$

Proof. For $\varepsilon \in (0, \varepsilon^*]$, by (6.66) and Lyapunov's inequality,

$$\begin{aligned}\frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} &\leq \mathbb{E}\left[\left(\frac{1+D}{1+v} + e^{-U_\theta}\right)^\varepsilon\left(1 + \frac{Z}{1+v}\right)^\varepsilon\right] \\ &\leq \left(\mathbb{E}\left[\left(\frac{1+D}{1+v} + e^{-U_\theta}\right)^{\varepsilon^*}\left(1 + \frac{Z}{1+v}\right)^{\varepsilon^*}\right]\right)^{\varepsilon/\varepsilon^*}.\end{aligned}$$

The expectation in the last expression converges to $\mathbb{E}e^{-\varepsilon^* U_\theta} < 1$, when $v \to \infty$. It follows that there exists $V < \infty$ such that the last expression is less or equal than 1 for all $v \geq V$. □

Lemma 6.18. *Under the conditions of Lemma 6.17 there exist $\varepsilon_0 = \varepsilon_0(V) > 0$ and $\rho_0 = \rho_0(V) < 1$ such that*

$$\sup_{\varepsilon \leq \varepsilon_0}\sup_{v \leq V}\frac{\mathfrak{K}(1+v)^\varepsilon}{(1+v)^\varepsilon} \leq \rho_0. \tag{6.69}$$

Proof. By the definition of K,

$$\mathfrak{K}(1+v)^\varepsilon = \mathfrak{P}(1+v)^\varepsilon - h(v)\int_0^{d_*}(1+y)^\varepsilon\,G(dy) \tag{6.70}$$

The latter equality together with the inequality (6.66) and using that h is a decreasing function and G is a probability measure on $[0, d_*]$ (see (6.20) and (6.21)), yield that for all $\varepsilon \le \varepsilon^*$

$$\frac{\mathfrak{K}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le \mathbb{E}\big(1 + D + e^{-U_\theta}\big)^\varepsilon \mathbb{E}\big(1+Z\big)^\varepsilon - \frac{h(V)}{(1+V)^{\varepsilon^*}}.$$

Choose $\varepsilon_0 \in (0, \varepsilon^*)$ such that

$$\mathbb{E}\left(1 + D + e^{-U_\theta}\right)^{\varepsilon_0} \mathbb{E}(1+Z)^{\varepsilon_0} \le 1 + \frac{h(V)}{2(1+V)^{\varepsilon^*}}.$$

This proves the lemma with

$$\rho_0 := 1 - \frac{h(V)}{2(1+V)^{\varepsilon^*}}.$$

□

Lemma 6.19. *Assume that conditions of Lemma 6.17 hold. Take ε_0 as in Lemma 6.18 and choose $\varepsilon_* \in (0, \varepsilon_0)$. Then there exist $V^* = V^*(\varepsilon_*) < \infty$ and $\rho^* = \rho^*(\varepsilon_*) < 1$ such that*

$$\sup_{\varepsilon \in [\varepsilon_*, \varepsilon^*]} \sup_{v \ge V^*} \frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le \rho^*. \tag{6.71}$$

Proof. By Assumption 6.1, there exists $0 < \gamma < 1$ such that $\max_{\varepsilon \in [\varepsilon_*, \varepsilon^*]} \mathbb{E} e^{-\varepsilon U_\theta} \le \gamma$. Choose $\rho \in (\gamma, 1)$ and recall inequality (6.66). There exist $V_1(\varepsilon_*)$ and V_2 such that,

$$\begin{aligned} \sup_{\varepsilon \in [\varepsilon_*, \varepsilon^*]} \mathbb{E}\left[\left(\frac{1+D}{1+v} + e^{-U_\theta}\right)^\varepsilon\right] &\le \rho \qquad v \ge V_1, \\ \mathbb{E}\left[\left(1 + \frac{Z}{1+v}\right)^{\varepsilon^*}\right] &\le 2 - \rho \qquad v \ge V_2. \end{aligned}$$

The latter two inequalities and (6.66) yield (6.71) with $\rho^* = \rho(2-\rho) = 1 - (1-\rho)^2 < 1$ and $V^* = \max\{V_1, V_2\}$. The constants V_1 and V_2 can be calculated explicitly.

□

We prove that (6.65) holds with

$$w_1(v) = (1+v)^{R(v)}, \qquad v \ge 0 \tag{6.72}$$

with properly chosen χ and

$$R(v) = r(\log(1+v)), \qquad v \ge 0,$$

where r is defined in (6.56).

Lemma 6.20. *Under the conditions of Lemma 6.19 there exists $\chi > 0$ such that $\|\mathfrak{K}\|_{w_1} = \rho < 1$.*

Proof. The proof is similar to the one of Lemma 6.15 for deterministic investments. Write

$$f_v(z) = (1+z)^{R(v)}.$$

Then

$$\|\mathfrak{K}\|_{w_1} \le \sup_{v\ge 0} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} + \sup_{v\ge 0} \frac{(\mathfrak{K}(w_1 - f_v))(v)}{f_v(v)}. \tag{6.73}$$

We prove that the first supremum on the right-hand side is strictly less than 1, and the second one is arbitrarily small for a proper choice of χ.

First, we prove

$$\sup_{v\ge 0} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} \le \rho_1 < 1. \tag{6.74}$$

Take V as in Lemma 6.17, ρ_0 as in Lemma 6.18 and $V^*(\varepsilon_*)$, $\rho^*(\varepsilon_*)$ as in Lemma 6.19. Without loss of generality we assume that $V^* \ge V$. Let us choose χ so that

$$R(V^*) \le \varepsilon_0.$$

This holds if (see (6.56))

$$\chi \le \chi_1 := \frac{\varepsilon_0 - \varepsilon_*}{(\varepsilon^* - \varepsilon_0)\log(1+\log(1+V))}.$$

Then Lemma 6.18 yields

$$\sup_{v\le V} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} \le \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{v\le V} \frac{\mathfrak{K}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le \rho_0,$$

and from Lemma 6.19,

$$\sup_{v\ge V^*} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} \le \sup_{\varepsilon\in[\varepsilon_*,\varepsilon^*]} \sup_{v\ge V^*} \frac{\mathfrak{P}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le \rho^*.$$

Relation (6.70) together with Lemma 6.17 yields

$$\sup_{v\in[V,V^*]} \frac{(\mathfrak{K}f_v)(v)}{f_v(v)} \quad \le \quad \sup_{\varepsilon\in[0,\varepsilon_0]} \sup_{v\in[V,V^*]} \frac{\mathfrak{K}(1+v)^\varepsilon}{(1+v)^\varepsilon} \le 1 - \frac{h(V^*)}{w_{1\varepsilon_0}(V^*)}.$$

which proves (6.74) with

$$\rho_1 = \max\left\{\rho^*, \rho_0, 1 - \frac{h(V^*)}{w_{1\varepsilon_0}(V^*)}\right\}.$$

This proves the first part of the bound for (6.73).

We now turn to the estimate of

$$\sup_{v\ge 0} \frac{(\mathfrak{K}(w_1 - f_v))(v)}{f_v(v)}. \tag{6.75}$$

Because $K \le P$, and $R(y)$ is an increasing function, for $A > 0$,

$$\begin{aligned}\frac{\mathfrak{K}(w_1 - f_v)(v)}{f_v(v)} &= \int_0^\infty K(v,dy)\frac{(1+y)^{R(y)} - (1+y)^{R(v)}}{(1+v)^{R(v)}} \\ &\le \left(\int_v^{v\vee A} + \int_{v\vee A}^\infty\right) P(v,dy)\frac{(1+y)^{R(y)} - (1+y)^{R(v)}}{(1+v)^{R(v)}} = I_1 + I_2.\end{aligned}$$

Choose χ so small that

$$\chi \le 1 - \frac{C}{1+C} =: \chi_2.$$

Since $r(v)v$ is Lipschitz with constant ε^* (see (6.63) and the arguments following it in the proof of Lemma 6.15), it follows that

$$\begin{aligned}\frac{(1+y)^{R(y)} - (1+y)^{R(v)}}{(1+v)^{R(v)}} &= \frac{e^{r(\log(1+y))\log(1+y)} - e^{r(\log(1+v))\log(1+y)}}{e^{r(\log(1+v))\log(1+v)}} \\ &\le e^{\varepsilon^*|\log(1+y)-\log(1+v)|}.\end{aligned}$$

The latter inequality implies that we can choose $A = A(\delta_2)$ such that

$$\begin{aligned}I_2 &\le \int_{v\vee A}^\infty P(v,dy)\,\exp\left(\varepsilon^*\left(\log(1+y) - \log(1+v)\right)\right) \\ &= \mathbb{E}\left[\left(\frac{1+X_{-\theta}(v+Z)}{1+v}\right)^{\varepsilon^*}\mathbf{1}\left(X_{-\theta}(v+Z) > (v\vee A)\right)\right] \le \delta_2.\end{aligned}$$

For $v < A$ (when $I_1 \ne 0$), we estimate I_1 similarly to the corresponding integral I_1 (defined by (6.62)) in the proof of Lemma 6.15,

$$\begin{aligned}I_1 &\le \frac{e^{r(\log(1+A))\log(1+A)} - e^{r(\log(1+v))\log(1+A)}}{e^{r(\log(1+v))\log(1+v)}} \\ &\le (1+A)^\varepsilon\left[e^{\chi(\varepsilon^*-\varepsilon_*)\log(1+A)(1+\log(1+\log(1+A)))} - 1\right] \le \delta_1,\end{aligned}$$

for $\chi \le \chi_3$, where

$$\chi_3 = \frac{\log\left(1+\delta_1(1+A)^{-\varepsilon}\right)}{(\varepsilon^*-\varepsilon_*)\log(1+A)(\log(1+\log(1+A))+1)}.$$

Choosing $\chi = \min\{\chi_1, \chi_2, \chi_3\}$, (6.75) is bounded from above by $\delta_1 + \delta_2$ which can be made arbitrarily small. This proves the lemma. □

6.6 Appendix II

In this section we provide the upper bound for the quantity $\|\mathfrak{P} - \mathfrak{P}'\|_{w_0}$ for the model with deterministic investments, and similar bounds for $\|\mathfrak{P} - \mathfrak{P}'\|_{w_1}$ in the models considered in Examples 6.6 and 6.7. We use the technique of minimal metrics, see Section 3.5 for a short survey.

The next lemma refers to the general case when the original model is governed by the parameter $a = (\lambda, c, F_Z, \beta, U)$, and the governing parameter of the perturbed model is indicated with primes.

Lemma 6.21. *Let $w : \mathbb{R} \to \mathbb{R}_+$ be a monotone increasing function satisfying $w(0) = 1$ and*

$$\mathbb{E}\left(w(X'_{-\theta'}(v+z))\right) \leq w(v)f(z) \qquad z \geq 0,\ v \geq 0, \tag{6.76}$$

where f is bounded away from 0. Then,

$$\|\mathfrak{P} - \mathfrak{P}'\|_w \quad \leq \quad \|\mathfrak{P}^1 - \mathfrak{P}'^1\|_w + \|F_Z - F_{Z'}\|_f + h_{\lambda c}\left|\frac{\lambda}{c} - \frac{\lambda'}{c'}\right| + h_\beta\left|\frac{\lambda}{\beta} - \frac{\lambda'}{\beta'}\right| + w(D)\Delta,$$

where

$$\begin{aligned} \|\mathfrak{P}^1 - \mathfrak{P}'^1\|_w &:= \sup_{v\geq 0}\frac{1}{w(v)}Var_w\left(X^1_{-\theta}(v+Z), X'^1_{-\theta'}(v+Z)\right), &(6.77)\\ \Delta &:= \sup_{v\geq D}\left|\mathbf{P}(\theta > -\tau(v)) - \mathbf{P}(\theta' > -\tau'(v))\right| &(6.78)\end{aligned}$$

and the constants are given by

$$\begin{aligned} h_{\lambda c} &= 2w(D)\left((D-d) + \left(\frac{c}{\lambda}\wedge\frac{c'}{\lambda'}\right)\right), &(6.79)\\ h_\beta &= 4w(D)\left(\frac{\beta}{\lambda}\wedge\frac{\beta'}{\lambda'}\right). &(6.80)\end{aligned}$$

Proof. By the definition of the norm $\|\cdot\|_w$, see Section 3.5,

$$\begin{aligned} \|\mathfrak{P} - \mathfrak{P}'\|_w &= \sup_{v\geq 0}\frac{Var_w(X_{-\theta}(v+Z), X'_{-\theta'}(v+Z'))}{w(v)}\\ &\leq \sup_{v\geq 0}\frac{Var_w(X'_{-\theta'}(v+Z), X'_{-\theta'}(v+Z'))}{w(v)}\\ &\quad + \sup_{v\geq 0}\frac{Var_w(X_{-\theta}(v+Z), X'_{-\theta'}(v+Z))}{w(v)}\\ &= I_1 + I_2. &(6.81)\end{aligned}$$

Because the (weighted) total variation metric $Var_w(X,Y)$ is completely determined via its marginal distributions F_X and F_Y (so called *simple* metric; see Zolotarev [52], p. 36), one may freely choose the dependence structure of (X,Y). Therefore we assume without loss of generality here and in what follows that Z is independent of X' and θ'.

Let us start with I_1. From Corollary 3.32 and the relation

$$\{X'_{-\theta'}(v+Z) \neq X'_{-\theta'}(v+Z')\} \subset \{Z \neq Z'\},$$

it follows that

$$\begin{aligned} & Var_w(X'_{-\theta'}(v+Z), X'_{-\theta'}(v+Z'))\\ \leq\ & \frac{1}{2}\mathbb{E}\left[\left(w(X'_{-\theta'}(v+Z)) + w(X'_{-\theta'}(v+Z'))\right)\mathbf{1}\left(X'_{-\theta'}(v+Z) \neq X'_{-\theta'}(v+Z')\right)\right]\\ \leq\ & \frac{1}{2}\mathbb{E}\left[\left(w(X'_{-\theta'}(v+Z)) + w(X'_{-\theta'}(v+Z'))\right)\mathbf{1}(Z \neq Z')\right]\\ \leq\ & \frac{1}{2}\mathbb{E}\left[(f(Z) + f(Z'))\,\mathbf{1}(Z \neq Z')\right]w(v),\end{aligned}$$

where the last inequality follows from the condition (6.76). This yields

$$I_1 \le \min_{P_{ZZ'}\in\mathcal{P}_{ZZ'}} \frac{1}{2}\mathbb{E}\left[(f(Z)+f(Z'))\,\mathbf{1}(Z\neq Z')\right] = \|F_Z - F_{Z'}\|_f. \tag{6.82}$$

Now we turn to the term I_2. We have

$$\begin{aligned}
&\frac{Var_w(X_{-\theta}(v+Z), X'_{-\theta'}(v+Z))}{w(v)}\\
=\;& \frac{1}{w(v)}\left(\int_0^D + \int_D^\infty\right) w(x)\Big|\mathbf{P}\big(X_{-\theta}(v+Z)\in dx\big) - \mathbf{P}\big(X'_{-\theta'}(v+Z)\in dx\big)\Big|\\
=\;& I_{21}+I_{22}.
\end{aligned}$$

Notice, that I_{22} depends only on values of X above the level D. In this region process X coincides with X^1, see (6.4). Hence,

$$\begin{aligned}
I_{22} &\le \sup_{v\ge 0}\frac{1}{w(v)}\int_D^\infty w(x)\Big|\mathbf{P}\big(X^1_{-\theta}(v+Z)\in dx\big) - \mathbf{P}\big({X'}^1_{-\theta'}(v+Z)\in dx\big)\Big|\\
&= \sup_{v\ge 0}\frac{1}{w(v)} Var_w\left(X^1_{-\theta}(v+Z), {X'}^1_{-\theta'}(v+Z)\right) = \|\mathfrak{P}^1 - {\mathfrak{P}'}^1\|_w.
\end{aligned}$$

It remains to consider the term I_{21}. It deals with X below the level D where X is described by X^2 and X^3, defined in (6.5) and (6.6). Using that Z is independent of X' and θ', we have

$$\begin{aligned}
I_{21} &= \left(\int_0^{(D-v)_+} + \int_{(D-v)_+}^\infty\right)\int_0^D w(x)\Big|\mathbf{P}\big(X_{-\theta}(v+z)\in dx\big) - \mathbf{P}\big(X'_{-\theta'}(v+z)\in dx\big)\Big|\,dF_Z(z)\\
&= I_{211}+I_{212}.
\end{aligned} \tag{6.83}$$

From $X_{-t}(x)\le x$ for $x\le D$ and $t\ge 0$ it follows that

$$\begin{aligned}
I_{211} &= \int_{z=0}^{(D-v)_+}\int_{x=0}^{v+z} w(x)\Big|\mathbf{P}\big(X_{-\theta}(v+z)\in dx\big) - \mathbf{P}\big(X'_{-\theta'}(v+z)\in dx\big)\Big|\,dF_Z(z)\\
&\le w(D)\sup_{u\in(0,D]}\int_0^u \big|\mathbf{P}\big(X_{-\theta}(u)\in dx\big) - \mathbf{P}\big(X'_{-\theta'}(u)\in dx\big)\big|\\
&=: w(D)\,S.
\end{aligned} \tag{6.84}$$

By the 'lack of memory' of the exponential r.v.'s θ and θ', we have

$$\begin{aligned}
I_{212} &= \int_{(D-v)_+}^\infty\int_0^D w(x)\Big|\mathbf{P}\big(X_{-\theta}(D)\in dx\big)\mathbf{P}\big(\theta > -\tau(v+z)\big)\\
&\qquad -\mathbf{P}\big(X'_{-\theta'}(D)\in dx\big)\mathbf{P}\big(\theta' > -\tau'(v+z)\big)\Big|\,dF_Z(z)\\
&\le w(D)(\Delta + S).
\end{aligned} \tag{6.85}$$

It now remains to estimate S. For $u \le d$ using (6.6) we have

$$\begin{aligned} S_d &:= \sup_{u\in(0,d]} \int_0^u \left|\mathbf{P}\big(X_{-\theta}(u) \in dx\big) - \mathbf{P}\big(X'_{-\theta'}(u) \in dx\big)\right| \\ &= \sup_{u\in(0,d]} \int_0^u \frac{1}{u} \left| \frac{\lambda}{\beta}\left(\frac{x}{u}\right)^{\frac{\lambda}{\beta}-1} - \frac{\lambda'}{\beta'}\left(\frac{x}{u}\right)^{\frac{\lambda'}{\beta'}-1}\right| dx \;=\; \int_0^1 \left|\frac{\lambda}{\beta} y^{\frac{\lambda}{\beta}-1} - \frac{\lambda'}{\beta'} y^{\frac{\lambda'}{\beta'}-1}\right| dy \\ &\le \min\left(\frac{\beta}{\lambda}, \frac{\beta'}{\lambda'}\right)\left|\frac{\lambda}{\beta} - \frac{\lambda'}{\beta'}\right| + \min\left(\frac{\lambda}{\beta}, \frac{\lambda'}{\beta'}\right) \int_0^1 \left| y^{\frac{\lambda}{\beta}-1} - y^{\frac{\lambda'}{\beta'}-1}\right| dy \\ &= 2\min\left(\frac{\beta}{\lambda}, \frac{\beta'}{\lambda'}\right)\left|\frac{\lambda}{\beta} - \frac{\lambda'}{\beta'}\right|. \end{aligned} \tag{6.86}$$

Let now $u > d$. Denote

$$\left(\sup_{u\in(d,D]}\int_0^d + \sup_{u\in(d,D]}\int_d^u\right)\left|\mathbf{P}\big(X_{-\theta}(u) \in dx\big) - \mathbf{P}\big(X'_{-\theta'}(u) \in dx\big)\right| \quad =: \quad A_1 + A_2.$$

Now,

$$\begin{aligned} A_2 &= \sup_{u\in(d,D]} \int_d^u \left|\frac{\lambda}{c}\exp\left(-\frac{\lambda}{c}(u-x)\right) - \frac{\lambda'}{c'}\exp\left(-\frac{\lambda'}{c'}(u-x)\right)\right| dx \\ &\le (D-d)\left|\frac{\lambda}{c} - \frac{\lambda'}{c'}\right|. \end{aligned} \tag{6.87}$$

and by the 'lack of memory' of the exponential distribution,

$$\begin{aligned} A_1 &= \sup_{u\in(d,D]} \int_0^d \left|\exp\left(-\frac{\lambda}{c}(u-d)\right)\mathbf{P}\big(X_{-\theta}(d) \in dx\big) - \exp\left(-\frac{\lambda'}{c'}(u-d)\right)\mathbf{P}\big(X'_{-\theta'}(d) \in dx\big)\right| \\ &\le \sup_{u\in(d,D]}\left|\exp\left(-\frac{\lambda}{c}(u-d)\right) - \exp\left(-\frac{\lambda'}{c'}(u-d)\right)\right| + S_d, \\ &\le \min\left(\frac{c}{\lambda}, \frac{c'}{\lambda'}\right)\left|\frac{\lambda}{c} - \frac{\lambda'}{c'}\right| + S_d. \end{aligned}$$

The latter together with (6.86) and (6.87) yields

$$S \;\le\; \left((D-d) + \left(\frac{c}{\lambda} \wedge \frac{c'}{\lambda'}\right)\right)\left|\frac{\lambda}{c} - \frac{\lambda'}{c'}\right| + 2\left(\frac{\beta}{\lambda} \wedge \frac{\beta'}{\lambda'}\right)\left|\frac{\lambda}{\beta} - \frac{\lambda'}{\beta'}\right|.$$

This and $I_{21} \le w(D)(\Delta + 2S)$ which follows from (6.83)–(6.85) complete the proof. □

6.6.1 Deterministic investments

We consider the model with deterministic investments, i.e., $U_t = \alpha t$. This model depends on the parameters $\lambda, F_Z, c, \alpha, \beta$ (see conditions (C1)-(C2) and (6.23)). In order to keep calculations simple we only investigate the stability with respect to $a = (\lambda, F_Z)$ assuming that α, c, β are the same for all models (perturbed and non-perturbed). The general case, i.e. stability with respect to all parameters, can be investigated, for example, by using Lemma 6.21 (with w replaced by w_0 in Lemma 6.15).

Lemma 6.22. *Let w_0 as defined in (6.58) satisfy Lemma 6.15 and assume that $\mathbb{E}e^{\varepsilon^* Z}$ and $\mathbb{E}e^{\varepsilon^* Z'}$ are finite. Then*

$$\|\mathfrak{P} - \mathfrak{P}'\|_{w_0} \leq \|F_Z - F'_Z\|_{f_0} + h_\lambda|\lambda - \lambda'|,$$

where $f_0(v) = \exp(\varepsilon^* v)$ and

$$h_\lambda = \frac{1}{\lambda \vee \lambda'}\left(\mathbb{E}f_0(Z) + \mathbb{E}f_0(Z')\right). \tag{6.88}$$

Proof. By definition of the norm $\|\cdot\|_{w_0}$ (see (3.42)),

$$\|\mathfrak{P} - \mathfrak{P}'\|_{w_0} = \sup_{v\geq 0}\frac{1}{w_0(v)}Var_{w_0}(X_{-\theta}(v+Z), X'_{-\theta'}(v+Z')).$$

It follows from Corollary 3.32 that,

$$\begin{aligned} & Var_{w_0}(X_{-\theta}(v+Z), X'_{-\theta'}(v+Z')) \\ \leq\ & \frac{1}{2}\mathbb{E}\Big[\left(w_0(X_{-\theta}(v+Z)) + w_0(X'_{-\theta'}(v+Z'))\right)\mathbf{1}\left(X_{-\theta}(v+Z) \neq X'_{-\theta'}(v+Z')\right)\Big] =: I. \end{aligned}$$

From the relation

$$\{X_{-\theta}(v+Z) \neq X'_{-\theta'}(v+Z')\} \subset \{\theta \neq \theta'\} \cup \{Z \neq Z'\},$$

and the inequality

$$w_0(X_{-\theta}(v+Z)) \leq w_0(v+Z) \leq f_0(Z)w_0(v),$$

(which follows from $X_{-\theta}(v+Z) \leq v+Z$ and the Lipschitz property of $\log(w_0)$) we obtain

$$\begin{aligned} I &\leq \frac{1}{2}\mathbb{E}\Big[\left(w_0(X_{-\theta}(v+Z)) + w_0(X_{-\theta'}(v+Z'))\right)\mathbf{1}\left(Z \neq Z'\right)\Big] \\ &+ \frac{1}{2}\mathbb{E}\Big[\left(w_0(X_{-\theta}(v+Z)) + w_0(X_{-\theta'}(v+Z'))\right)\mathbf{1}\left(\theta \neq \theta'\right)\Big] \\ &\leq \frac{1}{2}\mathbb{E}\Big[(f_0(Z) + f_0(Z'))\mathbf{1}(Z \neq Z')\Big]w_0(v) + \frac{1}{2}\mathbb{E}(f_0(Z) + f_0(Z'))\mathbf{P}\left(\theta \neq \theta'\right)w_0(v). \end{aligned}$$

From Corollary 3.32,

$$\begin{aligned} \inf_{P_{ZZ'}\in\mathcal{P}_{ZZ'}}\frac{1}{2}\mathbb{E}\left((f_0(Z) + f_0(Z'))\mathbf{1}(Z \neq Z')\right) &= \|F_Z - F'_Z\|_{f_0}; \\ \inf_{P_{\theta\theta'}\in\mathcal{P}_{\theta\theta'}}\frac{1}{2}\mathbf{P}\left(\theta \neq \theta'\right) = \frac{1}{2}Var(\theta, \theta') &\leq \frac{1}{\lambda \vee \lambda'}|\lambda - \lambda'|. \end{aligned}$$

This proves the lemma. $\square$

6.6.2 The general model

In this section we find explicit upper bounds for $\|\mathfrak{P}-\mathfrak{P}'\|_{w_1}$ for the models defined in Examples 6.6 and 6.7.

Lemma 6.23. *Take w_1 as defined by (6.72) and assume that it satisfies Lemma 6.20. Then relation (6.76) holds with w replaced by w_1 and with*

$$f(z) = (1+z)^{\varepsilon^*}\mathbb{E}\left(1+D+e^{-U'_{\theta'}}\right)^{\varepsilon^*}.$$

In particular, $\|F_Z - F_{Z'}\|_f = h_Z\,\|F_Z - F_{Z'}\|_{f_1}$, where

$$f_1(v) \;:=\; (1+v)^{\varepsilon^*} \qquad \text{and} \qquad h_Z \;:=\; \mathbb{E}\left(1+D+e^{-U'_{\theta'}}\right)^{\varepsilon^*}. \tag{6.89}$$

Proof. Notice that for $v_1, v_2 \geq 0$,

$$|\log(w_1(v_1)) - \log(w_1(v_2))| \leq \varepsilon^*|\log(1+v_1) - \log(1+v_2)|.$$

It follows that for $v, z \geq 0$,

$$\begin{aligned}
\frac{w_1(X'_{-\theta'}(v+z))}{w_1(v)} &\leq \frac{w_1(v+D+(v+z)e^{-U'_{\theta'}})}{w_1(v)} \\
&\leq \exp\left(\varepsilon^*\left|\log\left(\frac{1+v+D+(v+z)e^{-U'_{\theta'}}}{1+v}\right)\right|\right) \\
&= \exp\left(\varepsilon^*\left|\log\left((1+z)\frac{1+v+D+(v+z)e^{-U'_{\theta'}}}{(1+z)(1+v)}\right)\right|\right) \\
&\leq \left(1+D+e^{-U'_{\theta'}}\right)^{\varepsilon^*}(1+z)^{\varepsilon^*}. \qquad (6.90)
\end{aligned}$$

Relation (6.76) follows by taking expectations on both sides of (6.90). □

Example 6.24. The Poisson case

Let U be defined by (6.31), and assume that (6.33) holds. We fix the constants α, c and α_i, $i = 1,\ldots,m$. Then $a = (\lambda, F_Z, \beta, \lambda_1, \ldots, \lambda_m)$ is the governing parameter.

Lemma 6.25. *Assume that $\mathbb{E}Z^{\varepsilon^*} < \infty$ and $\mathbb{E}(Z')^{\varepsilon^*} < \infty$, and let w_1 be as in (6.72) such that Lemma 6.20 holds. Then,*

$$\|\mathfrak{P}^1 - \mathfrak{P}'^1\|_{w_1} \;\leq\; h^*_\lambda|\lambda-\lambda'| + \sum_{k=1}^{m} h^*_{\lambda k}\,|\lambda_k - \lambda'_k|,$$

where the constants h^*_λ and $h^*_{\lambda k}$ are from (6.98) and (6.97).

Proof. From (6.77),

$$\begin{aligned}\|\mathfrak{P}^1 - \mathfrak{P}'^1\|_{w_1} &\le \sup_{v\ge 0} \frac{1}{w_1(v)} Var_{w_1}\left(X'^1_{-\theta}(v+Z), X'^1_{-\theta'}(v+Z)\right) \\ &\quad + \sup_{v\ge 0} \frac{1}{w_1(v)} Var_{w_1}\left(X^1_{-\theta}(v+Z), X'^1_{-\theta}(v+Z)\right) \\ &= I_1 + I_2. \end{aligned} \tag{6.91}$$

We first consider I_2. Let $m = 1$. Recall from (6.31),

$$U_t = \alpha t + \alpha_1 P_t(\lambda_1), \qquad \text{and} \qquad U'_t = \alpha t + \alpha_1 P_t(\lambda'_1).$$

Without loss of generality we assume that $P(\lambda_1)$ and $P(\lambda'_1)$ satisfy the following relation (see also the remark following (6.81)),

$$P(\lambda_1 \vee \lambda'_1) = P(\lambda_1 \wedge \lambda'_1) + P(|\lambda_1 - \lambda'_1|), \qquad a.s. \tag{6.92}$$

where the two processes on the right-hand side are supposed to be independent, and independent of θ, θ, Z. It follows (see (6.4)) that

$$\begin{aligned}\left\{X^1_{-\theta}(v+Z) \neq X'^1_{-\theta}(v+Z)\right\} &\subset \left\{P_\theta\left(\lambda_1\right) \neq P_\theta\left(\lambda'_1\right)\right\} \\ &= \left\{P_\theta\left(|\lambda_1 - \lambda'_1|\right) \neq 0\right\}.\end{aligned}$$

This relation, Corollary 3.32 and inequality (6.90) yield

$$\begin{aligned}&\frac{1}{w_1(v)} Var_{w_1}\left(X^1_{-\theta}(v+Z), X'^1_{-\theta}(v+Z)\right) \\ \le\ & \frac{1}{2}\mathbb{E}\left[\left(\frac{w_1\left(X^1_{-\theta}(v+Z)\right)}{w_1(v)} + \frac{w_1\left(X'^1_{-\theta}(v+Z)\right)}{w_1(v)}\right) \mathbf{1}\left(X^1_{-\theta}(v+Z) \neq X'^1_{-\theta}(v+Z)\right)\right] \\ \le\ & \frac{1}{2}\mathbb{E}(1+Z)^{\varepsilon^*}\, \mathbb{E}\left[\left(\left(1+D+e^{-U_\theta}\right)^{\varepsilon^*} + \left(1+D+e^{-U'_\theta}\right)^{\varepsilon^*}\right) \mathbf{1}\left(P_\theta\left(|\lambda_1 - \lambda'_1|\right) \neq 0\right)\right]\end{aligned} \tag{6.93}$$

Let for definiteness $\lambda_1 > \lambda'_1$. Since the processes $U' = P(lambda'_1)$ and $P\left(|\lambda_1 - \lambda'_1|\right)$ are independent, and

$$\begin{aligned}&\mathbb{E}\left[\left(1+D+e^{-U'_\theta}\right)^{\varepsilon^*} \mathbf{1}\left(P_\theta\left(|\lambda_1 - \lambda'_1|\right) \neq 0\right)\right] \\ =\ & \mathbb{E}\left[\left(1+D+e^{-U'_\theta}\right)^{\varepsilon^*} \left(1 - e^{-\theta|\lambda_1 - \lambda'_1|}\right)\right] \\ \le\ & \mathbb{E}\left[\left(1+D+e^{-U'_\theta}\right)^{\varepsilon^*} \theta|\lambda_1 - \lambda'_1|\right] \\ \le\ & |\lambda_1 - \lambda'_1| \max(1, 2^{\varepsilon^*-1}) \left(\frac{(1+D)^{\varepsilon^*}}{\lambda} + \frac{\lambda}{(\lambda - g'(\varepsilon^*))^2}\right).\end{aligned} \tag{6.94}$$

Here we have used inequality $(a+b)^\varepsilon \le \max(1,2^{\varepsilon-1})(a^\varepsilon+b^\varepsilon)$ for positive a,b,ε, and equality $\mathbb{E}\left(\theta\exp(-\varepsilon^* U'_\theta)\right) = \lambda/(\lambda-g'(\varepsilon^*))^2$, where the function g' is defined by (6.32) with λ_is replaced by λ'_is.

Similarly,

$$\begin{aligned} & \mathbb{E}\left[\left(1+D+e^{-U_\theta}\right)^{\varepsilon^*} \mathbf{1}\left(P_\theta\left(|\lambda_1-\lambda'_1|\right)\neq 0\right)\right] \\ \le\ & \max(1,2^{\varepsilon^*-1})\left[|\lambda_1-\lambda'_1|\frac{(1+D)^{\varepsilon^*}}{\lambda} + \mathbb{E}\left(e^{-\varepsilon^* U_\theta}\,\mathbf{1}\left(P_\theta\left(|\lambda_1-\lambda'_1|\right)\neq 0\right)\right)\right]. \end{aligned} \tag{6.95}$$

We now insert expression (6.92) and consider

$$\begin{aligned} & \mathbb{E}\left(e^{-\varepsilon^*\alpha_1 P_\theta(|\lambda_1-\lambda'_1|)}\,\mathbf{1}\left(P_\theta\left(|\lambda_1-\lambda_1|\right)\neq 0\right)\,\middle|\,\theta\right) \\ =\ & \mathbb{E}\left(e^{-\varepsilon^*\alpha_1 P_\theta(|\lambda_1-\lambda'_1|)}\,\middle|\,\theta\right) - e^{-\theta|\lambda_1-\lambda'_1|} \\ =\ & \exp\left(\theta|\lambda_1-\lambda'_1|\left(e^{-\varepsilon^*\alpha_1}-1\right)\right)\left(1-\exp\left(-\theta|\lambda_1-\lambda'_1|e^{-\varepsilon^*\alpha_1}\right)\right) \\ \le\ & \theta|\lambda_1-\lambda'_1|e^{-\varepsilon^*\alpha_1}\mathbb{E}\left(e^{-\varepsilon^*\alpha_1 P_\theta(|\lambda_1-\lambda'_1|)}\,\middle|\,\theta\right). \end{aligned}$$

This yields

$$\mathbb{E}\left(e^{-\varepsilon^* U_\theta}\,\mathbf{1}\left(P_\theta\left(|\lambda_1-\lambda_1|\right)\neq 0\right)\right) \quad \le \quad |\lambda_1-\lambda'_1|\,\frac{\lambda e^{-\varepsilon^*\alpha_1}}{(\lambda-g(\varepsilon^*))^2}. \tag{6.96}$$

A similar argument applies in the general case $m\ge 1$. Thus, (6.93) through (6.96) yield

$$I_2 \le \sum_{k=1}^m |\lambda_k-\lambda'_k|\,h^*_{\lambda k},$$

where

$$h^*_{\lambda k} \quad = \quad \frac{1\vee 2^{\varepsilon^*-1}}{2}\mathbb{E}(1+Z)^{\varepsilon^*}\left(\frac{2(1+D)^{\varepsilon^*}}{\lambda} + \frac{\lambda e^{-\alpha_k\varepsilon^*\mathbf{1}(\lambda_k>\lambda'_k)}}{(\lambda-g(\varepsilon^*))^2} + \frac{\lambda e^{-\alpha'_k\varepsilon^*\mathbf{1}(\lambda'_k>\lambda_k)}}{(\lambda-g'(\varepsilon^*))^2}\right). \tag{6.97}$$

We now turn to I_1. Relation

$$\left\{X'^{1}_{-\theta}(v+Z)\neq X'^{1}_{-\theta'}(v+Z)\right\} \quad\subset\quad \{\,\theta\neq\theta'\,\},$$

Corollary 3.32 and inequality (6.90) imply

$$\begin{aligned} I_1 \quad &\le \quad \inf_{F_{\theta\theta'}\in\mathcal{P}_{\theta\theta'}} \frac{1}{2}\mathbb{E}(1+Z)^{\varepsilon^*}\mathbb{E}\left[\left(\left(1+D+e^{-U'_\theta}\right)^{\varepsilon^*} + \left(1+D+e^{-U'_{\theta'}}\right)^{\varepsilon^*}\right)\mathbf{1}(\theta\neq\theta')\right] \\ &\le \quad \mathbb{E}(1+Z)^{\varepsilon^*}\int_0^\infty \mathbb{E}\left(1+D+e^{-U'_t}\right)^{\varepsilon^*}\left|\lambda e^{-\lambda t}-\lambda' e^{-\lambda' t}\right|\,dt \\ &\le \quad \mathbb{E}(1+Z)^{\varepsilon^*}\int_0^\infty \mathbb{E}\left(1+D+e^{-U'_t}\right)^{\varepsilon^*} e^{-(\lambda\wedge\lambda')t/2}\,2\,|\lambda-\lambda'|\,dt. \end{aligned}$$

Inequality

$$\int_0^\infty \mathbb{E}\left(1+D+e^{-U'_t}\right)^{\varepsilon^*} e^{-(\lambda\wedge\lambda')t/2}\,dt \quad\leq\quad (1\vee 2^{\varepsilon^*-1})2\left(\frac{(1+D)^{\varepsilon^*}}{\lambda\wedge\lambda'}+\frac{1}{(\lambda\wedge\lambda')-2g'(\varepsilon^*)}\right)$$

completes the proof with

$$h^*_\lambda \quad=\quad \mathbb{E}(1+Z)^{\varepsilon^*}\frac{4(1\vee 2^{\varepsilon^*-1})}{\lambda\wedge\lambda'}\left((1+D)^{\varepsilon^*}+\frac{\lambda\wedge\lambda'}{(\lambda\wedge\lambda')-2g'(\varepsilon^*)}\right). \tag{6.98}$$

□

Lemma 6.26. *Under the conditions of Lemma 6.25,*

$$\Delta\equiv\sup_{v\geq D}\left|\mathbf{P}(\theta>-\tau(v))-\mathbf{P}(\theta'>-\tau'(v))\right| \leq \frac{2}{\lambda}\sum_{i=1}^m|\lambda_i-\lambda'_i|. \tag{6.99}$$

Proof. By definition of $\tau(v)$ and X^1 (see (6.8) and (6.4)),

$$\mathbf{P}\Big(\theta\geq-\tau(v)\Big) \quad=\quad \mathbf{P}\Big(X^1_{-\theta}(v)\leq D\Big) = \mathbf{P}\left(\frac{v-D}{c}<\int_0^\theta e^{U_u}\,du\right).$$

Let $m=1$. Without loss of generality (since Δ depends only on the marginal distributions) we assume (6.92) and the independence conditions given after (6.92). Let $U^*=U$ if $\lambda_1<\lambda'_1$ and $U^*=U'$ if $\lambda'_1<\lambda_1$. Then,

$$\Delta \quad=\quad \left|\mathbf{P}\left(\frac{v-D}{c}<\int_0^\theta e^{U^*_u+\alpha_1 P_u(|\lambda_1-\lambda'_1|)}\,du\right)-\mathbf{P}\left(\frac{v-D}{c}<\int_0^\theta e^{U^*_u}\,du\right)\right|.$$

In the last expression, the first probability is greater than the second one for $\alpha_1>0$, and smaller for $\alpha_1<0$. This together with the inequality $P_u(|\lambda_1-\lambda'_1|)\leq P_\theta(|\lambda_1-\lambda'_1|)$ a.s. for $u\leq\theta$ implies that

$$\Delta \quad\leq\quad \left|\mathbf{P}\left(\frac{v-D}{c}<e^{\alpha_1 P_\theta(|\lambda_1-\lambda'_1|)}\int_0^\theta e^{U^*_u}\,du\right)-\mathbf{P}\left(\frac{v-D}{c}<\int_0^\theta e^{U^*_u}\,du\right)\right|.$$

Using that $\{U^*_u\}_{u\in(0,\theta)}$ and $P_\theta(|\lambda_1-\lambda'_1|)$ are independent conditionally on θ, we obtain

$$\begin{aligned}\Delta &\leq \mathbb{E}\left[\mathbf{P}\left(\frac{v-D}{c}<\int_0^\theta e^{U^*_u}\,du\,\Big|\,\theta\right)\left|\mathbf{P}\Big(P_\theta(|\lambda_1-\lambda'_1|)=0\,\big|\,\theta\Big)-1\right|+\mathbf{P}\Big(P_\theta(|\lambda_1-\lambda'_1|)>0\,\big|\,\theta\Big)\right]\\ &\leq 2\mathbb{E}\left[\mathbf{P}\Big(P_\theta(|\lambda_1-\lambda'_1|)>0\,\big|\,\theta\Big)\right]\\ &= 2\mathbb{E}\Big(1-\exp\big(-\theta|\lambda_1-\lambda'_1|\big)\Big)\\ &\leq \frac{2}{\lambda}|\lambda_1-\lambda'_1|.\end{aligned}$$

In the case $m>1$ similar arguments yield (6.99). □

Lemmas 6.21, 6.23 and 6.26 prove the following corollary.

Corollary 6.27. *Assume the conditions of Lemma 6.25 hold. Then*

$$\|\mathfrak{P}-\mathfrak{P}'\|_{w_1} \leq h_\lambda|\lambda-\lambda'| + h_\beta\left|\frac{\lambda}{\beta}-\frac{\lambda'}{\beta'}\right| + h_Z\|F_Z-F_Z'\|_{f_1} + \sum_1^m h_{\lambda i}|\lambda_i-\lambda_i'|, \tag{6.100}$$

where h_β is from (6.80), h_Z and f_1 are from (6.89), $h_\lambda = h_\lambda^* + h_{\lambda c}/c$, and $h_{\lambda i} = h_{\lambda i}^* + 2w(D)/\lambda$, with $h_{\lambda c}$, $h_{\lambda i}^*$ from (6.79), (6.97).

Example 6.28. The Brownian case

Let U be defined by (6.34). We assume the governing parameter in this case is given by $a = (\lambda, B, c, \sigma, \beta)$.

Lemma 6.29. *Let function w_1 be as in (6.72) and satisfy Lemma 6.20. Assume that $\mathbb{E}Z^{\varepsilon^*} < \infty$ and $\mathbb{E}(Z')^{\varepsilon^*} < \infty$ and that condition (6.39) holds. Then,*

$$\begin{aligned}\|\mathfrak{P}-\mathfrak{P}'\|_{w_1} \leq\ & h_{\lambda c}\left|\frac{\lambda}{c}-\frac{\lambda'}{c'}\right| + h_\beta\left|\frac{\lambda}{\beta}-\frac{\lambda'}{\beta'}\right| + h_Z\|F_Z-F_Z'\|_{f_1}\\ &+ h_{\alpha\sigma}\left|\frac{\alpha}{\sigma^2}-\frac{\alpha'}{\sigma'^2}\right| + h_{\lambda\sigma}\left|\frac{\lambda}{\sigma^2}-\frac{\lambda'}{\sigma'^2}\right| + h_{c\sigma}\left|\frac{c}{\sigma^2}-\frac{c'}{\sigma'^2}\right|^\delta,\end{aligned}$$

where $\delta < 1/2$, constant $h_{\lambda c}$ and h_β are from (6.79) and (6.80), h_Z and f_1 are from (6.89), and $h_{\alpha\sigma}$, $h_\lambda\sigma$, $h_{c\sigma}$ are given in (6.112) in the proof below.

Proof. Following the lines of the proof of Lemma 6.21, it suffices to bound $\|\mathfrak{P}^1-\mathfrak{P}'^1\|_{w_1} + w(D)\Delta$.

We start with some preliminary calculations. From $\{U_{-t}\} \stackrel{d}{=} \{-U_t\}$, $\{\sigma W_u\} \stackrel{d}{=} \{W_{\sigma^2 u}\}$ and $\sigma^2\theta \sim Exp(\lambda/\sigma^2)$, it follows that

$$\begin{aligned}X_{-\theta}^1(v) &\stackrel{d}{=} (v-D)e^{-U_\theta} + D - c\int_0^\theta e^{-U_u}\,du\\ &\stackrel{d}{=} (v-D)\exp(-\alpha_1\theta_1 - W_{\theta_1}) + D - c_1\int_0^{\theta_1}\exp(-\alpha_1 u - W_u)\,du,\end{aligned}$$

where

$$\alpha_1 = \alpha/\sigma^2, \qquad c_1 = c/\sigma^2, \qquad \theta_1 \stackrel{d}{=} \sigma^2\theta \sim Exp(\lambda_1), \quad \text{with } \lambda_1 = \lambda/\sigma^2$$

and the corresponding relations hold for the perturbed model. Denote

$$X = \exp(-\alpha_1\theta_1 - W_{\theta_1}), \qquad \text{and} \qquad Y = \int_0^{\theta_1}\exp(-\alpha_1 u - W_u)\,du.$$

Then (X, Y) has density (see [9], p. 208) given by

$$p(x,y) = \frac{2\lambda_1}{y}\exp\left(-\alpha_1 x - 2\frac{1+e^{-x}}{y}\right) I_{2\sqrt{2\lambda_1+\alpha_1^2}}\left(\frac{4e^{-x/2}}{y}\right),$$

where $I_\nu(x)$ is the modified Bessel function of order ν. The density of (X', Y') for the perturbed model is denoted by p'.

Let us consider Δ defined in (6.78). For any $v > D$,

$$\begin{aligned}\mathbf{P}\left(\theta > -\tau(v)\right) &= \mathbf{P}\left(X < \frac{c_1}{v-D}Y\right) = \int_{y=0}^{\infty}\int_{x=0}^{\frac{c_1}{v-D}y} p(x,y)\,dx\,dy \\ &= \frac{v-D}{c_1}\int_{y=0}^{\infty}\int_{x=0}^{y} p\left(x, y\frac{v-D}{c_1}\right)\,dy\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta &\le \sup_{v\ge D}\int_{y=0}^{\infty}\int_{x=0}^{y}\left|\frac{v-D}{c_1}p\left(x, y\frac{v-D}{c_1}\right) - \frac{v-D}{c_1'}p'\left(x, y\frac{v-D}{c_1'}\right)\right|\,dy\,dx \\ &\le \int_0^\infty\int_0^\infty \left|p\big(x,y\big) - \gamma p'\Big(x, \gamma y\Big)\right|\,dy\,dx, \qquad (6.101)\end{aligned}$$

where $\gamma = c_1/c_1'$.

We now turn to $\|\mathfrak{P}^1 - \mathfrak{P}'^1\|_{w_1}$. Corollary 3.32, inequality (6.90) and the inclusion

$$\{X_{\theta'}(v+Z) \neq X'_{\theta'}(v+Z)\} \quad \subset \quad \{X \neq X'\} \cup \{c_1 Y \neq c_1' Y'\}$$

yield

$$\|\mathfrak{P}^1 - \mathfrak{P}'^1\|_{w_1} \le \mathbb{E}w(Z)\int_0^\infty\int_0^\infty (1+D+x)^{\varepsilon^*}\left|p\big(x,y\big) - \gamma p'\big(x,\gamma y\big)\right|\,dy\,dx. \qquad (6.102)$$

Hence, (6.101) and (6.102) imply

$$\|\mathfrak{P}^1 - \mathfrak{P}'^1\|_{w_1} + w(D)\Delta \le \int_0^\infty\int_0^\infty g(x^{\varepsilon^*})\left|p\big(x,y\big) - \gamma p'\big(x,\gamma y\big)\right|\,dy\,dx, \qquad (6.103)$$

where

$$g(x) = w(D)\left(1 + (1\vee 2^{\varepsilon^*-1})\mathbb{E}w(Z)\right) + \left(1 \vee 2^{\varepsilon^*-1}\right)\mathbb{E}w(Z)x.$$

We will estimate (6.103) using the explicit form of $p(x,y)$. Writing $\nu = 2\sqrt{2\lambda_1 + \alpha_1^2}$ and ν' for the perturbed model, as well as

$$F[a_1,a_2,a_3,a_4,a_5](x,y) := \frac{2a_1}{y}\exp\left(-a_2 x - 2\frac{1+e^{-x}}{a_3 y}\right) I_{a_5}\left(\frac{4e^{-x/2}}{a_4 y}\right),$$

we obtain

$$p\big(x,y\big) = F[\lambda_1, \alpha_1, 1, 1, \nu](x,y), \qquad \gamma p'\big(x,\gamma y\big) = F[\lambda_1', \alpha_1', \gamma, \gamma, \nu'](x,y).$$

Then,

$$\begin{aligned}
& \left|p(x,y) - \gamma p'(x,\gamma y)\right| \\
\le\ & |F[\lambda_1,\alpha_1,1,1,\nu] - F[\lambda_1',\alpha_1,1,1,\nu]|\,(x,y) \ +\ |F(\lambda_1',\alpha_1,1,1,\nu] - F[\lambda_1',\alpha_1',1,1,\nu]|\,(x,y) \\
& \quad + |F[\lambda_1',\alpha_1',1,1,\nu] - F[\lambda_1',\alpha_1',\gamma,\gamma,\nu]|\,(x,y) + |F[\lambda_1',\alpha_1',\gamma,\gamma,\nu] - F[\lambda_1',\alpha_1',\gamma,\gamma,\nu']|\,(x,y) \\
=:\ & (J_1 + J_2 + J_3 + J_4)\,(x,y). \qquad (6.104)
\end{aligned}$$

For any $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ write

$$Q(f) := \int_0^\infty \int_0^\infty g(x^{\varepsilon^*}) f(x,y)\,dx\,dy. \qquad (6.105)$$

It now remains to bound the quantities $Q(J_i)$, $i = 1,\ldots,4$. From Assumption 6.1 we have

$$Q(J_1) \ \le\ \frac{|\lambda_1 - \lambda_1'|}{\lambda_1}\mathbb{E}g\left(e^{-U_\theta}\right) \ \le\ \left|\frac{\lambda}{\sigma^2} - \frac{\lambda'}{\sigma'^2}\right| \frac{g(1)}{\lambda_1} \ =:\ \left|\frac{\lambda}{\sigma^2} - \frac{\lambda'}{\sigma'^2}\right| h_1. \qquad (6.106)$$

Let $\alpha_* = \min(\alpha_1,\alpha_1')$ and a positive η satisfy $\eta < \min\{\alpha_*, \alpha_* + \sqrt{2\lambda_1 + \alpha_1^2} - 2\alpha_1\}$. It is proved in Lemma 6.30 below that under condition (6.39),

$$Q(J_2) \ \le\ \left|\frac{\alpha}{\sigma^2} - \frac{\alpha'}{\sigma'^2}\right| \frac{2\lambda_1'}{\eta(2\lambda_1 + \alpha_1^2 - (\alpha_* - \eta)^2)} g\left(h_2^*\right) \ =:\ \left|\frac{\alpha}{\sigma^2} - \frac{\alpha'}{\sigma'^2}\right| h_2, \qquad (6.107)$$

where

$$h_2^* = \frac{2\lambda_1 + \alpha_1^2 - (\alpha_* - \eta)^2}{2\lambda_1 + \alpha_1^2 - (\alpha_* - \eta)^2 + 2\varepsilon^*(\alpha_* - \eta) - \varepsilon^{*2}}.$$

Let us fix a positive constant $\delta < 1/2$. Lemmas 6.31–6.34 below prove that under restriction (6.39),

$$Q(J_3) \ \le\ \left|\frac{c}{\sigma^2} - \frac{c'}{\sigma'^2}\right|^\delta h_3, \qquad (6.108)$$

where

$$\begin{aligned}
h_3 \ =\ & \left(c_1^{-\delta} \vee c_1^{-1}\right) \lambda_1' g\left(\alpha_1'^{\,\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right) \\
& \times \left[\frac{2}{\sqrt{\pi}}\left(\frac{\gamma \vee 1}{2}\right)^\delta \Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta} \ +\ \frac{2(\gamma\wedge 1)^\delta}{\sqrt{\pi}}\Gamma\binom{\delta, 1/2-\delta}{1-\delta}\right. \\
& \qquad \left. + 2^\delta(\gamma\wedge 1)^\delta\Gamma(\delta) + \frac{2^{\zeta+1}(\gamma\wedge 1)^\zeta}{\pi(\nu-\zeta)}\Gamma(\zeta)\right]. \qquad (6.109)
\end{aligned}$$

Here the positive constant ζ satisfies $\zeta < \min(1,\nu)$, and $\Gamma\binom{a_1, a_2}{b} := \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b)}$, where $\Gamma(b) = \int_0^\infty t^{b-1}e^{-t}\,dt$ $(b > 0)$ is the gamma function.

Lemma 6.35 proves that

$$Q(J_4) \ \le\ \left(\frac{|\lambda_1 - \lambda_1'|}{\sqrt{2(\lambda_1 \wedge \lambda_1')}} + |\alpha_1 - \alpha_1'|\right) h_4, \qquad (6.110)$$

where

$$h_4 = 2\lambda_1' g\left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right) \times\left[\left(\frac{\pi}{\cos(\delta\pi)}\left(\frac{\Gamma(1/2)}{\delta\sqrt{\pi}}+1\right)+\frac{\pi}{\nu}\right)\left(1+\frac{\pi|\nu-\nu'|}{2}\right)+\frac{1}{\pi\xi((\nu\vee\nu')-\xi)}\right]. \quad (6.111)$$

and $0<\xi<\min(\nu,\nu')$.

Thus, the lemma holds with

$$h_{\alpha\sigma} := h_2 + h_4, \qquad h_{\lambda\sigma} := h_1 + \frac{h_4}{\sqrt{2(\lambda_1\wedge\lambda_1')}}, \qquad h_{c\sigma} := h_3. \quad (6.112)$$

□

Supplement to the proof of Lemma 6.29

Here we provide the proofs of inequalities (6.107), (6.108) and (6.110).

Lemma 6.30. *Let (6.39) hold. Then we have the bound (6.107).*

Proof. From (6.104),

$$J_2(x,y) = \frac{2\lambda_1'}{y}\left|e^{-\alpha_1 x} - e^{\alpha_1' x}\right| \exp\left(-2\frac{1+e^{-x}}{y}\right) I_\nu\left(\frac{4e^{-x/2}}{y}\right).$$

Let $\alpha_* = \min(\alpha_1,\alpha_1')$ and $\eta\in(0,\alpha_*)$. Inequality $|e^{-\alpha_1 x}-e^{-\alpha_1' x}| \le \frac{|\alpha_1-\alpha_1'|}{\eta}e^{-(\alpha_*-\eta)}$ yields

$$J_2(x,y) \le \frac{|\alpha_1-\alpha_1'|}{\eta}\frac{\lambda_1'}{\lambda_1+\frac{\alpha_1^2-(\alpha_*-\delta)^2}{2}} F\left[\lambda_1+\frac{\alpha_1^2-(\alpha_*-\delta)^2}{2},\alpha_*-\delta,1,1,\nu\right],$$

from which it follows,

$$Q(J_2) \le \frac{|\alpha_1-\alpha_1'|}{\eta}\frac{\lambda_1'}{\lambda_1+\frac{\alpha_1^2-(\alpha_*-\eta)^2}{2}}\mathbb{E}g\left(e^{-(\alpha_*-\delta)\theta_*-W_{\theta_*}}\right), \quad (6.113)$$

where $\theta_*\sim Exp\left(\lambda_1+(\alpha_1^2-(\alpha_*-\delta)^2)/2\right)$. The expectation in (6.113) is finite if and only if

$$\varepsilon^* < \alpha_* - \eta + \sqrt{2\lambda_1+\alpha_1^2}. \quad (6.114)$$

From Assumption 6.1 we have that $\varepsilon^* < 2\alpha/\sigma^2 \equiv 2\alpha_1$ (see Example 6.7). it follows that (6.114) holds if $\eta \le \alpha_* + \sqrt{2\lambda_1+\alpha_1^2} - 2\alpha_1$ ((6.39) ensures that the latter is positive). Thus, for

$$\eta < \min\{\alpha_*, \alpha_* + \sqrt{2\lambda_1+\alpha_1^2} - 2\alpha_1\}$$

the bound (6.113) is finite and

$$\mathbb{E}\left(e^{-(\alpha_*-\delta)\varepsilon^*\theta_*-\varepsilon^* W_{\theta_*}}\right) = \frac{2\lambda_1+\alpha_1^2-(\alpha_*-\eta)^2}{2\lambda_1+\alpha_1^2-(\alpha_*-\eta)^2+2\varepsilon^*(\alpha_*-\eta)-\varepsilon^{*2}} \equiv h_2^*,$$

which proves the lemma. □

We will use the following expression of the modified Bessel function $I_\nu(u)$ (see [20] p. 958, formula 8.431.5),

$$I_\nu(u) = \frac{1}{\pi}\int_0^\pi e^{u\cos(v)}\cos(\nu v)\,dv - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty \exp\left(-u\cosh(s) - \nu s\right)\,ds, \tag{6.115}$$

and the result from [43] p. 305: for $b > 0$ and $-\nu < \delta < 1/2$,

$$\int_0^\infty x^{\delta-1}e^{-bx}I_\nu(bx)\,dx = \frac{(2b)^{-\delta}}{\sqrt{\pi}}\Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta}. \tag{6.116}$$

Let the positive constants δ and ζ satisfy $\delta < 1/2$ and $\zeta < \min(\nu, \nu')$.

We first consider case $\gamma \leq 1$. Let us write

$$\begin{aligned} J_3(x,y) &\leq \left|F[\lambda_1', \alpha_1', 1, 1, \nu] - F[\lambda_1', \alpha_1', \gamma, 1, \nu]\right|(x,y) + \left|F[\lambda_1', \alpha_1', \gamma, 1, \nu] - F[\lambda_1', \alpha_1', \gamma, \gamma, \nu]\right|(x,y) \\ &=: J_{31}(x,y) + J_{32}(x,y). \end{aligned} \tag{6.117}$$

Lemma 6.31. *Let $\gamma < 1$. Then*

$$Q(J_{31}) \leq \frac{2\lambda_1'}{\sqrt{\pi}}\left(\frac{1}{2}\right)^\delta\left(\frac{1}{\gamma}-1\right)^\delta\Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta}g\left(\alpha_1'^{\,\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right).$$

Proof. From (6.117) and (6.104),

$$\begin{aligned} J_{31} &= \frac{2\lambda_1'}{y}e^{\alpha_1' x}\left|\exp\left(-2\frac{1+e^{-x}}{y}\right) - \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right)\right| I_\nu\left(\frac{4e^{-x/2}}{y}\right) \\ &\leq \frac{2\lambda_1'}{y}e^{\alpha_1' x}\exp\left(-2\frac{1+e^{-x}}{y}\right)\left(2\frac{1+e^{-x}}{y}\left(\frac{1}{\gamma}-1\right)\right)^\delta I_\nu\left(\frac{4e^{-x/2}}{y}\right). \end{aligned}$$

Using (6.116) and the fact that $I_\nu(x)$ is increasing in x (see the series representation of in ([20]), p. 961, formula 8.445), we obtain

$$\begin{aligned} &\int_0^\infty y^{-1-\delta}\exp\left(-2\frac{1+e^{-x}}{y}\right)I_\nu\left(\frac{4e^{-x/2}}{y}\right)dy \\ &= \int_0^\infty u^{\delta-1}\exp\left(-2(1+e^{-x})u\right)I_\nu\left(4e^{-x/2}u\right)\,du \\ &\leq \int_0^\infty u^{\delta-1}\exp\left(-2(1+e^{-x})u\right)I_\nu\left(-2(1+e^{-x})u\right)\,du \\ &= \frac{\left(4(1+e^{-x})\right)^{-\delta}}{\sqrt{\pi}}\Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta}. \end{aligned}$$

This yields

$$Q(J_{31}) \leq \frac{2^{1-\delta}\lambda_1'}{\sqrt{\pi}}\left(\frac{1}{\gamma}-1\right)^\delta\Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta}\int_0^\infty g(x^{\varepsilon^*})e^{-\alpha_1' x}\,dx.$$

The identity

$$\int_0^\infty x^{\varepsilon^*} e^{-\alpha_1' x}\, dx \quad = \quad {\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1). \tag{6.118}$$

proves the lemma. □

Notice that condition (6.39) yields $\gamma \geq 1/2$.

Lemma 6.32. *Let* $\gamma \in [1/2, 1]$. *Then*

$$\begin{aligned} Q(J_{32}) \quad &\leq \quad \lambda_1' g \left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right)\left[\frac{2\gamma^\delta}{\sqrt{\pi}}\Gamma\begin{pmatrix}\delta, 1/2-\delta\\ 1-\delta\end{pmatrix} + (2\gamma)^\delta\Gamma(\delta) + \frac{2(2\gamma)^\zeta}{\pi(\nu-\zeta)}\Gamma(\zeta)\right] \\ &\quad \times \left|\frac{c}{\sigma^2} - \frac{c'}{\sigma'^2}\right| \max\left(c_1^{-\delta}, c_1^{-1}\right) \end{aligned}$$

Proof. From (6.117) and (6.104),

$$J_{32}(x,y) \quad = \quad \frac{2\lambda_1'}{y} e^{\alpha_1' x} \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right)\left| I_\nu\left(\frac{4e^{-x/2}}{y}\right) - I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right)\right|. \tag{6.119}$$

Using representation (6.115) we have

$$\begin{aligned} \left| I_\nu\left(\frac{4e^{-x/2}}{y}\right) - I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right)\right| \quad &\leq \quad \frac{1}{\pi}\left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right)\left|\left(e^{\frac{4e^{-x/2}}{\gamma y}\cos(v)} - e^{\frac{4e^{-x/2}}{y}\cos(v)}\right)\right| dv \\ &\quad + \frac{1}{\pi}\int_0^\infty \left| e^{\frac{4e^{-x/2}}{\gamma y}\cosh(s)} - e^{\frac{4e^{-x/2}}{y}\cosh(s)}\right| e^{-\nu s}\, ds \\ &=: \quad (A_1 + A_2 + B)(x,y). \end{aligned}$$

Now,

$$\begin{aligned} A_1(x,y) \quad &= \quad \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{4e^{-x/2}}{\gamma y}\cos(v)\right)\left|1 - \exp\left(-\frac{4e^{-x/2}}{y}\cos(v)\left(\frac{1}{\gamma}-1\right)\right)\right| dv \\ &\leq \quad \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{4e^{-x/2}}{\gamma y}\cos(v)\right)\left(\frac{4e^{-x/2}}{y}\cos(v)\left(\frac{1}{\gamma}-1\right)\right)^\delta dv \\ &\leq \quad \left(\frac{4}{y}\right)^\delta \left(\frac{1}{\gamma}-1\right)^\delta I_0\left(\frac{4e^{-x/2}}{\gamma y}\right). \end{aligned} \tag{6.120}$$

Similarly,

$$\begin{aligned} A_2(x,y) \quad &= \quad \frac{1}{\pi}\int_{\pi/2}^{\pi} \exp\left(\frac{4e^{-x/2}}{y}\cos(v)\right)\left|1 - \exp\left(-\frac{4e^{-x/2}}{y}\cos(v)\left(\frac{1}{\gamma}-1\right)\right)\right| dv \\ &\leq \quad \frac{1}{\pi}\int_{\pi/2}^{\pi} \exp\left(\frac{4e^{-x/2}}{y}\cos(v)\right)\left(\frac{4e^{-x/2}}{y}\cos(v)\left(\frac{1}{\gamma}-1\right)\right)^\delta dv \\ &\leq \quad \frac{1}{2}\left(\frac{4}{y}\right)^\delta \left(\frac{1}{\gamma}-1\right)^\delta. \end{aligned} \tag{6.121}$$

Let positive constant $\zeta < \min(1,\nu)$. Then,

$$\begin{aligned}
B(x,y) &= \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{4e^{-x/2}}{y}\cosh(s)\right)\left|1-\exp\left(\frac{4e^{-x/2}}{y}\cosh(s)\left(\frac{1}{\gamma}-1\right)\right)\right| e^{-\nu s}\,ds \\
&\le \frac{1}{\pi}\int_0^\infty \exp\left(-\frac{4e^{-x/2}}{y}\cosh(s)\right)\left(\frac{4e^{-x/2}}{y}\cosh(s)\right)^{1-\zeta+\zeta} e^{-\nu s}\left(\frac{1}{\gamma}-1\right)ds \\
&\le \frac{1}{\pi}\left(\frac{4e^{-x/2}}{y}\right)^{\zeta}\left(\frac{1}{\gamma}-1\right)\int_0^\infty (\cosh(s))^{\zeta} e^{-\nu s}\,ds \\
&\le \frac{1}{\pi}\left(\frac{4}{y}\right)^{\zeta}\left(\frac{1}{\gamma}-1\right)\frac{1}{\nu-\zeta}. \qquad (6.122)
\end{aligned}$$

In order to bound $Q(J_{32})$, we first have to investigate the following integrals. From (6.119) and (6.120),

$$\begin{aligned}
&\int_0^\infty y^{-1-\delta}\exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) I_0\left(\frac{4e^{-x/2}}{\gamma y}\right)dy \\
\le\ &\int_0^\infty u^{\delta-1}\exp\left(-2(1+e^{-x})\frac{u}{\gamma}\right) I_0\left(2(1+e^{-x})\frac{u}{\gamma}\right)du \\
=\ &\left(4\frac{1+e^{-x}}{\gamma}\right)^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\begin{pmatrix}\delta, 1/2-\delta \\ 1-\delta\end{pmatrix} \\
\le\ &\frac{1}{\sqrt{\pi}}\left(\frac{\gamma}{4}\right)^{\delta}\Gamma\begin{pmatrix}\delta, 1/2-\delta \\ 1-\delta\end{pmatrix}
\end{aligned}$$

From (6.121) and (6.119),

$$\int_0^\infty y^{-1-\delta}\exp\left(-2\frac{1+e^{-x}}{\gamma y}\right)dy \ \le\ \left(\frac{\gamma}{2}\right)^{\delta}\Gamma(\delta),$$

and the same integral with δ replaced by ζ comes from (6.122) and (6.119). Equality (6.118) yields

$$\begin{aligned}
Q(J_{32}) &\le \lambda_1' g\left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right)\left[\frac{2\gamma^{\delta}}{\sqrt{\pi}}\Gamma\begin{pmatrix}\delta, 1/2-\delta \\ 1-\delta\end{pmatrix} + (2\gamma)^{\delta}\Gamma(\delta) + \frac{2(2\gamma)^{\zeta}}{\pi(\nu-\zeta)}\Gamma(\zeta)\right] \\
&\quad \times \max\left(\left|\frac{1}{\gamma}-1\right|^{\delta}, \left|\frac{1}{\gamma}-1\right|\right).
\end{aligned}$$

For $\gamma \ge 1/2$, we have $|1-1/\gamma|^{\delta} \ge |1-1/\gamma|$. This together with $\gamma = c_1/c_1'$ complete the proof. □

Let $\gamma > 1$. Write

$$\begin{aligned}
J_3(x,y) &\le |F(\lambda_1',\alpha_1',1,1,\nu] - F[\lambda_1',\alpha_1',1,\gamma,\nu]|\,(x,y) + |F[\lambda_1',\alpha_1',1,\gamma,\nu] - F[\lambda_1',\alpha_1',\gamma,\gamma,\nu]|\,(x,y) \\
&=: J_{31}'(x,y) + J_{32}'(x,y). \qquad (6.123)
\end{aligned}$$

Lemma 6.33. *Let* $\gamma > 1$*. Then*

$$\begin{aligned} Q(J'_{32}) &\leq \lambda'_1 g\left({\alpha'_1}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right)\left[\frac{2}{\sqrt{\pi}}\Gamma\binom{\delta, 1/2-\delta}{1-\delta} + (2)^\delta\Gamma(\delta) + \frac{2^{\zeta+1}}{\pi(\nu-\zeta)}\Gamma(\zeta)\right] \\ &\quad\times\left|\frac{c}{\sigma^2} - \frac{c'}{\sigma'^2}\right| \max\left(c^{-\delta}, c^{-1}\right) \end{aligned}$$

Proof. The proof is similar to the one of Lemma 6.32 From (6.123) and (6.117), brown

$$J'_{32}(x,y) = \frac{2\lambda'_1}{y}e^{\alpha'_1 x}\exp\left(-2\frac{1+e^{-x}}{y}\right)\left|I_\nu\left(\frac{4e^{-x/2}}{y}\right) - I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right)\right|. \tag{6.124}$$

Using representation (6.115) we have

$$\begin{aligned} \left|I_\nu\left(\frac{4e^{-x/2}}{y}\right) - I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right)\right| &\leq \frac{1}{\pi}\left(\int_0^{\pi/2} + \int_{\pi/2}^{\pi}\right)\left|\left(e^{\frac{4e^{-x/2}}{\gamma y}\cos(v)} - e^{\frac{4e^{-x/2}}{y}\cos(v)}\right)\right| dv \\ &\quad + \frac{1}{\pi}\int_0^\infty \left|e^{\frac{4e^{-x/2}}{\gamma y}\cosh(s)} - e^{\frac{4e^{-x/2}}{y}\cosh(s)}\right| e^{-\nu s}\,ds \\ &=: (A'_1 + A'_2 + B')(x,y). \end{aligned}$$

The following estimates are obtained similar to (6.120)–(6.122) in the proof of Lemma 6.32,

$$\begin{aligned} A'_1(x,y) &\leq \left(\frac{4}{y}\right)^\delta\left(1-\frac{1}{\gamma}\right)^\delta I_0\left(\frac{4e^{-x/2}}{y}\right), \\ A'_2(x,y) &\leq \frac{1}{2}\left(\frac{4}{y}\right)^\delta\left(1-\frac{1}{\gamma}\right)^\delta, \\ B'(x,y) &\leq \frac{1}{\pi}\left(\frac{4}{y}\right)^\zeta\left(\frac{1}{\gamma}-1\right)\frac{1}{\nu-\zeta}. \end{aligned}$$

Relations

$$\begin{aligned} \int_0^\infty y^{-1-\delta}\exp\left(-2\frac{1+e^{-x}}{y}\right) I_0\left(\frac{4e^{-x/2}}{y}\right) dy &\leq 4^{-\delta}\frac{1}{\sqrt{\pi}}\Gamma\binom{\delta, 1/2-\delta}{1-\delta}, \\ \int_0^\infty y^{-1-\delta}\exp\left(-2\frac{1+e^{-x}}{y}\right) dy &\leq 2^{-\delta}\Gamma(\delta) \end{aligned}$$

together with (6.118) yield the desired bound. □

Lemma 6.34. *Let* $\gamma > 1$*. Then*

$$Q(J'_{31}) \leq \frac{2\lambda'_1}{\sqrt{\pi}}\left(\frac{\gamma}{2}\right)^\delta\left(1-\frac{1}{\gamma}\right)^\delta\Gamma\binom{\delta+\nu, 1/2-\delta}{1+\nu-\delta} g\left({\alpha'_1}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right).$$

Proof. This proof is similar to the one of Lemma 6.31

$$\begin{aligned} J'_{31} &= \frac{2\lambda'_1}{y} e^{\alpha'_1 x} \left| \exp\left(-2\frac{1+e^{-x}}{y}\right) - \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) \right| I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right) \\ &\leq \frac{2\lambda'_1}{y} e^{\alpha'_1 x} \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) \left(2\frac{1+e^{-x}}{y}\left(1-\frac{1}{\gamma}\right)\right)^\delta I_\nu\left(\frac{4e^{-x/2}}{y}\right), \end{aligned}$$

Similarly as in the proof of Lemma 6.31,

$$\int_0^\infty y^{-1-\delta} \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right) dy \leq \frac{(1+e^{-x})^{-\delta}}{\sqrt{\pi}} \left(\frac{\gamma}{4}\right)^\delta \Gamma\begin{pmatrix}\delta+\nu, 1/2-\delta \\ 1+\nu-\delta\end{pmatrix}.$$

Relation (6.118) completes the proof. □

Lemma 6.35. *Relation (6.110) holds.*

Proof.

$$J_4 = \frac{2\lambda'_1}{y} e^{\alpha'_1 x} \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) \left| I_\nu\left(\frac{4e^{-x/2}}{\gamma y}\right) - I_{\nu'}\left(\frac{4e^{-x/2}}{\gamma y}\right) \right|. \tag{6.125}$$

From (6.115),

$$\begin{aligned} |I_\nu(u) - I_{\nu'}(u)| &\leq \frac{1}{\pi}\int_0^\pi e^{u\cos(x)} |\cos(\nu x) - \cos(\nu' x)|\, dx \\ &\quad + \frac{|\sin(\nu\pi) - \sin(\nu'\pi)|}{\pi} \int_0^\infty e^{-u\cosh(s) - \nu s}\, ds \\ &\quad + \frac{1}{\pi}\int_0^\infty e^{-u\cosh(s)} \left| e^{-\nu s} - e^{-\nu' s} \right| ds \\ &=: (A+B+C)(u). \end{aligned}$$

Hence, choosing $u = 4e^{-x/2}/(\gamma y)$, we have

$$\begin{aligned} J_4 &\leq \frac{2\lambda'_1}{y} e^{\alpha'_1 x} \exp\left(-2\frac{1+e^{-x}}{\gamma y}\right) (A+B+C)\left(\frac{4e^{-x/2}}{\gamma y}\right) \\ &=: (J_{41} + J_{42} + J_{43})(x,y). \end{aligned} \tag{6.126}$$

Let us consider A. Using $1-\cos(x) \leq x^2/2$ and $\sin(x) \leq x$, we obtain

$$|\cos(\nu x) - \cos(\nu' x)| \leq \pi|\nu - \nu'|\left(1 + \frac{|\nu-\nu'|\pi}{2}\right),$$

from where it follows

$$
\begin{aligned}
A(u) &\leq \pi|\nu-\nu'|\left(1+\frac{\pi|\nu-\nu'|}{2}\right)\frac{1}{\pi}\int_0^\pi e^{u\cos x}\,dx \\
&\leq \pi|\nu-\nu'|\left(1+\frac{\pi|\nu-\nu'|}{2}\right)\frac{1}{\cos(\delta\pi)} \\
&\quad\times\left(\frac{1}{\pi}\int_0^\pi e^{u\cos x}\cos(\delta x)\,dx \pm \frac{\sin(\delta\pi)}{\pi}\int_0^\infty e^{-u\cosh(s)-\delta s}\,ds\right) \\
&\leq \pi|\nu-\nu'|\left(1+\frac{\pi|\nu-\nu'|}{2}\right)\frac{1}{\cos(\delta\pi)}\big(I_\delta(u)+1\big). \qquad (6.127)
\end{aligned}
$$

Collecting y-terms in (6.125) and (6.127) and again choosing $u=4e^{-x/2}/(\gamma y)$, we have

$$
\begin{aligned}
&\int_0^\infty y^{-1}\exp\left(-2\frac{1+e^{-x}}{\gamma y}\right)\left(I_\delta\left(\frac{4e^{-x/2}}{\gamma y}\right)+1\right)dy \\
\leq\ &\int_0^\infty u^{-1}\exp\left(-2\frac{1+e^{-x}}{\gamma}u\right)I_\delta\left(-2\frac{1+e^{-x}}{\gamma}u\right)du+\int_0^\infty u^{-1}e^{-\frac{2}{\gamma}u}\,du \\
=\ &\frac{1}{\sqrt{\pi}}\Gamma\binom{\delta,1/2}{1+\delta}+\Gamma(1) \\
=\ &\frac{\Gamma(1/2)}{\delta\sqrt{\pi}}+1. \qquad (6.128)
\end{aligned}
$$

From (6.125), (6.127) and (6.128) it follows that

$$
Q\left(J_{41}\right) \leq |\nu-\nu'|\frac{2\lambda_1'\pi}{\cos(\delta\pi)}\left(\frac{\Gamma(1/2)}{\delta\sqrt{\pi}}+1\right)\left(1+\frac{\pi|\nu-\nu'|}{2}\right)g\left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right). \qquad (6.129)
$$

We now turn to B. For any $u\geq 0$,

$$
B(u) \leq \pi|\nu-\nu'|\left(1+\frac{\pi|\nu-\nu'|}{2}\right)\frac{1}{\nu},
$$

which yields

$$
Q(J_{42}) \leq |\nu-\nu'|\frac{2\lambda_1'\pi}{\nu}\left(1+\frac{\pi|\nu-\nu'|}{2}\right)g\left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right). \qquad (6.130)
$$

It remains to consider C. Let ξ be a positive constant such that $\xi<\min(\nu,\nu')$. Then, for any $u\geq 0$,

$$
\begin{aligned}
C(u) &\leq \frac{1}{\pi}\int_0^\infty e^{-u\cosh(s)}e^{-\xi s}\left|e^{-(\nu-\xi)s}-e^{-(\nu'-\xi)s}\right|ds \\
&\leq \left|1+\frac{(\nu\wedge\nu')-\xi}{(\nu\vee\nu')-\xi}\right|\frac{1}{\pi\xi} = |\nu-\nu'|\frac{1}{\pi\xi((\nu\vee\nu')-\xi)}.
\end{aligned}
$$

It follows

$$
Q(J_{43})\leq|\nu-\nu'|\frac{2\lambda_1'}{\pi\xi((\nu\vee\nu')-\xi)}g\left({\alpha_1'}^{\varepsilon^*-1}\Gamma(\varepsilon^*+1)\right). \qquad (6.131)
$$

Combining relations (6.129)–(6.131) one obtains

$$\begin{aligned} Q(J_4) &\leq |\nu - \nu'|\, 2\lambda_1' g \left({\alpha_1'}^{\varepsilon^*-1} \Gamma(\varepsilon^* + 1) \right) \\ &\times \left[\left(\frac{\pi}{\cos(\delta\pi)} \left(\frac{\Gamma(1/2)}{\delta\sqrt{\pi}} + 1 \right) + \frac{\pi}{\nu} \right) \left(1 + \frac{\pi|\nu - \nu'|}{2} \right) + \frac{1}{\pi\xi((\nu \vee \nu') - \xi)} \right], \end{aligned}$$

which together with the inequality

$$|\nu - \nu'| = \left| \sqrt{2\lambda_1 + \alpha_1^2} - \sqrt{2\lambda_1' + {\alpha_1'}^2} \right| \leq \frac{|\lambda_1 - \lambda_1'|}{\sqrt{2(\lambda_1 \wedge \lambda_1')}} + |\alpha_1 - \alpha_1'|.$$

proves (6.110). □

Chapter 7

Model with Markov modulation and interest

In this chapter we derive stability bounds for the ruin probability in the Markov modulated risk model with investments. We will use the techniques developed by Kalashnikov [30], where similar bounds were obtained for the classical risk model and the model with random investments.

The risk process $\{R_n\}_{n\geq}$ is as in Section 2.2.4. In Section 7.1 we construct the reversed process $\{V_n\}_{n\geq 0}$ and express the ruin probability in terms of this process. The main steps of the approach and the results of this chapter are presented in Section 7.2: the special case without investments ($U \equiv 0$) is presented in Subsection 7.2.1, the model with deterministic investments ($U_t = \alpha t$) is given in Subsection 7.2.2 and in Subsection 7.2.3 we consider the general case (U is a genuine Lévy process). Technically more involved proofs are given in the Appendix.

7.1 The reversed process

Applying the construction (2.14) and (2.15) to the risk process $\{R_n\}$ defined in (2.29), we obtain the following reversed process:

$$V_{n+1} = \xi_{n+1}^{I_{n+1}} \left(V_n + \eta_{n+1}^{I_n, I_{n+1}} \right)_+, \qquad V_0 = 0, \tag{7.1}$$

where

$$\begin{aligned} \xi_{n+1}^{i} &= \exp\left(U_{T_{n+1}}^{i} - U_{T_n}^{i} \right), \\ \eta_{n+1}^{ji} &= \delta_{ji} Z_{n+1}^{i} - c_i \int_{T_{n+1}}^{T_n} \exp\left(U_{T_{n+1}}^{i} - U_u^{i} \right) du. \end{aligned}$$

The finite-time and ultimate ruin probabilities $\psi^{(n)}$ and ψ, respectively, with given parameter a governing the risk process, satisfy the relations

$$\psi_a(x) = \lim_{n\to\infty} \mathbf{P}(V_n > x) \quad \text{and} \quad \psi_a^{(n)}(x) = \mathbf{P}(V_n > x). \tag{7.2}$$

For fixed $i, j \in \mathbb{E} = \{1, \cdots, m\}$ the random vectors $(\xi_n^i, \eta_n^{j,i})_{n\geq 1}$ are i.i.d. It is convenient to introduce the generic vector

$$\left(\xi^i, \eta^{ji}\right) = \left(\exp\left(-U^i_{\theta^i}\right),\ \delta_{ji} Z^i - c_i \int_0^{\theta^i} \exp\left(U^i_{\theta^i} - U^i_u\right) du\right). \tag{7.3}$$

Throughout the paper $\mathbf{P}_i$ and $\mathbb{E}_i$ denote the conditional probability and conditional expectation, respectively, given $I_0 = i$, $i \in \mathbb{E}$.

Governing parameter. The governing parameter a consists of the parameters corresponding to the processes R^i and also of the transition probabilities $\{p_{ij}\}_{i,j\in\mathbb{E}}$ which fully describe the Markov chain $\mathcal{I}$ since its initial distribution is the stationary one. Similarly as in [30], we assume that for any $a, a' \in \mathbb{A}$ there exist constants $\beta_i \in \mathbb{R}_+$ such that the Lévy processes U^i and $U^{i'}$ satisfy

$$\{U'^i_t\}_{t\geq 0} \stackrel{d}{=} \{U^i_{\beta_i t}\}_{t\geq 0}, \qquad i \in \mathbb{E}. \tag{7.4}$$

Further we consider two risk models: the non-perturbed and perturbed ones, governed by the parameters a and a', respectively. All quantities corresponding to the perturbed model are marked with a prime. The parameters a and a' will often be skipped in the notation and the corresponding ruin probabilities are ψ, $\psi^{(n)}$ and $\psi', \psi'^{(n)}$, etc.

7.2 Stability bounds

In this section we present the main results leaving out most of the technical details. Stability bounds are given for the three particular cases:

(i) *no investments*: $U^i \equiv 0$ for all $i \in \mathbb{E}$;

(ii) *deterministic investments*: $U^i_t = \alpha_i t$ for all $i \in \mathbb{E}$;

(iii) *the general case*: U^i is a general Lévy process for all $i \in \mathbb{E}$.

The complete proofs are given in the Appendix.

Let w be a given weight function (i.e., it is increasing, typically to ∞, and bounded away from 0). We will derive the stability bounds under the following conditions:

(C1) There exists a constant C_w such that for all $a \in \mathbb{A}$,

$$\sup_{n\geq 0} \mathbb{E}_i w(V_n) \leq C_w, \qquad i \in \mathbb{E}. \tag{7.5}$$

(C2) For some appropriate metric d in $\mathbb{A}$ and for some weight functions $w_i \geq w$ the following 'contraction' property holds

$$\begin{aligned} &\sum_i q_{si}^{(n+1)} \sup_{u\geq 0} w_i(u)\ \left|\mathbf{P}_s(V_{n+1} \leq u \,|\, I_{n+1} = i) - \mathbf{P}_s(V'_{n+1} \leq u \,|\, I'_{n+1} = i)\right| \\ \leq\ & \kappa \left(\sum_j q_{sj}^{(n)} \sup_{u\geq 0} w_j(u)\ \left|\mathbf{P}_s(V_n \leq u \,|\, I_n = j) - \mathbf{P}_s(V'_n \leq u \,|\, I'_n = j)\right|\right) + d(a, a'), \end{aligned} \tag{7.6}$$

for every $s \in \mathbb{E}$ and some constant $\kappa < 1$.

These conditions will be checked in the following section separately for each of the three cases listed above. In what follows we always consider the Markov modulated risk process R with Lévy processes driven investments introduced in Section 2.2.4. It is governed by the parameter

$$a = (c_i, \lambda_i, F_{Z^i}, \beta_i, q_{ij}, i, j \in \mathbb{E}).$$

We assume that all governing parameters a, a' (corresponding to the perturbed and non-perturbed models) belong to the set

$$\mathbb{A}_q = \{a : \min_{i,j} q_{ij} \geq \underline{q}\}, \tag{7.7}$$

where $0 < \underline{q} < 1$ is a given constant.

Theorem 7.1. *If $a, a' \in \mathbb{A}_q$ satisfy conditions* **(C1)** *and* **(C2)***) then the following stability bound holds*

$$|\psi_a - \psi_{a'}|_w \leq C_w \left(2 + \frac{1}{\underline{q}}\right) \Delta_q + \frac{d(a, a')}{1 - \kappa}, \tag{7.8}$$

where κ and $d(a, a')$ are from (7.6) and Δ_q is given in (7.10).

Proof. From (7.2) we have

$$\psi(x) = \lim_{n \to \infty} \sum_s \pi_s \, \mathbf{P}_s(V_n > x).$$

It follows that

$$\begin{aligned} |\psi - \psi'|_w &= \sup_{x \geq 0} w(x) |\psi(x) - \psi'(x)| \\ &\leq \sup_{x \geq 0} \sup_{n \geq 0} \sum_i \pi_i \left[w(x) \left| \mathbf{P}_i(V_n > x) - \mathbf{P}_i(V'_n > x) \right| \right] \\ &\quad + \sup_{x \geq 0} \sup_{n \geq 0} \sum_i |\pi_i - \pi'_i| \left[w(x) \mathbf{P}_i(V'_n > x) \right] \\ &=: S_1 + S_2. \end{aligned} \tag{7.9}$$

Markov's inequality and (7.5) yield $w(x)\mathbf{P}_i(V'_n > x) \leq C_w$. Thus,

$$\begin{aligned} S_2 &\leq C_w \sum_i |\pi_i - \pi'_i| \\ &\leq C_w \sup_{n \geq 0} \|q_s^{(n)} - {q'}_s^{(n)}\|, \end{aligned}$$

where $\|q_s^{(n)} - {q'}_s^{(n)}\|$ is the total variation distance between the distributions $\{q_{ij}^{(n)}\}$ and $\{{q'}_{ij}^{(n)}\}$, i.e.,

$$\|q_s^{(n)} - {q'}_s^{(n)}\| = \sum_{i \in \mathbb{E}} |q_{si}^{(n)} - {q'}_{si}^{(n)}|.$$

Recall from Example 3.23 that if $\max_i \|q_i - q_i'\| \le 1$, then

$$\sup_{n\ge 0} \|q_s^{(n)} - {q'}_s^{(n)}\| \le h \max_i \|q_i - q_i'\| \log\left(\frac{e}{\|q_i - q_i'\|}\right) := \Delta_q \tag{7.10}$$

for some constant h which can be calculated. This yields

$$S_2 \le C_w\ \Delta_q.$$

Next we bound S_1. Let $s \in \mathbb{E}$ be fixed. Skipping the dependence on s, we denote

$$\begin{aligned} G_n(u) &= \mathbf{P}_s(V_n \le u), \\ G_n^i(u) &= \mathbf{P}_s(V_n \le u \,|\, I_n = i). \end{aligned}$$

By the total probability formula w.r.t. I_{n+1} and I'_{n+1} we have

$$\begin{aligned} &\sup_{x\ge 0,\ n\ge 0} w(x) \left|\overline{G}_{n+1}(x) - \overline{G'}_{n+1}(x)\right| \\ = &\sup_{x\ge 0,\ n\ge 0} w(x) \left|\sum_i \left(q_{si}^{(n+1)}\overline{G^i}_{n+1}(x) - {q'}_{si}^{(n+1)}\overline{G'^i}_{n+1}(x)\right)\right| \\ \le &\sup_{n\ge 0} \sum_i q_{si}^{(n+1)} \left|G_{n+1}^i - {G'}^i_{n+1}\right|_w + \sup_{x\ge 0,\ n\ge 0} \sum_i \left|q_{si}^{(n+1)} - {q'}_{si}^{(n+1)}\right| \left[w(x)\overline{G'^i}_{n+1}(x)\right], \end{aligned} \tag{7.11}$$

where $\bar{F} = 1 - F$ for any distribution function F. Inequalities (7.5) and (7.10) yield

$$\sum_i \left|q_{si}^{(n+1)} - {q'}_{si}^{(n+1)}\right| \left[w(x)\overline{G'^i}_{n+1}(x)\right] \le \frac{C_w}{\underline{q}}\ \Delta_q.$$

Plugging the latter in (7.11) and using the condition **(C2)** we obtain

$$\sup_{x\ge 0,\ n\ge 0} w(x) \left|\overline{G}_{n+1}(x) - \overline{G'}_{n+1}(x)\right| \le C_w \left(1 + \frac{1}{\underline{q}}\right) \Delta_q + \frac{d(a,a')}{1-\kappa}.$$

Hence,

$$S_1 + S_2 \le C_w \left(2 + \frac{1}{\underline{q}}\right) \Delta_q + \frac{d(a,a')}{1-\kappa},$$

which proves the theorem. □

Corollary 7.2. *Under the conditions of Theorem 7.1 the finite time ruin probabilities* $\psi_a^{(n)}$ *and* $\psi_{a'}^{(n)}$*, $n \ge 1$ satisfy the same stability bound (7.8).*

Proof. By (7.2) we have

$$\begin{aligned} |\psi_a^{(n)} - \psi_{a'}^{(n)}|_w &= \sup_{x\ge 0} w(x)w(x)|\psi_a^{(n)}(x) - \psi_{a'}^{(n)}(x)| \\ &\le \sup_{x\ge 0} \sum_i \pi_i \left[w(x)\,|\mathbf{P}_i(V_n > x) - \mathbf{P}_i(V_n' > x)|\right] \\ &\quad + \sup_{x\ge 0} \sum_i |\pi_i - \pi_i'| \left[w(x)\mathbf{P}_i(V_n' > x)\right] \\ &\le S_1 + S_2, \end{aligned}$$

where S_1 and S_2 are from (7.9). The rest of the proof coincides with that of Theorem 7.1. □

7.2.1 Model without investments

In this section we assume

$$U^i \equiv 0, \qquad \text{for all } i \in \mathbb{E},$$

i.e., we consider the Markov modulated risk model as in Section 2.2.2. Stability bounds using the Markov chain approach were obtained in Chapter 5.

Define the matrix $Q(\varepsilon) = (q_{ij}(\varepsilon))$ by

$$q_{ij}(\varepsilon) := q_{ij}\,\mathbb{E}\exp\left(\varepsilon\eta^{ij}\right) = \begin{cases} q_{ii}\mathbb{E}e^{\varepsilon(Z^i - c_i\theta^i)}, & i = j, \\ q_{ij}\mathbb{E}e^{-\varepsilon c_j\,\theta^j}, & i \neq j, \end{cases} \tag{7.12}$$

see (7.3). Let $\|Q(\varepsilon)\|_{SP}$ be the spectral radius of $Q(\varepsilon)$. It follows from the Frobenius theory for positive matrices that $\|Q(\varepsilon)\|_{SP}$ is the maximal eigenvalue $d(\varepsilon)$ of $Q(\varepsilon)$; see Gantmacher [16]. The corresponding eigenvector $\gamma(\varepsilon) = (\gamma_1(\varepsilon), \ldots, \gamma_m(\varepsilon))^t$ is such that $\min_i \gamma_i(\varepsilon) = 1$. Denote

$$\overline{\gamma}(\varepsilon) = \max_i \gamma_i(\varepsilon).$$

Assumption 7.3. *There exist $\varepsilon > 0$ and $\rho < 1$ such that*

$$\|Q(\varepsilon)\|_{SP} \;\leq\; \rho. \tag{7.13}$$

It follows from (7.13) that there exists a constant C_Z such that for all $i \in \mathbb{E}$

$$\mathbb{E}\exp(\varepsilon Z^i) \leq C_Z. \tag{7.14}$$

We consider the following set of governing parameters,

$$\mathbb{A} = \{a : \text{ relations (7.13) and (7.14) hold}\} \cap \mathbb{A}_q, \tag{7.15}$$

where $\mathbb{A}_q$ is defined in (7.7). We require the constants ρ and C_Z to be the *same* for all models with governing parameters in $\mathbb{A}$ in order to have *uniform* bounds over this set since the constants ρ and C_Z are involved in the final bounds.

Note that Assumption 7.3 is equivalent to Assumption 5.1 in Section 5. As in Section 5, we will use the weight function

$$w(v) = \exp(\varepsilon v). \tag{7.16}$$

for the comparison of ruin probabilities.

Corollary 7.4. *For any $a, a' \in \mathbb{A}$ the corresponding ruin probabilities ψ_a and $\psi_{a'}$ satisfy conditions* **(C1)** *and* **(C2)**, *and*

$$|\psi_a - \psi_{a'}|_w \leq \frac{C_w}{(1-\kappa)\,\underline{q}}\left(\max_j |F_{Z^j} - F_{Z'^j}|_w + \max_j \frac{4C_Z}{(\lambda_j \wedge \lambda'_j)}\left|\frac{\lambda_j}{c_j} - \frac{\lambda'_j}{c'_j}\right| + C_Z\overline{\gamma}(\varepsilon)h_q\Delta_q\right),$$

where $h_q = 1 + m\underline{q} + m$ and the constant C_w is given in (7.44) in the Appendix.

The same stability bound holds for the finite time ruin probabilities $\psi_a^{(n)}$ and $\psi_{a'}^{(n)}$ and any $n \geq 1$.

7.2.2 Deterministic investments

Let

$$U_t^i = \alpha_i t, \qquad t \geq 0, \qquad i \in \mathbb{E}, \tag{7.17}$$

where the constants α_i are positive. According to (7.4), the perturbed Lévy processes are ${U'}_t^i = \alpha_i \beta_i t$. We write $\alpha_i' := \alpha_i \beta_i$ for this model. The randomness of the investments is caused by the changes of states of the modulating chain.

Now the main assumption is the following.

Assumption 7.5. *There exist constants C_Z and $\varepsilon > 0$ such that for all $i \in \mathbb{E}$*

$$\mathbb{E} \exp\left(\varepsilon Z^i\right) \leq C_Z. \tag{7.18}$$

The set of admissible parameters is

$$\mathbb{A} = \{a : \quad (7.18) \text{ holds } \} \cap \mathbb{A}_q. \tag{7.19}$$

Let

$$w(v) = \exp\left(\varepsilon(v \vee \overline{v})\right), \tag{7.20}$$

where the constant $\overline{v} > 0$ is defined in (7.56) in the proof of Lemma 7.16.

Corollary 7.6. *For any $a, a' \in \mathbb{A}$ the corresponding ruin probabilities ψ_a and $\psi_{a'}$ satisfy the following bound:*

$$\begin{aligned} |\psi_a - \psi_{a'}|_w \quad \leq \quad & \frac{C_w}{(1-\kappa)\, \underline{q}} \left[e^{-\varepsilon \overline{v}} \max_j |F_{Z^j} - F_{Z'^j}|_w + \max_j \frac{4 C_Z}{(\lambda_j \wedge \lambda_j' \beta_j)} \left| \frac{\lambda_j}{\alpha_j} - \frac{\lambda_j'}{\alpha_j'} \right| \right. \\ & \left. + \max_j \frac{C_Z \lambda_j'}{\alpha_i \wedge \alpha_i'} \left| \frac{1}{\alpha_j} - \frac{1}{\alpha_j'} \right| + (C_Z h_q + \underline{q} + 1) \Delta_q \right], \end{aligned}$$

where $h_q = 1 + m\underline{q} + m$ and the constant C_w is given in (7.74) in the Appendix.

The corresponding finite time ruin probabilities $\psi_a^{(n)}$ and $\psi_{a'}^{(n)}$, $n \geq 1$ also satisfy the same stability bound.

7.2.3 General model

We now consider the model from Section 2.2.4 without assuming any particular form for the processes U^i, i.e., U^i are independent Lévy processes.

Recall the random vectors (ξ^i, η^{ji}) defined in (7.3). We define the matrix $Q(\varepsilon) = (q_{ij}(\varepsilon))$ as follows,

$$q_{ij}(\varepsilon) = q_{ij}\, \mathbb{E}\left(\xi^j\right)^{\varepsilon}. \tag{7.21}$$

Its spectral radius is the maximal eigenvalue $d(\varepsilon)$ of $Q(\varepsilon)$ and the corresponding eigenvector $\gamma(\varepsilon) = (\gamma_1(\varepsilon), \ldots, \gamma_m(\varepsilon))^t$ is such that $\min_i \gamma_i(\varepsilon) = 1$. We denote

$$\overline{\gamma}(\varepsilon) = \max_{i \in \mathbb{E}} \gamma_i(\varepsilon).$$

Assumption 7.7. There exist constants $\varepsilon > 0$ and $\rho_1, \rho_2 < 1$ such that

$$d(\varepsilon) \equiv \|Q(\varepsilon)\|_{SP} \leq \rho_1, \qquad \mathbb{E}(\xi^i)^{\varepsilon-1} \leq \rho_2, \quad \mathbb{E}(\xi^i)^{-1} \leq 1 \quad \forall i \in \mathbb{E}. \tag{7.22}$$

Assumption 7.8. *All r.v.'s Z^i have densities p_{Z^i} and there exist constants $C_Z < \infty$ and $C_p < \infty$ such that for every $i \in \mathbb{E}$*

$$\mathbb{E}\left(Z^i\right)^{\varepsilon} \leq C_Z, \qquad \sup_{u\geq 0} w(u) p_{Z_i}(u) \leq C_p. \tag{7.23}$$

Assumption 7.9. For any $M > 0$ there exists $q_M > 0$ such that

$$\mathbf{P}\left(\xi^i \leq 1, \eta^{ii} \leq -M\right) \geq q_M, \qquad i \in \mathbb{E}, \tag{7.24}$$

Example 7.10. Let U^i be the Brownian motion with drift. Then the vector (ξ^i, η^{ii}) has a density, see [9], p. 208. Thus, Assumption 7.9 is satisfied.

Remarks. The previously investigated models satisfy Assumptions 7.7 and 7.9. We studied them separately because (under the additional restriction that the claim sizes have finite exponential moments) we can prove stability bounds in the metric weighted with an exponential weight, while in the general case the stability bounds are stated in the metric weighted with a power function. In this sense the previous bounds are more tight.

We also notice that the asymptotic behavior of the ruin probability in this general case and also in the deterministic case have not been investigated yet.

We will consider the following set of parameters:

$$\mathbb{A} = \{a : \quad \text{relations (7.22)–(7.24) hold } \} \cap \mathbb{A}_q. \tag{7.25}$$

As a weight function we choose

$$w(v) = 1 + \alpha\left(v^{\varepsilon} \vee \overline{v}^{\varepsilon}\right), \tag{7.26}$$

where the constants α and $\overline{v}$ are given in the proof of Lemma 7.20 in Section 7.3.3, see the discussion following the relation (7.81).

Corollary 7.11. *For any $a, a' \in \mathbb{A}$ the corresponding ruin probabilities ψ_a and $\psi_{a'}$ satisfy*

$$\begin{aligned} |\psi_a - \psi_{a'}|_w \quad \leq \quad & \left(\frac{C_q h_q}{(1-\kappa)\underline{q}} + 1\right)\Delta_q + \frac{1}{1-\kappa}\Bigg(h_Z \max_j |F_{Z^j} - F_{Z'^j}|_w \\ & + \max_j h_{\lambda j}\left|\lambda_i - \frac{\lambda_i'}{\beta_i}\right| + \max_j h_{cj}\left|c_i - \frac{c_i'}{\beta_i}\right|\Bigg), \end{aligned}$$

see (7.115) and (7.116) in the Appendix for all the constants.

7.3 Appendix

In this section we prove the conditions (C1) and (C2) for all three models introduced above. The proofs use ideas from Kalashnikov [30], Section 6.1.

We first introduce the following notation (see also [30]):

$$\begin{aligned} f(v,\xi,\eta) &:= \xi(v+\eta)_+, \\ g_i(v,z,t) &:= \left((v+z)\exp(-U_t^i) - c_i\int_0^t \exp(-U_s^i)\,ds,\right)_+, \\ g_i'(v,z,t) &:= \left((v+z)\exp(-U_t^i) - \frac{c_i'}{\beta_i}\int_0^t \exp(-U_s^i)\,ds\right)_+. \end{aligned}$$

Then conditionally given $I_n = j,\ I_{n+1} = i$ ($I_n' = j,\ I_{n+1}' = i$) the r.v. V_{n+1} (and similarly V_{n+1}'), see (7.1) and (7.3), satisfies the following identities in law:

$$\begin{aligned} V_{n+1} &\overset{d}{=} f\left(V_n, \xi^i, \eta^{ji}\right) \overset{d}{=} g_i\left(V_n,\, \delta_{ji}Z^i,\, \theta^i\right), \\ \Big(V_{n+1}' &\overset{d}{=} f\left(V_n', {\xi'}^i, {\eta'}^{ji}\right) \overset{d}{=} g_i'\left(V_n',\, \delta_{ji}{Z'}^i,\, \beta_i{\theta'}^i\right)\Big). \end{aligned}$$

Let us consider the condition **(C2)**. Conditioning on I_n and using that $\mathbf{P}_s(I_n = j \,|\, I_{n+1} = i) = q_{sj}^{(n)} q_{ji}/q_{si}^{(n+1)}$ we have

$$\overline{G^i}_{n+1}(u) = \sum_j \frac{q_{sj}^{(n)} q_{ji}}{q_{si}^{(n+1)}} \mathbf{P}_s\left(f(V_n,\xi^i,\eta^{ji}) > u | I_n = j\right).$$

This yields

$$\begin{aligned} &\sum_i q_{si}^{(n+1)} \left| G_{n+1}^i - {G'}_{n+1}^i \right|_{w_i} \\ \le\ & \sum_j q_{sj}^{(n)}\left(A_j + B_j\right) + \left(\sup_{j,i} D_{ji}\right) \sum_{i,j} \left| \frac{q_{sj}^{(n)}}{{q'}_{sj}^{(n)}} q_{ji} - \frac{q_{si}^{(n+1)}}{{q'}_{si}^{(n+1)}} q_{ji}' \right|, \end{aligned} \tag{7.27}$$

where

$$\begin{aligned} A_j &= \sum_i q_{ji} \sup_{v\ge 0} w_i(v) \left|\mathbf{P}_s\left(f(V_n,\xi^i,\eta^{ji}) > v \,|\, I_n = j\right) - \mathbf{P}_s\left(f(V_n',\xi^i,\eta^{ji}) > v \,|\, I_n' = j\right)\right|, \\ B_j &= \sum_i q_{ji} \sup_{v\ge 0} w_i(v) \left|\mathbf{P}_s\left(f(V_n',\xi^i,\eta^{ji}) > v \,|\, I_n' = j\right) - \mathbf{P}_s\left(f(V_n',{\xi'}^i,{\eta'}^{ji}) > v \,|\, I_n' = j\right)\right|, \\ D_{ji} &= \sup_{v\ge 0, n\ge 0} {q'}_{sj}^{(n)} w_i(v)\, \mathbf{P}_s\left(f(V_n',{\xi'}^i,{\eta'}^{ji}) > v \,|\, I_n' = j\right). \end{aligned}$$

Using relation (7.10), the last sum in (7.27) can be bounded as follows:

$$\begin{aligned} &\sum_{i,j} \left| \frac{q_{sj}^{(n)}}{{q'}_{sj}^{(n)}} q_{ji} - \frac{q_{si}^{(n+1)}}{{q'}_{si}^{(n+1)}} q_{ji}' \right| \\ \le\ & \sum_j \left| \frac{q_{sj}^{(n)}}{{q'}_{sj}^{(n)}} - 1 \right| \sum_i q_{ji} + \sum_{i,j} \left| q_{ji} - q_{ji}' \right| + \sum_i \left| \frac{q_{si}^{(n+1)}}{{q'}_{si}^{(n+1)}} - 1 \right| \sum_j q_{ji}' \\ \le\ & \frac{\Delta_q}{\underline{q}}\left(1 + m\underline{q} + m\right). \end{aligned} \tag{7.28}$$

We now turn to the specific cases.

7.3.1 No investments

We consider the risk model as in Section 7.2.1. Recall the weight function w from (7.16) and let

$$w_i(v) := \gamma_i(\varepsilon)\exp(\varepsilon v), \tag{7.29}$$

where $(\gamma_1(\varepsilon), \ldots, \gamma_m(\varepsilon))$ is the eigenvector of the matrix $Q(\varepsilon)$ corresponding to its spectral radius $\|Q(\varepsilon)\|_{SP}$ such that $\min_i \gamma_i(\varepsilon) = 1$, see Section 7.2.1.

Lemma 7.12. *For $a, a' \in \mathbb{A}$ we have*

$$A_j \le \rho\, |G^j_n - {G'}^j_n|_{w_j}, \tag{7.30}$$

where the constant ρ is from Assumption 7.3.

Proof. Recall ξ^i and η^{ji} from (7.3). Then,

$$\begin{aligned} A_j &\le \sum_i q_{ji} \sup_{v\ge 0} \left[w_i(v)\mathbb{E}\left|\overline{G^j}_n(v/\xi^i - \eta^{ji}) - \overline{{G'}^j}_n(v/\xi^i - \eta^{ji})\right|\right] \\ &\le |G^j_n - {G'}^j_n|_{w_j} \sum_i q_{ji} \sup_{v\ge 0} A_{ji}(v), \end{aligned}$$

where

$$A_{ji}(v) := \mathbb{E}\left(\frac{w_i(v)}{w_j(v/\xi^i - \eta^{ji})}\mathbf{1}\left(v - \xi^i\eta^{ji} \ge 0\right)\right). \tag{7.31}$$

In this case $\xi^i \equiv 1$ and $\eta^{ji} = \delta_{ji}Z^i - c_i\theta^i$. Using Assumption 7.3,

$$\begin{aligned} \sum_i q_{ji} \sup_{v\ge 0} A_{ji}(v) &\le \sum_i q_{ji} \sup_{v\ge 0} \mathbb{E}\left(\frac{\gamma_i(\varepsilon)\exp(\varepsilon v)}{\gamma_j(\varepsilon)\exp(\varepsilon(v - \delta_{ji}Z^i + c_i\theta^i))}\right) \\ &= \frac{1}{\gamma_j(\varepsilon)} \sum_i q_{ji}(\varepsilon)\gamma_i(\varepsilon) \;\le\; \rho. \end{aligned}$$

This completes the proof. □

We now consider B_j. Obviously,

$$B_j \;\le\; q_{jj}B^1_j + \sum_{i\in\mathbb{E}} q_{ji}\, B^2_{ji},$$

where

$$B^1_j := \sup_{v\ge 0} w_j(v)\left|\mathbf{P}_s(g_j(V'_n, Z^j, \theta^i) > v | I'_n = j) - \mathbf{P}_s(g_j(V'_n, {Z'}^j, \theta^i) > v | I'_n = j)\right|, \tag{7.32}$$

$$B^2_{ji} := \sup_{v\ge 0} w_i(v)\left|\mathbf{P}_s(g_i(V'_n, \delta_{ji}{Z'}^i, \theta^i) > v | I'_n = j) - \mathbf{P}_s(g'_i(V'_n, \delta_{ji}{Z'}^i, \beta_i{\theta'}^i) > v | I'_n = j)\right|, \tag{7.33}$$

Lemma 7.13. *If $a, a' \in \mathbb{A}$ then,*

$$B_j^1 \le \frac{C_V}{\underline{q}} \, |F_{Z^j} - F_{Z'^j}|_{w_j}, \tag{7.34}$$

where C_V is the constant satisfying (7.35); see (7.38).

Proof. We have

$$\mathbf{P}(g_j(u, Z^j, \theta^j) > v) \;=\; \mathbb{E}\mathbf{P}\left(Z^j > v + c_j\theta^j - u \,|\, \theta^j\right) \;=\; \mathbb{E}\overline{F}_{Z^j}\left((v + c_j\theta^j - u)_+\right).$$

This yields

$$\begin{aligned}
B_j^1 &= \sup_{v\ge 0} w_j(v) \left| \int_{u=0}^{\infty} \mathbb{E}\left(\overline{F}_{Z^j}(v + c_j\theta^j - u) - \overline{F}_{Z'^j}(v + c_j\theta^j - u)\right) dG'^j_n(u) \right| \\
&\le |F_{Z^j} - F_{Z'^j}|_{w_j} \sup_{v\ge 0} \int_0^\infty \mathbb{E}\left(\frac{w_j(v)}{w_j(v + c_j\theta^j - u)}\right) dG'^j_n(u) \\
&\le |F_{Z^j} - F_{Z'^j}|_{w_j} \int_0^\infty e^{\varepsilon u}\, dG'^j_n(u) \\
&\le |F_{Z^j} - F_{Z'^j}|_{w_j} \frac{1}{\underline{q}} \sup_{n\ge 0} \mathbb{E}_s \exp(\varepsilon V_n) \\
&\le |F_{Z^j} - F_{Z'^j}|_{w_j} \, C_V/\underline{q}.
\end{aligned}$$

In the last step we used that for any $a \in \mathbb{A}$ the corresponding reversed process V satisfies condition

$$\sup_{n\ge 0} \mathbb{E}_s \exp(\varepsilon V_n) \;\le\; C_V \qquad \forall s \in \mathbb{E}. \tag{7.35}$$

Thus we proved (7.34).

The remainder of the proof is dedicated to show (7.35). Let

$$\tau \;=\; \min\{n > 0,\, V_n = 0\}.$$

Then, by the regenerative property of (V_n, I_n),

$$\begin{aligned}
\mathbb{E}_s\left(e^{\varepsilon V_n}\right) &= \sum_{k=0}^{n}\sum_{i=1}^{m} q_{si}^{(k)}\, \mathbf{P}_s(V_k = 0 \,|\, I_k = i)\, \mathbb{E}_i\left(e^{\varepsilon V_{n-k}}\, \mathbf{1}\left(V_1 > 0, \ldots, V_{n-k} > 0\right)\right) \\
&\le 1 + \sum_{i=1}^{m} \mathbb{E}_i \left(\sum_{1\le k<\tau} e^{\varepsilon V_k} \right) \\
&= 1 + \sum_{i=1}^{m} \mathbb{E}_i \left(\mathbb{E}_i \left(\sum_{1\le k<\tau} e^{\varepsilon V_k} \,\Big|\, V_1, I_1 \right)\right)
\end{aligned} \tag{7.36}$$

Let $\mathcal{A}$ be the generating operator of the Markov chain (V_n, I_n). Choose the test function

$$\varphi(v, i) = \begin{cases} 0, & \text{if } v = 0, \\ \gamma_i(\varepsilon) e^{\varepsilon v}, & \text{if } v > 0. \end{cases}$$

Recall $q_{ij}(\varepsilon)$ defined in (7.12). Then for $v > 0$,

$$\begin{aligned} \mathcal{A}\varphi(v,i) &= q_{ii}\,\gamma_i(\varepsilon)\,\mathbb{E}\left(e^{\varepsilon(v+Z^i-c_i\theta^i)}\,\mathbf{1}\left(v+Z^i-c_i\theta^i>0\right)\right) \\ &\quad + \sum_{j\neq i} q_{ij}\gamma_j(\varepsilon)\mathbb{E}\left(e^{\varepsilon(v-c_j\theta^j)}\mathbf{1}\left(v-c_j\theta^i>0\right)\right) - \gamma_i(\varepsilon)e^{\varepsilon v} \\ &\leq \sum_{j=1}^{m} q_{ij}(\varepsilon)\,\gamma_j(\varepsilon)e^{\varepsilon v} - \gamma_i(\varepsilon)e^{\varepsilon v} \\ &\leq -(1-\rho)e^{\varepsilon v}, \end{aligned} \tag{7.37}$$

where the latter inequality follows from Assumption 7.3. By Dynkin's formula,

$$\begin{aligned} 0 &= \mathbb{E}_i\left(\varphi(V_\tau, I_\tau)\,\middle|\, V_1, I_1\right) \\ &= \varphi(V_1,I_1) + \mathbb{E}_i\left(\sum_{1\leq k<\tau} \mathcal{A}\varphi(V_k,I_k)\,\middle|\, V_1, I_1\right) \\ &\leq \varphi(V_1,I_1) - (1-\rho)\mathbb{E}_i\left(\sum_{1\leq k<\tau} e^{\varepsilon V_k}\,\middle|\, V_1, I_1\right). \end{aligned}$$

The latter together with (7.36) implies

$$\mathbb{E}_s \exp\left(\varepsilon V_n\right) \leq 1 + \frac{1}{1-\rho}\sum_{i=1}^{m}\mathbb{E}_i\varphi(V_1,I_1) \leq 1 + \frac{\overline{\gamma}(\varepsilon)C_Z}{1-\rho}\sum_{i=1}^{m} q_{ii} =: C_V \tag{7.38}$$

In the last step we used

$$\mathbb{E}_i\varphi(V_1,I_1) \;=\; q_{ii}\gamma_i(\varepsilon)\mathbb{E}\left[e^{\varepsilon(Z^i-c_i\theta_i)}\mathbf{1}(Z^i-c_i\theta_i>0)\right] \;\leq\; q_{ii}\overline{\gamma}(\varepsilon)C_Z.$$

This proves (7.35) and concludes the proof of the lemma. □

Remark 7.14. It follows from the proof of Lemma 7.13 that MC $\{V_n\}$ has a unique stationary distribution Π: for a fixed $i \in \mathbb{E}$ take a set $A = \{0\} \cup i$ and a probability measure φ s.t. $\varphi(A) = 1$. Clearly, the set A is an accessible atom for a MC $\{V_n\}$. From Proposition 3.24 and relation (7.37) it follows that $\mathbb{E}_x\tau_A < \infty$, for all $x \geq 0$, which, by Proposition 3.6, implies the existence of a unique stationary distribution Π.

We now turn to the term B^2_{ji}. Recall that $\beta_i = 1$ for all $i \in \mathbb{E}$ in the current case with no investments.

Lemma 7.15. *If $a, a' \in \mathbb{A}$ then,*

$$B^2_{ji} \leq \frac{4C_Z C_V \overline{\gamma}(\varepsilon)}{\underline{q}\left(\lambda_i/c_i \,\wedge\, \lambda'{}_i/c'_i\right)}\,\left|\lambda^*_i - \lambda'^*_i\right|, \tag{7.39}$$

where $\lambda^*_i := \lambda_i/c_i$ and $\lambda'^*_i := \lambda'_i/c'_i$.

Proof. Notice that $c_i\theta^i \sim Exp(\lambda_i/c_i)$. Using the inequality

$$\left|\lambda_i^* e^{-\lambda_i^* t} - {\lambda'}_i^* e^{-{\lambda'}_i^* t}\right| \le 2|\lambda_i^* - {\lambda'}_i^*|\, e^{-(\lambda_i^* \wedge {\lambda'}_i^*)t/2},$$

we have

$$\begin{aligned}
B_{ji}^2 &\le \sup_{v\ge 0} w_i(v) \int_0^\infty \mathbf{P}\left(g_i(V_n', {Z'}^i, t) > v \,|\, I_n' = j\right) \left|\lambda_i^* e^{-\lambda_i^* t} - {\lambda'}_i^* e^{-{\lambda'}_i^* t}\right| dt \\
&\le 2|\lambda_i^* - {\lambda'}_i^*| \sup_{v\ge 0} \left\{ w_i(v) \int_0^\infty \mathbf{P}\left(V_n' + {Z'}^i > v \,|\, I_n' = j\right) \exp\left(-(\lambda_i^* \wedge {\lambda'}_i^*)t/2\right) dt \right\} \\
&\le 4\frac{|\lambda_i^* - {\lambda'}_i^*|}{\lambda_i^* \wedge {\lambda'}_i^*} \sup_{v\ge 0} \left\{ w_i(v) \mathbf{P}\left(V_n' + {Z'}^i > v \,|\, I_n' = j\right) \right\} \\
&\le 4\frac{|\lambda_i^* - {\lambda'}_i^*|}{\lambda_i^* \wedge {\lambda'}_i^*} \mathbb{E}\left[w_i\left(V_n' + {Z'}^i\right) | I_n' = j \right]. \qquad (7.40)
\end{aligned}$$

The latter relation follows from Markov's inequality.

Relations (7.35) and (7.14) yield

$$\begin{aligned}
\mathbb{E}\left[w_i(V_n' + {Z'}^i) | I_n' = j\right] &= \gamma_i(\varepsilon) \mathbb{E}\left[\exp\left(\varepsilon(V_n' + {Z'}^i)\right) | I_n' = j\right] \\
&\le \overline{\gamma}(\varepsilon) C_Z \frac{\mathbb{E}\exp(\varepsilon V_n')}{{q'}_{sj}^{(n)}} \\
&\le \frac{\overline{\gamma}(\varepsilon) C_Z C_V}{\underline{q}}. \qquad (7.41)
\end{aligned}$$

This and (7.40) prove (7.39). □

From Lemmas 7.13 and 7.15 we have,

$$B_j \le \frac{C_V \overline{\gamma}_j(\varepsilon)}{\underline{q}} |F_{Z^j} - F_{{Z'}^j}|_w + \frac{4 C_Z C_V \overline{\gamma}(\varepsilon)}{\underline{q}} \max_i \left(\frac{c_i}{\lambda_i} \vee \frac{c_i'}{{\lambda'}_i}\right) \left|\frac{\lambda_i'}{c_i'} - \frac{\lambda_i}{c_i}\right| \qquad (7.42)$$

Finally we bound D_{ji}. By Markov's inequality,

$$\begin{aligned}
D_{ji} &\le \mathbb{E}_s\left[w_i(f(V_n', {\xi'}^i, {\eta'}^{ji})) \,|\, I_n' = j\right] \\
&\le \mathbb{E}_s\left[w_i\left({\xi'}^i (V_n' + {Z'}^i)\right) \,|\, I_n' = j\right].
\end{aligned}$$

In the current case ${\xi'}^i \equiv 1$. Condition (7.14) and (7.35) yield

$$D_{ji} \le C_Z C_V \overline{\gamma}(\varepsilon). \qquad (7.43)$$

Now collecting the bounds for A_j from Lemma 7.12, for B_j from (7.42) and for D_{ji} from (7.43) and using relation (7.28), we conclude from (7.27) that **(C2)** holds with

$$\begin{aligned}
\kappa &:= \rho, \\
d(a, a') &:= \frac{C_V \overline{\gamma}(\varepsilon)}{\underline{q}} \left[\max_i |F_{Z^i} - F_{{Z'}^i}|_w + 4C_Z \max_i \left(\frac{c_i}{\lambda_i} \vee \frac{c_i'}{{\lambda'}_i}\right) \left|\frac{\lambda_i'}{c_i'} - \frac{\lambda_i}{c_i}\right| + C_Z h_q \Delta_q\right],
\end{aligned}$$

where $h_q = 1 + m + m\underline{q}$ (see (7.28)).

Condition **(C1)** follows from (7.35) with ,

$$C_w = \overline{\gamma}(\varepsilon) C_V, \tag{7.44}$$

where the constant C_V is given in (7.38).

7.3.2 Deterministic interest

We consider the model introduced in Section 7.2.2. First we show that **(C2)** holds. In this case we can choose $w_i = w$, where $w(v) = \exp(\overline{v} \vee v)$. As before, we need to estimate A_j, B_j and D_{ji}. We start with A_j.

Lemma 7.16. *For $a, a' \in \mathbb{A}$ we have*

$$A_j \le \kappa \, |G^j_n - {G'}^j_n|_w, \tag{7.45}$$

where the constant $\kappa < 1$ is given in (7.47).

Proof. Following the lines of the proof of Lemma 7.12, it suffices to bound the quantities $\sup_{v \ge 0} A_{ji}(v)$ defined in (7.31). In the current risk model

$$\xi^i = \exp(-\alpha_i \theta^i) \qquad \text{and} \qquad \eta^{ji} = \delta_{ji} Z^i - \frac{c_i}{\alpha_i}(\exp(\alpha_i \theta^i) - 1),$$

see (7.3).

For any v we have

$$\begin{aligned} A_{ji}(v) &= \mathbb{E}\left(\exp\left(\varepsilon v(1 - e^{\alpha_i \theta^i}) + \varepsilon \eta^{ji}\right) \mathbf{1}\left(\xi^i \eta^{ji} \le v\right)\right) \\ &\le \mathbb{E}\left(\exp\left(\varepsilon v(1 - e^{\alpha_i \theta^i}\right) \exp(\varepsilon Z^i)\right) \\ &\le C_Z \mathbb{E} \exp\left(\varepsilon v(1 - e^{\alpha_i \theta^i})\right). \end{aligned}$$

Here we have used the condition (7.18). The latter expression tends to 0 as $v \to \infty$. Therefore for any given $\rho_1 < 1$ we can find a constant $\overline{v}_1 = \overline{v}_1(\rho_1)$ such that for $v \ge \overline{v}_1$:

$$A_{ji}(v) \le \rho_1. \tag{7.46}$$

Now let $v \le \overline{v}$ where the constant $\overline{v} \ge \overline{v}_1$ will be chosen later. By construction of w, $w(v) = w(\overline{v})$ for $v \le \overline{v}$. Hence,

$$\begin{aligned} A_{ji}(v) &= \mathbb{E}\left(\frac{w(\overline{v})}{w(v/\xi^i - \eta^{ji})} \mathbf{1}\left(\xi^i \eta^{ji} \le v,\, \eta^{ji} < -2\overline{v}\right)\right) + \mathbb{E}\left(\frac{w(\overline{v})}{w(v/\xi^i - \eta^{ji})} \mathbf{1}\left(\xi^i \eta^{ji} \le v,\, \eta^{ji} \ge -2\overline{v}\right)\right) \\ &\le \mathbb{E}\left(\frac{w(\overline{v})}{w(-\eta^{ji})} \mathbf{1}\left(\xi^i \eta^{ji} \le v,\, \eta^{ji} < -2\overline{v}\right)\right) + \mathbb{E}\left(\frac{w(\overline{v})}{w(\overline{v})} \mathbf{1}\left(\xi^i \eta^{ji} \le v,\, \eta^{ji} \ge -2\overline{v}\right)\right) \\ &\le \frac{w(\overline{v})}{w(2\overline{v})} \mathbf{P}\left(\eta^{ji} < -2\overline{v}\right) + \mathbf{P}\left(\eta^{ji} \ge -2\overline{v}\right) \\ &\le 1 - \min_i \mathbf{P}\left(\eta^{ii} < -2\overline{v}\right)\left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right) \\ &=: \kappa. \end{aligned} \tag{7.47}$$

Choosing $\rho_1 := \kappa$ and combining (7.46) for $v \geq \overline{v}$ and (7.47) for $v \leq \overline{v}$ we conclude that

$$\sum_i \sup_{v\geq 0} q_{ji} A_{ji}(v) \leq \kappa.$$

This proves the lemma. ☐

Following the lines in the previous section we now bound the term B_j^1 defined in (7.32).

Lemma 7.17. *If $a, a' \in \mathbb{A}$ then,*

$$B_j^1 \leq \frac{C_V}{\underline{q}} |F_{Z^j} - F_{Z'^j}|_w, \tag{7.48}$$

where the constant C_V is given in (7.63).

Proof. The proof is similar to the one of Lemma 7.13. We have

$$\mathbf{P}(g_j(u, Z^j, \theta^j) \geq v) \quad = \quad \mathbb{E}\mathbf{P}\left(Z^j \geq Y_j(u,v) \,|\, \theta^j\right) \; = \; \mathbb{E}\overline{F}_{Z^j}\left((Y_j(u,v))_+\right),$$

where

$$Y_j(u,v) \quad := \quad v\exp(\alpha_j\theta^j) + \frac{c_j}{\alpha_j}(\exp(\alpha_j\theta^j) - 1) - u. \tag{7.49}$$

This yields

$$\begin{aligned}
B_j^1 &= \sup_{v\geq 0} w(v) \left| \int_{u=0}^{\infty} \mathbb{E}\left(\overline{F}_{Z^j}(Y_j(u,v)) - \overline{F}_{Z'^j}(Y_j(u,v))\right) dG'^j_n(u) \right| \\
&\leq |F_{Z^j} - F_{Z'^j}|_w \sup_{v\geq 0} \int_0^{\infty} \mathbb{E}\left(\frac{w(v)}{w(Y_j(u,v))}\right) dG'^j_n(u) \\
&\leq |F_{Z^j} - F_{Z'^j}|_w \sup_{v\geq 0} \int_0^{\infty} \frac{\exp(\varepsilon(\overline{v}\vee v))}{\exp(\varepsilon(\overline{v}\vee(v-u)))} dG'^j_n(u) \\
&\leq |F_{Z^j} - F_{Z'^j}|_w \int_0^{\infty} e^{\varepsilon u}\, dG'^j_n(u) \\
&\leq |F_{Z^j} - F_{Z'^j}|_w \sup_{n\geq 0} \mathbb{E}\exp(\varepsilon V'_n)/\underline{q}.
\end{aligned}$$

The proof of (7.48) is finished if we can show that there exists $C_V < \infty$ such that for any $a \in \mathbb{A}$ the corresponding process V satisfies

$$\sup_{n\geq 0} \mathbb{E}\exp(\varepsilon V_n) \leq C_V. \tag{7.50}$$

The remainder of the proof is devoted to show it.

Define the test function

$$\varphi(v) = \varphi(v,i) := \begin{cases} 0, & v = 0, \\ 1 + bv + de^{\varepsilon v}, & v > 0, \end{cases}$$

where the positive constants b and d will be chosen later.

From condition (7.18) and inequality $\varepsilon Z \le \exp(\varepsilon Z)$ a.s. we have

$$\mathbb{E} Z^i \le \frac{C_Z}{\varepsilon}, \qquad \forall i \in \mathbb{E}. \tag{7.51}$$

Recall that $\mathcal{A}$ is the generating operator of the Markov chain (V_n, I_n), i.e.,

$$\begin{aligned} \mathcal{A}\varphi(v,i) &= \sum_j q_{ij} \mathbb{E}\Big[\big(1 + b\xi^j(v+\eta^{ij}) + d\,\exp(\varepsilon \xi^j(v+\eta^{ij}))\big)\mathbf{1}(v+\eta^{ij}>0)\Big] \\ &\quad -1 - bv - d\,e^{\varepsilon v}. \end{aligned} \tag{7.52}$$

We first consider the case $v > \bar{v}$ for $\overline{v}$ sufficiently large chosen later. Using that $\xi^i = \exp(-\alpha_i \theta^i) \le 1$, $\eta^{ij} \le Z^j$ a.s. and (7.51), we have

$$\begin{aligned} \mathcal{A}\varphi(v,i) &\le \sum_j q_{ij} \mathbb{E}\Big[1 + b(v+Z^j) + d\exp\left(\varepsilon \xi^j(v+Z^j)\right)\Big] - 1 - bv - d\exp(\varepsilon v) \\ &\le \frac{bC_Z}{\varepsilon} + d\,C_Z \sum_j q_{ij} \mathbb{E}\exp\left(\varepsilon v \xi^j\right) - d\,\exp(\varepsilon v) \\ &= \frac{bC_Z}{\varepsilon} + d\,\exp(\varepsilon v)\left(C_Z \sum_j q_{ij} \mathbb{E}\exp\left(-\varepsilon v(1-\xi^j)\right) - 1\right) \\ &\le \frac{bC_Z}{\varepsilon} - \frac{d}{2}\exp(\varepsilon v). \end{aligned} \tag{7.53}$$

In the latter inequality we choose $\overline{v} \ge \overline{v}_2$, where $\overline{v}_2$ is such that

$$C_Z \sum_j q_{ij} \mathbb{E}\exp\left(-\varepsilon v(1-\xi^j)\right) - 1 \le -1/2, \qquad v \ge \overline{v}_2, \tag{7.54}$$

which is possible since the left-hand side converges to -1 when $v \to \infty$. We rewrite (7.53) as follows,

$$\mathcal{A}\varphi(v,i) \le \varphi(v,i) f(v) \qquad \text{where} \qquad f(v) := \left(\frac{bC_Z}{\varepsilon} - \frac{d}{2}\exp(\varepsilon v)\right)/\varphi(v,i).$$

We choose the constant $\overline{v}_3$ so that for $v \ge \overline{v}_3$, $f(v)$ is a decreasing function. In particular,

$$f'(v) = \frac{1}{\varphi^2(v,i)}\left[\frac{d}{2}e^{\varepsilon v}\left(-\varepsilon - b\varepsilon v + b\right) - b\frac{C_Z}{\varepsilon}(b + d\,\varepsilon e^{\varepsilon v})\right],$$

from where one can choose

$$\overline{v}_3 = \frac{1}{\varepsilon}. \tag{7.55}$$

Thus, if we set

$$\overline{v} := \max\{\overline{v}_1, \overline{v}_2, \overline{v}_3\}, \tag{7.56}$$

where $\overline{v}_1$ is defined in (7.46), $\overline{v}_2$ in (7.54) and $\overline{v}_3$ in (7.55), we have

$$\mathcal{A}\varphi(v,i) \le \varphi(v,i) f(\overline{v}), \qquad v \ge \overline{v}. \tag{7.57}$$

Now we turn to the case $0 < v \leq \bar{v}$. From (7.52),

$$\begin{aligned}
\mathcal{A}\varphi(v,i) &\leq \sum_j q_{ij}\mathbb{E}\Big[\big(1+b(v+Z^j)+d\,\exp(\varepsilon(v+Z^j))\big)\big(1-\mathbf{1}(v+\eta^{ij}\leq 0)\big)\Big]-1-bv-d\,e^{\varepsilon v}\\
&\leq \sum_j q_{ij}\Big[-(1+bv)\mathbf{P}\left(v+\eta^{ij}\leq 0\right)+b\frac{C_Z}{\varepsilon}+d\,C_Z\exp(\varepsilon\bar{v})-d\mathbf{P}\left(v+\eta^{ij}\leq 0\right)\Big]\\
&= -(1+bv+d)\sum_j q_{ij}\mathbf{P}\left(\overline{v}+\eta^{ij}\leq 0\right)+b\frac{C_Z}{\varepsilon}+d\,C_Z\exp(\varepsilon\bar{v}).
\end{aligned}$$

Let us denote

$$p := e^{-\varepsilon\overline{v}}\sum_j q_{ij}\mathbf{P}\left(\overline{v}+\eta^{ij}\leq 0\right).$$

Using that $(1+bv+d)/(1+bv+d\,\exp(\varepsilon\overline{v}))\geq \exp(-\varepsilon\overline{v})$ for $v\leq\overline{v}$, we obtain

$$\begin{aligned}
\mathcal{A}\varphi(v,i) &\leq \varphi(v,i)\left(-\frac{1+bv+d}{1+bv+d\exp(\varepsilon\overline{v})}\sum_j q_{ij}\mathbf{P}\left(\overline{v}+\eta^{ij}\leq 0\right)+b\frac{C_Z}{\varepsilon}+d\,C_Z\exp(\varepsilon\overline{v})\right)\\
&\leq \varphi(v,i)\left(-p+b\frac{C_Z}{\varepsilon}+d\,C_Z e^{\varepsilon\bar{v}}\right)\\
&=: \varphi(v,i)g(\overline{v}). \qquad (7.58)
\end{aligned}$$

In order to make the quantities $f(\overline{v})$ and $g(\overline{v})$ negative, we choose constants $d=d(\overline{v})$ and $b=b(d,\overline{v})$ so that $d\,C_Z\exp(\varepsilon\overline{v})\leq p/4$ and $bC_Z/\varepsilon\leq d\exp(\varepsilon\overline{v})/4$, in particular one can take

$$d = \frac{p}{4C_Z e^{\varepsilon\overline{v}}} \qquad \text{and} \qquad b=\frac{\varepsilon d e^{\varepsilon\overline{v}}}{4C_Z}=\frac{\varepsilon p}{16C_Z^2}. \qquad (7.59)$$

Then, with such b and d,

$$\begin{aligned}
f(\overline{v}) &= \left(\frac{bC_Z}{\varepsilon}-\frac{d}{2}e^{\varepsilon\overline{v}}\right)/\varphi(\overline{v},i)\leq -\frac{d}{4\varphi(\overline{v},i)}e^{\varepsilon\overline{v}}=-\frac{p}{16C_Z\varphi(\overline{v},i)}=:-\chi,\\
g(\overline{v}) &= -p+b\frac{C_Z}{\varepsilon}+d\,C_Z e^{\varepsilon\bar{v}}\leq -p+2d\,C_Z e^{\varepsilon\bar{v}}\leq -\frac{p}{2}\leq -\chi.
\end{aligned}$$

Combining the latter together with (7.57) for $v\geq\overline{v}$ and (7.58) for $v\leq\overline{v}$, we have that for any $v\geq 0$,

$$A\varphi(v,i)\leq -\chi\varphi(v,i). \qquad (7.60)$$

Similar to the proof of Lemma 7.13 we want to use Dynkin's formula. First we introduce the stopping time

$$\tau = \min\{n>0,\, V_n=0\}.$$

Then, by the regenerative property of (V_n, I_n),

$$\begin{aligned}
\mathbb{E}_s\left(e^{\varepsilon V_n}\right) &= \sum_{k=0}^{n}\sum_{i=1}^{m} q_{si}^{(k)}\,\mathbf{P}_s(V_k = 0 \mid I_k = i)\,\mathbb{E}_i\left(e^{\varepsilon V_{n-k}}\,\mathbf{1}\left(V_1 > 0, \dots, V_{n-k} > 0\right)\right) \\
&\le 1 + \sum_{i=1}^{m}\mathbb{E}_i\left(\sum_{1\le k<\tau} e^{\varepsilon V_k}\right) \\
&= 1 + \sum_{i=1}^{m}\mathbb{E}_i\left(\mathbb{E}_i\left(\sum_{1\le k<\tau} e^{\varepsilon V_k}\,\middle|\, V_1, I_1\right)\right) \qquad (7.61)
\end{aligned}$$

By Dynkin's formula we obtain

$$\begin{aligned}
0 &= \mathbb{E}_i\left(\varphi(V_\tau, I_\tau)\,\middle|\, V_1, I_1\right) \\
&= \varphi(V_1, I_1) + \mathbb{E}_i\left(\sum_{1\le k<\tau}\mathcal{A}\varphi(V_k, I_k)\,\middle|\, V_1, I_1\right) \\
&\le \varphi(V_1, I_1) - \chi\mathbb{E}_i\left(\sum_{1\le k<\tau}\varphi(V_k, I_k)\,\middle|\, V_1, I_1\right). \qquad (7.62)
\end{aligned}$$

Now (7.61), (7.62) and $\exp(\varepsilon v) \le (\varphi(v,i) - 1)/d$ imply

$$\mathbb{E}_s\left(e^{\varepsilon V_n}\right) \le 1 + \frac{1}{\chi d}\sum_{i=1}^{m}\mathbb{E}_i(\varphi(V_1, I_1) - 1) \le 1 + \frac{C_Z}{\chi d}\left(\frac{b}{\varepsilon} + d\right)\sum_{i=1}^{m} q_{ii} =: C_V. \qquad (7.63)$$

In the last step we used

$$\begin{aligned}
\mathbb{E}_i\varphi(V_1, I_1) - 1 &= q_{ii}\mathbb{E}\left[\left(b\xi^i(Z^i - c_i\theta^i) + d\exp\left(\varepsilon\xi^i(Z^i - c_i\theta^i)\right)\right)\mathbf{1}(Z^i - c_i\theta^i > 0)\right] \\
&\le q_{ii}C_Z\left(\frac{b}{\varepsilon} + d\right).
\end{aligned}$$

This completes the proof of (7.50) and of the lemma. □

Now we turn to the term B_{ji}^2 defined in (7.33). It is convenient to decompose it as follows,

$$B_{ji}^2 \le B_{ji}^{21} + B_{ji}^{22},$$

where the notation

$$B_{ji}^{21} := \sup_{v\ge0} w_i(v)\left|\mathbf{P}_s(g_i(V_n', \delta_{ji}Z'^i, \theta^i) > v | I_n' = j) - \mathbf{P}_s(g_i(V_n', \delta_{ji}Z'^i, \beta_i\theta'^i) > v | I_n' = j)\right|, \qquad (7.64)$$

$$B_{ji}^{22} := \sup_{v\ge0} w_i(v)\left|\mathbf{P}_s(g_i(V_n', \delta_{ji}Z'^i, \beta_i\theta'^i) > v | I_n' = j) - \mathbf{P}_s(g_i'(V_n', \delta_{ji}Z'^i, \beta_i\theta'^i) > v | I_n' = j)\right| \qquad (7.65)$$

will also be used in the next section. The next lemma bounds B_{ji}^{21}.

Lemma 7.18. *If $a, a' \in \mathbb{A}$ then,*

$$B^{21}_{ji} \leq \frac{4C_V C_Z \exp(\varepsilon\overline{v})}{\underline{q}(\lambda_i/\alpha_i \wedge {\lambda'}_i/\alpha'_i)} \left|\frac{\lambda_i}{\alpha_i} - \frac{\lambda'_i}{\alpha'_i}\right|. \tag{7.66}$$

Proof. The proof is similar to the one of Lemma 7.15. In this case we have,

$$g_i(V'_n, \delta_{ji}{Z'}^i, \theta^i) = \left(V'_n + \delta_{ji}{Z'}^i\right)\exp(-\alpha_i\theta^i) - \frac{c_i}{\alpha_i}\left(1 - \exp(-\alpha_i\theta^i)\right).$$

Now, $\alpha_i\theta^i \sim Exp(\lambda_i/\alpha_i)$. Let us denote $\lambda^*_i := \lambda_i/\alpha_i$ and ${\lambda'}^*_i = \lambda'_i/\alpha'_i$.

Following the lines of the proof of Lemma 7.15 with $\lambda^*_i, {\lambda'}^*_i$ defined above and using that $\exp(-\alpha_i t) \leq 1$, we arrive at the inequality (7.40), i.e.,

$$B^2_{ji} \leq 4\frac{|\lambda^*_i - {\lambda'}^*_i|}{\lambda^*_i \wedge {\lambda'}^*_i}\mathbb{E}\left[w\left(V'_n + {Z'}^i\right)|I'_n = j\right].$$

Now, conditions (7.18) and (7.50) yield

$$\mathbb{E}\left[w\left(V'_n + {Z'}^i\right)|I'_n = j\right] \leq \frac{C_V C_Z \exp(\varepsilon\overline{v})}{\underline{q}}. \tag{7.67}$$

This proves the lemma. □

Lemma 7.19. *Let $a, a' \in \mathbb{A}$. Then*

$$B^{22}_{ji} \leq \left|\frac{1}{\alpha_i} - \frac{1}{\alpha'_i}\right| \frac{\lambda'_i C_V C_Z \exp(\varepsilon\overline{v})}{\underline{q}(\alpha_i \wedge \alpha'_i)}. \tag{7.68}$$

Proof. Let $c^*_i = \max\{c_i/\alpha_i, c_i/\alpha'_i\}$ and $c_{i*} = \min\{c_i/\alpha_i, c_i/\alpha'_i\}$. Thus,

$$B^{22}_{ji} = \sup_{v\geq 0} w(v)\left|\mathbf{P}_s\left(\frac{1}{\alpha_i}\log\left(\frac{V'_n + \delta_{ji}{Z'}^i + c^*_i}{v + c^*_i}\right) \leq \alpha'_i{\theta'}^i < \frac{1}{\alpha_i}\log\left(\frac{V'_n + \delta_{ji}{Z'}^i + c_{i*}}{v + c_{i*}}\right)|I'_n = j\right)\right|.$$

Using that $\alpha'_i{\theta'}^i \sim Exp({\lambda'}^*_i)$, where ${\lambda'}^*_i = \lambda'_i/\alpha'_i$ as in the previous lemma, we have

$$B^{22}_{ji} = \sup_{v\geq 0} w(v)\int_{u=v}^{\infty}\left[\left(\frac{v + c^*_i}{u + c^*_i}\right)^{{\lambda'}^*_i/\alpha_i} - \left(\frac{v + c_{i*}}{u + c_{i*}}\right)^{{\lambda'}^*_i/\alpha_i}\right] d\mathbf{P}_s(V'_n + \delta_{ji}{Z'}^i \leq u \,|\, I'_n = j).$$

Inserting the bound (for $u \geq v$)

$$\left(\frac{v + c^*_i}{u + c^*_i}\right)^{{\lambda'}^*_i/\alpha_i} - \left(\frac{v + c_{i*}}{u + c_{i*}}\right)^{{\lambda'}^*_i/\alpha_i} \leq \frac{{\lambda'}^*_i}{\alpha_i c_{i*}}|c^*_i - c_{i*}| = \frac{\lambda'_i}{\alpha_i \wedge \alpha'_i}\left|\frac{1}{\alpha_i} - \frac{1}{\alpha'_i}\right| \tag{7.69}$$

and using the monotonicity of w, we obtain

$$B^{22}_{ji} \leq \frac{\lambda'_i}{\alpha_i \wedge \alpha'_i}\left|\frac{1}{\alpha_i} - \frac{1}{\alpha'_i}\right| \sup_{v\geq 0} I_{ji}(v),$$

where

$$I_{ji}(v) \quad = \quad \int_{u=v}^{\infty} w(u)\, d\mathbf{P}_s(V_n' + \delta_{ji} Z'^i \le u \,|\, I_n' = j).$$

For $j \neq i$, using (7.50), we have

$$I_{ji}(v) \quad = \quad \int_{u=v}^{\infty} w(u)\, dG'^i_n(u) \;\le\; \frac{e^{\varepsilon\overline{v}} C_V}{q'^{(n)}_{ji}} \;\le\; \frac{e^{\varepsilon\overline{v}} C_V}{\underline{q}}. \tag{7.70}$$

If $j = i$ then similarly,

$$\begin{aligned} I_{ji}(v) \quad &\le \quad \int_{u=0}^{\infty} w(u) \int_{x=0}^{u} dG'^i_n(u-x)\, dF_{Z'^i}(x) \\ &\le \quad e^{\varepsilon\overline{v}} \int_{x=0}^{\infty} e^{\varepsilon x}\, dF_{Z'^i}(x) \int_{y=0}^{\infty} e^{\varepsilon y}\, dG'^i_n(y) \\ &\le \quad \frac{e^{\varepsilon\overline{v}} C_Z C_V}{\underline{q}}. \end{aligned} \tag{7.71}$$

Now (7.68) follows from (7.69)–(7.71). □

Thus, Lemmas 7.17–7.19 yield

$$\begin{aligned} B_j \quad \le \quad & \frac{C_V}{\underline{q}} |F_{Z^j} - F_{Z'^j}|_w \\ & + \frac{C_V C_Z \exp(\varepsilon\overline{v})}{\underline{q}} \left(4 \max_i \left(\frac{\alpha_i}{\lambda_i} \vee \frac{\alpha_i'}{\lambda_i'} \right) \left| \frac{\lambda_i}{\alpha_i} - \frac{\lambda_i'}{\alpha_i'} \right| + \max_i \frac{\lambda_i'}{(\alpha_i \wedge \alpha_i')} \left| \frac{1}{\alpha_i} - \frac{1}{\alpha_i'} \right| \right) \end{aligned} \tag{7.72}$$

It remains to bound D_{ji}. Similar to the previous case, using Markov's inequality and $\xi'^i \le 1$ a.s., we have

$$D_{ji} \quad \le \quad \mathbb{E}_s \left[w\left((V_n' + Z'^i) \right) \,|\, I_n' = j \right].$$

Conditions (7.18) and (7.50) yield

$$D_{ji} \le C_Z C_V \exp(\varepsilon\overline{v}). \tag{7.73}$$

Collecting the bounds for A_j, B_j, D_{ji} from Lemma 7.16, (7.72) and (7.73) respectively, and using the relation (7.28) we conclude from (7.27) that **(C2)** holds with

$$\begin{aligned} \kappa \quad &:= \quad 1 - \min_i \mathbf{P}\left(\eta^{ii} < -2\overline{v} \right) \left(1 - \frac{w(\overline{v})}{w(2\overline{v})} \right), \\ d(a, a') \quad &:= \quad \frac{C_V \exp(\varepsilon\overline{v})}{\underline{q}} \Bigg[\exp(-\varepsilon\overline{v}) \max_j |F_{Z^j} - F_{Z'^j}|_w + 4 C_Z \max_j \left(\frac{\alpha_j}{\lambda_j} \vee \frac{\alpha_j'}{\lambda_j'} \right) \left| \frac{\lambda_j}{\alpha_j} - \frac{\lambda_j'}{\alpha_j'} \right| \\ & \quad + \; C_Z \max_j \frac{\lambda_j'}{(\alpha_j \wedge \alpha_j')} \left| \frac{1}{\alpha_j} - \frac{1}{\alpha_j'} \right| + C_Z h_q \Delta_q \Bigg] \end{aligned}$$

where $h_q = 1 + m + mq$.

From (7.50) and $w(v) = \exp(v \vee \overline{v})$ we have that condition **(C1)** holds with

$$C_w := C_V \exp(\varepsilon\overline{v}) = \left(1 + \frac{C_Z}{\chi d}\left(\frac{b}{\varepsilon} + d\right)\sum_{i=1}^{m} q_{ii}\right)\exp(\varepsilon\overline{v}), \tag{7.74}$$

where the constant C_V is defined in (7.63), $\overline{v}$ in (7.56) and the constants b and d are given in (7.59).

7.3.3 Random interest

We will prove that conditions **(C1)** and **(C2)** hold for the risk model defined in Section 7.2.3. Recall the weight function w from (7.26). The constants α and $\overline{v}$ will be defined in the course of the proof, see the lines following (7.82) in the proof of Lemma 7.20. Set

$$w_i(v) = \begin{cases} 1 + \alpha\overline{v}^{\varepsilon}, & 0 \le v \le \overline{v}, \\ 1 + \alpha\gamma_i(\varepsilon)v^{\varepsilon}, & v > \overline{v}. \end{cases} \tag{7.75}$$

First we prove the condition **(C2)**. As before, we need to bound the quantities A_j, B_j and D_{ji}.

Lemma 7.20. *For $a, a' \in \mathbb{A}$ we have*

$$A_j \le \kappa\, |G^j_n - {G'}^j_n|_{w_j},$$

where the constant $\kappa < 1$ is given in (7.84).

Proof. Following the lines of the proof of Lemma 7.12, we have to bound the quantity $\sum_i q_{ji} \sup_{v \ge 0} A_{ji}(v)$, where the A_{ji}'s are defined in (7.31). Recall (ξ^i, η^{ji}) from (7.3):

$$\left(\xi^i, \eta^{ji}\right) \;=\; \left(\exp\left(-U^i_{\theta^i}\right),\ \delta_{ji}Z^i - c_i\int_0^{\theta^i} \exp\left(U^i_{\theta^i} - U^i_u\right)\, du\right).$$

We extend the functions w_i to the negative real line by setting $w_i(v) = 1$ for $v < 0$. Then,

$$\begin{aligned} A_{ji}(v) \;&\le\; \mathbb{E}\left(\frac{w_i(v)}{w_j\left(v/\xi^i - Z^i\right)}\mathbf{1}\left(\xi^i Z^i > v\right)\right) + \mathbb{E}\left(\frac{w_i(v)}{w_j\left(v/\xi^i - Z^i\right)}\mathbf{1}\left(\xi^i Z^i \le v\right)\right) \\ &=:\; A^1_{ji}(v) + A^2_{ji}(v). \end{aligned} \tag{7.76}$$

We first consider $v > \overline{v}$. By Markov's inequality and Assumption 7.8,

$$\begin{aligned} A^1_{ji}(v) \;&=\; (1 + \alpha\gamma_i\varepsilon v^{\varepsilon})\mathbf{P}(\xi^i Z^i > v) \\ &\le\; \left(\frac{1}{\overline{v}^{\varepsilon}} + \alpha\overline{\gamma}(\varepsilon)\right) C_Z \mathbb{E}(\xi^i)^{\varepsilon}. \end{aligned}$$

For a given $\delta \in (0,1)$ to be chosen later we take $\overline{v} = \overline{v}(\delta)$ and $\alpha_1 = \alpha_1(\delta)$ such that

$$\left(\frac{1}{\overline{v}^{\varepsilon}} + \alpha_1\overline{\gamma}(\varepsilon)\right) C_Z \le \frac{\delta}{\overline{\gamma}(\varepsilon)}. \tag{7.77}$$

Then using Assumption 7.7, for any $\alpha \le \alpha_1$,

$$\begin{aligned}\sum_i q_{ji} \sup_{v \ge \overline{v}} A^1_{ji}(v) &\le \frac{\delta}{\overline{\gamma}(\varepsilon)} \sum_i q_{ji} \mathbb{E}(\xi^i)^\varepsilon \gamma_i(\varepsilon) \\ &\le \frac{\delta}{\overline{\gamma}(\varepsilon)} \rho_1 \gamma_j(\varepsilon) \\ &\le \delta \rho_1.\end{aligned}$$

We now turn to the term

$$A^2_{ji}(v) = \mathbb{E}\left(\frac{(\xi^i)^\varepsilon \,(1 + \alpha\gamma_i v^\varepsilon)}{(\xi^i)^\varepsilon + \alpha\gamma_j(\varepsilon)\,(v - \xi^i Z^i)^\varepsilon} \mathbf{1}\left(\xi^i Z^i \le v\right)\right). \tag{7.78}$$

For $\xi^i Z^i \le v$, the inequality

$$\left(\frac{\xi^i Z^i}{Z^i}\right)^\varepsilon + \alpha\gamma_j(\varepsilon)\left(v - \xi^i Z^i\right)^\varepsilon \ge \left(\frac{V^{ji}}{Z^i}\right)^\varepsilon$$

holds with

$$V^{ji} = \frac{(\alpha\gamma_j(\varepsilon))^{1/\varepsilon} v}{(\alpha\gamma_j(\varepsilon))^{1/\varepsilon} + 1/Z^i}.$$

This yields,

$$A^2_{ji}(v) \le \mathbb{E}\left(1 + Z^i(\alpha\gamma_j(\varepsilon))^{1/\varepsilon}\right)^\varepsilon \mathbb{E}\left(\xi^i\right)^\varepsilon \frac{1 + \alpha\gamma_i(\varepsilon) v^\varepsilon}{\alpha\gamma_j(\varepsilon) v^\varepsilon}.$$

We choose $\alpha_2 = \alpha_2(\delta)$ such that

$$\mathbb{E}\left(1 + Z^i(\alpha_2\overline{\gamma}(\varepsilon))^{1/\varepsilon}\right)^\varepsilon \le 1 + \delta. \tag{7.79}$$

From Assumption 7.7 we obtain that for any $\alpha \le \alpha_2$

$$\begin{aligned}\sum_i q_{ji} \sup_{v \ge \overline{v}} A^2_{ji}(v) &\le (1+\delta)\frac{1 + \alpha v^\varepsilon}{\alpha\gamma_j(\varepsilon) v^\varepsilon} \sum_i q_{ji}\gamma_i(\varepsilon)\mathbb{E}\left(\xi^i\right)^\varepsilon \\ &\le (1+\delta)\rho_1\left(\frac{1}{\alpha\overline{v}^\varepsilon} + 1\right).\end{aligned}$$

Now we require that $\overline{v} = \overline{v}(\delta, \alpha)$ in addition to (7.77) also satisfies

$$\frac{1}{\alpha\,\overline{v}^\varepsilon} \le \delta. \tag{7.80}$$

Choose δ such that

$$\rho_1(\delta + (1+\delta)^2) \le \frac{1+\rho_1}{2}. \tag{7.81}$$

Thus, we choose the constants in the following order. First we fix δ from the condition (7.81), then take $\alpha = \min\{\alpha_1, \alpha_2\} = \alpha(\delta)$ from the conditions (7.77) and (7.79) and finally take $\overline{v} = \overline{v}(\delta, \alpha)$ such that (7.77) and (7.80) hold.

We arrive at the bound

$$\sum_i q_{ji} \sup_{v \geq \overline{v}} A_{ji}(v) \leq \frac{1+\rho_1}{2}. \tag{7.82}$$

Now let $v \leq \bar{v}$. Then

$$\begin{aligned}
A_{ji}(v) &= \mathbb{E}\left(\frac{w_i(\overline{v})}{w_j\left(\frac{v}{\xi^i}-\eta^{ji}\right)}\mathbf{1}\left(\xi^i\eta^{ji}<v, \xi^i \leq 1, \eta^{ji} \leq -2\overline{v}\right)\right)\\
&\quad + \mathbb{E}\left(\frac{w_i(\overline{v})}{w_j\left(\frac{v}{\xi^i}-\eta^{ji}\right)}\mathbf{1}\left(\xi^i\eta^{ji}<v, \{\xi^i > 1\} \cup \{\eta^{ji} > -2\overline{v}\}\right)\right)\\
&\leq \frac{w_i(\overline{v})}{w_j(2\overline{v})}\mathbf{P}\left(\xi^i \leq 1,\, \eta^{ji} \leq -2\overline{v}\right) + 1 - \mathbf{P}\left(\xi^i \leq 1,\, \eta^{ji} \leq -2\overline{v}\right)\\
&\leq 1 - \left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right)\mathbf{P}\left(\xi^i \leq 1,\, \eta^{ji} \leq -2\overline{v}\right)
\end{aligned}$$

This together with Assumption 7.9 yields

$$\begin{aligned}
\sum_i q_{ji} \sup_{v \leq \overline{v}} A_{ji}(v) &\leq 1 - \left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right)\sum_i q_{ji}\mathbf{P}\left(\xi^i \leq 1,\, \eta^{ji} \leq -2\overline{v}\right)\\
&\leq 1 - q_{2\overline{v}}\left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right).
\end{aligned} \tag{7.83}$$

Thus, combining (7.82) and (7.83) we have

$$\sum_i \sup_{v \geq 0} q_{ji} A_{ji}(v) \leq \max\left\{\frac{1+\rho_1}{2}, 1 - q_{2\overline{v}}\left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right)\right\} =: \kappa, \tag{7.84}$$

where ρ_1 and $q_{2\overline{v}}$ are from Assumptions 7.7 and 7.9. This completes the proof. □

Lemma 7.21. *If $a, a' \in \mathbb{A}$ then,*

$$B_j^1 \leq h_Z |F_{Z^j} - F_{Z'^j}|_w, \tag{7.85}$$

where the constant h_Z is given by (7.93).

Proof. The proof is similar to the ones of the corresponding Lemmas 7.13 and 7.17 in the previous sections. We have

$$\mathbf{P}(g_j(u, Z^j, \theta^j) \geq v) \;=\; \mathbb{E}\mathbf{P}\left(Z^j \geq Y_j(u,v) \,|\, \theta^j\right) = \mathbb{E}\overline{F}_{Z^j}\left((Y_j(u,v))_+\right),$$

where

$$Y_j(u,v) := v \exp\left(U^j_{\theta^j}\right) + c_j \int_0^{\theta^j} \exp\left(U^j_{\theta^j} - U^j_s\right) ds - u. \tag{7.86}$$

This yields

$$\begin{aligned} B_j^1 &= \sup_{v\geq 0} w_j(v) \left| \int_{u=0}^{\infty} \mathbb{E}\left(\overline{F}_{Z^j}(Y_j(u,v)) - \overline{F}_{Z'^j}(Y_j(u,v))\right) dG'^j_n(u) \right| \\ &\leq \overline{\gamma}(\varepsilon)|F_{Z^j} - F_{Z'^j}|_w \sup_{v\geq 0} \int_0^{\infty} \mathbb{E}\left(\frac{w(v)}{w(Y_j(u,v))}\right) dG'^j_n(u). \end{aligned}$$

We first consider the expression

$$\frac{w(v)}{w(Y_j(u,v))} = \frac{w(v)}{w(Y_j(u,v)+u)} \frac{w(Y_j(u,v)+u)}{w(Y_j(u,v))}. \tag{7.87}$$

Using the inequality

$$\frac{1+\alpha(y^\varepsilon \vee u^\varepsilon)}{1+\alpha u^\varepsilon} \leq 1 + (1 \vee 2^{\varepsilon-1})(1 \vee \alpha u^\varepsilon), \tag{7.88}$$

we bound the second term by

$$\frac{w(Y_j(u,v)+u)}{w(Y_j(u,v))} \leq 1 + (1 \vee 2^{\varepsilon-1})(1+\alpha u^\varepsilon). \tag{7.89}$$

The expectation of the first term can be bounded as follows,

$$\begin{aligned} \mathbb{E}\left(\frac{w(v)}{w(Y_j(u,v)+u)}\right) &\leq \mathbb{E}\left(\frac{1+\alpha(\overline{v}^\varepsilon \vee v^\varepsilon)}{1+\alpha(\overline{v}^\varepsilon \vee (v/\xi^j)^\varepsilon)}\right) \\ &\leq \mathbb{E}\left(\frac{(\xi^j)^\varepsilon + \alpha(\overline{v}^\varepsilon \vee v^\varepsilon)(\xi^j)^\varepsilon}{(\xi^j)^\varepsilon + \alpha v^\varepsilon}\right) \\ &\leq 1 + \alpha\overline{v}^\varepsilon + \mathbb{E}(\xi^j)^\varepsilon. \end{aligned} \tag{7.90}$$

By Assumption 7.7,

$$\mathbb{E}(\xi^j)^\varepsilon \leq \frac{1}{\underline{q}\gamma_j(\varepsilon)} \sum_i q_{ji}\mathbb{E}(\xi^i)^\varepsilon \gamma_i(\varepsilon) \leq \frac{1}{\underline{q}},$$

which together with (7.87)–(7.90) yields

$$\int_0^{\infty} \mathbb{E}\left(\frac{w(v)}{w(Y_j(u,v))}\right) dG'^j_n(u) \leq \left(1 + \alpha\overline{v}^\varepsilon + \frac{1}{\underline{q}}\right)\left(1 + (1 \vee 2^{\varepsilon-1})\left(1 + \frac{\alpha}{\underline{q}}\mathbb{E}_s(V'_n)^\varepsilon\right)\right). \tag{7.91}$$

We will prove that for some constant $C_V < \infty$ and any $a \in \mathbb{A}$,

$$\sup_{n\geq 0} \mathbb{E}_s V_n^\varepsilon \leq C_V. \tag{7.92}$$

This proves (7.85) with

$$h_Z = \overline{\gamma}(\varepsilon)\left(1 + \alpha\overline{v}^\varepsilon + \frac{1}{\underline{q}}\right)\left(1 + (1 \vee 2^{\varepsilon-1})\left(1 + \frac{\alpha C_V}{\underline{q}}\right)\right). \tag{7.93}$$

Thus, it remains to prove (7.92). Repeating exactly the same arguments as in (7.36) in the proof of Lemma 7.13, we obtain

$$\mathbb{E}_s V_n^\varepsilon \quad \leq \quad \sum_{i=1}^m \mathbb{E}_i \left(\mathbb{E}_i \left(\sum_{1 \leq k < \tau} V_k^\varepsilon \big| V_1, I_1 \right) \right) \tag{7.94}$$

Take the test function

$$\varphi(v,i) = \begin{cases} 0, & \text{if } v = 0, \\ 1 + \alpha_0 \gamma_i(\varepsilon) v^\varepsilon & \text{if } v > 0, \end{cases}$$

and choose $v^* > 0$ such that for all $j \in \mathbb{E}$

$$\mathbb{E}\left(1 + \frac{Z^j}{v^*}\right)^\varepsilon \leq \frac{1+\rho_1}{2\rho_1}. \tag{7.95}$$

Then for $v \geq v^*$,

$$\begin{aligned} \mathbb{E}\left(\varphi(V_{n+1}, I_{n+1}) \,|\, V_n = v, I_n = i\right) &= 1 + \alpha_0 \mathbb{E}\left(\gamma_{I_{n+1}}(\varepsilon)(\xi^{I_{n+1}})^\varepsilon \left(v + \eta^{iI_{n+1}}\right)_+^\varepsilon \mathbf{1}\left(v + \eta^{iI_{n+1}} > 0\right) \,|\, I_n = i\right) \\ &= 1 + \alpha_0 \mathbb{E}\left(\gamma_{I_{n+1}}(\varepsilon) \left(\xi^{I_{n+1}}\right)^\varepsilon \left(v + \eta^{iI_{n+1}}\right)_+^\varepsilon \,|\, I_n = i\right) \\ &\leq 1 + \alpha_0 \sum_j q_{ij} \gamma_j(\varepsilon) \mathbb{E}\left(\xi^j\right)^\varepsilon \mathbb{E}\left(1 + \frac{Z^j}{v}\right)^\varepsilon v^\varepsilon \\ &\leq 1 + \alpha_0 v^\varepsilon \frac{1+\rho_1}{2\rho_1} \sum_j q_{ij} \gamma_j(\varepsilon) \mathbb{E}\left(\xi^j\right)^\varepsilon \\ &\leq 1 + \alpha_0 v^\varepsilon \frac{1+\rho_1}{2} \gamma_i(\varepsilon) \\ &= \frac{1 + \alpha_0 \gamma_i(\varepsilon) v^\varepsilon (1+\rho_1)/2}{1 + \alpha_0 \gamma_i(\varepsilon) v^\varepsilon} \varphi(v,i) \\ &\leq \frac{1 + \alpha_0 v^{*\varepsilon}(1+\rho_1)/2}{1 + \alpha_0 v^{*\varepsilon}} \varphi(v,i). \end{aligned}$$

This yields

$$\mathcal{A}\varphi(v,i) \quad \leq \quad -\frac{\alpha_0 \; v^{*\varepsilon}(1-\rho_1)}{2\left(1 + \alpha_0 v^{*\varepsilon}\right)} \varphi(v,i), \qquad \text{for } v \geq v^*. \tag{7.96}$$

Consider $0 < v \le v^*$. Recall $q_{v^*} \le \mathbf{P}(\xi^j \le 1, \eta^{ij} < -v^*)$ from (7.24). Then,

$$\begin{aligned}
\mathbb{E}\Big[\varphi(V_{n+1}, I_{n+1}) \,|\, V_n = v, I_n = i\Big] &\le \mathbb{E}\Big[\varphi(\xi^{I_{n+1}}(v + Z^{I_{n+1}}), I_{n+1}) \,|\, I_n = i\Big] \\
&\quad -\mathbb{E}\Big[\varphi(\xi^{I_{n+1}}(v + Z^{I_{n+1}}), I_{n+1})\mathbf{1}\left(\eta^{iI_{n+1}} \le v_*\right) \,|\, I_n = i\Big] \\
&\le \sum_j q_{ij}\big(1 + \alpha_0\gamma_j(\varepsilon)\mathbb{E}(\xi^j)^\varepsilon \mathbb{E}(v + Z^j)^\varepsilon\big) - \sum_j q_{ij}\mathbf{P}(\eta^{ij} \le -v^*) \\
&\le 1 + \alpha_0 \sum_j q_{ij}\mathbb{E}(\xi^j)^\varepsilon \gamma_j(\varepsilon)\mathbb{E}\left(1 + \frac{Z^j}{v^*}\right)^\varepsilon v^{*\varepsilon} - q_{v^*} \\
&\le 1 + \alpha_0\gamma_i(\varepsilon)\frac{1+\rho_1}{2}v^{*\varepsilon} - q_{v^*}.
\end{aligned}$$

In the last step we used Assumption 7.7 and the relation (7.95). Take

$$\alpha_0 = \frac{q_{v^*}}{\overline{\gamma}(\varepsilon)v^{*\varepsilon}(1+\rho_1)}.$$

Then,

$$\mathcal{A}\varphi(v,i) \le -\frac{q_{v^*}}{2}\varphi(v,i), \qquad \text{for } v \in (0, v^*). \tag{7.97}$$

Combining (7.96) for $v \ge v^*$ and (7.97) for $v \in (0, v^*)$, we have that for $v > 0$,

$$A\varphi(v,i) \le -\chi\varphi(v,i), \qquad \chi = \min\left\{\frac{\alpha_0 v^{*\varepsilon}(1-\rho_1)}{2\,(1+\alpha_0 v^{*\varepsilon})}, \frac{q_{v^*}}{2}\right\}. \tag{7.98}$$

Similar to the proof of Lemma 7.17, using the regeneration property of (V_n, I_n), and denoting $\tau = \min\{n > 0,\, V_n = 0\}$, we have

$$\begin{aligned}
\mathbb{E}_s V_n^\varepsilon &= \sum_{k=0}^{n}\sum_{i=1}^{m} q_{si}^{(k)}\,\mathbf{P}_s(V_k = 0 \,|\, I_k = i)\,\mathbb{E}_i\left(V_{n-k}^\varepsilon\,\mathbf{1}\,(V_1 > 0, \ldots, V_{n-k} > 0)\right) \\
&\le \sum_{i=1}^{m}\mathbb{E}_i\left(\sum_{1\le k<\tau} V_k^\varepsilon\right) \\
&= \sum_{i=1}^{m}\mathbb{E}_i\left(\mathbb{E}_i\left(\sum_{1\le k<\tau} V_k^\varepsilon \,\big|\, V_1, I_1\right)\right)
\end{aligned} \tag{7.99}$$

An application of Dynkin's formula together with (7.98) yields

$$0 \le \varphi(V_1, I_1) - \chi\mathbb{E}_i\left(\sum_{1\le k<\tau}\varphi(V_k, I_k)\,\big|\, V_1, I_1\right).$$

This together with (7.99) implies

$$\mathbb{E}_s V_n^\varepsilon \le \frac{1}{\chi\alpha_0}\sum_i \mathbb{E}_i\varphi(V_1, I_1). \tag{7.100}$$

The bound

$$\begin{aligned}\mathbb{E}_i\varphi(V_1,I_1) &\leq \mathbb{E}_i\left[\left(1+\alpha_0\gamma_{I_1}(\xi^{I_1})^\varepsilon Z^{I_1}\right)\delta_{i,I_1}\right]\\ &\leq q_{ii}+\alpha_0 q_{ii}\gamma_i(\varepsilon)\mathbb{E}(\xi^i)^\varepsilon C_Z\\ &\leq q_{ii}+\alpha_0\overline{\gamma}(\varepsilon)C_Z\end{aligned}$$

together with (7.100) proves (7.92) with

$$C_V=\frac{\sum_i q_{ii}+m\alpha_0\overline{\gamma}(\varepsilon)C_Z}{\chi\alpha_0}. \tag{7.101}$$

and completes the proof of the lemma. □

Remark 7.22. Condition **(C1)** immediately follows from (7.92) with

$$C_w := 1+\alpha\left(\overline{v}^\varepsilon+C_V\right), \tag{7.102}$$

where the constant C_V is given in (7.101).

Now we consider the term B^{21}_{ji} defined in (7.64).

Lemma 7.23. *If $a,a'\in\mathbb{A}$ then,*

$$B^{21}_{ji}\leq h_{\lambda i}\left|\lambda_i-\frac{\lambda'_i}{\beta_i}\right|, \tag{7.103}$$

where $h_{\lambda i}$ is given by (7.105).

Proof. From (7.64) we have

$$B^{21}_{ji} \leq \sup_{v\geq 0} w_i(v)\int_0^\infty \mathbf{P}\left(g_i(V'_n,Z'^i,t)\geq v\,|\,I'_n=j\right)\left|\lambda_i e^{-\lambda_i t}-(\lambda'_i/\beta_i)e^{-(\lambda'_i/\beta_i)t}\right|dt$$

Set $\underline{\lambda}_i:=\min(\lambda_i,\lambda'_i/\beta_i)$ and $\overline{\lambda}_i:=\max(\lambda_i,\lambda'_i/\beta_i)$. Then,

$$\begin{aligned}\left|\lambda_i e^{-\lambda_i t}-(\lambda'_i/\beta_i)e^{-(\lambda'_i/\beta_i)t}\right| &\leq \left|\overline{\lambda}_i-\underline{\lambda}_i\right|e^{-\overline{\lambda}_i t}+\left|1-e^{-(\overline{\lambda}_i-\underline{\lambda}_i)t}\right|\underline{\lambda}_i e^{-\underline{\lambda}_i t}\\ &\leq \left|\overline{\lambda}_i-\underline{\lambda}_i\right|e^{-\overline{\lambda}_i t}+\left|\overline{\lambda}_i-\underline{\lambda}_i\right|t\underline{\lambda}_i e^{-\underline{\lambda}_i t}.\end{aligned}$$

Let the r.v. $\overline{\theta}^i\sim Exp(\overline{\lambda}_i)$ and the r.v.'s $\underline{\theta}^i_1$ and $\underline{\theta}^i_2$ be i.i.d. with distribution $Exp(\underline{\lambda}_i)$ (all independent of the process U^i). Then $\underline{\theta}^i_1+\underline{\theta}^i_2$ follows a $\Gamma(2,\underline{\lambda}_i)$ distribution. This together with the Markov inequality yields

$$\begin{aligned}B^{21}_{ji} &\leq \frac{|\overline{\lambda}_i-\underline{\lambda}_i|}{\underline{\lambda}_i}\sup_{v\geq 0}w_i(v)\left\{\mathbf{P}\left((V'_n+Z'^i)e^{-U^i_{\overline{\theta}^i}}\geq v\,|\,I'_n=j\right)+\mathbf{P}\left((V'_n+Z'^i)e^{-U^i_{\underline{\theta}^i_1+\underline{\theta}^i_2}}\geq v\,|\,I'_n=j\right)\right\}\\ &\leq \frac{|\overline{\lambda}_i-\underline{\lambda}_i|}{\underline{\lambda}_i}\mathbb{E}\left[w_i\left((V'_n+Z'^i)\exp\left(-U^i_{\overline{\theta}^i}\right)\right)+w_i\left((V'_n+Z'^i)\exp\left(-U^i_{\underline{\theta}^i_1+\underline{\theta}^i_2}\right)\right)|I'_n=j\right]\\ &\leq \frac{|\overline{\lambda}_i-\underline{\lambda}_i|}{\underline{\lambda}_i}\left(2+\alpha\gamma_i(\varepsilon)\right)\overline{v}^\varepsilon\mathbb{E}\left[\left(V'_n+Z'^i\right)^\varepsilon|I'_n=j\right]\left(\mathbb{E}\exp\left(-\varepsilon U^i_{\overline{\theta}^i}\right)+\mathbb{E}\exp\left(-\varepsilon U^i_{\underline{\theta}^i_1+\underline{\theta}^i_2}\right)\right).\end{aligned} \tag{7.104}$$

Assumption 7.7 implies that

$$\mathbb{E}\exp\left(-\varepsilon U^i_{\theta^i}\right) \le \frac{1}{\underline{q}\gamma_i(\varepsilon)}\sum_j q_{ij}\mathbb{E}\exp\left(-\varepsilon U^j_{\theta^j}\right)\gamma_j(\varepsilon) \le \frac{1}{\underline{q}},$$

and similarly, $\mathbb{E}\exp\left(-\varepsilon {U'}^i_{\theta^i}\right) \le 1/\underline{q}$ for the perturbed process. Since either $U^i_{\theta^i} \overset{d}{=} U^i_{\underline{\theta}^i_1}$ (if $\underline{\lambda}_i = \lambda_i$) or ${U'}^i_{\theta^i} \overset{d}{=} U^i_{\underline{\theta}^i_1}$ (if $\underline{\lambda}_i = \lambda'_i/\beta_i$)), it follows by the stationary and independent increments of the process U^i that

$$\mathbb{E}\exp\left(-\varepsilon U^i_{\underline{\theta}^i_1+\underline{\theta}^i_2}\right) = \left(\mathbb{E}\exp\left(-\varepsilon U^i_{\underline{\theta}^i_1}\right)\right)^2 \le \underline{q}^{-2}.$$

Inserting the latter in (7.104) and using inequality $\mathbb{E}\left(V'_n + {Z'}^i\right)^\varepsilon \le \max(1, 2^{\varepsilon-1})\left(C_V/\underline{q} + C_Z\right)$ we obtain (7.103) with

$$h_{\lambda i} = \frac{2 + \alpha\gamma_i(\varepsilon)\overline{v}^\varepsilon(1\vee 2^{\varepsilon-1})\left(C_V/\underline{q}+C_Z\right)(1+\underline{q})/\underline{q}^2}{\lambda_i \wedge (\lambda'_i/\beta_i)} \tag{7.105}$$

which completes the proof of the lemma. □

Lemma 7.24. *For any $a, a' \in \mathbb{A}$,*

$$B^{22}_{jj} \le h_{cj}\left|c_j - \frac{c'_j}{\beta_j}\right|,$$

where the constant h_{cj} is given in (7.108).

Proof. From (7.65) we have

$$B^{22}_{jj} = \sup_{v\ge 0} w_j(v)\mathbf{P}_s\left(\underline{v}_j < V'_n + {Z'}^j \le \overline{v}_j \mid I'_n = j\right),$$

where

$$\begin{aligned} \underline{v}_j &:= v\exp\left(U^j_{\beta_j{\theta'}^j}\right) + (c_j \wedge c'_j/\beta_j)\int_0^{\beta_j{\theta'}^j}\exp\left(U^j_{\beta_j{\theta'}^j} - U^j_u\right)du \quad (7.106)\\ \overline{v}_j &:= v\exp\left(U^j_{\beta_j{\theta'}^j}\right) + (c_j \vee c'_j/\beta_j)\int_0^{\beta_j{\theta'}^j}\exp\left(U^j_{\beta_j{\theta'}^j} - U^j_u\right)du. \quad (7.107)\end{aligned}$$

Using the condition (7.23) and the inequality (7.88) we obtain

$$\begin{aligned} B^{22}_{jj} &= \sup_{v\ge 0} w_j(v)\mathbb{E}\int_{\underline{v}_j}^{\overline{v}_j}\int_0^t p_{{Z'}^j}(t-u)\,d{G'}^j_n(u)\,dt\\ &\le \gamma_j(\varepsilon)C_p\sup_{v\ge 0} w(v)\mathbb{E}\int_{\underline{v}_j}^{\overline{v}_j}\int_0^t \frac{1}{w(t-u)}\,d{G'}^j_n(u)\,dt\\ &\le \gamma_j(\varepsilon)C_p\sup_{v\ge 0} w(v)\mathbb{E}\int_{\underline{v}_j}^{\overline{v}_j}\frac{1}{w(t)}\int_0^t (1+(1\vee 2^{\varepsilon-1})w(u))\,d{G'}^j_n(u)\,dt\\ &\le \gamma_j(\varepsilon)C_p\left(1+(1\vee 2^{\varepsilon-1})\frac{C_w}{\underline{q}}\right)\sup_{v\ge 0}\mathbb{E}\left(\frac{w(v)}{w(\underline{v}_j)}\left|\overline{v}_j - \underline{v}_j\right|\right)\\ &\le \gamma_j(\varepsilon)C_p\left(1+(1\vee 2^{\varepsilon-1})\frac{C_w}{\underline{q}}\right)\left|c_j - \frac{c'_j}{\beta_j}\right|\mathbb{E}\left[\left(1+\alpha\overline{v}^\varepsilon + (\xi^j)^\varepsilon\right)\int_0^{\beta_j{\theta'}^j}\exp\left(U^j_{\beta_j{\theta'}^j} - U^j_s\right)ds\right].\end{aligned}$$

In the last inequality we have used relations (7.90). This completes the proof with

$$h_{cj} := \gamma_j(\varepsilon)C_p \left(1 + (1 \vee 2^{\varepsilon-1})\frac{C_w}{\underline{q}}\right) \mathbb{E}\left[\left(1 + \alpha\overline{v}^{\varepsilon} + \exp\left(-\varepsilon U^j_{\beta_j\theta'^j}\right)\right)\int_0^{\beta_j\theta'^j} \exp\left(U^j_{\beta_j\theta'^j} - U^j_s\right) ds\right]. \tag{7.108}$$

□

Lemma 7.25. *For any $a, a' \in \mathbb{A}$,*

$$B^{22}_{ji} \leq h^*_{ci}\left|c_i - \frac{c'_i}{\beta_i}\right|,$$

where the constant h^*_{ci} is given in (7.112).

Proof. Similarly to the proof of the previous lemma, we have

$$B^{22}_{ji} = \sup_{v\geq 0} w(v)\mathbf{P}_s\left(\underline{v}_i < V'_n \leq \overline{v}_i \,|\, I'_n = j\right),$$

where the r.v.'s $\underline{v}_i$ and $\overline{v}_i$ are defined in (7.106) and (7.107), respectively (with i instead of j).

Next we will prove that for $v > 0$ there exists a density $\frac{dG'^i_n(v)}{dv}$ s.t. for all $n \geq 0$,

$$\sup_{v\geq 0} w(v)\frac{dG'^i_n(v)}{dv} \leq C_p, \tag{7.109}$$

for some constant $C_p < \infty$ and for all $i \in \mathbb{E}$. Relation (7.109) holds for $n = 0, 1$. Assume, it holds for $n \leq k$ $(k \geq 1)$.

$$\begin{aligned}
w(v)\frac{dG'^i_{k+1}(v)}{dv} &= w(v)\sum_j \mathbf{P}_s(I_k = j \,|\, I_{k+1} = i)\,\frac{d\,\mathbf{P}_s\left(\xi^i(V_k + \eta^{ji}) \leq v \,|\, I_k = j\right)}{dv} \\
&= w(v)\sum_j \mathbf{P}_s(I_k = j \,|\, I_{k+1} = i)\left[\frac{d\,\mathbf{P}_s\left(V_k \leq (\xi^i)^{-1}v - \eta^{ji} \,|\, I_k = j\right)}{dv}\right] \\
&= w(v)\sum_j \mathbf{P}_s(I_k = j \,|\, I_{k+1} = i)\,\mathbb{E}_s\left[\frac{d\,G^j_k((\xi^i)^{-1}v - \eta^{ji}))}{dv}\right] \\
&\leq C_p\sum_j \mathbf{P}_s(I_k = j \,|\, I_{k+1} = i)\,\mathbb{E}_s\left[\frac{w(v)(\xi^i)^{-1}}{w((\xi^i)^{-1}v - Z^i)}\right].
\end{aligned} \tag{7.110}$$

Note that

$$\mathbf{P}_s(V_k \leq (\xi^i)^{-1}v - \eta^{ji} \,|\, I_k = j) = \mathbf{P}_s(V_k \leq (\xi^i)^{-1}v - \eta^{ji},\, (\xi^i)^{-1}v - \eta^{ji} > 0 \,|\, I_k = j),$$

which proves that $G'^i_{k+1}(v)$ is differentiable. The expectation on the right-hand side of (7.110) is bounded uniformly in v, and the expression under the expectation converges to $(\xi^i)^{\varepsilon-1}$, when $v \to \infty$. Thus, by

Lebesque's dominated convergence theorem and Assumption 7.7, there exists constant $v_1 < \infty$ s.t. for all $i \in \mathbb{E}$ and $v \geq v_1$,

$$\mathbb{E}_s\left[\frac{w(v)(\xi^i)^{-1}}{w((\xi^i)^{-1}v - Z^i)}\right] \leq \frac{\rho_2+1}{2}. \tag{7.111}$$

Further we require that $\overline{v} \geq v_1$ (recall the construction of the weight function $w(v) = 1 + (\overline{v} \vee v)^\varepsilon$ from Lemma 7.78). For $v \leq \overline{v}$, we have

$$\mathbb{E}\left[\frac{w(v)(\xi^i)^{-1}}{w((\xi^i)^{-1}v - Z^i)}\right] \leq \mathbb{E}(\xi^i)^{-1} \leq 1.$$

Thus, the sum in (7.110) does not exceed one and $\sup_{v\geq 0} w(v)\frac{dG'^i_{k+1}(v)}{dv} \leq C_p$.

Now, using (7.109), we proceed similarly as in the previous lemma,

$$\begin{aligned}
B^{22}_{ji} &= \sup_{v\geq 0} w_i(v)\mathbb{E}_s \int_{\underline{v}_i}^{\overline{v}_i} dG'^j_n(u) \\
&\leq \gamma_i(\varepsilon)C_p \sup_{v\geq 0} w(v)\mathbb{E}\int_{\underline{v}_i}^{\overline{v}_i} \frac{1}{w(u)}\,du \\
&\leq \gamma_i(\varepsilon)C_p \sup_{v\geq 0} \mathbb{E}\left(\frac{w(v)}{w(\underline{v}_i)}\,|\overline{v}_i - \underline{v}_i|\right) \\
&\leq \gamma_i(\varepsilon)C_p \left|c_j - \frac{c'_i}{\beta_i}\right| \mathbb{E}\left[\left(1 + \alpha\overline{v}^\varepsilon + (\xi^i)^\varepsilon\right)\int_0^{\beta_i\theta'^i} \exp\left(U^i_{\beta_i\theta'^i} - U^i_s\right)\,ds\right],
\end{aligned}$$

proving the lemma with

$$h^*_{ci} := C_p\,\mathbb{E}\left[\left(1 + \alpha\overline{v}^\varepsilon + \exp\left(-\varepsilon U^i_{\beta_i\theta'^i}\right)\right)\int_0^{\beta_i\theta'^i} \exp\left(U^i_{\beta_i\theta'^i} - U^i_s\right)\,ds\right]. \tag{7.112}$$

□

Remark 7.26. Conditions $\mathbb{E}(\xi^i)^{\varepsilon-1} \leq \rho_2$ and $\mathbb{E}(\xi^i)^{-1} \leq 1$ in Assumption 7.7 are only used in the proof of Lemma 7.25. Hopefully, these technical conditions can be relaxed.

Combining Lemmas 7.21–7.25 and using that $h_{ci} \geq h^*_{ci}$ (see (7.108) and (7.112)), we obtain

$$B_j \leq h_Z|F_{Z^j} - F_{Z'^j}|_w + h_{\lambda j}\left|\lambda_i - \frac{\lambda'_i}{\beta_i}\right| + \max_i h_{ci}\left|c_i - \frac{c'_i}{\beta_i}\right|, \tag{7.113}$$

where the constants $h_Z, h_{\lambda j}$ and h_{ci} are given in (7.93), (7.105) and (7.108), respectively.

It remains to bound D_{ji}. By Markov's inequality and using that $\mathbb{E}(\xi'^i)^\varepsilon \leq 1/\underline{q}$, we have

$$\begin{aligned}
D_{ji} &\leq \mathbb{E}_s\left[w(f(V'_n, \xi'^i, \eta'^{ji}))\,|\,I'_n = j\right] \\
&\leq \mathbb{E}_s\left[w\left(\xi'^i(V'_n + Z'^i)\right)\,|\,I'_n = j\right] \\
&\leq 1 + \frac{\alpha(1 \vee 2^{\varepsilon-1})}{\underline{q}}\left(\frac{C_V}{\underline{q}} + C_Z\right) =: C_q.
\end{aligned} \tag{7.114}$$

Collecting the bounds for A_j, B_j, D_{ji} from Lemma 7.20, (7.113) and (7.114), respectively, and using relation (7.28) we conclude from (7.27) that **(C2)** holds with

$$\kappa = \max\left\{\frac{1+\rho_1}{2}, 1 - q_{2\overline{v}}\left(1 - \frac{w(\overline{v})}{w(2\overline{v})}\right)\right\}, \tag{7.115}$$

$$d(a,a') = h_Z \max_j |F_{Z^j} - F_{Z'^j}|_w + \max_j h_{\lambda j}\left|\lambda_i - \frac{\lambda_i'}{\beta_i}\right| + \max_j h_{cj}\left|c_i - \frac{c_i'}{\beta_i}\right| + \frac{C_q h_q}{\underline{q}}\Delta_q, \tag{7.116}$$

where the constants $h_Z, h_{\lambda j}, h_{cj}$ and C_q are given in (7.93), (7.105), (7.108) and (7.114), respectively, and $h_q = 1 + m + m\underline{q}$.

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