On the stability of spherically symmetric self-gravitating classical and quantum systems

PhD thesis

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Abstract

We study the energetic stability of spherically symmetric self-gravitating systems beginning with an extensive review of the literature on perfect fluid bodies in Newtonian gravity with a particular focus on existence and uniqueness results for solutions of the Chandrasekhar equation. Moving on to the description of the corresponding systems in the setting of general relativity, it is shown, that the Tolman-Oppenheimer-Volkoff equation can be obtained from a suitable variation of the total energy. We prove a previously unnoticed energetic instability of the model. Staying in the general relativistic setting, we examine the self-gravitating massive free scalar field. It is shown, by proving suitable differentiability properties of the occurring functionals, that Einstein's equations in this setting can again be obtained by a constrained variation of the total mass as defined by Arnowitt, Deser and Misner. As for the perfect fluid, we prove energetic instability and conclude our investigations by constructing a naive quantum version of the free massive scalar field, that also suffers energetic instability.

Resumé

I denne afhandling undersøger vi stabiliteten af sfærisk symmetriske selvtiltrækkende systemer. Vi begynder med en omfangsrig oversigt over den eksisterende litteratur om selv-tiltrækkende perfekte væske i Newtons mekanik, med særlig hensyn til eksistens- og entydighedsresultater om løsninger til Chandrasekhar ligningen. Vi fortsætter med beskrivelsen af de tilsvarende systemer i den generelle relativitetsteori. Vi forklarer, hvordan Tolman-Oppenheimer-Volkoff ligningen kan afledes fra en variation af den totale energi og viser, at systemet er ustabil. Derefter studerer vi en massiv fri skalar felt, og viser at man kan igen får Einsteins feltligninger, idet man varier massen som defineret af Arnowitt, Deser, and Misner. Vi afslutter ved at konstruere en naiv kvanteversion af den frie skalare feltteori, som ligeledes er ustabil.

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Contents

1. Introduction

In the first part of this introduction we establish the notion of stability, that is going to be central in the following discussions. Also, we explain the concept of an equation of state, a crucial notion in the description of perfect fluids. The remaining part is an outlook on the chapters to come and concludes with some perspectives on future questions of interest related to the findings of this work.

1.1. Stability

Our main concern in this thesis are questions of energetic stability. Provided one can obtain the equations describing a physical system as the ones describing the stationary points of a suitable variation of the total energy, possibly with constraints, the notion of energetic stability is defined as follows:

A model of a physical system is said to be energetically stable if the total energy is bounded from below.

A prototypical quantum mechanical model with states described by some subset of the unit ball in $L^2\left(\mathbb{R}^d\right)$ with energy functional $\mathcal{E}$ is said to be stable, if $\mathcal{E}_0 = \inf\left\{\mathcal{E}(\psi) : \|\psi\|_2 = 1\right\}$ is finite. This is also referred to as stability of the first kind.

One can illustrate that this is a physically sensible notion of stability by the following thought: Imagine, that the energy of a model is unbounded from below, as an example one could consider an atom with point nucleus described in classical electromagnetism. Then on could extract an infinite amount of energy from this given system, even though this would be fantastic for solving humankind's demand for a sustainable energy source, infinite energy sources are violating the first law of thermodynamics and are thus generally considered unphysical[1].

In the course of this thesis the energy functionals are usually a priori bounded from below, they are non-negative. In this case, for the notion of energetic stability not to be trivial, one adapts it as follows:

Definition 0: A model with a non-negative energy functional $\mathcal{E}$ is energetically stable if $\mathcal{E}$ is positively bounded from below.

In a relativistic setting this states, that the system cannot evaporate into nothing.

Turning the definition of energetic stability around, the non-existence of a bound as above on the energy functional implies, that solutions to the variational equations can at most be local minima and the model can therefore be regarded unstable.

[1]Which also seems well supported by the fact that no perpetuum mobile has been found, albeit the great efforts made in the search.

1.2. Equations of state

Sometimes the description of a physical system in terms of its fundamental building blocks is not feasible, for example because the system is too big rendering the description impossible as often the case in many-body physics. At other times it might simply not be necessary because the detailed microscopic behavior is not of interest. In such cases one can often use statistical methods to model the macroscopic properties of the system, either to be able to learn anything about the system at all, or because those might be of interest in the first place.

These macroscopic properties are also referred to as thermodynamical ones and include for example volume, pressure, entropy, and various types of energies.

A relation referred to as an equation of state for a given thermodynamical system relates the thermodynamical properties necessary to completely characterize the macroscopic state of the system[2].

For a classical ideal gas with a fixed particle number n these are pressure p, the volume V and the temperature T. With R being the ideal gas constant the equation of state reads

$$pV = n\,\mathrm{R}\,T.$$

In our considerations of a perfect fluid subject to Newtonian gravity and the general relativistic fluid, the equations of state is a reminiscence to the respective underlying microscopic theory and will play a prominent role. The relevance of the equations of state is nicely illustrated by Chandrasekhar's celebrated article on ideal white dwarfs, self-gravitating completely degenerate Fermi gasses, cf. [8], in which the equation of state is the single place, where quantum mechanics enters the discussion and still one obtains a significant improvement of the purely classical results on the upper bound for the mass.

1.3. In Newtonian gravity

Models of self-gravitating systems in Newtonian gravity have been extensively studied in the past. Among the literature is the famous work [8] of Chandrasekhar, who computed a bound on the mass of stable ideal white dwarfs. Reviews on this and related topics are for example given in in the textbooks by Weinberg [49] and Landau, Lifshitz [28].

Apart from the vast astrophysical literature on the topic, there also exists an immense amount of more mathematically oriented work, out of which the results of the work [31] by Lieb and Yau are most relevant for our considerations.

They are considering self-gravitating fermionic, as well as bosonic many-body quantum systems and obtain remarkable results. First they prove, that the quantum mechanical model converges to the model considered by Chandrasekhar in a suitable

[2] In non-equilibrium thermodynamics one furthermore has to take into account the history of a system in terms of process dependent quantities such as work and heat.

limit and a similar result in the bosonic case. Second, energetic stability of the respective systems is proven for sub-critical particle numbers. To this avail the problem is formulated as a variation of the energy constrained to a fixed particle number. It is then shown, that for any sub-critical particle number there is a density minimizing the energy. In the fermionic case the minimizing density is furthermore compactly supported and unique up to translations.

The relevant equations in the purely classical treatment following the approach of Chandrasekhar are the so called Lane-Emden equations, cf. [49]. These are a special case in a well understood class of non-linear elliptic equations, a more detailed review on the results is given in section 2.2.1.

Our motivation to study the Newtonian case stem from our numerical analysis of a self-gravitating fluid subject to general relativity. There we used the solutions to the Newtonian equation as a plausibility test for the results in the more general setting.

To find the solutions to the Chandrasekhar equation, or more precisely suitable initial conditions for such, we used the following procedure: Transforming the system of first order equations

$$p'(r) = -Gr^{-2}m(r)\varrho(r), \text{ and}$$
$$m'(r) = 4\pi r^2\varrho(r)$$

describing the hydrostatic equilibrium of a perfect fluid with an equation of state given by $\varrho(p)$ into a second order equation, one obtains for some suitable function[3] $F : \mathbb{R}_+ \to \mathbb{R}_+$ and $Q(r) = rF(p(r))$ an equation

$$Q''(r) = -4\pi Gr\varrho\left(F^{-1}\left(r^{-1}Q(r)\right)\right).$$

Fixing Q to vanish with a negative derivative at some radius R there exists a unique solution on some interval $[r, R]$ by the existence and uniqueness theorems for regular ordinary differential equations, cf. [1].

The remaining problem is to find the second initial condition, i.e. a mass $M(R)$, such that the solution Q to the second order equation vanishes at the origin and is positive on $(0, R)$,

The derivative of Q at R is proportional to the mass. Our numerical analysis showed, that for large enough masses Q drops to 0 again close to R. Decreasing the mass moves this root towards the origin until it eventually reaches it for some mass $M(R)$. The curves in figure 1.1 show the parameter pairs $(R, M(R))$, obtained numerically via the above procedure for four exemplary equations of state. In section 2.2.2 the same strategy is used to reprove the existence of solutions Q to the second order equation above in an elementary way, for a small class of equations of state and the relevant points necessary for a possible generalization are pointed out. In addition the exact behavior the curves in the figure is proven in section 2.3.1 by a scaling argument that is significantly simpler than the common argument via the asymptotic behavior of expansions of the Lane-Emden functions – that is solutions to the Lane-Emden equations.

[3] $\mathbb{R}_+ = [0, \infty)$

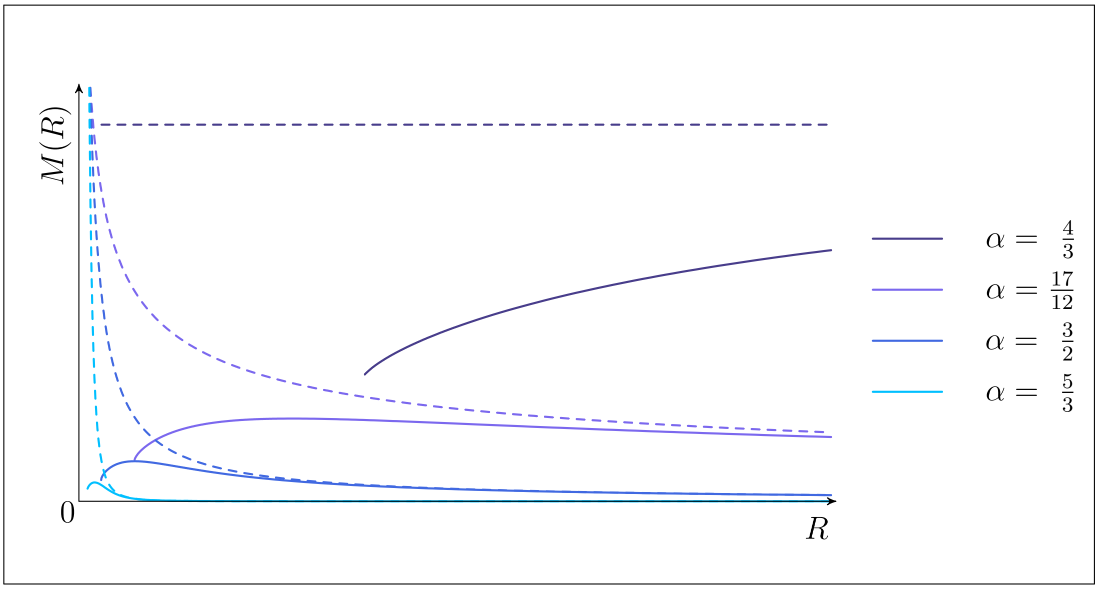

Figure 1.1.: The total mass $M(R)$ of solutions to the Chandrasekhar (dashed-lines) and TOV equation as a function of the radius R at which the solution vanishes. The respective equations of state are $\varrho(p) = p^\alpha$ in the first case and $\varrho(p) = p^\alpha + 3p$ in the latter.

It is an appealing aspect of these arguments, that the central density or alternatively the central pressure are eliminated as an initial condition.

1.4. In the general relativistic setting

Within the framework of general relativity we will consider two kinds of systems, starting with a perfect fluid and moving on to free scalar fields.

Perfect Fluid

The equation describing the stationary states of a spherically symmetric self-gravitating fluid with an equation of state $\varrho(p)$ is the Tolman-Oppenheimer-Volkoff (TOV) equation, see [44],[38], or [49], and is our object of interest in chapter 3. As in the case of Newtonian gravity it can be formulated as a differential equation for the pressure p:

$$p'(r) = -Gr^{-2}m(r)\varrho(r)\left(1+\frac{p(r)}{\varrho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{m(r)}\right)\left(1-\frac{2Gm(r)}{r}\right)^{-1} \quad \text{with}$$
$$m'(r) = 4\pi r^2 \varrho(r).$$

Comparing the equation to the one for the Newtonian setting given on the previous page, the right hand side has a number of additional factors. In the limit, where gravitational forces are weak, that is when the densities are small relative to the pressure, the above equation reduces to the Newtonian one.

Originally this equation was derived by considering Einstein's equations for a perfect fluid and a spherically symmetric metric, see section 3.1.1, but it can also be obtained from a variational principle. Varying the density, the equation is the condition for either the total mass being stationary for a fixed particle number or vice versa. This is proven in Weinberg's book [49] and in section 3.1.2 we present an extended version of the proof including the technical details to make the statement precise.

Existence of solutions to the TOV equation is proven for equations of state $\varrho(p)$ that are smooth non-negative monotonically increasing functions for $p > 0$ by Rendall and Schmidt in [42].

Another proof of existence and uniqueness of solutions to the TOV equation is given in [39] by Pfister. The fixed point arguments used there are based on those used in [43] for the Newtonian case and hold for equations of state $\varrho(p)$ that are non-decreasing, Lipschitz continuous, and satisfy $c_1\sqrt{p} \geq \varrho(p) > c_2 p$ for small p and some $c_1, c_2 > 0$.

Regarding variational formulations in the spirit of Lieb and Yau we prove the non-existence of a global minimizer to the total mass for any fixed particle number. Consequently any solution to the TOV equation can at most be a local minimum. One would expect, that the solutions can nevertheless be obtained as minimizers to a variational problem by adding an additional constraint, for example on the central pressure, but we have not been able to prove so.

Some of the results of a numerical analysis of the problem are shown in figure 1.1. To determine the initial conditions we used a procedure very similar to the one described above in the Newtonian setting, but looking at the root of the mass function $m(r)$. The equations of state used for the numerics are motivated by the following approximations: In the relativistic case, when the contribution of the momentum dominates the one by the mass, the equation of state approximately satisfies $\varrho(p) \propto p^{\frac{3}{4}}$. When the kinetic energy of the fermions in the fluid increases further, one refers to this as the ultra-relativistic regime, the equation of state becomes linear, i.e. $\varrho(p) = 3p$. A simplified model for an equation of state that takes into account both scenarios, is simply the sum of the two. Detailed derivations are given in chapter V in [28] or chapter 11 in [49], where in addition the physical processes regarding stellar models are described.

Note, that the solutions corresponding masses and radii such that the derivative of $M(R)$ with respect to R is positive are generally considered unstable, cf. page 321 in Weinberg. In the light of our non-existence result of a global minimum any solution can at most be a local minimum, though possibly with huge energy barriers. For an illustration of the instability by results of our numerical investigation see appendix A.3.

The classical complex scalar field

One might argue, that a classical complex scalar field by itself is not a reasonable model regarding physical application, but it is the classical counterpart to a quantized real scalar field , which is the prime example of a quantum field theory, and as such it bears great importance.

In chapter 4 we investigate the stability of a classical complex scalar field on a spherically symmetric asymptotically flat space-time. The relevant equations for this system are Einstein's equations, expressed in terms of the geometric Einstein-tensor $G_{\mu\nu}$ and the stress-energy tensor $\mathcal{T}_{\mu\nu}$ of the scalar field as

$$G_{\mu\nu} = 8\pi G \mathcal{T}_{\mu\nu}.$$

Numerically solutions to these have been described by Friedberg, Lee, and Pang [16] and by Lee, and Pang [29] and their existence was proven by Christodoulou in [10], and Bizoń and Wasserman in [6].

Concerning the stability, we first prove that Einstein's equations can be obtained as the equations describing the stationary points of the total mass functional, as defined by Arnowitt, Deser, and Misner (ADM) in [2], when varied with respect to the field for a fixed particle number. This formulation is equivalent to the description in terms of Hamiltonian mechanics.

Finally we show the non-existence of a global minimizer to the variational problem. Both of the above results also hold true for multipartite scalar fields.

The scalar quantum field

Based on our findings for the classical complex scalar field we are then investigating the quantized version in chapter 5. The scalar quantum field has been extensively studied and very well understood in various settings. Among those the one of "quantum field theory on curved space-times" seems most appropriate for our purposes.

In this approach one considers quantum field theories coupled to gravity, the latter described by the general theory of relativity. It is regarded as a semi-classical theory, as no attempt is made to quantize gravity. As such one would expect this theory to be a suitable limit of a full quantum theory of gravity, when gravitational effects are small. General introductions can for example be found in the books by Haag [22], Birrell and Davies [5], and Wald [48]. The necessary concepts for our purposes are described in section 5.1.

Within this Framework it is possible to treat the question of back-reaction, that is how the quantum field influences the geometry of space-time. In [15] the authors review the procedure how to treat the question of back-reaction in the cosmological setting, that is mostly Friedman-Lemaître-Robertson-Walker (FLRW) space-times, where the theory been successfully applied, see e.g. [11], and [23].

The equation ruling back-reaction effects is the so-called semi-classical Einstein equation,

$$G_{\mu\nu} = 8\pi G \omega(\hat{\mathcal{T}}_{\mu\nu}).$$

The left-hand side is given by the Einstein tensor and the right hand side is the expectation value of a suitable quantum stress-energy tensor operator in the state ω. To give meaning to this equation one has to overcome a number of conceptual difficulties, such as the problem of defining the quantum field theory under consideration on all space-times simultaneously. This can be resolved taking the “general local covariant” approach as introduced by Brunetti, Fredenhagen, and Verch in [7]. Furthermore one has to define a suitable quantum-stress energy tensor. This leads to questions of regularization, that can be overcome by choosing a suitable class of states, the Hadamard-states, cf. section 5.2.3 for some elaboration of this aspect.

In order to keep the connection to our results in the classical case, we diverge from the above procedure and take a rather naive approach allowing for a less restrictive class of states – we comment on the drawbacks in the end of the corresponding section 5.2.

In this setting we are able to prove, that one can construct for a massive scalar quantum field theory a manifold and a state with a fixed particle number, such that the ADM-mass is arbitrarily small and thus the system is unstable in the sense definition 0.

1.5. Future perspectives

In the light of the work done in the course of writing this thesis, there are a number of aspects that suggest themselves for further investigation.

In the case of the general relativistic fluid, a further study of the variational problem could shed some light on the details of the instability. Provided one proves, that the solutions are in fact local minima in the variation, it could be an interesting question to study the energy barriers around the minima to get a more detailed picture.

The most interesting open questions lie however in the quantum field theoretic setting. There the first step would be to depart from the analogy to the classical theory and treat the problem within the procedure reviewed for cosmological applications in [15]. This would in the first place involve an analysis of the physically suitable regularization procedure in the spherically symmetric setting. In comparison with the FLRW space-times, there is no time dependence, but in return one has more spatial degrees of freedom in the metric components. Regarding the questions concerning the states on could start by considering maximally symmetric Hadamard states, the expectation value of the regularized stress-energy tensor operator is shown to have the form of the classical stress-energy tensor of a perfect fluid by Olbermann in [37]. In a similar fashion one should investigate, whether spherically symmetric states on a space-time with the same symmetry also have this property. If this was the case, the solutions to the corresponding semiclassical Einstein equation could become accessible via the results on the TOV equation. The states of low energy introduced there and in particular their construction via a local minimization of the stress-energy expectation value, might also offer potential for a physically sound analysis of the question of energetic stability.

2. Newtonian fluid

In this chapter, we review a number of results on the spherically symmetric perfect fluid[1] subject to Newtonian gravity. In addition we give a novel elementary proof of the existence of solutions for a small class of equations states.

2.1. Derivation of the equilibrium equation[2]

The derivation of the equation describing a static state of a perfect fluid subject to Newtonian gravity, cf. for example Chandrasekhar [8], departs from the assumption, that the system under consideration is in thermodynamic equilibrium. This implies in particular the equality of the gravitational pressure and the pressure of the fluid.

One derives the gravitational pressure in the Newtonian setting by considering the force $\mathrm{d}F$ exerted on a spherically symmetric fluid shell of thickness thickness $\mathrm{d}r$ at radius r by the fluid enclosed by it. Denoting the spherically symmetric density by $\varrho(r)$, the mass $\mathrm{d}m$ of the shell is given by

$$\mathrm{d}m = 4\pi r^2 \varrho(r)\, \mathrm{d}r.$$

Based on this one introduces the following quantities associated to the fluid ball:

Definition 2.1: On the space of density functions

$$\tilde{\mathcal{D}} = \left\{ \varrho \in C(\mathbb{R}_+) \;\middle|\; \varrho \text{ non-negative and } \int_0^\infty r^2 \varrho(r)\, \mathrm{d}r < \infty \right\}$$

one defines the mass

$$M(\varrho) = 4\pi \int_0^\infty \mathrm{d}r\; r^2 \varrho(r), \tag{2.1}$$

and the mass up to the radius r

$$m(r) = 4\pi \int_0^r \mathrm{d}s\; s^2 \varrho(s). \tag{2.2}$$

[1] In the literature the systems considered here are usually referred to as "ideal gas spheres", but for the sake of homogeneity in our terminology over the different chapters we will use the synonymous name "perfect fluid spheres", which is the term customarily used in the general relativistic setting.

[2] The presented Version of the derivation follows [25] by Hainzl.

By Newton's law of gravity and denoting the gravitational constant by G the gravitational force on the fluid shell by the enclosed fluid is given by

$$\mathrm{d}F = -G\frac{m(r)}{r^2}\,\mathrm{d}m = -4\pi G m(r)\varrho(r)\,\mathrm{d}r.$$

The gravitational pressure p is defined to be the gravitational force per area exerted on a spherical shell by the enclosed fluid. Its differential thus satisfies

$$\mathrm{d}p = -Gr^{-2}m(r)\varrho(r)\,\mathrm{d}r.$$

Rewriting the above as a differential equation for the pressure one obtains the equation

$$p'(r) = -Gr^{-2}m(r)\varrho(r). \tag{C}$$

Following the nomenclature of Lieb and Yau in [31] we will refer to this equation as the Chandrasekhar equation. As described in the introduction, the microscopic behavior or the fluid is effectively modeled by an equation of state.

Definition 2.2: We define the space $\overline{\mathcal{D}}$ of equations of state as

$$\overline{\mathcal{D}} = \left\{\varrho : \mathbb{R}_+ \to \mathbb{R}_+ \;\middle|\; \varrho(0) = 0,\ \forall\, p \in \mathbb{R}_+ : \varrho(p) < \infty \text{ and } \int_0^p \frac{1}{\varrho(s)}\,\mathrm{d}s < \infty\right\}.$$

We call an equation of state polytropic, or a polytrope, if it has the form[3]

$$\varrho(p) = c_p p^\alpha$$

for $c_p > 0$ and $\alpha \in (0, 1)$.

Note that this space includes the physically relevant polytropic equations of state, i.e. $\alpha = \frac{3}{4}$ for a highly relativistic completely degenerate Fermi gas and $\alpha = \frac{3}{5}$ in the non-relativistic limit.

The Chandrasekhar equation (C) together with the mass $m(r)$ from definition 2.1 and an equation of state $\varrho(p)$ form the integro-differential system

$$\begin{aligned} &p'(r) = -Gr^{-2}m(r)\varrho(p(r)), \quad m(r) = 4\pi\int_0^r s^2\varrho(p(s))\,\mathrm{d}s \\ &p(R) = 0, \quad m(R) = M, \end{aligned} \tag{Cs}$$

describing the static states of a self-gravitating perfect fluid in the Newtonian setting.

[3] The constant c_p carries the physically relevant units, to ensure the agreement of the left- and right-hand side of the equation as physical quantities.

2.2. On the existence and uniqueness of solutions

In this section we review some existence and uniqueness results for solutions to the system (Cs). But before turning our attention to these, we will first reformulate the problem as one second order differential equation and then rewrite it once more to obtain a more convenient form of the equation.

Definition 2.3: Given an equation of state $\varrho(p) \in \overline{\mathcal{D}}$, define

$$F(p) = \int_0^p \frac{1}{\varrho(s)} \, \mathrm{d}s,$$

and set $P(r) = F(p(r))$. We will refer to P as the transformed pressure.

With the previous definition, we can write equation (C) as

$$r^2 P'(r) = -Gm(r).$$

As $F(p)$ is strictly monotonic by the non-negativity of ϱ, the inverse function F^{-1} exists. Taking another derivative and using equation (2.2) yields

$$(rP(r))'' = -4\pi G r \varrho \left(F^{-1}\big(P(r)\big)\right). \tag{2.3}$$

With initial conditions $P(R) = 0$ and $P'(R) = -GMR^{-2}$, the above equation is equivalent to (Cs). In the following we will denote $\varrho(F^{-1}(P))$ simply as $\varrho(P)$, whereas $\varrho(p)$ remains the original equation of state.

Example: Polytropic equations of state

Given a polytropic equation of state, i.e.

$$\varrho(p) = c_p p^\alpha,$$

one obtains $F(p) = c_p^{-1}(1-\alpha)^{-1}p^{1-\alpha}$ and $\varrho(P) = c_p^{1-\alpha}(1-\alpha)^\alpha P^{\frac{\alpha}{1-\alpha}}$. Consequently for $l = \alpha(1-\alpha)^{-1}$ equation (2.3) reads

$$(rP(r))'' = -4c_p^{1-\alpha}(1-\alpha)^\alpha \pi G r^{1-l}\big(rP(r)\big)^l \tag{LE}$$

in these cases. This equation is customarily referred to as the Lane-Emden equation.

2.2.1. Literature survey

In the following we will give an overview of some results that are relevant for our discussion. The first two theorems that will be quoted hold in a far more general setting, than the one we require for our purposes.

As most of the following statements are formulated in terms of the Laplacian Δ acting on functions from $\mathbb{R}^d$ to $\mathbb{R}$. , let us briefly comment on the relation of equation (2.3) to the elliptic equation

$$\Delta P = -f(P), \tag{2.4}$$

where f is some sufficiently regular function on $\mathbb{R}$.

In the three dimensions, the Laplacian in the spherically symmetric case of interest reduces to $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}$. Consequently equation (2.4) can then be written as

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}P}{\mathrm{d}r} = -f(P) \quad \Leftrightarrow \quad \frac{\mathrm{d}^2(rP)}{\mathrm{d}r^2} = -rf(P).$$

As our notation suggests, one obtains the differential equation (2.3) for the transformed pressure and with $f(P)$ playing the role of $\varrho(P)$.

The first result is due to Gidas, Ni, and Nirenberg, who prove the symmetry and monotonicity of solutions to equation (2.4) in a very general setting:

Theorem 2.4 (Theorem 1 in [19])**:** In the interior of the closed ball $B_R^n(0)$ of radius R in $\mathbb{R}^n$, and for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$, let P solution of equation (2.4) in $C^2(B_R(0))$ with initial conditions $P|_{r=R} = 0$ that is positive on the interior of $B_R(0)$.

Then P is rotationally symmetric and

$$\frac{\mathrm{d}P}{\mathrm{d}r} < 0 \quad \text{for} \quad 0 < r < R.$$

The existence of positive solutions to equation (2.4) on a ball is proven by de Figueiredo, Lions and Nussbaum for a wide range of functions f. The theorem presented here is a special case of the one proven in the reference [12]. As we are interested in rotationally symmetric solutions, we restrict the support of the solutions of interest to be a ball of radius R. The original theorem however holds for more general bounded, convex sets.

Theorem 2.5 (Theorem 2.1 in [12])**:** Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ fulfill the following conditions:

- The function f is locally Lipschitz continuous.
- For the first eigenvalue λ_1 of $-\Delta$ acting on $H_0^1\Big(\mathrm{int}\left(B_R^n(0)\right)\Big)$, f satisfies the inequalities

$$\liminf_{t\to+\infty} \frac{f(t)}{t} > \lambda_1,$$
$$\lim_{t\downarrow 0} \frac{f(t)}{t} < \lambda_1 \text{ and } f(0) = 0,$$

  where $\lambda_1 > 0$.

- For $\sigma = (n+2)(n-2)^{-1}$, $(\sigma < \infty,\ \text{if } n = 2)$

  $$\lim_{t\to+\infty} \frac{f(t)}{t^\sigma} = 0.$$

- With $F(t) = \int_0^t f(s)\,\mathrm{d}s$ it holds that

  $$\limsup_{t\to+\infty} \frac{tf(t) - \theta F(t)}{t^2 f(t)^{2/n}} \leq 0$$

  for some $0 \leq \theta \leq 2n(n-2)^{-1}$.

Then the equation (2.4) has at least one solution P, that is positive on the interior of $B_R^n(0)$.

In the previously cited article Gidas et al. furthermore prove the following lemma:

Lemma 2.6 (Lemma 2.3 in [19])**:** Let P_1 and P_2 be positive solutions of

$$\frac{\mathrm{d}^2 P}{\mathrm{d}r^2} + \frac{(n-1)}{r}\frac{\mathrm{d}P}{\mathrm{d}r} + P^l = 0, \quad 0 \leq r < R, \quad \left.\frac{\mathrm{d}P}{\mathrm{d}r}\right|_{r=0} = 0, \quad P(R) = 0. \tag{2.5}$$

Then

$$P_1(r) = \kappa^{2(l-1)^{-1}} P_2(\kappa r)$$

for $\kappa^{2(l-1)^{-1}} = P_1(0)/P_2(0)$.

This lemma implies the uniqueness of the positive solutions to equation (2.5), which in the 3 dimensional case equals the transformed Chandrasekhar equation for polytropic equations of state (LE) multiplied by an appropriate constant. The arguments are the following: Assume there exist two distinct solutions P_1 and P_2 to (2.5) vanishing at $r = R$ and assume without loss of generality that $P_2(0) > P_1(0)$ – if the two solutions were equal at the origin, they would be identical by lemma 2.6. One has

$$0 = P_1(R) = \kappa^{2(p-1)^{-1}} P_2(\kappa R) \quad \Rightarrow \quad P_2(\kappa R) = 0$$

by lemma 2.6. But as $\kappa < 1$ this contradicts the assumption that $P_2(r) > 0$ for $0 < r < R$. Hence the solution that is positive on $(0, R)$ and vanishes at $r = R$ is unique.

In a later work Ni [35] proves the uniqueness of radially symmetric solutions to equation (2.4) with Dirichlet boundary conditions on a ball in 3-dimensions for functions f that are not necessarily monomials:

Theorem 2.7 (Theorem 1.6 in [35])**:** For $n = 3$ consider equation (2.4) with Dirichlet boundary conditions for a ball. Let P be a solution to the equation, that is positive on the interior of the Ball, then P is unique in the class of all positive functions on the open ball provided f is rotationally symmetric, locally Lipschitz continuous, and satisfies

$$3tf'(t) > f(t) \geq tf'(t) \quad \text{for } t > 0.$$

In the polytropic case, that is the restriction of f in equation (2.4) to polytropes, Ni furthermore proves uniqueness for radial solutions to the equation for Dirichlet boundary conditions on an annulus $\mathcal{A} = \{x \in \mathbb{R}^n \mid 0 < R_1 \leq \|x\| \leq R_2 < \infty\} \subset \mathbb{R}^n,\ n \geq 2$:

Theorem 2.8 (Theorem 1.2 in [35])**:** Let P be a positive rotationally symmetric solution of

$$\Delta P = -P^l \quad \text{in } \mathcal{A}, \quad P|_{\partial\mathcal{A}} = 0.$$

Then P is unique in the class of all positive rotationally symmetric functions if

$$\begin{cases} 1 \leq l \leq \frac{n+2}{n-2} & \text{for } n \geq 3, \\ 1 \leq l < \infty & \text{for } n = 2. \end{cases}$$

In a subsequent work [36] by Ni together with Nussbaum these results are refined and generalized. Theorem 2.7 is generalized to dimensions greater than or equal to 3, cf. theorem 1.6 in the reference, while the statement of theorem 2.8 is proven to be true for $l > 1$ and dimension greater than or equal to 2.

A survey of the results for elliptic equations of the form (2.4) in more general scenarios can be found in [32].

An additional rather general treatment of existence and uniqueness of solutions to equation (2.4) in the physically relevant 3 dimensional case can be found in the work by Schaudt [43].

In a very different spirit Lieb and Yau prove the existence of solutions in a variational setting in [31], where the equations of state are derived from quantum mechanical considerations.

2.2.2. The uniqueness proof for solutions on an annulus

In the following we will elaborate the uniqueness proof for solutions on an annulus following the works of Ni [35] and Ni and Nussbaum [36].

A large part of the uniqueness proofs of positive solutions vanishing outside an annulus, in both of the aforementioned references, is based on comparison identities from Wronskians and we are thus required to assume a stricter regularity for the equation of state, namely $\varrho \in C^1(\mathbb{R}_+)$, from here on.

With $Q = rP(r)$ and $\Theta = \frac{\partial}{\partial M}Q(r)$ equation (2.3) reads

$$Q''(r) = -4\pi G r \varrho\left(r^{-1}Q(r)\right) \tag{2.6}$$

and the initial conditions are

$$Q(R) = 0, \quad Q'(R) = -\frac{GM}{R}. \tag{2.7}$$

A solution to equation (2.6) with initial conditions (2.7) satisfies

$$[Q'\Theta - Q\Theta']' = -4\pi G r \left(\varrho(P) - P\frac{\partial\varrho(P)}{\partial P}\right)\Theta, \tag{2.8}$$

$$[Q''\Theta - Q'\Theta']' = -4\pi G \left(\varrho(P) - P\frac{\partial\varrho(P)}{\partial P}\right)\Theta, \quad \text{and} \tag{2.9}$$

$$\begin{aligned}\left[(rQ' + bQ)'\,\Theta - (rQ' + bQ)\,\Theta'\right]' \\ = -4\pi G r \left(2\varrho(P) + (b+1)\left(\varrho(P) - P\frac{\partial\varrho(P)}{\partial P}\right)\right)\Theta\end{aligned} \tag{2.10}$$

for all $b \in \mathbb{R}$. Using those one proves the lemma below.

Lemma 2.9 (Ni, Nussbaum in [36])**:** Let $Q(r)$ be a solution to the equation 2.6 with initial conditions (2.7). If $Q(R)$ has a second root at $R_0 \in (0, R)$ for some $M_0 > 0$, then R_0 is a monotonically increasing function of M in a neighborhood of M_0 if for all $P > 0$

$$\varrho(P) < P\frac{\partial\varrho(P)}{\partial P} \le 3\varrho(P). \tag{2.11}$$

Proof. The proof of the lemma is based on the three comparison identities (2.8), (2.9), and (2.10), which will be applied in steps 1, 2, and 3 of the proof respectively.

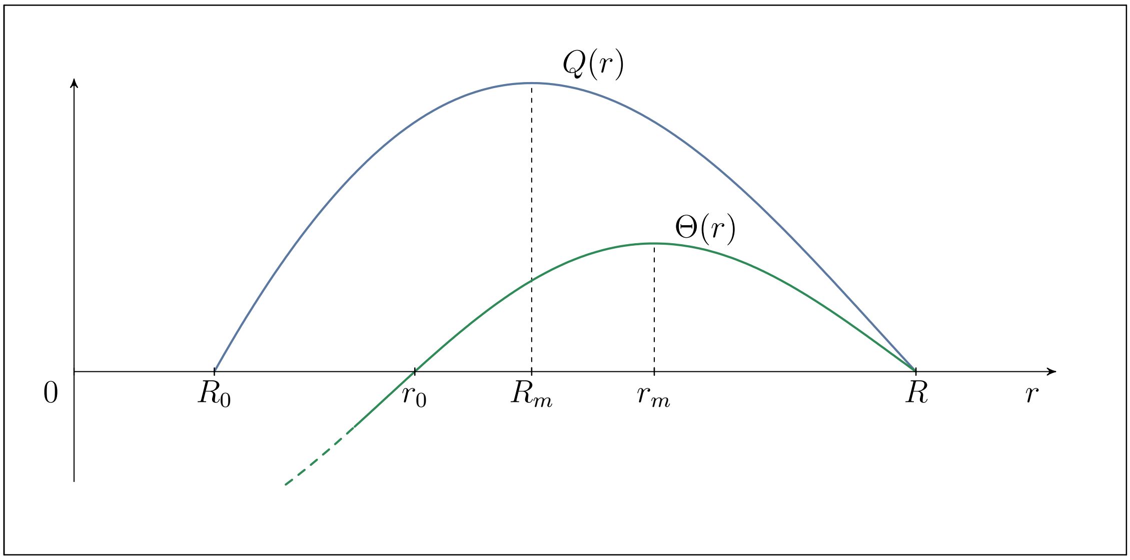

Figure 2.1.: Schematic drawing of the graphs of $Q(r)$ and $\Theta(r)$, to illustrate the nomenclature and the relations of the distinguished points. Note that we do not prove any details on the behaviour of $\Theta(r)$ for $R_0 \le r < r_0$ other than that negativity.

Step 0: We note that $Q(r)$ is a convex function whenever positive. Thereby it has a unique maximum at $R_m \in (R_0, R)$. The function $\Theta(r)$ is strictly convex whenever it is positive, as it satisfies

$$\Theta''(r) = -4\pi G \frac{\partial \varrho(P)}{\partial P} \Theta(r)$$

and $\frac{\partial \varrho(P)}{\partial P} > 0$ for $P > 0$. In addition the initial conditions give $\Theta(R) = 0$ and $\Theta'(R) = -GR^{-1} < 0$.

Step 1: We begin to prove $\Theta(R_0) < 0$ by showing that $\Theta(r)$ cannot be a positive function on $[R_0, R)$.

The integral of the first comparison identity from R_0 to R yields

$$-Q'(R_0)\Theta(R_0) = -4\pi G \int_{R_0}^{R} r \left(\varrho(P) - P \frac{\partial \varrho(P)}{\partial P} \right) \Theta \, \mathrm{d}r.$$

Since $\Theta(r)$ was assumed to be positive on (R_0, R), the right hand side is positive, the left hand side non-positive as $Q'(R_0) > 0$. Hence $\Theta(r)$ must have at least one root. Let $r_0 \in [R_0, R)$ be the largest of the roots of $\Theta(r)$. Then $\Theta(r)$ has a unique local maximum at $r_m \in (r_0, R)$ by convexity.

Integrating the first comparison identity from r_m to R we obtain

$$-Q'(r_m)\Theta(r_m) > 0 \quad \Longrightarrow Q'(r_m) < 0. \tag{2.12}$$

The above implies $r_m > R_m$.

Step 2: Then the integral of the second comparison identity (2.9) from r_0 to r_m yields

$$Q'(r_0)\Theta'(r_0) + Q''(r_m)\Theta(r_m) > 0.$$

Assuming $Q'(r_0) < 0$ contradicts the above and therefore $r_0 < R_m$.

Step 3: Now we assume, that $\Theta(r)$ has a second root r_1 in $[R_0, r_0]$ and integrate the final comparison identity (2.10) from r_1 to r_0. As a consequence of the upper bound on the equation of state, we can pick the constant b non-negative, such that the integral of the right hand side vanishes. The remaining equation reads

$$\big(r_1 Q'(r_1) + bQ(r_1)\big)\Theta'(r_1) = \big(r_0 Q'(r_0) + bQ(r_0)\big)\Theta'(r_0).$$

The parenthesis containing Q and Q' are positive on both sides by the previous arguments. As the $\Theta'(r)$ can not have the same sign at two consecutive roots of Θ, the second one, i.e. r_1, cannot exist and therefore $\Theta(R_0) < 0$.

Computing the derivative of $Q(R_0) = 0$ with respect to M, one finds

$$\frac{\mathrm{d}Q(R_0)}{\mathrm{d}M} = \frac{\partial Q(R_0)}{\partial M} + Q'(R_0)\frac{\mathrm{d}R_0}{\mathrm{d}M} = 0,$$

and as $Q'(R_0) > 0$, the above can be rewritten as

$$\frac{\mathrm{d}R_0}{\mathrm{d}M} = -\frac{1}{Q'(R_0)}\frac{\mathrm{d}Q(R_0)}{\mathrm{d}M} > 0.$$

This concludes the proof of the lemma. ■

It is noteworthy, that the above lemma also holds with the two roots of the solution reversed, i.e. the larger root is a monotonically decreasing function of the solutions derivative at the smaller root.

The lemma furthermore directly implies the following uniqueness results:

Corollary 2.10: Given an equation of state satisfying condition (2.11) a positive solution to equation (2.6) on an interval with Dirichlet boundary conditions is unique.

Corollary 2.11: Given an equation of state satisfying condition (2.11) and let P be a compactly supported solution of (2.3) that is regular at $r = 0$. Then P is unique.

Regarding the assumptions of the lemma 2.9 and Ni in [35] proves the following oscillatory behavior of the solutions

Theorem 2.12 (Theorem 2.2 in [35] restricted to three dimensions)**:** For $l \in (1, 3]$ let Q be a solution of equation (LE) with initial conditions

$$Q(R_0) = 0, \quad Q'(R_0) = K > 0.$$

Then there exists a finite $R > R_0$, such that $Q(R) = 0$ and $Q > 0$ on (R_0, R).

We will in the course of the following discussion obtain a similar result.

2.2.3. Results regarding the existence of solutions

In the coming discussion we will provide an elementary proof of the existence of solutions to the problem given by equations (2.6) and (2.7), that vanish at some positive $R_0 < R$ and are positive on (R, R_0).

We explain, how the above result can lead to solutions, that are positive on $(0, R)$ and have finite derivative at the origin. Solutions of this kind correspond to positive compactly supported solutions of the Chandrasekhar equation (C). The admissible transformed equations of state will be continuously differentiable and satisfy the conditions

(S1) $\varrho(0) = 0$, and ϱ is monotonically increasing[4],

(S2) $\varrho(P) < P\frac{\partial\varrho(P)}{\partial P} \leq (2-\varepsilon)\varrho(P)$ for $P > 0,\ 0 < \varepsilon < 1$,

(S3) $\lim\limits_{P\to\infty} \frac{\varrho(P)}{P^k} = c$ for some positive constants c and $k > 1$,

[4]The monotonicity condition here is obsolete, if (S2) is assumed, but we are not always going to assume (S1) and (S2).

(S4) $\lim\limits_{P\to 0} \frac{\varrho(P)}{P} = 0$.

As it turns out to be quite convenient for our purposes, we will rewrite the equation in an integral form.

Integral representation of the equations

To arrive at the desired equation, we integrate equation (2.6) with initial conditions (2.7). The first integration yields

$$Q'(r) = 4\pi G \int_r^R s\varrho\Big(s^{-1}Q(s)\Big)\,\mathrm{d}s - \frac{GM}{R}.$$

The second one gives

$$Q(r) = -4\pi G \int_r^R \int_{s'}^R s\varrho\Big(s^{-1}Q(s)\Big)\,\mathrm{d}s\,\mathrm{d}s' + GM\frac{R-r}{R}.$$

Then by an integration by parts we find the following integral equations equivalent to equation (2.6) with initial conditions (2.7):

$$Q(r) = GM\frac{R-r}{R} - 4\pi G \int_r^R (s-r)s\varrho\Big(s^{-1}Q(s)\Big)\,\mathrm{d}s. \tag{2.13}$$

The corresponding equation for the mass of the solution reads

$$m(r) = M - 4\pi \int_r^R s^2\varrho\Big(r^{-1}Q(s)\Big)\,\mathrm{d}s. \tag{2.14}$$

Example: polytropic equations of state

Setting $k = 4c_p^{1-\alpha}(1-\alpha)^\alpha \pi G$. The integral equation version of equation (LE) – replacing $rP(r)$ by $Q(r)$ – with initial conditions (2.7) reads

$$Q(r) = k\int_r^R (r-s)s^{-l+1}Q^l(s)\,\mathrm{d}s + \frac{GM(R-r)}{R}. \tag{2.15}$$

Bounds on local solutions

From the previously derived integral representation (2.13) of equation (2.6) with initial conditions (2.7), we can infer upper and lower bounds on the local solutions, which exist by the regularity of the equation away from the origin, cf. for example [1] for general results on ordinary differential equations. These bounds will be essential ingredients to our discussion.

The convexity implies that the tangent $q(r)$ to the solution at $r = R$, given by

$$q(r) = \frac{GM(R-r)}{R} \tag{2.16}$$

bounds $Q(r)$ from above on $[R_0, R]$.

Proposition 2.13: If the transformed equation satisfies conditions (S1) and (S3), there exists an $s < R$ such that for $r \in (s, R)$ it holds that $Q(r) > 0$ and in particular

$$Q(r) \geq \frac{GM(R-r)}{R}\left[1 - 4\pi R^2 \frac{(R-r)}{GM}\varrho\left(\frac{GM(R-r)}{Rr}\right)\right]. \tag{2.17}$$

Proof. The first thing to note is, that the integral term in the integral representation (2.13), is monotonic in the following sense:

Let f_1 and f_2 be two functions on $\mathbb{R}_+$ with $f_i(R) = 0$ and $f_i'(R) < 0$, such that $f_1 \geq f_2$ on some interval $[r, R]$. Then

$$\int_r^R (s-r)s\varrho\left({}^{-1}f_1(s)\right)\mathrm{d}s \geq \int_r^R (s-r)s\varrho\left(s^{-1}f_2(s)\right)\mathrm{d}s.$$

As $Q(r) \leq q(r)$ by previous arguments, the monotonicity of the integral term implies

$$Q(r) \geq GM\frac{(R-r)}{R} - 4\pi G\int_r^R (s-r)s\varrho\left(s^{-1}q(s)\right)\mathrm{d}s.$$

From this bound we can construct an upper bound s on the biggest root of $Q(r)$ inside $[0, R]$ as follows:

By the monotonicity of $q(r)$ we obtain

$$\begin{aligned}
Q(r) &\geq GM\frac{(R-r)}{R} - 4\pi G\int_r^R (R-r)R\varrho\left(r^{-1}q(r)\right)\mathrm{d}s, \\
&\geq GM\frac{(R-r)}{R} - 4\pi G(R-r)^2 R\varrho\left(r^{-1}q(r)\right) \\
&= GM\frac{(R-r)}{R}\left[1 - 4\pi\frac{(R-r)R}{M}\varrho\left(GM\frac{(R-r)}{rR}\right)\right]
\end{aligned}$$

By the assumptions on ϱ the right hand side of

$$1 = 4\pi\frac{(R-r)R}{M}\varrho\left(GM\frac{(R-r)}{rR}\right), \tag{2.18}$$

is a monotonically decreasing function of r. Furthermore it diverges for r going to zero and vanishing at R. Consequently there exists an $s \in (0, R]$ that solves (2.18). This concludes the proof of the lemma. ■

Proposition 2.14: Assume ϱ to satisfy conditions (S1) and (S3), and for $1 > \alpha > \frac{k-1}{k+1}$ set $r_\alpha = \frac{(GM)^\alpha R}{1+(GM)^\alpha}$. Then there exist $\tilde{M} > 0$ and $\delta_{\tilde{M}} < 1$ such that for all $M \geq \tilde{M}$

$$Q(r_\alpha) \geq (GM)^{1-\alpha}(1 - \delta_{\tilde{M}}) \equiv q_\alpha^M.$$

Proof. We choose $\tilde{M}$ large enough, such that

$$4\pi R^2 (GM)^{-\alpha-1} \varrho\left(\frac{(GM)^{(1-\alpha)}}{R}\right) < \frac{1}{2} \quad \text{for } M \geq \tilde{M}.$$

This is possible by our assumption on the asymptotic behavior of ϱ. For large enough masses the left-hand side of the above inequality behaves proportionally to

$$(GM)^{k(1-\alpha)-\alpha-1},$$

and by our choice of α the exponent is negative, hence the expression will eventually become arbitrarily small. Furthermore it will become monotonically decreasing, which is the second requirement on the size of $\tilde{M}$. Note, that the value of $\tilde{M}$ solely depends on the explicit given equation of state.

Given an $\tilde{M}$ large enough, such that both requirements are fulfilled, we set $\delta_{\tilde{M}}$ to be the left-hand side of the previous inequality. Inserting r_α, $\tilde{M}$ and $\delta_{\tilde{M}}$ into (2.17) finishes the proof. ■

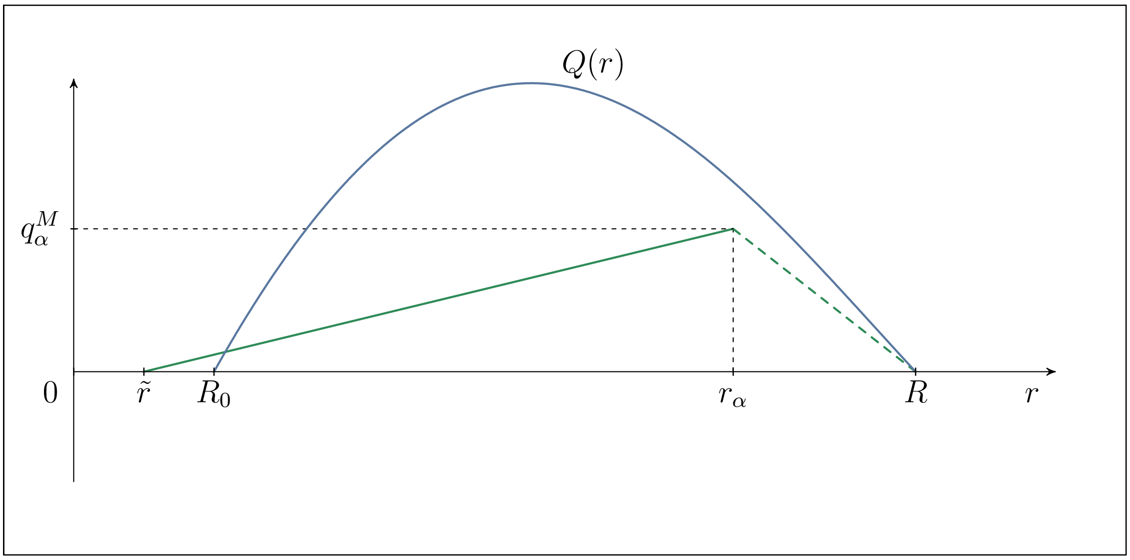

Figure 2.2.: Schematic drawing of the graph of $Q(r)$ and the candidate for a lower bound to illustrate the contradiction used in the proof of lemma 2.15.

We are now going to use the preceding propositions to prove the essential lemma, which will be used to prove the existence of solutions to the Chandrasekhar equation, starting with the advertised result on the existence of a further root of solutions to the transformed equation:

Lemma 2.15: Given a transformed equation of state satisfying (S1) and (S3), there exists a mass $M > 0$ for every $\tilde{r} \in (0, R)$, such that the solution $Q(r)$ to the equation (2.13) has a root in $[\tilde{r}, R)$.

Proof. We are going to prove the statement of the lemma by contradiction. To this avail we assume that $Q(r) > 0$ for all $r \in [\tilde{r}, R)$ and any $M > 0$. We fix $\tilde{r} \in [0, R]$ and choose M large enough, such that the following conditions are satisfied:

- Proposition 2.14 applies, i.e. $M > \tilde{M}$,
- $r_\alpha > \tilde{r}$,
- $q_\alpha^M = (GM)^{1-\alpha}(1 - \delta_{\tilde{M}}) > R$.

To begin with we bound $Q(\tilde{r})$ by restricting the range of the integration,

$$Q(\tilde{r}) \leq q(\tilde{r}) - 4\pi G \int_{\tilde{r}}^{r_\alpha} s(s - \tilde{r})\varrho\left(s^{-1}Q(s)\right) \mathrm{d}s$$

Then we use the positivity assumption, which implies via the convexity of $Q(\tilde{r})$, that $Q(\tilde{r})$ is bounded from below by the straight line connecting the points $(\tilde{r}, 0)$ and $(r_\alpha, q_<(r_\alpha))$, cf. figure 2.2:

$$Q(\tilde{r}) \leq q(\tilde{r}) - 4\pi G \int_{\tilde{r}}^{r_\alpha} s(s - \tilde{r})\varrho\left(\frac{q_\alpha^M}{s(r_\alpha - \tilde{r})}(s - \tilde{r})\right) \mathrm{d}s.$$

Increasing the lower integration limit to $r_* = \tilde{r} + \frac{1}{2}(r_\alpha - \tilde{r})$ yields

$$\begin{aligned} Q(\tilde{r}) &\leq q(\tilde{r}) - 4\pi G \int_{r_*}^{r_\alpha} s(s - \tilde{r})\varrho\left(\frac{q_\alpha^M}{s(r_\alpha - \tilde{r})}(s - \tilde{r})\right) \mathrm{d}s \\ &\leq q(\tilde{r}) - 4\pi G \int_{r_*}^{r_\alpha} r_*(r_* - \tilde{r})\varrho\left(\frac{q_\alpha^M}{R(r_\alpha - \tilde{r})}(r_* - \tilde{r})\right) \mathrm{d}s \\ &= q(\tilde{r}) - 4\pi G(r_\alpha - r_*)r_*(r_* - \tilde{r})\varrho\left(\frac{q_\alpha^M}{R(r_\alpha - \tilde{r})}(r_* - \tilde{r})\right) \\ &\leq q(\tilde{r})\left[1 - \frac{\pi G R(r_\alpha - \tilde{r})^2(r_\alpha + \tilde{r})}{2GM(R - \tilde{r})}\varrho\left(\frac{(GM)^{1-\alpha}(1 - \delta_{\tilde{M}})}{2R}\right)\right] \end{aligned}$$

For increasing M the negative summand inside the square brackets in the above inequality will for positive constants c_1 and c_2 behave as

$$-c_1(GM)^{-1}\varrho\left(c_2(GM)^{1-\alpha}\right).$$

With $\alpha = \frac{k-1+\varepsilon}{k+1}$ the exponent of GM asymptotically becomes

$$k(1-\alpha) - 1 = \frac{k-1}{k+1} - \varepsilon \frac{k}{k+1}$$

Choosing α such that $\varepsilon < \min\left\{\frac{k-1}{k}, \frac{2}{k+1}\right\}$ the exponent becomes positive and consequently the upper bound on $Q(\tilde{r})$ becomes negative. This contradicts the positivity assumption and proves, that there exists a mass, such that the solution has a root in $[\tilde{r}, R)$. ■

We restrict the transformed equations to those, for which the monotonicity of the biggest root in $(0, R]$ has previously been proven, cf. lemma 2.9.

Proposition 2.16: For a transformed equation of state satisfying (S1) a solution $Q(r)$ to equation (2.13), that has a root at $R_0 \in (0, R)$ and is positive on (R_0, R) has finite derivative at R_0.

Proof. As $Q(r) > 0$ for all $r \in (R_0, R)$, one can bound $Q'(R_0)$ by arguments analogous to those in the proof of the previous proposition to obtain:

$$Q'(R_0) \leq \frac{GM}{R}\left[4\pi \frac{R^3}{M} \varrho\left(\frac{GM}{R_0}\right) - 1\right] < \infty. \qquad ■$$

Proposition 2.17: Let ϱ be a transformed equation of state for which condition (S4) is satisfied and $Q(r)$ a solution to equation (2.13) that has a root R_0 in $(0, R)$. Then $Q'(R_0)$ is bounded from below by some positive constant P_0.

Proof. By proposition 2.16, $Q'(R_0)$ is finite. By convexity it holds for all $r \in [R_0, R]$,

$$Q(r) \leq Q'(R_0)\,(r - R_0) \leq Q'(R_0) r.$$

By the integral representation of the equation we have

$$\begin{aligned} Q'(R_0) &\leq 4\pi G \int_{R_0}^{R} s\varrho\left(Q'(R_0)\right) \mathrm{d}s - \frac{GM}{R} \\ &\leq 4\pi G R (R - R_0) \varrho\left(Q'(R_0)\right). \end{aligned}$$

We rewrite the above as

$$1 \leq 4\pi G R^2 \frac{\varrho\left(Q'(R_0)\right)}{Q'(R_0)}.$$

By the assumptions on ϱ this would be violated, if $Q'(R_0)$ was arbitrarily small. ■

On the existence of solutions to the Chandrasekhar equation

By a combination of the previously obtained bounds on the local solutions, we give an elementary proof of the existence of solutions to equation (2.13) that are positive on $(0, R)$ and vanish at the origin and at radius R.

The existence results holds for equations of state satisfying (S1)-(S4). To illustrate the role of condition (S2), which one should in principal be able to weaken, we split the statement into a lemma that holds, if (S1), (S3), and (S4) and an extra assumption (S2*) are satisfied and a proposition that (S2) implies (S2*).

Lemma 2.18: Let ϱ be a continuously differentiable transformed equation of state satisfying (S1),(S3), (S4), as well as

(S2*) There exists a finite $\tilde{P}_0 > 0$, such that for all masses smaller than some $\tilde{M}$ and the corresponding solutions to equation (2.13) it holds that $Q'(R_0) < \tilde{P}_0$ provided R_0 exists, such that $Q(R_0) = 0$ and $Q(r)$ positive on (R_0, R).

Then there exist a mass M and a corresponding solution $Q_M(r)$ to the equation (2.13) such that $Q_M(r) > 0$ on $(0, R)$ and $Q_M(0) = 0$.

Proof. For some large enough mass M_1 a solution to the equation has a root at $R_0 \in (0, R)$ by lemma 2.15. By the regularity of the ordinary differential equation (2.6) equivalent to the integral equation (2.13), we know that R_0 is a continuous function of the initial conditions, in particular the mass. Thus changing the mass will change R_0 continuously as long as $R_0 > 0$.

First, we are going to show, that there cannot exist a finite radius R_* such that $R_0 > R_*$ for all $M < M_1$. The second step will be to prove, that R_0 reaches the origin for a finite mass.

As it will be used in both steps, we recall that proposition 2.17 states, that there exists a fixed lower bound $P_0 > 0$ on $Q'(R_0)$. This bound is in particular independent of the mass.

Using the integral representation of the equations once more, but integrating from R_0 outwards, we have for $r \in [R_0, R]$:

$$Q(r) = Q'(R_0)(r - R_0) - 4\pi G \int_{R_0}^{r} s(r - s)\varrho\left(\frac{Q(s)}{s}\right) \mathrm{d}s. \tag{2.19}$$

Now, we assume the existence of $R_* \in (0, R)$, such that $R_0 > R_*$ for all $M < M_1$. Then using the tangent $q(r)$ to $Q(r)$ at $r = R$ as an upper bound on $Q(r)$,

$$Q(R) \geq P_0(R - R_0) - 4\pi G R(R - R_*)^2 \varrho\left(\frac{GM(R - R_*)}{RR_*}\right). \tag{2.20}$$

Decreasing the mass, the second summand on the right hand side becomes arbitrarily small. This results in a positive lower bound for $Q(R)$ contradicting $Q(R) = 0$.

Hence the first step is complete, as the existence of R_* as above is excluded. Thus R_0 eventually becomes arbitrarily small when decreasing the mass.

Finally, we assume, that $R_0 > 0$ for all $M > 0$. Then by inserting the upper bound by the tangent to $Q(r)$ at R_0 into equation (2.19) we find,

$$\begin{aligned} Q(r) &\geq P_0(r - R_0) - 4\pi G r (r - R_0)^2 \varrho\left(Q'(R_0)\right) \\ &\geq P_0(r - R_0)\left[1 - 4\pi G r^2 \frac{\varrho(Q'(R_0))}{P_0}\right] \\ &\geq P_0(r - R_0)\left[1 - 4\pi G r^2 \frac{\varrho(\tilde{P}_0)}{P_0}\right]. \end{aligned}$$

As P_0 and $\tilde{P}_0$ are independent of the mass when it is sufficiently small, we can for some M_2 small enough pick an $r \in (R_0, R)$, such that the right hand side of the last inequality becomes positive. We have therefore found a fixed lower bound to $Q(r)$ for all masses smaller than M_2.

But from the tangent $q(r)$ to $Q(r)$ at $r = R$ we have an upper bound, that decreases with the mass. Hence we can choose M small enough, such that this upper bound will be smaller than the fixed lower bound we just constructed. This contradiction proves the existence of an $M > 0$, such that for corresponding solution $Q_M(r)$ satisfies $Q_M(0) = 0$ and $Q(r) > 0$ for $r \in (0, R)$. ■

Corollary 2.19: Let ϱ be a transformed equation of state as in lemma 2.18, that in addition satisfies the inequality (2.11), then the solution $Q_M(r)$ to equation (2.13) obtained in lemma 2.18 is unique.

Proof. By the finiteness of $Q'_M(R_0)$ the solution $Q_M(r)$ has a linear upper bound by its tangent at the origin. Therefore $Q''(0) = 0$ by equation (2.6). The additional assumption implies the finiteness of all the integrals necessary to prove lemma 2.9. Consequently the statement of the lemma holds, i.e.

$$\frac{\mathrm{d}Q_M(0)}{\mathrm{d}M} < 0.$$

Thus $Q_M(r)$ is unique. ■

We are now going to show, that the assumptions lemma 2.18 can be satisfied for equations of state satisfying (S1)-(S4). The more general existence results suggest that one should also be able to satisfy them for a greater class of equations of state.

Proposition 2.20: Let ϱ be a transformed equation of state satisfying (S1)-(S4), then there exists a unique solution $Q(r)$ to equation (2.13) that is positive on $(0, R)$ with $Q(0) = Q(R) = 0$, i.e. (S2) implies (S2*).

Proof. By lemma 2.15 there exists a mass, such that the corresponding solution $Q(r)$ has a root at R_0. Computing the derivative of equation (2.13) at R_0, one finds

$$\begin{aligned} Q'(R_0) &= 4\pi G \int_{R_0}^{R} r\varrho\left(\frac{Q(r)}{r}\right) \mathrm{d}r - \frac{GM}{R} \\ &\leq 4\pi G \int_{R_0}^{R} r\varrho\left(\frac{GM}{r}\right) \mathrm{d}r. \end{aligned} \tag{2.21}$$

The conditions on ϱ imply the assumptions of lemma 2.9. Thus R_0 is a decreasing function of M. Furthermore the inequality (2.20) holds with R_* replaced by R_0. From that we can conclude, that R_0 goes to 0 faster than linear as a function of M, otherwise one would obtain a positive lower bound for $Q(R)$. Therefore we can choose $\tilde{M}$ such that $R_0 < G\tilde{M} < R$ and split the integral in inequality (2.21) as follows:

$$Q'(R_0) \leq 4\pi G \int_{R_0}^{G\tilde{M}} r\varrho\left(\frac{G\tilde{M}}{r}\right) \mathrm{d}r + 4\pi G \int_{G\tilde{M}}^{R} r\varrho\left(\frac{G\tilde{M}}{r}\right) \mathrm{d}r.$$

Condition (S2) implies

$$\begin{cases} \varrho(1)x > \varrho(x) \geq \varrho(1)x^{2-\varepsilon} & \text{for } 0 < x \leq 1, \\ \varrho(1)x < \varrho(x) \leq \varrho(1)x^{2-\varepsilon} & \text{for } x > 1. \end{cases}$$

With this we obtain

$$\begin{aligned} Q'(R_0) &\leq 4\pi G\varrho(1)\left(\int_{R_0}^{G\tilde{M}} r\left(\frac{G\tilde{M}}{r}\right)^{2-\varepsilon} \mathrm{d}r + G\tilde{M}\left(R - G\tilde{M}\right)\right) \\ &\leq 4\pi G\varrho(1)\left(\varepsilon^{-1}\left(G\tilde{M}\right)^{2-\varepsilon}\left(\left(G\tilde{M}\right)^{\varepsilon} - R_0^{\varepsilon}\right) + G\tilde{M}R\right) \\ &\leq 4\pi G\varrho(1)\left(\varepsilon^{-1}\left(G\tilde{M}\right)^{2} + G\tilde{M}R\right). \end{aligned}$$

Note, that this bound is a monotonically increasing function of $\tilde{M}$ and consequently a smaller bound of the same form holds for all $M < \tilde{M}$. With this bound, the equation of state satisfies the assumptions of lemma 2.18 and proposition 2.19. Thereby there exists a unique solution to equation (2.13) with the desired properties. ■

2.3. The relation between the mass and the radius

In this section we will go back to the polytropic equations of state[5] and establish the connection between the Mass and the Radius of the solutions to the Chandrasekhar equation (C). For the corresponding numerical results see figure 1.1 in the introduction.

2.3.1. The scaling behavior of the mass with the radius for polytropic equations of state

In the case of polytropic equations of state one can obtain an explicit scaling behavior of the mass as a function of the radius.

Lemma 2.21: Given a polytropic equation of state for $\alpha \in \left(\frac{1}{2}, \frac{5}{6}\right)$,

$$\varrho(p) = c_p p^\alpha,$$

it holds for the mass M of the solution with radius R of the Chandrasekhar equation (C) that[6]

$$M \propto R^{\frac{l-3}{l-1}}.$$

Proof. To begin with, note that the choices of the exponent α in the equation of state correspond to transformed equations of the form (LE) with exponents $l \in (1, 5]$. Therefore the existence and uniqueness results, cf. theorem 2.5 and page 2.8, hold.

Now, we consider the Chandrasekhar equation in the integral form (2.15). Changing to a dimensionless variable $x = \frac{r}{R}$ one obtains

$$Q(Rx) = GM_{R,l}(1-x) - kR^{3-l} \int_x^1 (y-x) y^{1-l} Q^l(Ry)\,\mathrm{d}y. \tag{2.22}$$

Next, let $\beta \in \mathbb{R}$ and multiply the above equation (2.22) by R^β. This yields an equation for the function $Q_\beta(x) = R^\beta Q(Rx)$:

$$Q_\beta(x) = GM_{R,l}R^\beta(1-x) - kR^{3-l-\beta(l-1)} \int_x^1 (y-x) y^{1-l} {Q_\beta}^l(y)\,\mathrm{d}y. \tag{2.23}$$

We set $\beta = (3-l)(l-1)^{-1}$. Then equation (2.23) coincides with (2.22) for the case $R = 1$ up to the constant in the first summand.

---

[5]Note, that these are not the transformed equations of state, that have been central in the previous section.

[6]Recall $l = \frac{\alpha}{1-\alpha}$.

By the uniqueness of the solutions of equation (2.22) there exists a unique mass $M_{1,l}$ for the solution that vanishes at $x = 0$, $x = 1$, and is positive in between. As this equation coincides with the rescaled equation (2.23) for the appropriate exponent, it follows that

$$M_{R,l} = M_{1,l} R^{\frac{l-3}{l-1}}.$$

This concludes the proof of the lemma. ■

It is a nice feature of the preceding discussion that one can compute the scaling of the mass with the radius without actually solving the equation and without reference to the central density $\varrho(p(0))$. A significantly more complicated way to obtain the scaling above, involving the Lane-Emden functions as well as the central density, is discussed in chapter 11 of [49].

3. General relativistic fluid

The model we are considering in this chapter, is again a self-gravitating fluid but this time in a general relativistic setting. Therefore the governing equations will be Einstein's equation and in particular the so called Tolman-Oppenheimer-Volkoff (TOV) equation.

3.1. TOV equation

This section is devoted to the (TOV) equation describing the static states of a self-gravitating perfect fluid, subject to general relativity, and in particular the different routes on can take in deriving it.

3.1.1. Conventional derivation

One of the possible ways, likely the most common one, to derive the TOV equation for spherically symmetric self-gravitating perfect fluids is to combine the components Einstein's equation, relating the geometric Einstein tensor $G_{\mu\nu}$ to the stress-energy tensor $T_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{EE}$$

Being interested in spherically symmetric fluid bodies we will describe the geometry by a spherically symmetric Lorentzian metric on $\mathbb{R}^{1+3}$. The corresponding line element is determined by two functions u and v, whose dependence will be specified , and reads[1]

$$\mathrm{d}s^2 = -e^{2u}\,\mathrm{d}t^2 + e^{2v}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}a^2 + \sin^2(a)\,\mathrm{d}\beta^2\right). \tag{3.1}$$

Within this chapter we are interested in a static metric and therefore we will assume u and v to be functions of the radial coordinate r only, i.e. $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$.

The stress-energy tensor $\mathcal{T}_f$ of a perfect fluid at rest in terms of the density ϱ and the pressure p, is given by its components

$$(\mathcal{T}_f)^t{}_t(r) = -\varrho(r), \quad (\mathcal{T}_f)^r{}_r = (\mathcal{T}_f)^a{}_a = (\mathcal{T}_f)^\beta{}_\beta = p(r). \tag{3.2}$$

[1] Our notation follows [29], while the derivation of the Tolman-Oppenheimer-Volkoff equation loosely follows the presentation in [42].
We will later on frequently refer to a corresponding manifold as a spherically symmetric space-time. Furthermore we call the t variable time and the r variable the radial coordinate or simply the radius.

Inserting the metric (3.1) and the stress-energy tensor (3.2) into Einstein's equations (EE)[2], , the non-trivial components are

$$G^t{}_t = r^{-2}\frac{\partial}{\partial r}r(e^{-2v}-1) = -8\pi G\varrho \tag{3.3}$$

$$G^r{}_r = r^{-2}\left(e^{-2v}-1+2e^{-2v}r\frac{\partial u}{\partial r}\right) = 8\pi G p \tag{3.4}$$

$$G^a{}_a = G^\beta{}_\beta = e^{-2v}\left[\frac{\partial^2 u}{\partial r^2}+\left(\frac{\partial u}{\partial r}+\frac{1}{r}\right)\left(\frac{\partial u}{\partial r}-\frac{\partial v}{\partial r}\right)\right] = 8\pi G p. \tag{3.5}$$

As a consequence of the Bianchi identities, the Einstein tensor satisfies $\nabla^\mu G_{\mu\nu}$. By the equations (3.3)- (3.5) this implies the covariant conservation of the stress-energy tensor[3],

$$\nabla^\mu(\mathcal{T}_f)_{\mu\nu} = 0. \tag{3.6}$$

This reduces to one (scalar) equation, as all but the $\nu = r$ component are trivially satisfied by the assumptions on the symmetry. The only non-trivial component of equation (3.6) reads

$$p'(r) = -u'(r)\left(p(r)+\varrho(r)\right). \tag{3.7}$$

Up to the specification of an equation of state, any three of the four equations (3.3)- (3.7) describe the self-gravitating fluid.

To obtain the TOV equation, one eliminates the geometric degrees of freedom u and v from the set of equations (3.3), (3.4), and (3.7).

One has to restrict the equation of state to solve the Einstein equations (3.3) and (3.4). Given such a restricted equation of state $\varrho \in \overline{\mathcal{D}}$, cf. definition 2.2, one solves equation (3.3) for $e^{-2v(r)}$ and finds

$$e^{-2v(r)} = 1 - \frac{8\pi G}{r}\int_0^r s^2\varrho(p(s))\,\mathrm{d}s, \tag{3.8}$$

as well as equation (3.7) for $u(r)$ to get

$$u(r) = \int_0^r \frac{p'(s)}{p(s)+\varrho(p(s))}\,\mathrm{d}s = -\int_{p(0)}^{p(r)} (p+\varrho(p))^{-1}\,\mathrm{d}p. \tag{3.9}$$

Up to this point, one has solved two of three equations necessary to describe the static self-gravitating fluid. As the final equation one chooses equation (3.4) and eliminates $v(r)$ and $u'(r)$ by the solution (3.8) and equation (3.7) to obtain an equation

[2]Cf. section A.1 for further details on the relevant geometric quantities.

[3]Repeated greek indices are to be understood as being summed over here and in the following.

for $p(r)$. Using the quantities from definition 2.1 these steps lead to the Tolman-Oppenheimer-Volkoff equation

$$p'(r) = -Gr\Big(p(r) + \varrho(r)\Big)\left(4\pi p(r) + \frac{m(r)}{r^3}\right)\left(1 - \frac{2Gm(r)}{r}\right)^{-1}. \tag{TOV}$$

Given a suitable equation of state, a solution to the integro-differential system consisting of the equations (TOV) and (2.1)[4] together with the solutions (3.8) and (3.9) completely describes the static self-gravitating fluid.

The connection to the Newtonian equation (C) can be made by writing the (TOV) equation as

$$p'(r) = -Gr^{-2}m(r)\varrho(r)\left(1 + \frac{p(r)}{\varrho(r)}\right)\left(1 + \frac{4\pi r^3 p(r)}{m(r)}\right)\left(1 - \frac{2Gm(r)}{r}\right)^{-1}.$$

The Newtonian equation can be regarded as the non-relativistic limit of the above expression.

3.1.2. Variational derivations

With regard to the question of stability of the solutions to the TOV equation it might be favorable to derive the above set of equations in a variational setting. To this avail we introduce the particle (or nucleon) number.

Definition 3.1: Set

$$\mathcal{D} = \left\{\varrho \in \tilde{D} \;\middle|\; \operatorname{supp}(\varrho) \text{ is compact, and } m(r) < \tfrac{r}{2G} \text{ for all } r > 0\right\}.$$

Let n be a non-negative increasing continuous function on $\mathbb{R}_+$, with $n(0) = 0$. Given a density $\varrho \in \mathcal{D}$, we obtain the number density by inserting ϱ into n. One defines the nucleon number

$$N(\varrho) = 4\pi \int_0^\infty \mathrm{d}r\; r^2 n(\varrho)\left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}}. \tag{3.10}$$

One of the possible ways to derive the (TOV) equation is to consider stationary points of the total mass defined as in (2.1) under variations $\delta\varrho$ of the density that leave the particle number (3.10) invariant. A second possibility is to consider the variational problem turned around, i.e. stationary points of the particle number with a fixed mass.

At this point one should make a comment on the (total) mass, $M(\varrho)$, as given in definition 2.1. In the general relativistic case under consideration the expression for

[4] When referring to solutions of the TOV equation later on, we will imply a solution of (TOV) with $m(r)$ defined by (2.14) as in the Newtonian case and (2.1) for some fixed M as the additional initial condition.

the total mass in terms of the density is identical to the one in the Newtonian case. They do however represent different quantities. In the Newtonian case formula (2.1) describes the actual mass of the fluid, it is the volume integral over the density.

In the present case however, the volume measure is given by $4\pi r^2 \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \mathrm{d}r$. Thus the expression in formula (2.1) is no longer the volume integral of the density, but same expression describes the total energy of the system, which does indeed include the total mass, but also takes into account the gravitational effects.

The following derivation is based on theorem 2 and the corresponding proof in chapter 11 of [49].

Proposition 3.2 (Theorem 2, page 306 in [49])**:** Given a continuously differentiable equation of state, $p(\varrho)$, and a twice continuously differentiable function n on $\mathbb{R}_+$ satisfying the condition of constant entropy per particle,

$$n'(\varrho) = \frac{n(\varrho)}{p(\varrho) + \varrho}, \quad \text{and } n(0) = 0, \tag{3.11}$$

the stationary points of the total Mass $M(\varrho)$ under variations $\delta\varrho$ of the density leaving the total particle number $N(\varrho) = N$ unchanged solve the (TOV) equation. The converse also holds, every solution to the (TOV) equation is a stationary point of the total mass with respect to the aforementioned variations of the density.

Proposition 3.3: Under the same assumptions as in proposition 3.2 the pressure $p(r) = p(\varrho)(r)$ satisfies the (TOV) equation if and only if ϱ is a stationary point of $N(\varrho)$ for a fixed mass $M(\varrho) = M$.

Before turning our attention to the proof of the propositions, we show the following technical statement:

Lemma 3.4: For $p(\varrho)$ and n satisfying the assumptions in propositions 3.2 and 3.3 the particle number is Fréchet differentiable on $\mathcal{D}$ equipped with the norm $\|\cdot\| = \int\limits_0^\infty r^2 \left|\cdot\right| \mathrm{d}r$ and the corresponding derivative reads[5]

$$\begin{aligned}\delta_\varrho N(\delta\varrho) = 4\pi \int\limits_0^\infty \mathrm{d}r \Bigg[& r^2 n'(\varrho) \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \delta\varrho \\ & + 4\pi G r n(\varrho) \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}} \int_0^r \mathrm{d}s\, s^2 \delta\varrho(s) \Bigg].\end{aligned} \tag{3.12}$$

Proof. We set

$$\Delta N(\varrho, \delta\varrho) = N(\varrho + \delta\varrho) - N(\varrho).$$

[5] In the calculation n and p are understood as functions of ϱ.

To prove the Fréchet differentiability of the particle number, one is required to show, that for $\|\delta\varrho\| \to 0$

$$\|\delta\varrho\|^{-1} \left|\Delta N(\varrho, \delta\varrho) - \delta_\varrho N(\delta\varrho)\right| \to 0, \tag{3.13}$$

and that $\delta_\varrho N(\delta\varrho)$ is bounded and linear in $\delta\varrho$.

The last condition is satisfied by the linearity of the integrations.

The boundedness requires some arguments in which we consider the two summands in (3.12) separately, starting with the first one.

The assumption $\varrho \in \mathcal{D}$ implies $0 < 1-\frac{2Gm(r)}{r} \leq 1$, as $\lim_{r\to 0} \frac{m(r)}{r} = 0$ and the expression $1 - \frac{2Gm(r)}{r}$ is monotonically increasing outside $\mathrm{supp}(\varrho)$. Thus it cannot get arbitrarily close to zero and consequently has a finite minimum.

Combining the preceding arguments with the continuity of n', and ϱ and the compact support of ϱ, we obtain the anticipated bound for the first summand in (3.12), that is for some positive $C \in \mathbb{R}$

$$4\pi \int_0^\infty \mathrm{d}r\; r^2 n'(\varrho) \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \delta\varrho < C \int_0^\infty \mathrm{d}r\; r^2 \left|\delta\varrho\right| = C \left\|\delta\varrho\right\|.$$

To bound the second summand in (3.12) we rewrite the integrand in the following way:

$$\left[G r^{-1} \left(1 - \frac{2Gm(r)}{r}\right)^{-1}\right] 4\pi r^2 n(\varrho) \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \int_0^r \mathrm{d}s\; s^2 \delta\varrho(s).$$

The integral factor in the integrand is bounded from above by $\|\delta\varrho\|$. To bound the remaining factors from above, we note, that as $m(r) < M$, there exists an R, such that for all $r > R$ it holds, that $r - 2Gm(r) > 1$, i.e. the factor enclosed in square brackets is bounded from above by G for all $r > R$.

Using the limiting behaviour of $r^{-1}m(r)$ for $r \to 0$ and taking into account the continuity of the remaining appearing functions the finiteness of the integral

$$\int_0^R r\, n(\varrho) \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}} \mathrm{d}r = C_2 < \infty \tag{3.14}$$

follows. Combining the preceding arguments we find

$$\begin{aligned}
&4\pi G \int_0^\infty \mathrm{d}r\, rn \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}} \int_0^r \mathrm{d}s\, s^2 \delta\varrho(s) \\
&\leq 4\pi G \left\|\delta\varrho\right\| \int_0^\infty \mathrm{d}r\, rn \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}} \\
&\leq \left(4\pi G C_2 + 4\pi G \int_R^\infty rn \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}}\right) \left\|\delta\varrho\right\| \\
&\leq \left(4\pi G C_2 + 4\pi G \int_R^\infty r^2 n \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}}\right) \left\|\delta\varrho\right\| \\
&\leq \left(4\pi G C_2 + N(\varrho)\right) \left\|\delta\varrho\right\| .
\end{aligned}$$

Thereby we have proven, that $\delta_\varrho N(\varrho, \delta\varrho)$ is linear and bounded in $\delta\varrho$.

To obtain the required limit (3.13), we will first compute two Taylor expansions with Lagrange remainder. For $x_0 \in (0,1)$ and some x_* in the open interval between x and x_0 the one of $(1-x)^{-\frac{1}{2}}$ around x_0 reads

$$(1-x)^{-\frac{1}{2}} = (1-x_0)^{-\frac{1}{2}} + \tfrac{1}{2}(1-x_0)^{-\frac{3}{2}}(x-x_0) + \tfrac{3}{8}(1-x_L)^{-\frac{5}{2}}(x-x_0)^2.$$

We write $\int_0^r s^2 \delta\varrho(s)\,\mathrm{d}r = \delta m(r)$. Then for $x_0 = \frac{2Gm(r)}{r}$, $x = \frac{2G(m(r)+\delta m(r))}{r}$, and some function $m_L(r)$ between $m(r)$ and $m(r) + \delta m(r)$ the above yields[6]

$$\begin{aligned}
\left(1 - \frac{2G\,(m+\delta m)}{r}\right)^{-\frac{1}{2}} = &\left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \\
&+ \frac{G}{r}\left(1 - \frac{2Gm}{r}\right)^{-\frac{3}{2}} \delta m \\
&+ \frac{3G^2}{2r^2}\left(1 - \frac{2Gm_L}{r}\right)^{-\frac{5}{2}} \delta m^2.
\end{aligned} \tag{3.15}$$

Using the assumption, that the number density satisfies equation (3.11) the expansion of $n(\varrho)$ around ϱ at $\varrho + \delta\varrho$ is for ϱ_L between ϱ and $\varrho + \delta\varrho$ given by

$$n(\varrho + \delta\varrho) = n(\varrho) + n'(\varrho)\delta\varrho + \tfrac{1}{2}n''(\varrho_L)\delta\varrho^2. \tag{3.16}$$

To make use of the above expansions, we introduce two non-negative continuous functions bounding m_L and ϱ_L from above:

$$\begin{aligned}
m_*(r) &:= \max\left\{m(r), m(r) + \delta m(r)\right\}, \\
\varrho_{**}(r) &:= \max\left\{\varrho(r), \varrho(r) + \delta\varrho(r)\right\}.
\end{aligned}$$

[6] In the following calculations, we drop the r dependence of $m(r)$, $\delta m(r)$, and $m_L(r)$.

The function ϱ_{**} is furthermore compactly supported and positive as ϱ and $\varrho + \delta\varrho$ are, while m_* satisfies $m_* < r$.

Using the two expansions (3.15) and (3.16) bounding m_L and ϱ_L from above by m_* and ϱ_{**} respectively, on gets the following estimate for $N(\varrho + \delta\varrho)$:

$$
\begin{aligned}
N(\varrho + \delta\varrho) \leq 4\pi \int_0^\infty \mathrm{d}r \Bigg[& r^2 n(\varrho) \left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \\
& + Grn(\varrho) \left(1 - \frac{2Gm}{r}\right)^{-\frac{3}{2}} \delta m \\
& + r^2 n'(\varrho) \left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \delta\varrho \\
& + \frac{3G^2}{2} n(\varrho) \left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}} \delta m^2 \\
& + Grn'(\varrho) \left(1 - \frac{2Gm}{r}\right)^{-\frac{3}{2}} \delta m \delta\varrho \\
& + \frac{r^2}{2} \left|n''(\varrho_{**})\right| \left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \delta\varrho^2 \\
& + \frac{3G^2}{2} n'(\varrho) \left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}} \delta\varrho \delta m^2 \\
& + \frac{Gr}{2} \left|n''(\varrho_{**})\right| \left(1 - \frac{2Gm}{r}\right)^{-\frac{3}{2}} \delta m \delta\varrho^2 \\
& + \frac{3G^2}{4} \left|n''(\varrho_{**})\right| \left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}} \delta\varrho^2 \delta m^2 \Bigg].
\end{aligned}
\tag{3.17}
$$

We note, that the first summand is equal to $N(\varrho)$, whereas the second and third add up to $\delta_\varrho N(\delta\varrho)$. Consequently the difference $\Delta N(\varrho, \delta\varrho) - \delta_\varrho N(\delta\varrho)$ is given by the remaining six summands.

We will proceed by showing appropriate bounds for the terms, that are of second order in the variations $\delta\varrho$. These are in particular prototypical for the higher order terms. To begin with we consider the term containing δm^2. As $|\delta m| \leq \|\delta\varrho\|$ the term is bounded from above by

$$
6G^2\pi \left\|\delta\varrho\right\|^2 \int_0^\infty \mathrm{d}r \; n(\varrho) \left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}}. \tag{3.18}
$$

The remaining integrand is compactly supported, because $n(0) = 0$ and $\mathrm{supp}(\varrho)$ is compact. Furthermore

$$
\left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}} \leq \max_{r \in \mathbb{R}} \left(1 - \frac{2Gm_*}{r}\right)^{-\frac{5}{2}} < \infty.
$$

The above maximum tends to the one with m_* replaced by m for the limit $\|\delta\varrho\| \to 0$, in which the whole term under consideration consequently vanishes.

Next, we consider the second order term containing both $\delta\varrho$ and δm. It is bounded from above by

$$G\,\|\delta\varrho\| \int_0^\infty \mathrm{d}r\; r n'(\varrho) \left(1 - \frac{2Gm}{r}\right)^{-\frac{3}{2}} |\delta\varrho| \tag{3.19}$$

We note, that the remaining integrand is compactly supported and continuous and vanishes for $\|\delta\varrho\| \to 0$, as in this case $\delta\varrho \to 0$ everywhere by continuity of $\delta\varrho$ itself.

The last second order term we bound by

$$\begin{aligned}
&\max_{r\in\mathbb{R}} \left(\frac{|n''(\varrho_{**})|}{2} \left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \delta\varrho \right) \int_0^\infty \mathrm{d}r\; r^2\, |\delta\varrho| \\
&= \max_{r\in\mathbb{R}} \left(\frac{|n''(\varrho_{**})|}{2} \left(1 - \frac{2Gm}{r}\right)^{-\frac{1}{2}} \delta\varrho \right) \|\delta\varrho\|\,.
\end{aligned} \tag{3.20}$$

The maximum exist, as is taken over a compactly supported continuous function, and goes to 0 for $\|\delta\varrho\| \to 0$ by continuity again.

The preceding arguments show, that the second order terms vanish in the limit $\|\delta\varrho\| \to 0$. By the same arguments used to obtain the necessary bounds, one can show the same for the third and fourth order terms. Hence we obtain the desired differentiability of $N(\varrho)$. ■

Having established the technical necessities, we continue by proving the propositions 3.2 and 3.3.

Proof of propositions 3.2 and 3.3. Using the method of Lagrange multipliers the statements of both propositions are proven, once the existence of a real constant Λ is shown, such that for all admissible variations $\delta\varrho$ the equation $\delta_\varrho M(\varrho) - \Lambda\delta_{\delta\varrho} N(\varrho) = 0$ holds.

$$\begin{aligned}
\delta_{\delta\varrho} N(\varrho) - \Lambda\delta_\varrho M(\varrho) = 4\pi \int_0^\infty \mathrm{d}r\; \Bigg[&- \Lambda r^2 \delta\varrho + r^2 n' \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} \delta\varrho \\
&+ 4\pi G r n \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{3}{2}} \int_0^r \mathrm{d}s\; s^2 \delta\varrho(s) \Bigg]
\end{aligned}$$

Using the compactness of the support of the density, we can integrate the last summand by parts and obtain

$$= 4\pi \int_0^\infty \mathrm{d}r\; r^2 \Bigg[-\Lambda + n' \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} + 4\pi G \int_r^\infty s\, n \left(1 - \frac{2Gm(s)}{s}\right)^{-\frac{3}{2}} \mathrm{d}s \Bigg] \delta\varrho$$

The condition for this expression to vanish for all compactly supported variations is

$$\Lambda = n' \left(1 - \frac{2Gm(r)}{r}\right)^{-\frac{1}{2}} + 4\pi G \int_r^\infty s\, n \left(1 - \frac{2Gm(s)}{s}\right)^{-\frac{3}{2}} \mathrm{d}s. \tag{3.21}$$

This in particular implies, that the right hand side of equation (3.21) is constant. Using the condition of constant entropy per particle (3.11) this is equivalent to the TOV-equation (TOV). ■

3.2. Results on the stability

In this section, we will prove the non-existence of a global minimizer of the mass for any fixed finite particle number.

Theorem 3.5: For any fixed $N_0 > 0$ it holds, that

$$\inf \left\{ M(\varrho) \;\middle|\; \varrho \in \mathcal{D},\; N(\varrho) = N_0 \right\} = 0.$$

With the same definitions as before, we note that$M$ and N are increasing functions of ϱ. If $\varrho_1 \leq \varrho_2$, then $M(\rho_1) \leq M(\rho_2)$ and $N(\rho_1) \leq N(\rho_2)$. Furthermore $M(0) = N(0) = 0$.

Proof. To begin with, we consider the family of non-continuous functions ϱ_ε on $\mathbb{R}_+$, which are for $\varepsilon \in (0,1)$ given by

$$\varrho_\varepsilon(r) = \begin{cases} \frac{1-\varepsilon}{8\pi G r^2} & \text{for } 0 < r \leq R, \\ 0 & \text{otherwise.} \end{cases}$$

For those one has

$$m(r) = \begin{cases} \frac{1-\varepsilon}{2G} r & \text{for } 0 \leq r \leq R, \\ \frac{1-\varepsilon}{2G} R & \text{otherwise,} \end{cases}$$

and under the additional assumption that $\lim_{r\to\infty} n(r) < \infty$,

$$N(\varrho_\varepsilon) = \frac{4\pi}{\sqrt{\varepsilon}} \int_0^R \mathrm{d}r\; r^2 n(\varrho_\varepsilon(r)).$$

Now we choose an arbitrary $M_0 > 0$ and set $R = R_\varepsilon = 2GM_0(1-\varepsilon)^{-\frac{1}{2}}$ accordingly. Then $M(\varrho_\varepsilon) = M_0$ for all $\varepsilon \in (0,1)$. By the continuity and the decay behavior assumptions on n we have

$$\lim_{\varepsilon\to 0} \sqrt{\varepsilon} N(\varrho_\varepsilon) = 4\pi \int_0^{2GM_0} \mathrm{d}r\; r^2 n(\varrho_0(r))$$

and consequently $N(\varrho_\varepsilon)$ diverges in the same limit and we can therefore choose an $\varepsilon > 0$ such that $N(\varrho_\varepsilon) > N_0$. As for a fixed density $\varrho \in \mathcal{D}$, $N(a\varrho) = N_\varrho(a) : [0,1] \to \mathbb{R}_+$ is a continuous increasing function vanishing for $a = 0$, there furthermore exists an $a \in (0,1)$ for which

$$N(a\varrho_\varepsilon) = N_0 \text{ and}$$
$$M(a\varrho_\varepsilon) \leq M(\varrho_\varepsilon) = M_0.$$

Since M_0 was chosen arbitrarily, this proves that

$$\inf\left\{M(a\varrho_\varepsilon)\;\middle|\; 0<\varepsilon<1,\; 0<a<1,\; N(a\varrho_\varepsilon) = N_0\right\} = 0.$$

It remains, to extend the arguments to continuous densities, which we will do in the following by a limiting argument.

Given $M_0 > 0$ and N_0, we choose a and ε, such that $M(a\varrho_\varepsilon) < M_0$ and $N(a\varrho_\varepsilon) = 2N_0$. For $K > 0$ we define

$$\varrho_\varepsilon^K(r) = \min\{\varrho_\varepsilon(r), K\}.$$

It follows, that $M(a\varrho_\varepsilon^K) < M_0$. In addition

$$N(a\varrho_\varepsilon^K) \xrightarrow{K\to\infty} 2N_0.$$

Thus we can choose a K for which $M(a\varrho_\varepsilon^k) < M_0$ and $N(a\varrho_\varepsilon^K) > N_0$.

The remaining discontinuity at $r = R_\varepsilon$ can be removed by defining

$$\varrho_\varepsilon^K = \begin{cases} \varrho_\varepsilon^K & \text{for } 0 < r \leq R_\varepsilon \\ \varrho_\varepsilon^K(R_\varepsilon)\delta^{-1}(R_\varepsilon + \delta - r) & \text{for } R_\varepsilon < r \leq R_\varepsilon + \delta \\ 0 & \text{otherwise.} \end{cases}$$

For $\delta > 0$ small enough $\varrho_\varepsilon^{K,\delta} \in \mathcal{D}$ and

$$N(a\varrho_\varepsilon^{K,\delta}) \geq N(a\varrho_\varepsilon^K) > N_0,$$
$$M(a\varrho_\varepsilon^{K,\delta}) \xrightarrow{\delta\to 0} M_0.$$

Given arbitrary $M_0 > 0$ and $N_0 > 0$, we can find ε, a, K and δ, such that

$$a\varrho_\varepsilon^{K,\delta} \in \mathcal{D}, \quad M(a\varrho_\varepsilon^{K,\delta}) < 2M_0, \quad \text{and } N(a\varrho_\varepsilon^{K,\delta}) > N_0.$$

An analogous scaling argument to the discontinuous case concludes the proof. ■

Recent results regarding different notions of stability, namely dynamic and thermodynamic stability, can be found in [20] and the references therein.

4. The classical complex scalar field

This chapter is devoted to the study of the stability of the self-gravitating classical complex scalar field in a general relativistic setting.

4.1. Lagrangian formalism

In the following section we will briefly review the Lagrangian formalism for scalar fields on spherically symmetric space-times and point out some features of this particular setting.

4.1.1. The Lagrangian

To obtain the gravitational action for a spherically symmetric Lorentzian metric on $\mathbb{R}^{1+3}$, which will again by described by the line element [1]

$$\mathrm{d}s^2 = -e^{2u}\,\mathrm{d}t^2 + e^{2v}\,\mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\alpha^2 + \sin^2\alpha\,\mathrm{d}\beta^2\right),$$

it is convenient to first consider a variational problem constrained to a subset homeomorphic to the direct product $\tilde{M} = I \times B_r^3$ of some closed interval I with a three dimensional closed ball B_r^3 of radius r. In this setting the usual Einstein-Hilbert action

$$S_{EH} = (16\pi G)^{-1}\int_{\tilde{M}} \mathrm{d}vol_{\tilde{M}}\ R$$

needs some modifications in order to be stationary for solutions of Einstein's equations and variations vanishing on the boundary but with possibly non-vanishing normal derivatives. To allow variations of this kind seems rather arbitrary in the Lagrangian setting, as considering only local variations gives a well defined theory without any modifications to the Lagrangian.

Without these variations, there arise however problems in the transition to the Hamiltonian formalism in the same setting as we shall see. In this case one would require some ad hoc modifications of the Hamiltonian in order to give meaningful variational equations, cf. [41].

In order to be able to make the transition in the canonical way and furthermore the equations of motion are unaffected, the changes are made already on the Lagrangian level here. Denote by $\mathfrak{n}$ the outwards pointing normal vector field on the boundary of $\tilde{M}$. Then an appropriate modification (cf. [18]) is the addition of the surface term

$$\int_{\partial\tilde{M}} \mathrm{d}vol_{\partial\tilde{M}} \nabla_\mu \mathfrak{n}^\mu.$$

[1] Cf. equation (3.1) for the first occurrence.

Finally, taking the Limit $r \to \infty$ and $I \to \mathbb{R}$ gives the action for the metric on the full space-time. The "modified" Einstein-Hilbert Lagrangian respectively reads

$$L_g = (2G)^{-1} \int_0^\infty \mathrm{d}r \, \left(e^{u+v} + e^{u-v} \left(2r \frac{\partial v}{\partial r} - 1 \right) + \frac{\partial}{\partial r} 2re^u \left(e^{-v} - 1 \right) \right). \tag{4.1}$$

The Lagrangian for a Klein-Gordon field on a spherically symmetric background is given by

$$L_m = \int_0^\infty \mathrm{d}r \, \mathcal{L}_m = 4\pi \int_0^\infty \mathrm{d}r \, r^2 e^{u+v} \left(-m^2 |\varphi|^2 - e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 + e^{-2u} \left| \frac{\partial \varphi}{\partial t} \right|^2 \right).$$

The total Lagrangian L is the sum of these two, $L = L_g + L_m$.

It is a remarkable feature of the spherical symmetric geometry, that the Lagrangian is independent of the time derivatives of u and v.

4.1.2. The Euler Lagrange equations

The Euler-Lagrange equation with respect to $\overline{\varphi}$,i.e. the Klein-Gordon equation, reads

$$\begin{aligned} 0 = e^{-2u} &\left[-\frac{\partial^2 \varphi}{\partial t^2} + \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right) \frac{\partial \varphi}{\partial t} \right] \\ &+ e^{-2v} \left[\frac{\partial^2 \varphi}{\partial r^2} + \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r} + \frac{2}{r} \right) \frac{\partial \varphi}{\partial r} \right] - m^2 \varphi. \end{aligned} \tag{4.2}$$

Variation with respect to φ gives the complex conjugate of the equation above.

Varying with respect to u one finds

$$r^{-2} \frac{\partial}{\partial r} r \left(e^{-2v} - 1 \right) = -8\pi G \left(e^{-2u} \left| \frac{\partial \varphi}{\partial t} \right|^2 + e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 + m^2 |\varphi|^2 \right). \tag{4.3}$$

The final Euler-Lagrange equation, i.e. the variational equation with respect to v, is

$$r^{-2} \left(e^{-2v} - 1 + 2re^{-2v} \frac{\partial u}{\partial r} \right) = 8\pi G \left(e^{-2u} \left| \frac{\partial \varphi}{\partial t} \right|^2 + e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 - m^2 |\varphi|^2 \right). \tag{4.4}$$

The last two equations will reappear as the constraint equations in the Hamiltonian formalism (cf. section 4.2.3). The only two non-zero canonical momenta are

$$\begin{aligned} \Pi &= 4\pi r^2 e^{v-u} \frac{\partial \overline{\varphi}}{\partial t} \\ \overline{\Pi} &= 4\pi r^2 e^{v-u} \frac{\partial \varphi}{\partial t} \end{aligned} \tag{4.5}$$

and consequently u as well as v are cyclic coordinates.

Einstein's equations

The stress-energy tensor density is defined as

$$\mathcal{T}^{\mu\nu} = -\frac{k}{8\pi\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}}.$$

For a generic Lorentzian metric $\tilde{g}_{\mu\nu}$ the Lagrangian density for a complex scalar field reads

$$\mathcal{L}_{KG} = -\sqrt{-\tilde{g}}\left[\tilde{g}_{\mu\nu}\left(\nabla^{\mu}\varphi\right)\nabla^{\nu}\overline{\varphi} + m^2\left|\varphi\right|^2\right].$$

Thus one obtains the following for the stress-energy tensor density:

$$\begin{aligned}\mathcal{T}^{\mu\nu} &= \frac{k}{8\pi\sqrt{-\tilde{g}}}\frac{\delta}{\delta\tilde{g}_{\mu\nu}}\sqrt{-\tilde{g}}\left[\tilde{g}_{\gamma\delta}\left(\nabla^{\gamma}\varphi\right)\nabla^{\delta}\overline{\varphi} + m^2\left|\varphi\right|^2\right] \\ &= -\frac{k}{16\pi}\tilde{g}^{\mu\nu}\left[\tilde{g}_{\gamma\delta}\left(\nabla^{\gamma}\varphi\right)\nabla^{\delta}\overline{\varphi} + m^2\left|\varphi\right|^2\right] \\ &\quad + \frac{k}{8\pi}\begin{cases}\left(\nabla^{\mu}\varphi\right)\nabla^{\nu}\overline{\varphi} & \text{if } \mu=\nu \\ \left(\nabla^{\mu}\varphi\right)\nabla^{\nu}\overline{\varphi} + \left(\nabla^{\nu}\varphi\right)\nabla^{\mu}\overline{\varphi} & \text{otherwise.}\end{cases}\end{aligned}$$

Setting $k = 16\pi$ (cf. [47] and [29]) and inserting the spherically symmetric metric from equation (3.1) one is left with the following non vanishing components:

$$\begin{aligned}T^{tt} &= e^{-2u}\left(m^2\left|\varphi\right|^2 + e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right), \\ T^{rr} &= e^{-2v}\left(-m^2\left|\varphi\right|^2 + e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right), \\ T^{\alpha\alpha} &= r^{-2}\left(-m^2\left|\varphi\right|^2 - e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right), \\ T^{\beta\beta} &= r^{-2}\sin^{-1}\alpha\left(-m^2\left|\varphi\right|^2 - e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right), \\ T^{tr} &= T^{rt} = e^{-2(u+v)}\left(\left(\frac{\partial\varphi}{\partial r}\right)\frac{\partial\overline{\varphi}}{\partial t} + \left(\frac{\partial\varphi}{\partial t}\right)\frac{\partial\overline{\varphi}}{\partial r}\right).\end{aligned}$$

Given these and the corresponding components of $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, cf. appendix A.1, we can write down Einstein's equations given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

The non-trivial equations read

$$
\begin{aligned}
G^t{}_t &= r^{-2}\frac{\partial}{\partial r} r(e^{-2v}-1) \\
&= -8\pi G\left(m^2|\varphi|^2 + e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right),
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
G^r{}_r &= r^{-2}\left(e^{-2v} - 1 + 2e^{-2v} r\frac{\partial u}{\partial r}\right) \\
&= 8\pi G\left(-m^2|\varphi|^2 + e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right),
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
G^\alpha{}_\alpha = G^\beta{}_\beta &= e^{-2u}\left[-\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right)\right] \\
&\quad + e^{-2v}\left[\frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial u}{\partial r} + \frac{1}{r}\right)\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial r}\right)\right] \\
&= 8\pi G\left(-m^2|\varphi|^2 - e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right),
\end{aligned}
\tag{4.8}
$$

$$
e^{2v}G^r{}_t = -e^{2u}G^t{}_r = 2r^{-1}\frac{\partial v}{\partial t} = 8\pi G\left(\left(\frac{\partial\varphi}{\partial r}\right)\frac{\partial\overline{\varphi}}{\partial t} + \left(\frac{\partial\varphi}{\partial t}\right)\frac{\partial\overline{\varphi}}{\partial r}\right).
\tag{4.9}
$$

Together with the Klein-Gordon equation $(\Box - m^2)\varphi = 0$, cf. equation (4.2) for the explicit form, these constitute the complete set of equations of motion for the scalar field coupled to the metric under consideration.

An immediate but noteworthy consequence of the equations above is the non-existence of ultra-static solutions, that is solutions u, v, φ to the equations with $u = 0$ everywhere, other than $u = 0$ and $v = 0$.

Theorem 4.1 (cf. [29])**:** Given a triple of regular solutions $(\tilde{\varphi}, \tilde{u}, \tilde{v})$ to the Euler-Lagrange equations for the Lagrangian

$$
\begin{aligned}
L = (2G)^{-1}&\int_0^\infty \mathrm{d}r\ \left(e^{u+v} + e^{u-v}\left(2r\frac{\partial v}{\partial r} - 1\right) + \frac{\partial}{\partial r}2re^u\left(e^{-v}-1\right)\right) \\
&+4\pi\int_0^\infty \mathrm{d}r\ r^2 e^{u+v}\left(-m^2|\varphi|^2 - e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 + e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2\right)
\end{aligned}
\tag{4.10}
$$

with respect to $\varphi, u,$ and v is equivalent to a triple of solutions to the full set of Einstein's equations

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}
$$

for the scalar field coupled to the spherically symmetric metric and the corresponding Klein-Gordon equation.

Proof. A direct comparison reveals of the equations (4.2)-(4.4) and (4.6)-(4.9) reveals, that the Euler-Lagrange equations are a subset of Einstein's equations and the Klein-Gordon equation.

To prove the theorem it is therefore sufficient to prove that a solution to the Euler-Lagrange equations solves the angular and off-diagonal Einstein equations. The essential ingredient for the proof is the conservation of energy $\nabla_\mu G^\mu{}_\nu = 0$ following from the Klein-Gordon equation.

Writing out $\nabla_\mu G^\mu{}_t = 0$ gives rise to the following identity for the components of $G^\mu{}_\nu$:

$$-r^2 \partial_t G^t{}_t = \partial_r \left(r^2 G^r{}_t \right).$$

By eq. (4.3), we can express the left hand side of this identity in terms of the field φ and its complex conjugate. For convenience define

$$W := \left(\frac{\partial \overline{\varphi}}{\partial t} \right) \frac{\partial \varphi}{\partial r} + \left(\frac{\partial \overline{\varphi}}{\partial r} \right) \frac{\partial \varphi}{\partial t}.$$

Using eq. (4.2), one obtains

$$\begin{aligned} \partial_t G^t{}_t = -8\pi G \Bigg[&- 2e^{-2u} \left(\frac{\partial v}{\partial t} \right) \left| \frac{\partial \varphi}{\partial t} \right|^2 + e^{-2v} \left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} + \frac{2}{r} \right) W \\ &- 2e^{-2v} \left(\frac{\partial v}{\partial t} \right) \left| \frac{\partial \varphi}{\partial r} \right|^2 \Bigg] - 8\pi G \frac{\partial}{\partial r} \left(e^{-2v} W \right) \end{aligned} \tag{4.11}$$

Summing the equations (4.3) and (4.4) and using the geometric expression for $G^r{}_t$, cf. the left hand side of equation (4.9), yields the following identity:

$$2e^{-2u} \left(\frac{\partial v}{\partial t} \right) \left| \frac{\partial \varphi}{\partial t} \right|^2 + 2e^{-2v} \left(\frac{\partial v}{\partial t} \right) \left| \frac{\partial \varphi}{\partial r} \right|^2 = r^2 \left(\frac{\partial v}{\partial r} + \frac{\partial u}{\partial r} \right) G^r{}_t$$

Using this, (4.11) becomes

$$\frac{\partial}{\partial r} r^2 e^{u+v} \left(8\pi G e^{-2v} W - G^r{}_t \right) = 0$$

Integrating this equation and using the regularity assumptions on the functions involved, one finds

$$G^r{}_t = 8\pi G e^{-2v} W$$

which is one of the missing components of Einstein's equations. Contraction with the metric yields the corresponding equation for G^t_r.

To obtain the $G^\beta{}_\beta$ component of the Einstein tensor in terms of the field, one uses the r component of the energy conservation equation

$$\nabla_\mu G^\mu{}_r = 0.$$

Writing out the contraction for the non-vanishing Christoffel symbols, this becomes

$$2r^{-1}G^{\beta}{}_{\beta} = \partial_t G^t_r + \partial_r G^r{}_r - \Gamma^t_{rt}\left(G^t{}_t - G^r{}_r\right) + \left(\Gamma^{\beta}_{r\beta} + \Gamma^{\alpha}_{r\alpha}\right) G^r{}_r + \Gamma^t_{tt} G^t{}_r + \Gamma^t_{rr} G^r{}_t$$

The right hand side depends only on components of the Einstein tensor that one can express in terms of the fields by the previous calculations. Using the Klein-Gordon equation to simplify the expressions, one finds

$$G^{\beta}{}_{\beta} = 8\pi G\left(e^{-2u}\left|\frac{\partial\varphi}{\partial t}\right|^2 - e^{-2v}\left|\frac{\partial\varphi}{\partial r}\right|^2 - m^2\left|\varphi\right|^2\right)$$

Due to the symmetries of the metric one has $G^{\alpha}{}_{\alpha} = G^{\beta}{}_{\beta}$ and thereby one obtains all non-trivial components of Einsteins equations. ■

4.2. Hamiltonian formalism

From now on we will assume the space-time to be static, i.e. u and v are functions of r only.

4.2.1. Asymptotics

The assumption of asymptotic flatness is usually formulated as the line element behaving like

$$-\left(1 - \frac{M}{8\pi r}\right)\mathrm{d}t^2 + \sum_{i,j=1}^{3}\left(\delta_{ij} + \frac{M}{8\pi r}\right)\mathrm{d}x_i\,\mathrm{d}x_j$$

at large distances from the origin. Further details can be found for example in [41, 29, 47].

For the consecutive arguments, we will use less restrictive asymptotic conditions summarized in the following definition.

Definition 4.2: By $\mathfrak{F}^g$ we denote the space of gravitational fields consisting of pairs of functions[2] $(u, v) \in C^1(\mathbb{R}_+, \mathbb{R}) \times C^1(\mathbb{R}_+, \mathbb{R})$ satisfying

(u) $\|u\|_\infty < \infty, \quad \|\partial_r u\|_1 < \infty$ and

(v) $\|v\|_\infty < \infty, \quad \|\partial_r(rv)\|_1 < \infty$.

[2] The space $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$ is the space of functions from $\mathbb{R}_+$ to $\mathbb{C}$, that are continuously differentiable with respect to the first argument and twice continuously differentiable in the second argument. If the regularity is the same in both variables or if the function only depends on a single variable, the second superscript is omitted.

Furthermore we define the space of matter fields $\mathfrak{F}^m$ as the space of pairs $(\varphi, \Pi) \in \mathfrak{F}^\varphi \times \mathfrak{F}^\Pi \subset C^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C}) \times C^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$ such that for all $t \in \mathbb{R}$ the following hold for the norms taken with respect to the second argument [3] r :

$$\left\| \left(\varphi(t,r), \Pi(t,r)\right) \right\|_2^2 := \|r\varphi(t,r)\|_2^2 + \|r\partial_r\varphi(t,r)\|_2^2 + (16\pi^2)^{-1} \left\| r^{-1}\Pi(t,r) \right\|_2^2 < \infty,$$

$$\left\| \left(\varphi(t,r), \Pi(t,r)\right) \right\|_\infty := \|\varphi(t,r)\|_\infty + \|\partial_r\varphi(t,r)\|_\infty + (4\pi)^{-1} \left\| r^{-2}\Pi(t,r) \right\|_\infty < \infty.$$

These definitions directly entail the well-definedness of the matter Hamiltonian

$$H^m = 4\pi \int_0^\infty \mathrm{d}r\, r^2 \left(|\varphi|^2 + e^{-2v} \left|\partial_r\varphi\right|^2 + (16\pi^2 r^4)^{-1} e^{-2v} \left|\Pi\right|^2 \right),$$

as well as further bounds on the gravitational fields collected in the following proposition.

Proposition 4.3: For v satisfying the conditions (v) in Def. 4.2, the following inequalities hold:

(v1) $\|rv\|_\infty \leq \|\partial_r(rv)\|_1$

(v2) $\|r\,(1 - e^{-v})\|_\infty \leq e^{\|v\|_\infty} \|\partial_r(rv)\|_1$

(v3) $\|2 - e^{-v} - e^{v}\|_1 \leq 2 \left(\|v\|_\infty^2 + \|\partial_r(rv)\|_1^2 \right) e^{\|v\|_\infty}$

(v4) $\|v\|_2^2 \leq \|v\|_\infty^2 + \|\partial_r(rv)\|_1^2$

(v5) $\|1 - e^{\pm v}\|_2^2 \leq e^{2\|v\|_\infty} \left(\|v\|_\infty^2 + \|\partial_r(rv)\|_1^2 \right)$

Proof.

(v1): $|rv(r)| = \left| \int_0^r \mathrm{d}r' \frac{\partial}{\partial r'}(r'v(r')) \right| \leq \int_0^r \mathrm{d}r' \left| \frac{\partial}{\partial r'}(r'v(r')) \right| \leq \|\partial_r(rv)\|_1$

(v2): $|r\,(1 - e^{-v})| \leq r \left| \sum_{n=1}^\infty \frac{(-v)^n}{n!} \right| \leq r \sum_{n=1}^\infty \frac{|v|^n}{n!} \leq r\,|v| \sum_{n=0}^\infty \frac{|v|^n}{(n+1)!}$

$|r\,(1 - e^{-v})| \leq |rv|\, e^{|v|} \leq e^{\|v\|_\infty} \|\partial_r(rv)\|_1$

(v3): $\|2 - e^{v} - e^{-v}\|_1 = \int_0^\infty \mathrm{d}r\, |2 - e^{v} - e^{-v}|$

$= \int_0^1 \mathrm{d}r\, |2 - e^{v} - e^{-v}| + \int_1^\infty \mathrm{d}r\, |2 - e^{v} - e^{-v}|$

Using $|2 - e^{v} - e^{-v}| \leq 2v^2 e^{|v|} \leq 2\, \|v\|_\infty^2 e^{\|v\|_\infty}$ we can evaluate the first, and as moreover $2v^2 e^{|v|} \leq 2\, \|rv\|_\infty^2 e^{\|v\|_\infty} r^{-2}$ also the second summand.

(v4): The statement follows by splitting up the integral as in the previous argument and the inequality (v1).

(v5): Again splitting up the integral as before and the previous inequalities give the desired result. ■

[3] All the norms of the matter fields in the following are to be understood to be taken with respect to the spatial argument r.

4.2.2. The Hamiltonian

It is important, that the two metric degrees of freedom, u and v, are cyclic coordinates in the Lagrangian formalism and consequently there are no associated momenta. Thus these degrees of freedom are not dynamical. The corresponding Euler-Lagrange equations (or the corresponding variational equations in the Hamiltonian formalism) should hence be regarded as constraint equations. The remaining dynamical variables are φ and Π and their complex conjugates. The equations of motion, i.e. Hamilton's equations read[4]

$$\frac{\delta\mathcal{H}}{\delta\varphi} = -\dot{\Pi}, \qquad \frac{\delta\mathcal{H}}{\delta\Pi} = \dot{\varphi}, \tag{4.12}$$

and accordingly for the complex conjugates.

The Hamiltonian corresponding to the previously discussed Lagrangian, with Π being the canonical momentum associated to the field φ, is then given as[5]

$$\begin{aligned} H = \int_0^\infty \mathrm{d}r \, \Bigg[& 4\pi e^{u+v} r^2 \left(m^2 |\varphi|^2 + e^{-2v} \left| \frac{\partial\varphi}{\partial r} \right|^2 + \left(16\pi^2 r^4\right)^{-1} e^{-2v} |\Pi|^2 \right) \\ & + (2G)^{-1} e^u \left(2 - e^v - e^{-v} + 2r\left(1 - e^{-v}\right) \frac{\partial u}{\partial r} \right) \Bigg], \end{aligned} \tag{4.13}$$

which is a well defined expression for $(\varphi, \Pi) \in \mathfrak{F}^m$ and $(u, v) \in \mathfrak{F}^g$ by the bounds in Def. 4.2 and corollary. 4.3 respectively.

Fréchet differentiability of the Hamiltonian

Using the definitions of $\mathfrak{F}^m$, $\mathfrak{F}^g$ and prop. 4.3 one can deduce the finiteness of the variations [6] of the gravitational Hamiltonian with respect to u and v,

$$\begin{aligned} \delta_u H^g = (2G)^{-1} \int_0^\infty \mathrm{d}r \, e^u \Bigg[& \left(2 - e^v - e^{-v} + 2r\left(1 - e^{-v}\right) \frac{\partial u}{\partial r} \right) \delta u \\ & + 2r\left(1 - e^{-v}\right) \frac{\partial \delta u}{\partial r} \Bigg] \quad \text{and} \end{aligned}$$

$$\delta_v H^g = (2G)^{-1} \int_0^\infty \mathrm{d}r \, e^u \left(e^{-v} - e^v + 2r e^{-v} \frac{\partial u}{\partial r} \right) \delta v$$

[4] Where the dot customarily denotes the time derivative.

[5] The field independent part is equal to the Lagrangian up to a sign, the different appearance in comparison with (4.10) is due to the expansion of the derivatives which will be convenient for some of the following computations.

[6] The variation of some functional F over a space of functions $X \ni u, \delta u$ is defined as $\delta_u F(\delta u) := \left. \frac{\partial F(u + s\delta u)}{\partial s} \right|_{s=0}$.

and with constants $C_u(u,v)$ and $C_v(u,v)$ depending on the relevant norms of u and v we find upper bounds of the form

$$\begin{aligned} |\delta_u H^g| &\leq C_u(u,v)\left(\|\delta u\|_\infty + \|\partial_r \delta u\|_1\right) \\ |\delta_v H^g| &\leq C_v(u,v)\left(\|\delta v\|_\infty + \|\partial_r (r\delta v)\|_1\right). \end{aligned}$$

The second inequality is a consequence of $|e^v - e^{-v}| \leq 2\,|v|\,e^v$ and $v, \delta v \in L^2(\mathbb{R}_+)$.

Now set[7]

$$\Delta H^g := H^g(u+\delta u, v+\delta v) - H^g(u,v)$$

Then for δu and δv such that their respective norms in def. 4.2 are less than or equal to one, Taylor expansion of the integrand $\mathcal{H}^g(u+\delta u, v+\delta v)$ around (u,v) and prop. 4.3, cf. appendix A.2 for detailed calculations, yield

$$\begin{aligned} &|\Delta H^g - \delta_u H^g - \delta_v H^g| \\ &\leq\; C^g(\|u\|_\infty, \|v\|_\infty, \|\partial_r u\|_1, \|\partial_r(rv)\|_1)\left(\|\delta u\|_\infty^2 + \|\delta v\|_\infty^2 + \|\partial_r \delta u\|_1^2 \right.\\ &\qquad\qquad \left. + \|\partial_r(r\delta v)\|_1^2\right) \end{aligned}$$

where C^g is a non-decreasing function of its arguments. As the variational differentials are furthermore bounded linear functionals of the variations, H^g is Fréchet differentiable on the space of gravitational fields.

Analogously one proves the corresponding statement for H^m on the space of matter fields. The detailed calculation can again be found in appendix A.2. The variational differentials are

$$\delta_\varphi H^m = 4\pi \int_0^\infty \mathrm{d}r\; r^2 e^{u+v}\left[m^2\overline{\varphi}\delta\varphi + m^2\varphi\overline{\delta\varphi} + e^{-2v}\left((\partial_r\overline{\varphi})\partial_r\delta\varphi + (\partial_r\varphi)\partial_r\overline{\delta\varphi}\right)\right],$$

$$\delta_\Pi H^m = (4\pi)^{-1} \int_0^\infty \mathrm{d}r\; r^{-2} e^{u-v}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right),$$

$$\delta_u H^m = 4\pi \int_0^\infty \mathrm{d}r\; r^2 e^{u+v}\left(m^2|\varphi|^2 + e^{-2v}|\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1} e^{-2v}|\Pi|^2\right)\delta u$$

$$\delta_v H^m = 4\pi \int_0^\infty \mathrm{d}r\; r^2 e^{u+v}\left(m^2|\varphi|^2 - e^{-2v}|\partial_r\varphi|^2 - (16\pi^2 r^4)^{-1} e^{-2v}|\Pi|^2\right)\delta v.$$

For $\|\delta u\|_\infty + \|\delta v\|_\infty \leq 1$, we find

$$\begin{aligned} &|\Delta H^m - \delta_\varphi H^m - \delta_\Pi H^m - \delta_u H^m - \delta_v H^m| \\ &\leq C^m(\|u\|_\infty, \|v\|_\infty, \|r\varphi\|_2, \|r\partial_r\varphi\|_2, \left\|r^{-1}\Pi\right\|_2)\Big(\|\delta u\|_\infty^2 + \|\delta v\|_\infty^2 + \|r\delta\varphi\|_2^2 \\ &\qquad\qquad + \|r\partial_r\delta\varphi\|_2^2 + \left\|r^{-1}\delta\Pi\right\|_2^2\Big). \end{aligned}$$

[7] We define ΔH^m and analogously.

The Fréchet differentiability follows by the same arguments as in the previous case. We summarize the preceding discussion in the following lemma.

Lemma 4.4: The Hamiltonian $H(u, v, \varphi, \Pi)$ is Fréchet differentiable with derivative

$$\delta H = \delta_u H^g + \delta_v H^g + \delta_u H^m + \delta_v H^m + \delta_\varphi H^m + \delta_\Pi H^m.$$

4.2.3. The Constraint equations

The equation, we refer to as the first constraint equation is the Euler-Lagrange equation with respect to u, as indicated in section 4.1.1. One can nevertheless also derive it as a variational equation with respect to local variations in u, i.e.

$$0 = \frac{\delta H}{\delta u}$$

Recalling the derivative of H from lemma 4.4, the stationary points with respect to u are given by the first constraint equation

$$8\pi G r^2 \left(m^2 |\varphi|^2 + e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 + \frac{1}{16\pi^2 r^4} e^{-2v} |\Pi|^2 \right) = \frac{\partial}{\partial r} r \left(1 - e^{-2v}\right) \tag{4.14}$$

The second constraint equation is the Euler-Lagrange equation with respect to local variations in v. Analogously to the first constraint equation one can again derive this equation from the Hamiltonian. The derivative of the gravitational part with respect to v, cf. lemma 4.4 once more, reads

$$\delta_v H^g = (2G)^{-1} \int_0^\infty \mathrm{d}r \, \left(e^{u-v} - e^{u+v} + 2r e^{u-v} \frac{\partial u}{\partial r} \right) \delta v.$$

The corresponding field-dependent part is given by

$$\delta_v H^m = 4\pi \int_0^\infty \mathrm{d}r \; e^{u+v} r^2 \left(m^2 |\varphi|^2 - e^{-2v} \left| \frac{\partial \overline{\varphi}}{\partial r} \right|^2 - \left(16\pi^2 r^4\right)^{-1} e^{-2v} |\Pi|^2 \right) \delta v$$

and consequently the second constraint equation is

$$\begin{aligned} &8\pi G r^2 \left(-m^2 |\varphi|^2 + e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 + \left(16\pi^2 r^4\right)^{-1} e^{-2v} |\Pi|^2 \right) \\ &= e^{-2v} - 1 + 2r e^{-2v} \frac{\partial u}{\partial r}. \end{aligned} \tag{4.15}$$

Solution to the first constraint equation

To solve the constraint equation (4.14), rewrite it as

$$\frac{\partial}{\partial r}\left(1-e^{-2v}\right)r+(8\pi Gr)r\left(1-e^{-2v}\right)\left|\frac{\partial\varphi}{\partial r}\right|^2+G\left(2\pi r^3\right)^{-1}r\left(1-e^{-2v}\right)|\Pi|^2$$
$$=\left(8\pi Gr^2\right)\left(m^2|\varphi|^2-\left|\frac{\partial\varphi}{\partial r}\right|^2-\left(16\pi^2r^4\right)^{-1}|\Pi|^2\right).$$

As first order ordinary differential equation for $(1-e^{-2v})$ one can solve this equation. The unique solution for $1-e^{-2v}$ that is regular at the origin is given as

$$1-e^{-2v}=\frac{8\pi G}{r}\left[\int_0^r \mathrm{d}r''(r'')^2\left(m^2|\varphi|^2+\left|\frac{\partial\varphi}{\partial(r'')}\right|^2+\frac{1}{16\pi^2(r'')^4}|\Pi|^2\right)\times\right.$$
$$\left.\exp\left(-8\pi G\int_{r''}^{r}\mathrm{d}r'(r')\left(\left|\frac{\partial\varphi}{\partial(r')}\right|^2+\frac{1}{16\pi^2(r')^4}|\Pi|^2\right)\right)\right]. \tag{4.16}$$

Due to its prominent role in the following, this solution is being given a name.

Definition 4.5: For $(\varphi,\Pi)\in\mathfrak{F}^m$ define the function $\Xi(\varphi,\Pi):\mathbb{R}_+\to\mathbb{R}_+$ to be the right hand side of (4.16) as a function of r.

Note, that the boundary condition excludes the Schwarzschild space-time, which for some $M\in\mathbb{R}$ is given by the line element

$$-\left(1-\frac{2MG}{r}\right)\mathrm{d}t^2+\left(1-\frac{2MG}{r}\right)^{-1}\mathrm{d}r^2++r^2\left(\mathrm{d}\alpha^2+\sin^2\alpha\,\mathrm{d}\beta^2\right),$$

i.e. $(1-e^{-2v})=\frac{2MG}{r}$, as a solution in the case of vanishing fields.

Apart from continuity and positivity of $\Xi(\varphi,\Pi)$, the properties of φ and Π, as well as the first constraint equation (4.14) directly entail the following:

Proposition 4.6:

(Ξ1) $\Xi(\varphi,\Pi)(r)\leq\frac{1}{3}r^2\left\|(\varphi,\Pi)\right\|_\infty^2$, in particular $\Xi(\varphi,\Pi)(r)=\mathcal{O}(r^2)$ for $r\to 0$,

(Ξ2) $\Xi(\varphi,\Pi)(r)\leq r^{-1}\left\|(\varphi,\Pi)\right\|_2^2$, in particular $\Xi(\varphi,\Pi)(r)=\mathcal{O}(r^{-1})$ for $r\to\infty$,

(Ξ3) $\Xi(\varphi,\Pi)$ is bounded,

(Ξ4) $\partial_r(r\Xi(\varphi,\Pi))$ is integrable.

Solution to the second constraint equation

Using the previously obtained solution, cf. definition 4.5, to the first constraint equation (4.14), we rewrite the second one (4.15) as

$$\partial_r u = 4\pi G r \left(|\partial_r \varphi|^2 + (16\pi^2 r^4)^{-1} |\Pi|^2 \right) - \frac{8\pi G m^2 r^2 |\varphi|^2 - \Xi(\varphi, \Pi)}{2r(1 - \Xi(\varphi, \Pi))}.$$

Provided $\Xi(\varphi, \Pi) \neq 1$ and $r > 1$, the unique solution $u(\varphi, \Pi)$ satisfying $u(\varphi, \Pi) \to 0$ for $r \to \infty$ is given by

$$u(\varphi, \Pi)(r) = -\int_r^\infty \mathrm{d}r' \left[4\pi G r' \left(|\partial_{r'} \varphi|^2 + (16\pi^2 r'^4)^{-1} |\Pi|^2 \right) - \frac{8\pi G m^2 r'^2 |\varphi|^2 - \Xi(\varphi, \Pi)}{2r'(1 - \Xi(\varphi, \Pi))} \right]. \tag{4.17}$$

Define

$$\mathfrak{B} := \{ (\varphi, \Pi) \in \mathfrak{F}^m \mid \|\Xi(\varphi, \Pi)\|_\infty < 1 \}.$$

The maps u and v defined by (4.17) and

$$v(\varphi, \Pi) = -\tfrac{1}{2} \log \left(1 - \Xi(\varphi, \Pi) \right) \tag{4.18}$$

respectively are well defined on $\mathfrak{B}$ and take values in the continuous functions on $\mathbb{R}_+$. Furthermore the elements of $v(\mathfrak{B})$ are non-negative and satisfy

$$v(\varphi, \Pi)(r) = \mathcal{O}(r^2) \text{ for } r \to 0 \tag{4.19}$$
$$v(\varphi, \Pi)(r) = \mathcal{O}(r^{-1}) \text{ for } r \to \infty \tag{4.20}$$

by (Ξ1) and (Ξ2) and the Taylor expansion of $\log(1-x)$ for $|x| < 1$, whereas for some real constant C the $u(\mathfrak{B})$ satisfy

$$|u(\varphi, \Pi)(r)| \leq \frac{C}{r} \text{ for } r \geq 1. \tag{4.21}$$

With positive real constants $\tilde{C}_1, C_1, C_{,2}$, and C_3 we can estimate $|u(\varphi, \Pi)(r)|$ as follows:

$$|u(\varphi, \Pi)(r)| \leq \int_r^\infty \mathrm{d}r' \left[\frac{\tilde{C}_1}{r} r'^2 \left(|\partial_{r'} \varphi|^2 + (16\pi^2 r'^4)^{-1} |\Pi|^2 \right) + \frac{8\pi G m^2 r'^2 |\varphi|^2 + |\Xi(\varphi, \Pi)|}{2r'(1 - \Xi(\varphi, \Pi))} \right].$$

As $(\varphi, \Pi) \in \mathfrak{B}$ and using (Ξ2), one finds

$$|u(\varphi,\Pi)(r)| \leq \frac{C_1}{r} + \frac{C_2}{r}\int\limits_r^\infty \mathrm{d}r' \left[r'^2\,|\varphi|^2 + \frac{C_3}{r'^2}\right] \leq \frac{C}{r}.$$

To obtain the value $u(\varphi,\Pi)(0)$, note that summing of the two constraint equations (4.14) and (4.15) one obtains

$$\frac{\partial(u+v)(\varphi,\Pi)}{\partial r} = 8\pi G r\left(|\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\Pi|^2\right) \tag{4.22}$$

and integration over $\mathbb{R}_+$ together with (4.19), (4.20), and (4.21) yields

$$u(\varphi,\Pi)(0) = -8\pi G\int\limits_0^\infty \mathrm{d}r\ r\left(|\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\Pi|^2\right).$$

Fréchet differentiability of the solutions

In order to address the issue of Fréchet differentiability of u and v, we make a statement about their ranges first.

Lemma 4.7: (u, v) is a continuous map from the open set $\mathfrak{B} \subseteq \mathfrak{F}^m$ into $\mathfrak{F}^g$.

Writing $\mathfrak{F}^g = \mathfrak{F}^u \times \mathfrak{F}^v$, we have that $\Xi : \mathfrak{F}^m \to \mathfrak{F}^V$ is continuous, hence $\mathfrak{B}$ is open. In the following statements, we will prove the differentiability of (u, v) and thereby its continuity.

Lemma 4.8: $\Xi : \mathfrak{F}^m \to \mathfrak{F}^v$ is a Fréchet differentiable map with derivative

$$\begin{aligned}
&\delta\Xi(\delta\varphi,\delta\Pi)(r)\\
&= 8\pi G r^{-1}\int\limits_0^r dr''\ r''^2\Bigg[\Big(m^2\overline{\varphi}\delta\varphi + m^2\varphi\overline{\delta\varphi} + (\partial_{r''}\overline{\varphi})\partial_{r''}\delta\varphi + (\partial_{r''}\varphi)\partial_{r''}\overline{\delta\varphi}\\
&\qquad\qquad + (16\pi^2 r''^4)^{-1}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right)\Big)\\
&\qquad -8\pi G\Big(m^2\,|\varphi|^2 + |\partial_{r''}\varphi|^2 + (16\pi^2 r''^4)^{-1}\,|\Pi|^2\Big)\\
&\qquad \times\int_{r''}^r \mathrm{d}r'\ r'\Big[(\partial_{r'}\overline{\varphi})\partial_{r'}\delta\varphi + (\partial_{r'}\varphi)\partial_{r'}\overline{\delta\varphi}\\
&\qquad\qquad + (16\pi^2 r'^4)^{-1}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right)\Big]\Bigg]\\
&\qquad \times\exp -8\pi G\int_{r''}^r \mathrm{d}r' r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}\,|\Pi|^2\right)
\end{aligned}$$

Proof. For each $(\varphi, \Pi) \in \mathfrak{F}^m$ $\delta\Xi$ is bounded and linear as a function mapping $(\delta\varphi, \delta\Pi) \in \mathfrak{F}^m$ into $\mathfrak{F}^v$ as one can bound $\|\delta\Xi(\delta\varphi, \delta\Pi)\|_\infty$ and $\|\partial_r(r\delta\Xi(\delta\varphi, \delta\Pi))\|_1$. To obtain the required bounds, note that

$$\tfrac{1}{r}\int_0^r \mathrm{d}r''\ r''^2\,|\varphi|^2 \le \int_0^1 \mathrm{d}r''\ \ |\varphi|^2 + \int_1^r \mathrm{d}r''\ r''^2\,|\varphi|^2 \le \|\varphi\|_\infty^2 + \|r\varphi\|_2^2\,,$$

and similarly

$$\int_{r''}^r \mathrm{d}r'\ r'\,|\partial_r\varphi|^2 \le \int_0^\infty \mathrm{d}r'\ r'\,|\partial_r\varphi|^2 \le \|\partial_r\varphi\|_\infty^2 + \|r\partial_r\varphi\|_2^2\,. \tag{4.23}$$

Analogously one obtains the corresponding bounds for the other occurring integrals and for some $C' \in \mathbb{R}$ one can bound both norms by

$$C'\left(\|(\varphi,\Pi)\|_\infty + \|(\varphi,\Pi)\|_2\right)\left(1 + \|(\varphi,\Pi)\|_\infty^2 + \|(\varphi,\Pi)\|_2^2\right) \\ \times\left(\|(\delta\varphi,\delta\Pi)\|_\infty + \|(\delta\varphi,\delta\Pi)\|_2\right).$$

For the sake of readability, we introduce the following notation:

$$\begin{aligned}
&T(r) := 8\pi G r^2\left(m^2\,|\varphi|^2 + |\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\Pi|^2\right),\\
&T_\delta(r) := 8\pi G r^2\left(m^2\overline{\varphi}\delta\varphi + m^2\varphi\delta\overline{\varphi} + (\partial_r\overline{\varphi})\partial_r\delta\varphi + (\partial_r\varphi)\partial_r\overline{\delta\varphi}\right.\\
&\qquad\qquad\left. +(16\pi^2 r^4)^{-1}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right)\right),\\
&T_{\delta^2}(r) := 8\pi G r^2\left(m^2\,|\delta\varphi|^2 + |\partial_r\delta\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\delta\Pi|^2\right),\\
&E(r) := -8\pi G r\left(|\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\Pi|^2\right),\\
&E_\delta(r) := -8\pi G r\Big((\partial_r\overline{\varphi})\partial_r\delta\varphi + (\partial_r\varphi)\partial_r\delta\overline{\varphi} + (16\pi^2 r^4)^{-1}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right)\Big)\\
&E_{\delta^2}(r) := -8\pi G r\left(|\partial_r\delta\varphi|^2 + (16\pi^2 r^4)^{-1}\,|\delta\Pi|^2\right).
\end{aligned}$$

With these we furthermore define

$$E(r, r'') := \int_{r''}^{r} \mathrm{d}r'\ E(r')$$

and $E_\delta(r, r'')$, $E_{\delta^2}(r, r'')$ analogously. Note that $\partial_r E(r, r'') = E(r)$. By slightly abusing notation we denote $T_{\delta^2+\delta+\cdot}(r) := T_{\delta^2}(r) + T_\delta(r) + T(r)$ and $E_{\delta^2+\delta+\cdot}$, and $E_{\delta^2+\delta}$ accordingly.

Along the lines of the proof of the Fréchet differentiability of the Hamiltonian, consider

$$\Xi(\varphi+\delta\varphi,\Pi+\delta\Pi)=\tfrac{1}{r}\int_0^r \mathrm{d}r''\Big[T(r'')\exp\left(E(r,r'')\right)$$
$$+T_\delta(r'')\exp\left(E(r,r'')\right)$$
$$+T(r'')E_\delta(r,r'')\exp\left(E(r,r'')\right)$$
$$+T_{\delta^2}(r'')\exp\left(E_{\delta^2+\delta+\cdot}(r,r'')\right)$$
$$+T(r'')\exp\left(E(r,r'')\right)\Big[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1$$
$$-E_\delta(r,r'')\Big]$$
$$+T_\delta(r'')\exp\left(E(r,r'')\right)\Big[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1\Big]\Big].$$

The first three summands are equal to $\Xi(\varphi,\Pi)+\delta\Xi(\varphi,\Pi)$ and will consequently be cancel in the expression for $|\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)-\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)|$. Hence we find using that E_δ, E_{δ^2} are non-positive

$$|\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)-\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)|$$
$$\leq\tfrac{1}{r}\int_0^r \mathrm{d}r''\ \Big[T_{\delta^2}(r''))+T(r'')\Big|\Big[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1-E_{\delta^2+\delta}(r,r'')\Big]\Big|$$
$$+T_\delta(r'')\Big|\Big[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1\Big]\Big|\Big].$$

As $|e^x-1|\leq|x|\,e^{|x|}$ and $|e^x-1-x|\leq|x|^2\,e^{|x|}$, assuming $\|(\delta\varphi,\delta\Pi)\|_\infty\leq 1$ as well as $\|(\delta\varphi,\delta\Pi)\|_2\leq 1$, and letting $C''(\varphi,\Pi)$ be some real constant, we arrive at

$$\begin{aligned}&|\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)-\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)|\\&\leq C''(\varphi,\Pi)\left(\|(\delta\varphi,\delta\Pi)\|_\infty^{\ 2}+\|(\delta\varphi,\delta\Pi)\|_2^2\right).\end{aligned}\tag{4.24}$$

In order to prove a bound of the same kind for $\|\partial_r r\left(\Delta\Xi-\delta\Xi\right)\|_1$ we will proceed similarly to the preceding considerations. First, we will however write out $\partial_r(r\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi))$ which reads

$$\begin{aligned}T_\delta(r)+\int_0^r \mathrm{d}r''\ \Big[\Big(&T_\delta(r'')E(r)+T(r'')E_\delta(r)\\&+T(r'')E_\delta(r,r'')E(r)\Big)\exp\left(E(r,r'')\right)\Big]\end{aligned}\tag{4.25}$$

Then rewriting $\partial_r(r\Xi(\varphi+\delta\varphi,\Pi+\delta\Pi))$ as

$$\begin{aligned}
&T_{\delta^2+\delta+\cdot}(r)\\
&+\int_0^r \mathrm{d}r'' \Big[T(r'')E(r)\exp\left(E(r,r'')\right)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1+1\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-E_\delta(r,r'')+E_\delta(r,r'')\right]\\
&\qquad\quad+T_\delta(r'')E(r)\exp\left(E(r,r'')\right)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1+1\right]\\
&\qquad\quad+T(r'')E_\delta(r)\exp\left(E(r,r'')\right)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1+1\right]\\
&\qquad\quad+\left(T_{\delta^2}(r'')E_{\delta^2+\delta+\cdot}(r)+T_\delta(r'')E_{\delta^2+\delta}(r)+T(r'')E_{\delta^2}(r)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\exp\left(E_{\delta^2+\delta+\cdot}(r,r'')\right)\Big]
\end{aligned}$$

and comparing this with (4.25) to determine the cancellations of terms containing $\delta\varphi$ and $\delta\Pi$ in $|\partial_r r\left(\Delta\Xi-\delta\Xi\right)|$, we can bound the latter by

$$\begin{aligned}
&T_{\delta^2}(r)\\
&+\int_0^r \mathrm{d}r'' \Big[T(r'')E(r)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1-E_\delta(r,r'')\right]\\
&\qquad\quad+T_\delta(r'')E(r)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1\right]\\
&\qquad\quad+T(r'')E_\delta(r)\left[\exp\left(E_{\delta^2+\delta}(r,r'')\right)-1\right]\\
&\qquad\quad+\left(T_{\delta^2}(r'')E_{\delta^2+\delta+\cdot}(r)+T_\delta(r'')E_{\delta^2+\delta}(r)+T(r'')E_{\delta^2}(r)\right)\Big].
\end{aligned}\tag{4.26}$$

By the same arguments as before and under the same assumptions with $C'''(\varphi,\Pi)$ some real constant, we can bound the $\|\cdot\|_1$-norm of (4.26) by

$$C'''(\varphi,\Pi)\left(\|(\delta\varphi,\delta\Pi)\|_\infty^2+\|(\delta\varphi,\delta\Pi)\|_2^2\right)$$

which concludes the proof of the lemma. ∎

Next, we will use Lemma 4.8 to prove the following one.

Lemma 4.9: $v:\mathfrak{B}\to\mathfrak{F}^v$ is Fréchet differentiable.

Proof. For $(\varphi,\Pi)\in\mathfrak{B}$ choose $\varepsilon>0$ such that for $0<\delta<1-\Xi(\varphi,\Pi)$ one has $\Xi(\varphi+\delta\varphi,\Pi+\delta\Pi)<1-\delta$ for $\|(\delta\varphi,\delta\Pi)\|_\infty<\varepsilon$ and $\|(\delta\varphi,\delta\Pi)\|_2<\varepsilon$. As

$$\begin{aligned}
\Delta v(\varphi,\delta\varphi,\Pi,\delta\Pi)&=-\tfrac{1}{2}\big(\log\left(1-\Xi(\varphi+\delta\varphi,\Pi+\delta\Pi)\right)-\log\left(1-\Xi(\varphi,\Pi)\right)\big)\\
&=-\tfrac{1}{2}\log\left(1-\frac{\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)}{1-\Xi(\varphi,\Pi)}\right)
\end{aligned}$$

and provided $\delta\varphi$ and $\delta\Pi$ are small enough, so that $|\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)|<1-\Xi(\varphi,\Pi)$, one can find a $c\le 1-\delta$ such that the above can be rewritten as

$$\Delta v(\varphi,\delta\varphi,\Pi,\delta\Pi)=\frac{\Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)(r)}{2(1-\Xi(\varphi,\Pi)(r))}+\frac{\left(\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)(r)\right)^2}{4(1-c)^2}.$$

In order to arrive at this, we compute the Taylor expansion of $\log\left(1-\frac{x}{k}\right)$ to first order with Lagrange remainder. With $x^* \in (0,x)$ this reads

$$\log\left(1-\frac{x}{k}\right) = -\frac{x}{k} - \frac{1}{2}\left(k-x^*\right)^{-2} x^2.$$

For $x = \Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)$, $k = 1-\Xi(\varphi,\Pi)$, and $c = x^* + \Xi(\varphi,\Pi)$this gives the identity above as for $\Delta\Xi \geq 0$ one finds $c < \Delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi) + \Xi(\varphi,\Pi) \leq 1-\delta$ and similarly for $\Delta\Xi < 0$ one has $c < \Xi(\varphi,\Pi) < 1-\delta$ as required.

With $\delta v(\varphi,\delta\varphi,\Pi,\delta\Pi) = \frac{1}{2}(1-\Xi(\varphi,\Pi))^{-1}(\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)$ one obtains

$$|\Delta v(r) - \delta v(r)| \leq \frac{|\Delta\Xi(r) - \delta\Xi(r)|}{1-\Xi(\varphi,\Pi)(r)} + \delta^{-2}\left|\Delta\Xi(r)\right|^2.$$

From Lemma 4.8 and the calculations in the corresponding proof it follows that for some constant $\tilde{C}(\varphi,\Pi)$,

$$\|\Delta v(r) - \delta v(r)\|_\infty \leq \tilde{C}(\varphi,\Pi)\left(\|(\varphi,\Pi)\|_\infty^2 + \|(\varphi,\Pi)\|_2^2\right).$$

It remains to find a bound of the same kind for the other relevant norm of $\Delta v(r) - \delta v(r)$.

We rewrite $2(\Delta v - \delta v)$ as follows[8]:

$$\begin{aligned}
&\int_0^1 \mathrm{d}t\, \left[-\frac{\partial}{\partial t}\log\left(1-(1-t)\Xi - t\Xi(\varphi+\delta\varphi,\Pi+\delta\Pi)\right) + \frac{\delta\Xi}{1-\Xi}\right] \\
&= \int_0^1 \mathrm{d}t\, \left[\frac{\Delta\Xi - \delta\Xi}{1-\Xi-t\Delta\Xi} + \frac{t\Delta\Xi\delta\Xi}{(1-\Xi)(1-\Xi-t\Delta\Xi)}\right]. \qquad (4.27)
\end{aligned}$$

Now considering $2\partial_r r\left(\Delta v - \delta v\right)$ we find

$$\int_0^1 \mathrm{d}t\, \left[\frac{\partial_r r(\Delta\Xi-\delta\Xi)}{1-\Xi-t\Delta\Xi} + \frac{\partial_r(\Xi+t\Delta\Xi)}{(1-\Xi-t\Delta\Xi)^2}(\Delta\Xi-\delta\Xi)r + \partial_r r\left(\frac{t\Delta\Xi\delta\Xi}{(1-\Xi)(1-\Xi-t\Delta\Xi)}\right)\right].$$

For $(\delta\varphi,\delta\Pi)$ small enough we can bound the above.

$$\begin{aligned}
2\partial_r r\left(\Delta v - \delta v\right) \leq\ & \delta^{-1}\left|\partial_r r(\Delta\Xi - \delta\Xi)\right| \\
&+ \delta^{-2}\max_{t\in[0,1]}\left|\left[\partial_r(\Xi + t\Delta\Xi)\right](\Delta\Xi-\delta\Xi)\right| r \qquad (4.28) \\
&+ \int_0^1 \mathrm{d}t\, \left[\partial_r r\left(\frac{t\Delta\Xi\delta\Xi}{(1-\Xi)(1-\Xi-t\Delta\Xi)}\right)\right]
\end{aligned}$$

[8] In the following calculations we will drop the arguments whenever they are the ones to be expected, i.e. Ξ denotes $\Xi(\varphi,\Pi)(r)$ and we write $\delta\Xi$ for $\delta\Xi(\varphi,\delta\varphi,\Pi,\delta\Pi)(r)$ and so forth.

A bound of the desired shape for the first summand on the right hand side follows directly from Lemma 4.8.

Note that $\|r(\Delta\Xi - \delta\Xi)\|_\infty \leq \|\partial_r r(\Delta\Xi - \delta\Xi)\|_1$, which we use to decompose the integral of the second summand.

$$\int_0^\infty \mathrm{d}r \;\, |\partial_r(\Xi + \Delta\Xi)|\, r\, |\Delta\Xi - \delta\Xi|$$
$$\leq \|(\Delta\Xi - \delta\Xi)\|_\infty \int_0^1 \mathrm{d}r \; r\, |\partial_r(\Xi + \Delta\Xi)| + \|r(\Delta\Xi - \delta\Xi)\|_\infty \int_1^\infty \mathrm{d}r \;\, |\partial_r(\Xi + \Delta\Xi)|$$

For the first integral we find

$$\begin{aligned}\int_0^1 \mathrm{d}r \; r\, |\partial_r(\Xi + \Delta\Xi)| &= \int_0^1 \mathrm{d}r \;\, |\partial_r r(\Xi + \Delta\Xi) - (\Xi + \Delta\Xi)| \\ &\leq \|\Xi\|_\infty + \|\Delta\Xi\|_\infty + \|\partial_r r\Xi\|_1 + \|\partial_r r\Delta\Xi\|_1 \,,\end{aligned}$$

whereas we estimate the second one as follows:

$$\begin{aligned}\int_1^\infty \mathrm{d}r \;\, |\partial_r(\Xi + \Delta\Xi)| &= \int_1^\infty \mathrm{d}r \; r^{-1}\, |\partial_r r(\Xi + \Delta\Xi) = \Xi + \Delta\Xi| \\ &\leq \|\partial_r r\Xi\|_1 + \|\partial_r r\Delta\Xi\|_1 + \|r(\Xi - \Delta\Xi)\|_\infty \\ &\leq 2\,\|\partial_r r\Xi\|_1 + 2\,\|\partial_r r\Delta\Xi\|_1\end{aligned}$$

Provided $\|(\delta\varphi, \delta\Pi)\|_\infty \leq 1$ and $\|(\delta\varphi, \delta\Pi)\|_2 \leq 1$ these bounds and Lemma 4.8 entail the desired bound for the second summand.

Finally we address the third summand in (4.28). Expanding the r-derivative the integrand reads

$$\begin{aligned}(1-\Xi)^{-1}(1-\Xi-t\Delta\Xi)^{-1}\Bigg(&\Delta\Xi\partial_r(r\delta\Xi) + r\delta\Xi\partial_r\Delta\Xi - \frac{rt\Delta\Xi\delta\Xi}{1-\Xi}\partial_r\Xi \\ &+\frac{rt\Delta\Xi\delta\Xi}{1-\Xi-t\Delta\Xi}\partial_r(\Xi + t\Delta\Xi)\Bigg)\,.\end{aligned} \tag{4.29}$$

As before the denominators can be bounded by δ^{-2} and δ^{-3} respectively making the t-integration trivial. Consequently the $\|\cdot\|_1$-norm of the first summand in (4.29) is bounded from above by $\delta^{-2}\,\|\Delta\Xi\|_\infty\,\|\partial_r r\delta\Xi\|_1$. To bound the second summand, we split the integral as before and obtain

$$\delta^{-2}\,\|r\delta\Xi\partial_r\Delta\Xi\|_1 \leq 2\,\|r\delta\Xi\|_\infty\,\|\partial_r r\Delta\Xi\|_1 + \|\delta\Xi\|_\infty\left(\|\partial_r r\Delta\Xi\|_1 + \|\Delta\Xi\|_\infty\right).$$

The two remaining summands can be bound analogously. As δV is linear as a function from $\mathfrak{F}^m$ to $\mathfrak{F}^v$, together with Lemma 4.8 and the assumptions $\|(\delta\varphi, \delta\Pi)\|_\infty \leq 1$ and $\|(\delta\varphi, \delta\Pi)\|_2 \leq 1$ the previously obtained bounds conclude the proof of the Lemma. ■

Lemma 4.10: The solution u to the second constraint equation is Fréchet differentiable as a function $u : \mathfrak{B} \to \mathfrak{F}^u$ with derivative

$$\delta u(\varphi, \delta\varphi, \Pi, \delta\Pi) = -\int_r^\infty \mathrm{d}r' \Bigg[4\pi G r' \left((\partial_r\overline{\varphi})\partial_r\delta\varphi + (\partial_r\varphi)\partial_r\delta\overline{\varphi}\right) + (16\pi^2 r'^4)^{-1}\left(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi}\right) - (2r'(1-\Xi))^{-1}\Big[8\pi G r'^2(\overline{\varphi}\delta\varphi + \varphi\overline{\delta\varphi}) - \delta\Xi - 2r'\left(8\pi G r'^2|\varphi|^2 - \Xi\right)(2r'(1-\Xi))^{-1}\delta\Xi\Big]\Bigg]$$

Proof. Integrating equation (4.22) and the decay behavior of u and v from equations (4.21) and (4.20) respectively gives

$$(\Delta u - \delta u) + (\Delta v - \delta v) = -8\pi G \int_r^\infty \mathrm{d}r'\, r' \left(|\partial_r\delta\varphi|^2 + (16\pi^2 r^4)^{-1}|\delta\varphi|^2\right).$$

Via (4.23) and analogous bound for the corresponding integral of $\delta\Pi$ and some positive constant $\hat{C}$ this entails

$$\|(\Delta u - \delta u)\|_\infty \le \|\Delta v - \delta v\|_\infty + \hat{C}\left(\|(\delta\varphi, \delta\Pi)\|_\infty^2 + \|(\delta\varphi, \delta\Pi)\|_2^2\right).$$

Lemma 4.9 then gives the sufficient bound.

For the second norm we split the relevant integral into two parts. For the first one we have

$$\int_0^1 \mathrm{d}r \left|\frac{\partial}{\partial r}\left[8\pi G\int_r^\infty \mathrm{d}r' \left(|\partial_r\delta\varphi|^2 + (16\pi^2 r^4)^{-1}|\delta\varphi|^2\right) - (\Delta v - \delta v)\right]\right| \le 8\pi G\|(\delta\varphi, \delta\Pi)\|_\infty^2 + \int_0^1 \mathrm{d}r\; |\partial_r(\Delta v - \delta v)|.$$

Rewriting $\Delta v - \delta v$ as in (4.27) the second summand becomes

$$\frac{1}{2}\int_0^1 \mathrm{d}r \left|\int_0^1 \mathrm{d}t\, \frac{\partial}{\partial r}\left[\frac{\Delta\Xi - \delta\Xi}{1 - \Xi - t\Delta\Xi} + \frac{t\Delta\Xi\delta\Xi}{(1-\Xi)(1-\Xi-t\Delta\Xi)}\right]\right|$$

Expanding the r-derivative and bounding the denominators under the same assumptions as in the proof of lemma 4.9, i.e. we choose $\varepsilon > 0$ such that for $0 < \delta < 1 - \Xi(\varphi, \Pi)$ one has $\Xi(\varphi + \delta\varphi, \Pi + \delta\Pi) < 1 - \delta$ for $\|(\delta\varphi, \delta\Pi)\|_\infty < \varepsilon$ and $\|(\delta\varphi, \delta\Pi)\|_2 < \varepsilon$, we obtain the following bound:

$$\frac{1}{2}\int_0^1 \mathrm{d}r\ \Big[\delta^{-1}\left|\partial_r(\Delta\Xi-\delta\Xi)\right| + \delta^{-3}\big(\left\|\Delta\Xi\right\|_\infty \left\|\delta\Xi\right\|_\infty (2\left|\partial_r\Xi\right| + \left|\partial_r\Delta\Xi\right|)\big) + \delta^{-2}\big(\left\|\Delta\Xi-\delta\Xi\right\|_\infty (\left|\partial_r\Xi\right| + \left|\partial_r\Delta\Xi\right|) + \left|\partial_r\Delta\Xi\delta\Xi\right|\big)\Big] \tag{4.30}$$

It can be shown in a straight forward manner, that for some constant $\check{C}(\varphi,\Pi)$ this can be bounded by

$$\check{C}(\varphi,\Pi)\left(\left\|(\delta\varphi,\delta\Pi)\right\|_\infty^2 + \left\|(\delta\varphi,\delta\Pi)\right\|_2^2\right).$$

As a prototypical example, we will prove a bound of this form for

$$\left\|\Delta\Xi-\delta\Xi\right\|_\infty \int_0^1 \mathrm{d}r\ \left|\partial_r\Xi\right|.$$

Considering the integral using the notation from the proof of lemma 4.8 as well as positive constants $\check{C}_1$, $\check{C}_2$ and $\check{C}_3$ one finds

$$\begin{aligned}\int_0^1 \mathrm{d}r\ \left|\partial_r\Xi\right| &= \int_0^1 \mathrm{d}r\ \bigg|\bigg[8\pi G r^{-1} T(r) + 8\pi G r^{-1}\int_0^r \mathrm{d}r''\ T(r'')E(r)\exp E(r,r'')\bigg]\bigg| \\ &\le \int_0^1 \mathrm{d}r\ \left[\check{C}_1 r\left\|(\varphi,\Pi)\right\|_\infty^2 + \check{C}_2 r^3\left\|(\varphi,\Pi)\right\|_\infty^4\right] \\ &\le \check{C}_3\left(\left\|(\varphi,\Pi)\right\|_\infty^2 + \left\|(\varphi,\Pi)\right\|_\infty^4\right).\end{aligned}$$

The estimate for $\left\|\Delta\Xi-\delta\Xi\right\|_\infty$ from (4.24) entails the desired bound.

The essential ingredients in this example are, that the occurring integrand can be bounded by a product of a positive power of r and some polynomial of $\left\|(\varphi,\Pi)\right\|_\infty$ and consequently the integration can be carried out.

With analogous arguments all the remaining terms in (4.30) can be shown to satisfy appropriate bounds for $\left\|(\delta\varphi,\delta\Pi)\right\|_\infty$ small enough.

We now address the remaining second part of the integral, which reads

$$\int_1^\infty \mathrm{d}r\ \left|\frac{\partial}{\partial r}\left[8\pi G\int_r^\infty \mathrm{d}r'\ \left(\left|\partial_r\delta\varphi\right|^2 + (16\pi^2 r^4)^{-1}\left|\delta\varphi\right|^2\right) - (\Delta v - \delta v)\right]\right|$$

Similarly to the previous arguments on obtains the upper bound

$$8\pi G \left\|(\delta\varphi, \delta\Pi)\right\|_2^2 + \int_1^\infty \mathrm{d}r \;\; \left|\partial_r(\Delta v - \delta v)\right| .$$

Using lemma 4.9 we can bound the remaining integral as follows:

$$\begin{aligned}\int_1^\infty \mathrm{d}r \;\; \left|\partial_r(\Delta v - \delta v)\right| &\leq \int_1^\infty \mathrm{d}r \;\; \left|(\partial_r r(\Delta v - \delta v)) - (\Delta v - \delta v)\right| \\ &\leq \left\|\partial_r r(\Delta v - \delta v)\right\|_1 + \left\|r(\Delta v - \delta v)\right\|_\infty \\ &\leq 2\left\|\partial_r r(\Delta v - \delta v)\right\|_1 .\end{aligned}$$

A further application of lemma 4.9 gives sufficient bounds and as $\delta_\varphi u$ and $\delta_\Pi u$ are linear in $\delta\varphi$ and $\delta\Pi$ respectively this concludes the proof. ■

4.2.4. The mass

Definition 4.11 (the mass)**:** Given $(\varphi, \Pi) \in \mathfrak{B}$ and the corresponding solutions $u(\varphi, \Pi)$ and $v(\varphi, \Pi)$ of the two constraint equations, define the mass $\mathfrak{M}$ to be

$$\mathfrak{M}(\varphi, \Pi) = H(u(\varphi, \Pi), v(\varphi, \Pi), \varphi, \Pi)$$

One obtains the first expression for the mass $\mathfrak{M}$ by inserting the solution of first the constraint equation (4.14) into the Hamiltonian (4.13). The field-part of the Hamiltonian can conveniently be expressed by the right hand side of (4.14). The Mass than reads[9]

$$\mathfrak{M} = \frac{1}{2G}\int_0^\infty \left[\left(e^v - e^{-v}\right)e^u + 2\left(1 - e^{-v}\right)\frac{\partial}{\partial r}\left(re^u\right) - e^{u+v}\frac{\partial}{\partial r}\left(re^{-2v} - r\right)\right] .$$

Integration by parts of the term containing the partial derivative of e^u results in a cancellation of the integrand, the boundary term however does not vanish in the limit $r \to \infty$.[10] Hence one is left with

$$\mathfrak{M} = \lim_{r\to\infty} G^{-1} r e^u \left(1 - e^{-v}\right) .$$

This is by the definition of Arnowitt, Deser and Misner [2] the total mass $\mathfrak{M}$ of the scalar field coupled to the background metric.

From the constraint equation, which is Einstein's equation describing the energy density, one can obtain different expressions for the mass by integration [16], the first

[9] Note that u and v denote the solutions to the constraint equations here.

[10] In the case of the usual assumptions for an asymptotically flat space-time, cf. section 4.2.1, the term gives the constant M.

being the integral of the left hand side of (4.14) with an additional factor of $(2G)^{-1}$ to be consistent with Einsteins equations.

$$\mathfrak{M} = 4\pi \int_0^\infty \mathrm{d}r \left(m^2 |\varphi|^2 + e^{-2v} \left| \frac{\partial \varphi}{\partial r} \right|^2 + \left(16\pi^2 r^4\right)^{-1} e^{-2v} |\Pi|^2 \right) r^2.$$

Whereas the previous one shows the positivity of the mass, the most convenient one for our purposes is the integral of the right hand side of (4.14)

$$\mathfrak{M} = (2G)^{-1} \lim_{r\to\infty} r \left(1 - e^{-2v}\right) = (2G)^{-1} \lim_{r\to\infty} r\Xi. \tag{4.31}$$

Variation of the mass

In the following we consider the equations arising from variations of the mass $\mathfrak{M}$. Based on the studies in the preceding section the equivalence of this variational problem and Hamilton's equations of motion will be proven.

Theorem 4.12: The variational problem $\delta\mathfrak{M} = 0$ is equivalent to the static version of Hamilton's equations of motion in the sense that $(\varphi, \Pi) \in \mathfrak{B}$ is a critical point of $\mathfrak{M}$ if and only if it is a critical point of H.

Proof. The statement of the lemma is a direct consequence of the Fréchet differentiability of the Hamiltonian, cf. lemma 4.4 and the solutions to the constraint equations, cf. lemma 4.9 and 4.10 respectively, which via the chain rule entail

$$\begin{aligned}
0 = \frac{\delta\mathfrak{M}}{\delta\varphi} &= \frac{\delta}{\delta\varphi} H(u(\varphi,\Pi), v(\varphi,\Pi), \varphi, \Pi) \\
&= \frac{\delta}{\delta u'} H(u', v(\varphi,\Pi), \varphi, \Pi)\bigg|_{u'=u(\varphi,\Pi)} \frac{\delta u(\varphi,\Pi)}{\delta\varphi} \\
&\quad + \frac{\delta}{\delta v'} H(u(\varphi,\Pi), v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi)} \frac{\delta v(\varphi,\Pi)}{\delta\varphi} \\
&\quad + \frac{\delta}{\delta\varphi} H(u', v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi), u'=u(\varphi,\Pi)}
\end{aligned}$$

and

$$\begin{aligned}
0 = \frac{\delta\mathfrak{M}}{\delta\Pi} &= \frac{\delta}{\delta\Pi} H(u(\varphi,\Pi), v(\varphi,\Pi), \varphi, \Pi) \\
&= \frac{\delta}{\delta u'} H(u', v(\varphi,\Pi), \varphi, \Pi)\bigg|_{u'=u(\varphi,\Pi)} \frac{\delta u(\varphi,\Pi)}{\delta\Pi} \\
&\quad + \frac{\delta}{\delta v'} H(u(\varphi,\Pi), v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi)} \frac{\delta v(\varphi,\Pi)}{\delta\Pi} \\
&\quad + \frac{\delta}{\delta\Pi} H(u', v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi), u'=u(\varphi,\Pi)} .
\end{aligned}$$

As $u(\varphi, \Pi)$ and $v(\varphi, \Pi)$ solve the constraint equations and have well defined derivatives, these equations are equal to

$$0 = \frac{\delta \mathfrak{M}}{\delta \varphi} = \frac{\delta}{\delta \varphi} H(u', v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi), u'=u(\varphi,\Pi)} \qquad \text{and}$$
$$0 = \frac{\delta \mathfrak{M}}{\delta \Pi} = \frac{\delta}{\delta \Pi} H(u', v', \varphi, \Pi)\bigg|_{v'=v(\varphi,\Pi), u'=u(\varphi,\Pi)} .$$

These last two equations together with the two constraint equations are the static version of Hamilton's equations of motion and as the equations are, there solutions are also identical. ■

4.3. Stability of the classical complex scalar field

In order to give a meaning to the question of stability, i.e. the existence of a positive lower bound of the mass, we will be investigating the variation of the mass with an additional constraint - namely a fixed total charge. To introduce this constraint we define the charge (or particle number).

Definition 4.13: Given $(\varphi, \Pi) \in \mathfrak{F}^m$ the charge / particle number N is defined as

$$N = i \int_0^\infty \mathrm{d}r \left(\overline{\varphi \Pi} - \varphi \Pi \right) . \tag{4.32}$$

As an immediate consequence of definition 4.2 the particle number is finite. Furthermore it is a conserved quantity if the field solves Hamilton's equations of motion[11]. Finally note, that a non-zero charge excludes static solutions, as the relation between φ and Π, cf. equation (4.5), implies that the momentum vanishes if the field is constant in time.

Fixing a value $\mathfrak{N}$ during the variation, amounts to a restriction of the solutions of the equations of motion to a corresponding subset of $\mathfrak{F}^m$.

Before we address the problem of the constrained variation, we will infer some properties of generic solutions from the equations.

Proposition 4.14: The absolute value $|\varphi(r,t)|$ of any solution φ is constant in time.

Proof. The sum of the two constraint equations (4.14) and (4.15) yields

$$(8\pi G m^2 r^2)^{-1} \left[\partial_r r \left(1 - e^{-2v}\right) - e^{-2v} + 1 - 2r(\partial_r u) e^{-2v} \right] = |\varphi|^2 .$$

As the left hand side only includes quantities, that are time-independent by assumption, the statement follows. ■

[11] The conservation is an immediate consequence of Hamilton's equations of motion

Consequently, the time behavior has to be determined by a potentially r dependent phase.

Lemma 4.15: For any solution to with a nonzero particle number, the phase is constant on connected parts $I_j,\ j \in \underline{n}$ of the support of f with characteristic functions χ_j. For some $\omega_j \in \mathbb{R}$ the solution can be written as

$$\varphi(t,r) = f(r) \sum_{j \in \underline{n}} \chi_j e^{i\omega_j t}.$$

Proof. By theorem 4.1 and as the space-time is assumed to be static, a solution of the problem solves in particular

$$0 = \left(\frac{\partial \varphi}{\partial r}\right) \frac{\partial \overline{\varphi}}{\partial t} + \left(\frac{\partial \overline{\varphi}}{\partial r}\right) \frac{\partial \varphi}{\partial t}. \tag{4.33}$$

By proposition 4.14 $\varphi(t,r) = e^{i\omega(r)t} f(r)$. Inserting this into equation (4.33), one obtains

$$0 = 2t\omega(r)\omega'(r) \left|f(r)\right|^2 - 2i\omega(r) \,\mathfrak{Re}\, \left(f'(r)\overline{f}(r)\right). \tag{4.34}$$

As the first summand is real and the second one is purely imaginary, hence they need to vanish individually. For the purpose of proving the lemma considering the first summand is sufficient. As equation (4.34) has to be satisfied for all t and r, one can discard the multiplication by t and is left with

$$0 = \omega'(r)\omega(r) \left|f(r)\right|^2 \tag{4.35}$$

For this to be true, there are the cases $\omega = 0$ and $f = 0$, which lead to vanishing charge and hence contradict the assumptions, furthermore there are the following four possibilities:

First, assume that the interior of the support of ω and f is disjoint[12]. This implies that the solution is time independent which is excluded by having a non-zero charge.

Second, let ω' and ω have disjoint support in the above sense, then $\omega = 0$. Again, this is ruled out by the charge requirement.

Third, consider ω' and f to have disjoint support as before. This amounts to a piecewise constant phase of the solution.

Finally, a constant phase, i.e. $\omega' = 0$, solves the equation.

Therefore we find, that the only possibilities compatible with the charge constraint are a constant, or a piecewise constant phase. A multiplication of $\exp(i\omega(r)t)$ by the sum of the characteristic functions χ_j of the connected components of the support of f yields the desired form of the solution, as

$$\begin{aligned}\varphi(r,t) = e^{i\omega(r)t} f(r) = e^{i\omega(r)t} \sum_j \chi_j f(r) = f(r) \sum_j \chi_j e^{i\omega(r)t} \\ = f(r) \sum_{j \in \underline{n}} \chi_j e^{i\omega_j t}.\end{aligned}$$

■

[12] The assumption $\left\|\partial_r \varphi\right\|_\infty$ implies, that φ vanishes on $\partial\left(\mathrm{supp}(\varphi)\right)$.

Proposition 4.16: The Fréchet derivative of the number constraint $(N - \mathfrak{N}) = G_{\mathfrak{N}} : \mathfrak{F}^m \to \mathbb{R}$ exists on $\mathfrak{F}^m$, is bounded, real linear, and has full range.

Proof. The Fréchet derivative of $G_{\mathfrak{N}}$ is given by

$$\delta_{\varphi,\Pi} G_{\mathfrak{N}}(\delta\varphi, \delta\Pi) = i \int_0^\infty \mathrm{d}r \, \left(\overline{\varphi\delta\Pi} + \overline{\Pi\delta\varphi} - \varphi\delta\Pi - \Pi\delta\varphi\right), \qquad (4.36)$$

as

$$\Delta G_{\mathfrak{N}} - \delta_{\varphi,\Pi} G_{\mathfrak{N}} = 0.$$

Furthermore one finds the following bound:

$$\begin{aligned} |\delta_{\varphi,\Pi} G_{\mathfrak{N}}| &\leq 2 \left(\|r\varphi\|_2 \left\|r^{-1}\delta\Pi\right\|_2 + \left\|r^{-1}\Pi\right\|_2 \|r\delta\varphi\|_2 \right) \\ &\leq C(\varphi, \Pi) \left(\left\|r^{-1}\delta\Pi\right\|_2 + \left\|r^{-2}\delta\Pi\right\|_\infty + \|r\delta\varphi\|_2 + \|\delta\varphi\|_\infty \right). \end{aligned}$$

As $\delta_{\varphi,\Pi} G_{\mathfrak{N}}$ is real linear in the variations, cf. equation (4.36), the previous bound entails continuity. For $c \in \mathbb{R}$ pick $\delta\varphi = 0$ and $\delta\Pi = i\frac{c}{2} r^2 \|r\varphi\|_2^{-2} \overline{\varphi}$. Then $\|r^{-1}\delta\Pi\|_2 = |c| \, \|r\varphi\|_2^{-1} < \infty$, and $\|r^{-2}\delta\Pi\|_\infty = \left|\frac{c}{2}\right| \|r\varphi\|_2^{-2} \|\varphi\|_\infty < \infty$, and thereby $\delta\Pi \in \mathfrak{F}^\Pi$. Furthermore $\delta\varphi \in \mathfrak{F}^\varphi$.

With these choices one finds

$$\delta_{\varphi,\Pi} G_{\mathfrak{N}}(\delta\varphi, \delta\Pi) = c. \qquad \blacksquare$$

The previous proposition allows us to use the following lemma on Lagrange multipliers:

Lemma 4.17: (Corollary 43.22. in [51])
Suppose the conditions

1. X and Y are real Banach spaces and $F : U(u_0) \subseteq X \to R$ is Fréchet differentiable at u_0,

2. $G : U(u_0) \subseteq X \to Y$ is Fréchet differentiable in an open neighborhood of u_0,

3. the Fréchet derivative of G has a closed range and is continuous at u_0,

are satisfied. Then the following holds: If F has a critical point with respect to the set $\{u \in D(G) : G(u) = 0\}$ at u_0, then there exist a λ_0 in $\mathbb{R}$ and a $\Lambda \in Y^*$, where not both are simultaneously equal to zero, such that for all $\delta u \in X$

$$\lambda_0 \delta_{u_0} F(\delta u) - \Lambda\left(\delta_{u_0} G(\delta u)\right) = 0.$$

If $\delta_{u_0} G$ has full range, on can choose $\lambda_0 = 1$.

Setting $U = \mathfrak{F}^m$,$X = L^\infty(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$, $Y = \mathbb{R}$, $F(\varphi, \Pi) = H(\varphi, \Pi)$, and $G(\varphi, \Pi) = G_{\mathfrak{N}}(\varphi, \Pi)$, the conditions in lemma 4.17 are satisfied by lemma 4.4 and proposition 4.16. As the Fréchet derivative of G furthermore hast full range, we can choose $\lambda_0 = 1$.

Thereby a non-trivial solution to the variational problem yields a solution to the constraint one.

The constraint Hamiltonian density is given by

$$\mathcal{H}_{\Lambda,\mathfrak{N}} = \mathcal{H} + \Lambda\left[i\left(\overline{\varphi\Pi} - \varphi\Pi\right) - \mathfrak{N}\right]$$

where $\Lambda \in \mathbb{R}$ is the Lagrange multiplier. As the modification is independent of the functions u and v, the respective variational equations will be unchanged. This implies in particular that the solutions to the constraint equations will be equal to the unconstraint case and that the mass $\mathfrak{M}_{\Lambda,\mathfrak{N}} = H_{\Lambda,\mathfrak{N}}(u(\varphi, \Pi), v(\varphi, \Pi), \varphi, \Pi)$ in the constraint case will equal $\mathfrak{M} + \Lambda[N - \mathfrak{N}]$. The variation of $\mathfrak{M}_{\Lambda,\mathfrak{N}}$ with respect to the field together with Hamilton's equations of motion yields

$$0 = \frac{\delta\mathcal{H}}{\delta\varphi} - i\Lambda\Pi = \partial_t\Pi - i\Lambda\Pi$$

and the analogues for the complex conjugate as well as for the momentum. The solutions to these equations are a radial function multiplied by a time dependent phase and we will refer to them as stationary solutions.

Proposition 4.18: Every non-static solution to Einstein's equations in the setting under consideration is stationary, i.e. is of the form

$$\varphi(t, r) = e^{i\omega t} f(r).$$

Proof. By lemma 4.15 every solution with non vanishing particle number is of the form

$$\varphi(t, r) = f(r) \sum_{j \in \underline{n}} \chi_j e^{i\omega_j t}.$$

Given a solution (φ, Π), we define $\mathfrak{N} = N(\varphi, \Pi)$. Then (φ, Π) solves the constraint problem and by lemma 4.17, φ solves the equations

$$0 = -\partial_t\varphi - i\Lambda\varphi$$

for a fixed $\Lambda \in \mathbb{R}$ which implies $\omega_j = -\Lambda$ for all $j \in \underline{n}$, as

$$-\partial_t\varphi(t, r) - i\Lambda\varphi(t, r) = -f(r) \sum_{j \in \underline{n}} i(\omega_j + \Lambda)\chi_j e^{i\omega_j t} = 0. \qquad \blacksquare$$

Lemma 4.19: For $(\varphi, \Pi) \in \mathfrak{F}^m$ the following equality holds:

$$\begin{aligned}
&\Xi(\varphi,\Pi)(r)\\
&= \frac{8\pi G}{r}\int_0^r \mathrm{d}r''\; r''^2 m^2 |\varphi|^2 \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right)\\
&\quad + 1 - \frac{1}{r}\int_0^r \mathrm{d}r''\; \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right).
\end{aligned}$$

Proof. Note that by integration by parts

$$\begin{aligned}
&\frac{8\pi G}{r}\int_0^r \mathrm{d}r''\; r''^2 \left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\\
&\qquad\qquad \times \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right)\\
&= \frac{1}{r}\left[r''\exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right)\right]_0^r\\
&\quad - \frac{1}{r}\int_0^r \mathrm{d}r''\; \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right)\\
&= 1 - \frac{1}{r}\int_0^r \mathrm{d}r''\; \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_{r'}\varphi|^2 + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right).
\end{aligned}$$

Insertion of the above into the defining equation (4.16) of $\Xi(\varphi, \Pi)(r)$ gives the desired equality. ■

The previous lemma entails:

Corollary 4.20: In the case of a massless scalar field $\mathfrak{B} = \mathfrak{F}^m$.

Proof. For $m = 0$ lemma 4.19 gives

$$\begin{aligned}
&\Xi(\varphi,\Pi)(r)\\
&= 1 - \frac{1}{r}\int_0^r \mathrm{d}r''\; \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\; r'\left(|\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}|\Pi|^2\right)\right)\\
&\le 1 - \exp\left[-\left(\|\partial_r\varphi\|_\infty^2 + \|\partial_r\varphi\|_2^2 + (16\pi^2)^{-1}\left(\left\|r^{-1}\Pi\right\|_\infty^2 + \left\|r^{-1}\Pi\right\|_2^2\right)\right)\right]\\
&< 1,
\end{aligned}$$

which proves the statement. ■

Lemma 4.21: Assume $(\varphi, \Pi) \in \mathfrak{F}^m$ have compact support in $[r_0, r_1]$, $0 \le r_0 < r_1 < +\infty$, and define for $\alpha < 1$ the rescaled functions

$$\varphi_\lambda(r) = \lambda^\alpha \varphi(\lambda r),$$
$$\Pi_\lambda(r) = \lambda^{1-\alpha}\Pi(\lambda r).$$

Then $(\varphi_\lambda, \Pi_\lambda) \in \mathfrak{B}$ if λ is large enough.

Proof. The two relevant integrals for Ξ as a function of the rescaled fields written as in lemma 4.19 read

$$\begin{aligned}
&\int_0^r \mathrm{d}r'' \; \exp\left(-8\pi G \int_{r''}^{r} \mathrm{d}r' \; r' \left(|\partial_{r'}\varphi_\lambda|^2 + (16\pi^2 r^4)^{-1} |\Pi_\lambda|^2\right)\right) \\
&= \int_0^r \mathrm{d}r'' \; \exp\left(-8\pi G \int_{r''}^{r} \mathrm{d}r' \; r' \left(\lambda^{2\alpha} |\partial_{r'}\varphi(\lambda r')|^2 + \lambda^{2-2\alpha}(16\pi^2 r^4)^{-1} |\Pi(\lambda r')|^2\right)\right) \\
&= \int_0^r \mathrm{d}r'' \; \exp\left(-8\pi G \int_{\lambda r''}^{\lambda r} \mathrm{d}r' \; r' \left(\lambda^{2\alpha} |\partial_{r'}\varphi(r')|^2 + \lambda^{4-2\alpha}(16\pi^2 r^4)^{-1} |\Pi(r')|^2\right)\right) \\
&= \tfrac{1}{\lambda} \int_0^{\lambda r} \mathrm{d}r'' \; \exp\left(-8\pi G \int_{r''}^{\lambda r} \mathrm{d}r' \; r' \left(\lambda^{2\alpha} |\partial_{r'}\varphi|^2 + \lambda^{4-2\alpha}(16\pi^2 r^4)^{-1} |\Pi|^2\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^r \mathrm{d}r'' \; r''^2 \, |\varphi_\lambda|^2 \exp\left(-8\pi G \int_{r''}^{r} \mathrm{d}r' \; r' \left(|\partial_{r'}\varphi_\lambda|^2 + (16\pi^2 r^4)^{-1} |\Pi_\lambda|^2\right)\right) \\
&= \int_0^r \mathrm{d}r'' \; \lambda^{2\alpha} r''^2 \, |\varphi(\lambda r'')|^2 \exp\left(-8\pi G \int_{\lambda r''}^{\lambda r} \mathrm{d}r' \; r' \Big(\lambda^{2\alpha} |\partial_{r'}\varphi|^2\right. \\
&\qquad\qquad\qquad\qquad\left. + \lambda^{4-2\alpha}(16\pi^2 r^4)^{-1} |\Pi|^2 \Big)\right) \\
&= \int_0^{\lambda r} \mathrm{d}r'' \; \lambda^{2\alpha-3} r''^2 \, |\varphi|^2 \exp\left(-8\pi G \int_{r''}^{\lambda r} \mathrm{d}r' \; r' \Big(\lambda^{2\alpha} |\partial_{r'}\varphi|^2\right. \\
&\qquad\qquad\qquad\qquad\left. + \lambda^{4-2\alpha}(16\pi^2 r^4)^{-1} |\Pi|^2 \Big)\right).
\end{aligned}$$

Using these expressions, one obtains

$$
\begin{aligned}
&\Xi(\varphi_\lambda, \Pi_\lambda)(r) \\
&= 1 - \tfrac{1}{\lambda r} \int\limits_0^{\lambda r} \mathrm{d}r'' \, \left(1 - \lambda^{2\alpha-2} 8\pi G r''^2 m^2 \left|\varphi\right|^2\right) \\
&\qquad\qquad \times \exp\left(-8\pi G \int_{r''}^{\lambda r} \mathrm{d}r' \, r' \left(\lambda^{2\alpha} \left|\partial_{r'}\varphi\right|^2 + \lambda^{4-2\alpha}(16\pi^2 r^4)^{-1} \left|\Pi\right|^2\right)\right) .
\end{aligned}
$$

Now choose λ_0 such that $\lambda_0^{2-2\alpha} \geq 8\pi G r_1^2 m^2 \left\|\varphi\right\|_\infty^2$. As $2\alpha - 2 < 0$ by assumption, the integrand is positive for all $\lambda > \lambda_0$ and consequently $\left\|\Xi(\varphi_\lambda, \Pi_\lambda)\right\|_\infty < 1$ which concludes the proof of the lemma. ■

We finally address the question of stability based on our preceding discussions that culminate in the following theorem:

Theorem 4.22 (Instability of a classical scalar field coupled to a spherically symmetric metric on $\mathbb{R}^{1+3}$)**:** The total mass $\mathfrak{M}$ of a classical complex scalar field (φ, Π) on a spherically symmetric space-time satisfying

$$\|(\varphi,\Pi)\|_2^2 = \|r\varphi\|_2^2 + \|r\partial_r\varphi\|_2^2 + (16\pi^2)^{-1}\left\|r^{-1}\Pi\right\|_2^2 < \infty \quad \text{and}$$
$$\|(\varphi,\Pi)\|_\infty = \|\varphi\|_\infty + \|\partial_r\varphi\|_\infty + (4\pi)^{-1}\left\|r^{-2}\Pi\right\|_\infty < \infty,$$

where the norms are taken with respect to the spatial variable, as well as

$$1 > \Xi(\varphi,\Pi) = \frac{8\pi G}{r}\int_0^r \mathrm{d}r'' r''^2 \left(m^2|\varphi|^2 + \left|\frac{\partial\varphi}{\partial r''}\right|^2 + \frac{1}{16\pi^2 r''^4}|\Pi|^2\right) \times \exp\left(-8\pi G\int_{r''}^r \mathrm{d}r' r' \left(\left|\frac{\partial\varphi}{\partial r'}\right|^2 + \frac{1}{16\pi^2 r'^4}|\Pi|^2\right)\right)$$

is given by

$$\mathfrak{M} = 4\pi\int_0^\infty \mathrm{d}r\, r^2\left(m^2|\varphi|^2 + |\partial_r\varphi|^2 + (16\pi^2 r^4)^{-1}|\Pi|^2\right) \times \exp\left(-8\pi G\int_r^\infty \mathrm{d}r'\, r'\left(|\partial_{r'}\varphi| + (16\pi^2 r'^4)^{-1}|\Pi|^2\right)\right)$$

and for a fixed total charge N there exists no positive lower bound for $\mathfrak{M}$ on $\mathfrak{B} \subseteq \mathfrak{F}^m$.

Proof of theorem 4.22. The conditions on the field correspond to $(\varphi, \Pi) \in \mathfrak{B}$ and by corollary 4.20 and lemma 4.21 there exist fields of this kind that are furthermore compactly supported.

For $(\varphi, \Pi) \in \mathfrak{B}$ the solutions (4.18) and (4.17) to the constraint equations (4.14) and (4.15) are well defined and Fréchet differentiable, cf. lemma 4.9 and 4.10. Thereby the total mass $\mathfrak{M}$ is well defined and given as

$$(2G)^{-1}\lim_{r\to\infty} r\Xi(\varphi,\Pi),$$

cf. equation (4.31) and the preceding discussion. By dominated convergence and writing out $\Xi(\varphi, \Pi)$ explicitly one obtains the expression in the theorem.

Now let $(\varphi, \Pi) \in \mathfrak{B}$ with compact support in $[r_0, r_1]$ with $0 \le r_0 < r_1 < \infty$ and $\alpha\mathbb{R}_+$ such that $\alpha < 1$. For $\lambda \in \mathbb{R}_+$ we define rescaled fields as in the proof of lemma 4.21:

$$\varphi_\lambda(r) = \lambda^\alpha\varphi(\lambda r) \quad \text{and}$$
$$\Pi_\lambda(r) = \lambda^{1-\alpha}\Pi(\lambda r).$$

Note that the rescaling leaves the total charge invariant as

$$\int_0^\infty \mathrm{d}r\ \varphi_\lambda \Pi_\lambda = \int_0^\infty \mathrm{d}r\ \lambda\varphi(\lambda r)\Pi(\lambda r) = \int_0^\infty \mathrm{d}r\ \varphi\Pi.$$

The mass of the rescaled fields reads

$$\begin{aligned}
&4\pi \int_0^\infty \mathrm{d}r\ r^2 \left(m^2 |\varphi_\lambda|^2 + |\partial_r \varphi_\lambda|^2 + (16\pi^2 r^4)^{-1} |\Pi_\lambda|^2\right) \\
&\qquad \times \exp\left(-8\pi G \int_r^\infty \mathrm{d}r'\ r' \left(|\partial_{r'}\varphi_\lambda| + (16\pi^2 r'^4)^{-1} |\Pi_\lambda|^2\right)\right) \\
&= 4\pi \int_0^\infty \mathrm{d}r\ r^2 \Big(m^2 \lambda^{2\alpha} |\varphi(\lambda r)|^2 + \lambda^{2\alpha} |\partial_r \varphi(\lambda r)|^2 \\
&\qquad + \lambda^{6-2\alpha} (16\pi^2 (\lambda r)^4)^{-1} |\Pi(\lambda r)|^2\Big) \\
&\qquad \times \exp\Bigg(- 8\pi G \int_r^\infty \mathrm{d}r'\ r' \Big(\lambda^{2\alpha} |\partial_{r'}\varphi(\lambda r')|^2 \\
&\qquad\qquad + \lambda^{6-2\alpha} (16\pi^2 (\lambda r')^4)^{-1} |\Pi(\lambda r')|^2 \Big)\Bigg) \\
&= 4\pi \int_0^\infty \mathrm{d}r\ r^2 \Big(\dots\Big) \exp\Bigg(- 8\pi G \int_{\lambda r}^\infty \mathrm{d}r'\ r' \Big(\lambda^{2\alpha} |\partial_{r'}\varphi(r')|^2 \\
&\qquad\qquad + \lambda^{4-2\alpha} (16\pi^2 (r')^4)^{-1} |\Pi(r')|^2 \Big)\Bigg) \\
&= 4\pi \int_0^\infty \mathrm{d}r\ r^2 \left(m^2 \lambda^{2\alpha-3} |\varphi(r)|^2 + \lambda^{2\alpha-1} |\partial_r \varphi(r)|^2 + \lambda^{3-2\alpha} (16\pi^2 r^4)^{-1} |\Pi(r)|^2\right) \\
&\qquad\qquad \times \exp\Bigg(- 8\pi G \int_r^\infty \mathrm{d}r'\ r' \Big(\dots\Big)\Bigg)
\end{aligned}$$

As the exponent of λ in the first summand is negative the latter can be bounded from above by $\lambda^{2\alpha-3} 4\pi m^2 \|r\varphi\|_2^2$. Including the support properties we will bound the remaining two summands

$$\begin{aligned}
&4\pi \int_0^{r_1} \mathrm{d}r\ r^2 \left(\lambda^{2\alpha-1} |\partial_r \varphi(r)|^2 + \lambda^{3-2\alpha} (16\pi^2 r^4)^{-1} |\Pi(r)|^2\right) \\
&\qquad \times \exp\Bigg(- 8\pi G \int_r^{r_1} \mathrm{d}r'\ r' \Big(\lambda^{2\alpha} |\partial_{r'}\varphi|^2 + \lambda^{4-2\alpha} (16\pi^2 r'^4)^{-1} |\Pi|^2 \Big)\Bigg).
\end{aligned}$$

An integration by parts turns the above into

$$\lambda^{-\alpha}(2G)^{-1}\, r \exp\left(-8\pi G \int_r^{r_1} \mathrm{d}r'\, r' \Big(\lambda^{2\alpha} \left|\partial_{r'}\varphi\right|^2 + \lambda^{4-2\alpha}(16\pi^2 r'^4)^{-1} \left|\Pi\right|^2 \Big)\right)\Bigg|_{r_0}^{r_1}$$

$$-\lambda^{-\alpha}(2G)^{-1} \int_0^{r_1} \mathrm{d}r\ \exp\left(-8\pi G \int_r^{r_1} \mathrm{d}r'\, r' \Big(\lambda^{2\alpha} \left|\partial_{r'}\varphi\right|^2 + \lambda^{4-2\alpha}(16\pi^2 r'^4)^{-1} \left|\Pi\right|^2 \Big)\right).$$

The above is then bounded by

$$\lambda^{-\alpha}(2G)^{-1} r_1.$$

Consequently the total mass of the rescaled fields is bounded by

$$\lambda^{-(\alpha+2\beta)} 4\pi m^2 \left\|r\varphi\right\|_2^2 + \lambda^{-\alpha}(2G)^{-1} r_1$$

which tends to 0 in the limit $\lambda \to \infty$ and as the total charge was unaffected by the rescaling this concludes the proof. ■

4.4. Stability of a multi-component scalar field

In the following we consider the same basic problem as before but with a multi-component scalar field, i.e. a field consisting of k components $\varphi_i,\ i \in \underline{k}$[13] with masses $m_i,\ i \in \underline{k}$. We will reproduce all the results proven for the single component field.

The matter Lagrangian in this case reads

$$L_k^m(u, v, \varphi_1, \dots, \varphi_k) = 4\pi \int_0^\infty \mathrm{d}r\ r^2 e^{u+v} \sum_{i=1}^{k} \Big(-m_i^2 \left|\varphi_i\right|^2 - e^{-2v} \left|\partial_r \varphi_i\right|^2 + e^{-2u} \left|\partial_t \varphi_i\right|^2 \Big) \tag{4.37}$$

whereas the gravitational Lagrangian (4.1) is unchanged. The canonical momenta corresponding to the fields φ_i are consequently given as

$$\Pi_i = 4\pi r^2 e^{v-u} \partial_t \overline{\varphi}_i, \qquad i \in \underline{k}$$

and accordingly in the complex conjugate cases. The Euler-Lagrange equations with respect to the fields are as in the single component field case given by Klein-Gordon

[13] For $k \in \mathbb{N}$ define $\underline{k} = \{1, \dots, k\}$.

equations (4.2) for the individual components and their complex conjugates respectively. As before the Euler-Lagrange equations with respect to u and v play the role of constraint equations. The first one is

$$r^{-2}\partial_r r\left(e^{-2v}-1\right) = -8\pi G\sum_{i=1}^{k}\left(m_i^2\left|\varphi_i\right|^2 + e^{-2v}\left|\partial_r\varphi_i\right|^2 + e^{-2u}\left|\partial_t\varphi_i\right|^2\right),$$

or equivalently written in terms of the fields and their momenta

$$\begin{aligned} &= -8\pi G\sum_{i=1}^{k}\left(m_i^2\left|\varphi_i\right|^2 + e^{-2v}\left|\partial_r\varphi_i\right|^2\right. \\ &\qquad\left. +(16\pi^2r^4)^{-1}e^{-2v}\left|\Pi_i\right|^2\right). \end{aligned} \tag{4.38}$$

Similarly on finds the second constraint equation for the multi-component system to be

$$\begin{aligned} r^{-2}\left(e^{-2v}-1+2re^{-2v}\partial_r u\right) = 8\pi G\sum_{i=1}^{k}\Big(&e^{-2v}\left|\partial_r\varphi_i\right|^2 + (16\pi^2r^4)^{-1}e^{-2v}\left|\Pi_i\right|^2 \\ &-m_i^2\left|\varphi_i\right|^2\Big). \end{aligned} \tag{4.39}$$

From the Lagrangian given by the sum of the gravitational Lagrangian (4.1) and the multi-component matter Lagrangian (4.37) one obtains the Hamiltonian

$$\begin{aligned} H_k = \int_0^\infty \mathrm{d}r\,\Bigg[&4\pi e^{u+v}r^2\sum_{i=1}^{k}\left(m^2\left|\varphi_i\right|^2 + e^{-2v}\left|\frac{\partial\varphi_i}{\partial r}\right|^2 + \left(16\pi^2r^4\right)^{-1}e^{-2v}\left|\Pi_i\right|^2\right) \\ &+(2G)^{-1}e^u\left(2-e^v-e^{-v}+2r\left(1-e^{-v}\right)\frac{\partial u}{\partial r}\right)\Bigg]. \end{aligned} \tag{4.40}$$

Definition 4.23: For $k \in \mathbb{N}$ we define the multi-component scalar field $\left(\underline{\varphi,\Pi}\right)$ as a collection

$$\left(\underline{\varphi,\Pi}\right) = (\varphi_1,\ldots\varphi_k,\Pi_1,\ldots\Pi_k)$$

of k pairs $(\varphi_i,\Pi_i) \in \mathfrak{F}^m$. We refer to the space of multi-component scalar fields with k components as $\mathfrak{F}^m_k$.

Lemma 4.24: Let $(u,v) \in \mathfrak{F}^g$ and $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{F}^m_k$, then the Hamiltonian (4.40) is well-defined and Fréchet differentiable with respect to u, v, and all fields φ_i, as well as the momenta Π_i .

Proof. The well definedness of the gravitational Hamiltonian follows from definition 4.2 and proposition 4.3.

Note that the purely gravitational part is unchanged and by lemma 4.4 Fréchet differentiable with respect to u and v. It remains to prove the statement for the field dependent part of the Hamiltonian.

As the matter Hamiltonian decomposes as follows:

$$H_k^m\left(u, v, \left(\underline{\varphi, \Pi}\right)\right) = \sum_{i=1}^{k} H^m(u, v, \varphi_i, \Pi_i),$$

we can apply the same steps as in the proof of lemma 4.4 to each of the summands of this decomposition and obtain the necessary bounds and conclude the as the required linearity properties of the variational differentials follow for any field component analogously to the single field case. ■

As for the single component scalar field we can solve the two constraint equations.

Lemma 4.25: For a multi-component scalar field $\left(\underline{\varphi, \Pi}\right) \in \mathfrak{F}_k^m$, the first constraint equation (4.38) has the following unique solution for $1 - e^{-2v}$, that is regular at the origin:

$$\tilde{\Xi}\left(\left(\underline{\varphi, \Pi}\right)\right) =$$

$$\frac{8\pi G}{r}\left[\int_0^r \mathrm{d}r''(r'')^2 \sum_{i=1}^{k}\left(m^2\left|\varphi_i\right|^2 + \left|\frac{\partial \varphi_i}{\partial(r'')}\right|^2 + \frac{1}{16\pi^2 (r'')^4}\left|\Pi_i\right|^2\right) \times \right.$$
$$\left. \exp\left(-8\pi G \int_{r''}^{r} \mathrm{d}r'(r') \sum_{j=1}^{k}\left(\left|\frac{\partial \varphi_j}{\partial(r')}\right|^2 + \frac{1}{16\pi^2 (r')^4}\left|\Pi_j\right|^2\right)\right)\right].$$

Furthermore $\tilde{\Xi}$ satisfies the bounds in proposition 4.6 with $\|(\varphi, \Pi)\|_\infty^2$ and $\|(\varphi, \Pi)\|_2^2$ replaced by the sums $\sum_{i=1}^k \|(\varphi_i, \Pi_i)\|_\infty^2$ and $\sum_{i=1}^k \|(\varphi_i, \Pi_i)\|_2^2$ respectively.

In addition $\tilde{\Xi}$ is Fréchet differential on the multi-component scalar fields $\mathfrak{F}_k^m$.

Proof. The prove that $\tilde{\Xi}$ solves the first constraint equation in the multi-component case, one proceeds analogously to the proof for the single component case, cf. section 4.2.3.

The bounds in proposition 4.6 follow directly from the definitions 4.23 and 4.2.

The proof of the Fréchet differentiability of $\tilde{\Xi}$ does not need more sophisticated arguments than the proof of lemma 4.8. The bounds for the original case are valid for any of the components and redefining the $T, T_\delta, T_{\delta^2}, E, E_\delta$, and E_{δ^2} appropriately one can recycle all the arguments. ■

Definition 4.26: Define the subset $\mathfrak{B}_k \subseteq \mathfrak{F}_k^m$ by

$$\mathfrak{B}_k = \left\{\left(\underline{\varphi, \Pi}\right) \in \mathfrak{F}_k^m \;\middle|\; \left\|\tilde{\Xi}\left(\left(\underline{\varphi, \Pi}\right)\right)\right\|_\infty < 1\right\}.$$

Lemma 4.27: For all $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{B}_k$ define the function $\tilde{v}: \left(\underline{\varphi,\Pi}\right) \to \mathfrak{F}^v$

$$\tilde{v}\left(\left(\underline{\varphi,\Pi}\right)\right)(r) = -\tfrac{1}{2}\log\left(1 - \tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)(r)\right).$$

Then $\tilde{v}$ is non-negative, bounded and Fréchet differentiable on $\mathfrak{B}_k$.

Proof. The non-negativity of $\tilde{v}$ is an immediate consequence of the definition and the boundedness follows from lemma 4.25 and Taylor expansion, cf. equations (4.19), (4.20) and the succeeding arguments.

The Fréchet differentiability follows by the arguments of the proof of lemma 4.9 with minor adaptions, which is based on the properties of Ξ which by lemma 4.25 also hold for $\tilde{\Xi}$. ■

Lemma 4.28: For $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{B}_k$ and $\tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)(r)$ as above the second constraint equation rewritten as

$$\partial_r u = 4\pi G r \sum_{i=1}^{k}\left(|\partial_r\varphi_i|^2 + (16\pi^2 r^4)^{-1}|\Pi_i|^2\right) - \frac{8\pi G m^2 r^2 \sum_{j=1}^{k}|\varphi_j|^2 - \tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)}{2r\left(1 - \tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)\right)}$$

has the following unique solution that vanishing for $r \to \infty$:

$$\begin{aligned}\tilde{u}\left(\left(\underline{\varphi,\Pi}\right)\right)(r) = -\int_r^\infty \mathrm{d}r' \Bigg[&4\pi G r' \sum_{i=1}^{k}\left(|\partial_{r'}\varphi_i|^2 + (16\pi^2 r'^4)^{-1}|\Pi_i|^2\right) \\ &- \frac{8\pi G m^2 r'^2 \sum_{j=1}^{k}|\varphi_j|^2 - \tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)}{2r'\left(1 - \tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)\right)}\Bigg],\end{aligned} \tag{4.41}$$

which is Fréchet differentiable as a function $\tilde{u}: \mathrm{B}_k \to \mathfrak{F}^u$.

Proof. Once more the arguments of the single component case do essentially apply. Using the appropriate constraint equations (4.38) and (4.39), as well as the decay behaviors of u and v on finds an expression that together with lemma 4.27 can be straightforwardly bounded in a suitable manner when considering the L_∞-norm.

The prove of the appropriate bound of the second norm, $\|\partial_r \cdot\|_1$, is essentially based on the properties of Ξ, which still hold by lemma 4.25, and can therefore be adapted. ■

Definition 4.29: We define the total mass of the multi-component scalar field $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{B}_k$ coupled to the spherically symmetric background as

$$\tilde{\mathfrak{M}} = H_k\left(\tilde{u}\left(\left(\underline{\varphi,\Pi}\right)\right), \tilde{v}\left(\left(\underline{\varphi,\Pi}\right)\right), \left(\underline{\varphi,\Pi}\right)\right).$$

Note that the expression for the mass used in the following will be the integral of the left hand side of the first constraint equation (4.38), which reads

$$\tilde{\mathfrak{M}} = (2G)^{-1} \lim_{r\to\infty} r\tilde{\Xi}. \tag{4.42}$$

Furthermore an adapted statement of theorem 4.12 is true in this case. Again, we introduce the total charge to study the stability.

Definition 4.30: We define the total charge or particle number of the multi-component scalar field $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{F}_k^m$ as

$$\tilde{N} = i \int_0^\infty \mathrm{d}r \sum_{i=1}^k \left(\overline{\varphi_i \Pi_i} - \varphi_i \Pi_i\right).$$

In case where the φ_i and Π_i solve Hamilton's equations of motion the total charge is a conserved quantity, just as in the single component case. To study the stability via a scaling argument as in the single component case we prove an analogous identity to the one in lemma 4.19.

Lemma 4.31: For $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{F}_k^m$ the following identity

$$\begin{aligned}
&\frac{8\pi G}{r} \int_0^r \mathrm{d}r''\, r''^2 \sum_{i=1}^k \left(|\partial_r \varphi_i|^2 + (16\pi^2 r^4)^{-1} |\Pi_i|^2\right) \\
&\qquad\qquad \times \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right) \\
&= 1 - \frac{1}{r} \int_0^r \mathrm{d}r''\, \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right).
\end{aligned}$$

holds.

Proof. The identity follows by an integration by parts as in the proof of lemma 4.19.

$$\begin{aligned}
&\frac{8\pi G}{r} \int_0^r \mathrm{d}r''\, r''^2 \sum_{i=1}^k \left(|\partial_r \varphi_i|^2 + (16\pi^2 r^4)^{-1} |\Pi_i|^2\right) \\
&\qquad\qquad \times \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right) \\
&= \frac{1}{r} \left[r'' \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right)\right]_0^r \\
&\quad - \frac{8\pi G}{r} \int_0^r \mathrm{d}r'' \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right) \\
&= 1 - \frac{1}{r} \int_0^r \mathrm{d}r''\, \exp\left(-8\pi G \int_{r''}^r \mathrm{d}r'\, r' \textstyle\sum_{j=1}^k \left(|\partial_r \varphi_j|^2 + (16\pi^2 r^4)^{-1} |\Pi_j|^2\right)\right) \qquad \blacksquare
\end{aligned}$$

As previously, this lemma implies the that $\mathfrak{B}_k$ is non-empty.

Proposition 4.32: Let $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{F}^m$. If $m_i = 0\ \forall i \in \underline{k}$, then $\mathfrak{B}_k = \mathfrak{F}_k^m$. In the case of one or more m_i being nonzero, choose $\lambda \in \mathbb{R}_+$ as well as $0 < \alpha < 1$ and define

$$\begin{aligned}\left(\underline{\varphi,\Pi}\right)_\lambda &= \left(\lambda^{1-\alpha}\varphi_1(\lambda\cdot),\dots,\lambda^{1-\alpha}\varphi_k(\lambda\cdot),\lambda^\alpha\Pi_1(\lambda\cdot),\dots,\lambda^\alpha\Pi_k(\lambda\cdot)\right)\\ &= (\varphi_{1,\lambda},\dots,\varphi_{k,\lambda},\Pi_{1,\lambda},\dots,\Pi_{k,\lambda})\end{aligned}$$

Then $\left(\underline{\varphi,\Pi}\right)_\lambda \in \mathfrak{B}_k$ for λ large enough.

Proof. The first statement follows directly from the previous lemma 4.31. To prove the second one, rewrite $\tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)_\lambda\right)(r)$ as

$$\begin{aligned}&\frac{8\pi G}{r}\int_0^r \mathrm{d}r''\, r''^2 \sum_{i=1}^k \left(m_i^2\left|\varphi_{i,\lambda}\right|^2\right)\\ &\qquad\times\exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\, r'\textstyle\sum_{j=1}^k\left(\left|\partial_{r'}\varphi_{j,\lambda}\right|^2 + (16\pi^2 r'^4)^{-1}\left|\Pi_{j,\lambda}\right|^2\right)\right)\\ &+1-\frac{1}{r}\int_0^r \mathrm{d}r''\,\exp\left(-8\pi G\int_{r''}^r \mathrm{d}r'\, r'\textstyle\sum_{j=1}^k\left(\left|\partial_{r'}\varphi_{j,\lambda}\right|^2 + (16\pi^2 r'^4)^{-1}\left|\Pi_{j,\lambda}\right|^2\right)\right)\\ &=1-\frac{1}{\lambda r}\int_0^{\lambda r}\mathrm{d}r''\left(1-\lambda^{2\alpha-2}8\pi G r''^2\sum_{i=1}^k\left(m_i^2\left|\varphi_{i,\lambda}\right|^2\right)\right)\\ &\qquad\times\exp\left(-8\pi G\int_{r''}^{\lambda r}\mathrm{d}r'\, r'\textstyle\sum_{j=1}^k\left(\left|\partial_{r'}\varphi_{j,\lambda}\right|^2+(16\pi'^2 r^4)^{-1}\left|\Pi_{j,\lambda}\right|^2\right)\right).\end{aligned}$$

For $\lambda^{2-2\alpha} > 8\pi G r''^2\sum_{i=1}^k\left(m_i\left\|\varphi_{i,\lambda}\right\|_\infty\right)^2$ the integrand is positive and consequently $\left\|\tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)_\lambda\right)\right\|_\infty < 1.$ ■

Theorem 4.33 (Instability of a multi-component scalar field coupled to a spherically symmetric metric on $\mathbb{R}^{1+3}$)**:** The total mass $\tilde{M}$ of a multi-component scalar field $\left(\underline{\varphi,\Pi}\right) \in \mathfrak{F}_k^m$ satisfying

$$\left\|\tilde{\Xi}\left(\left(\underline{\varphi,\Pi}\right)\right)\right\|_\infty < 1$$

is given by

$$\begin{aligned}\tilde{\mathfrak{M}} = 8\pi G\int_0^\infty \mathrm{d}r''\Bigg[&(r'')^2\sum_{i=1}^k\left(m^2\left|\varphi_i\right|^2+\left|\partial_{r''}\varphi_i\right|^2+(16\pi^2 r^4)^{-1}\left|\Pi_i\right|^2\right)\times\\ &\exp\left(-8\pi G\int_{r''}^{\lambda r}\mathrm{d}r'\, r'\textstyle\sum_{j=1}^k\left(\left|\partial_{r'}\varphi_j\right|^2+(16\pi^2 r'^4)^{-1}\left|\Pi_j\right|^2\right)\right)\Bigg].\end{aligned}$$

For a fixed total charge $\tilde{N}$ the mass has no positive lower bound.

Proof. One obtains the expression for the mass by inserting the expression for $\tilde{\Xi}$ from lemma 4.25 into (4.42). Note that the expression is well defined as $\big(\underline{\varphi, \Pi}\big) \in \mathfrak{B}_k$ and the limit exists by dominated convergence. Using the rescaled multi-component field from the proof of 4.32 an analogous calculation to the proof of theorem 4.22 entails a similar upper bound to the mass, that in the limit $\lambda \to \infty$ tends to zero. ■

5. An approach to a quantum version

In this chapter we are going to relate the results on the classical complex scalar field to a related quantum field theory. To this avail we will recall the basic notions and constructions for a free scalar field theory on a static background.

5.1. The quantization of a scalar field on stationary globally hyperbolic space-times

For the moment we are not only considering spherically symmetric space-times as in the previous chapter, but a far more general class.

Definition 5.1: Let (M, g) be a Lorentzian manifold and $O \subseteq M$ a subset.

1. We call O achronal, if every timelike curve intersects O at most in one point.
2. The (future / past) domain of dependence $D_{(\pm)}(O, M)$ of O is defined as the set of points $x \in M$ such that every (past / future directed) inextendible causal curve through x intersects O.
3. The causal future (past) $J^M_\pm(x)$ of a point $\mathbf{x} \in M$ is the set of points that is connected to x by future (past) directed causal curves. The causal future (past) of a subset of M is defined as the union of the causal futur (past) of all its elements. We denote the union of the causal future and the causal past by $J^M(\cdot)$.
4. A Cauchy surface Σ of M is an achronal, closed subset of M satisfying $D(\Sigma, M) = M$.
5. A Lorentzian manifold (M, g) containing a Cauchy surface is called globally hyperbolic.

In the following, all space times (M, g) will be globally hyperbolic. We will denote an arbitrary Cauchy surface by $\Sigma \subseteq M$ and the future directed unit normal vector field to Σ by $\mathfrak{n}^\mu$.

We will furthermore assume the space-times under consideration to be stationary, as this allows for a canonical Fock-space representation of the free quantum field algebra.

Definition 5.2: A globally hyperbolic space-time is said to be stationary, if it admits a one-parameter group of isometries with timelike orbits.

Before we turn our attention to the quantum world, let us briefly recapitulate, what the classical theory is, whose quantum version we wish to define.

The system under consideration, the minimally coupled free massive classical Klein-Gordon field, is on a generic globally hyperbolic space-time (M, g) conveniently characterized by the action functional

$$\mathfrak{S} = \int_{I\times\Sigma} \left(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + m^2 |\varphi|^2\right) \mathrm{d}\mu(M).$$

The space of configurations is given by the set of functions with spatially compact support[1], for which the functional $\mathfrak{S}$ is stationary. In the previous chapter we discussed the more concrete situation, where (M, g) was a spherically symmetric static space-time.

The space of states of the system is given by the set of stationary points of the functional. This can be equivalently described as the space of solutions to the Euler-Lagrange equations corresponding to $\mathfrak{S}$. This space can be endowed with a canonical symplectic structure. For our system of interest we have the following:

Definition 5.3: Define $\mathcal{S}$ to be the space of real solutions to the Klein-Gordon equation with spatially compact support. Endowing $\mathcal{S}$ with the symplectic form Ω, which for $\varphi_1,\ \varphi_2 \in \mathcal{S}$ is given as

$$\Omega(\varphi_1, \varphi_2) = \int\limits_\Sigma \mathrm{d}\mu(\Sigma)\ \left(\varphi_2 \mathfrak{n}^\mu\nabla_\mu\varphi_1 - \varphi_1\mathfrak{n}^\mu\nabla_\mu\varphi_2\right),$$

one obtains the symplectic space $(\mathcal{S}, \Omega)$.

The classical observables of the theory are given by real valued functions on $\mathcal{S}$.

One particularly important feature of globally hyperbolic space-times is, the existence of unique solutions to differential equations determined by a normally hyperbolic operators acting on sections of vector bundles over (M, g):

Theorem 5.4 (Theorem 3.2.11. in [4])**:** Consider a globally hyperbolic Lorentzian manifold (M, g) with a Cauchy surface $\Sigma \subset M$ with future directed timelike unit normal vector field $\mathfrak{n}$. Let E be a vector bundle over M and P be a normally hyperbolic operator acting on sections in E.

Then for all compactly supported smooth sections[2] u_0, u_1 in $E|_\Sigma$ and each compactly supported smooth section f in E, there exists a unique smooth section u in E satisfying

$$Pu = f, \quad u|_\Sigma = u_0, \quad \mathfrak{n}^\mu\nabla_\mu u|_\Sigma = u_1.$$

Moreover[3], $\mathrm{supp}(u) \subset J^M\big(\mathrm{supp}(u_0) \bigcup \mathrm{supp}(u_1) \bigcup \mathrm{supp}(f)\big)$.

[1]We say a function φ defined on M has spatially compact support, if for any Cauchy surface Σ the intersection $\mathrm{supp}(\varphi) \cap \Sigma$ is a compact subset of Σ.

[2]We denote the space of (compactly supported) smooth sections of E over M by $C^\infty_{(0)}(M)$.

[3]Here the supports of the functions u_0, u_1 and f are to be understood as embedded in the base manifold M, such that their union yields a subset of M.

Furthermore there exist unique advanced and retarded Green's operators $G_\pm$, cf. [4] pp. 88-92. Those are linear maps from $C_0^\infty(M,E)$ to $C^\infty(M,E)$, such that for $f \in C_0^\infty(M,E)$

- $PG_\pm f = f$ on the support of f and $PG_\pm f = 0$ elsewhere,
- $G_\pm P|_{C_0^\infty(M,E)} f = f$ on the support of f and $G_\pm P|_{C_0^\infty(M,E)} f = 0$ elsewhere,
- $\operatorname{supp}(G_\pm f) \subset J_\pm^M(\operatorname{supp}(f))$.

We will denote the causal propagator, i.e. the difference of advanced and retarded Green's operator, as G in the following. In particular those exist for the Klein-Gordon operator and thereby $\mathcal{S}$ is isomorphic to $C_0^\infty(\Sigma)\times C_0^\infty(\Sigma)$ for any Cauchy surface. This corresponds to the splitting of the phase space into the position and the momentum part for classical mechanical systems with finite degrees of freedom.

5.1.1. The free scalar field algebra

The construction of the quantized real scalar field on a stationary globally hyperbolic space-time mainly follows [3] with occasional variations based on the description in [48, 26].

To every $\varphi \in \mathcal{S}$ one associates an abstract object $F(\varphi)$, which is customarily referred to as the field operator, the reason being that the representations of interest of the algebra created by these elements will be those as operators on some Hilbert space.

Definition 5.5: The (free scalar quantum) field algebra $\mathfrak{A}$ is defined by equivalence classes of elements of the free[4] $*$-algebra restricted to products of finitely many elments over the complex numbers generated by the quantum fields[5] $F(\varphi)$, $\varphi \in \mathcal{S}$. The equivalence classes are defined by the following relations:

For $\varphi, \varphi_1, \varphi_2 \in \mathcal{S}$ and Id being the identity element in $\mathfrak{A}$:

(A1) $F(\varphi) = F^*(\varphi)$,

(A2) $\forall r \in \mathbb{R}: \quad F(\varphi_1 + r\varphi_2) = F(\varphi_1) + rF(\varphi_2)$,

(A3) $[F(\varphi_1), F(\varphi_2)] = -i\Omega(\varphi_1,\varphi_2)\,\mathrm{Id}$.

Definition 5.6: Let φ be an element of S and Σ be a Cauchy surface with unit normal vector field $\mathfrak{n}$. One defines the canonical conjugate field operators ϕ, ϖ as

$$\phi(\varphi) = F\left(G\left(0, \mathfrak{n}^\alpha \partial_\alpha \varphi|_\Sigma\right)\right),$$
$$\varpi(\varphi) = F\left(G\left(\varphi|_\Sigma, 0\right)\right).$$

[4]The free algebra over some commutative ring is in particylar associative and unital.

[5]We will often refer to the quantum fields as field operators, as one is mostly considering representations of this algebra as linear operators on some suitable Hilbert space.

By (A1) the canonical conjugate field operators are self-adjoint and the commutation relations (A3) entail

$$[\phi(\varphi_1), \varpi(\varphi_2)] = -i \left(\int_\Sigma \mathrm{d}\mu(\Sigma)\ \varphi_2 \mathfrak{n}^\alpha \partial_\alpha \varphi_1 \right) \mathrm{Id}\,.$$

5.1.2. The construction of a Fock space

To complete the construction of the free scalar quantum field theory, one needs a space of states, i.e. associations of expectation values to observables, on the previously defined field algebra.

Given $*$-algebra $\mathfrak{A}$ with unit Id , there is an algebraic notion of states:

Definition 5.7: One defines the space of states as the set of linear functionals ω on $\mathfrak{A}$ satifying

- $\omega(A^*A) \geq 0$ (Positivity),
- $\omega(\mathrm{Id}) = 1$ (Normalization).

This relates to the more common quantum mechanical picture of states corresponding to rays in a Hilbert space via the GNS-construction.

Theorem 5.8 (Gelfand-Naimark-Segal (GNS)-construction in the version presented in [14])**:** Given a state ω on a unital $*$-algebra $\mathfrak{A}$, there exists a pre-Hilbert space $(H, \langle\cdot,\cdot\rangle)$, a representation π of $\mathfrak{A}$ by linear operators on $(H, \langle\cdot,\cdot\rangle)$ and a unit vector $\psi \in (H, \langle\cdot,\cdot\rangle)$, such that for all $A \in \mathfrak{A}$

$$\omega(A) = \langle\psi, \pi(A)\psi\rangle$$

and furthermore ψ is cyclic for $\pi(\mathfrak{A})$, i.e.

$$H = \pi(\mathfrak{A})\psi.$$

As the GNS-construction above is given in terms of a $*$-algebra, rather then a C^*-algebra, the operators $\pi(A)$ are a priori only densely defined on the Hilbert space completion of H and might not be extendible.

On general globally hyperbolic space-times the Hilbert-space representations of the theory obtained from different algebraic states via the GNS-construction need not be unitarily equivalent. One ramification of this is the lack of a well-defined notion of particles in general curved space-times. The situation regarding this issue is somewhat better, if one restricts to stationary ones. In that case the Killing vector field of the one-parameter group of isometries distinguishes a preferred "time-direction", which in return singles out an equivalence class of unitarily equivalent Hilbert-space representations of the theory. These issues are discussed in some detail in sections 4.3 and 4.4 of [48].

Along the lines of the work of Ashtekar and Magnon [3][6], we are going to construct a Hilbert space representation of the scalar field theory from a pair $(\alpha, \beta) \in C_0^\infty(\mathbb{R}^3, \mathbb{C}) \times C_0^\infty(\mathbb{R}^3, \mathbb{C})$ that will play the role of the initial conditions for a complex classical solution to the Klein-Gordon equation.

To begin with, we will however recall their original construction of a Hilbert space representation. Hence we want to consider as the space of states the representation space of a $*$-representation of the field algebra $\mathfrak{A}$ by operators on a Hilbert space $\mathfrak{F}$ subject to certain conditions.

Definition 5.9: We call the representation space $\mathfrak{F}$ of a $*$-representation of the field algebra $\mathfrak{A}$ a space of states, if the space and the representation satisfy the conditions

(H1) The space $\mathfrak{F}$ is a symmetric Fock space based on a Hilbert space $\mathfrak{H}$.

(H2) There exists a dense subspace in $\mathfrak{H}$ that is isomorphic to $\mathcal{S}$ as a real vector space.

(H3) For any $\varphi \in \mathcal{S}$ the field operators $F(\varphi)$ are mapped to the sum of the concretely defined creation and annihilation operators on $\mathfrak{F}$, i.e. for $\mathrm{K} : \mathcal{S} \to \mathfrak{H}$ being the isomorphism of real vector spaces from (H2), $F(\varphi) \mapsto a^*(\mathrm{K}\,\varphi) + a(\mathrm{K}\,\varphi)$.

In the subsequent discussion, we will consider the complexification of $\mathcal{S}$ and construct the pre-Hilbert space from condition (H2) as a subspace. They choice of the subspace will be determined by a complex structure on the space of real solutions of the classical theory.

As before, let $\mathcal{S}$ be the space of spatially compactly supported real solutions of the Klein-Gordon equation and $\tilde{\mathfrak{H}}$ its complexification $\tilde{\mathfrak{H}} = \mathbb{C} \otimes \mathcal{S}$ endowed with complex bilinear symplectic form Ω, that one obtains via an extension of the canonical one on $\mathcal{S}$. Furthermore, let J be a linear complex structure on $\mathcal{S}$ and denote its complex linear extension to $\tilde{\mathfrak{H}}$ the same. To construct a Fock space representation, we require an isomorphism K of real vector spaces mapping $\mathcal{S} \subset \tilde{\mathfrak{H}}$ into some complex subspace $\mathrm{K}\,\mathcal{S} \subset \tilde{\mathfrak{H}}$. On $\mathrm{K}\,\mathcal{S}$ we need an hermitian[7] inner product $\langle \cdot, \cdot \rangle$. For $(\mathrm{K}\,\mathcal{S}, \langle \cdot, \cdot \rangle)$ to be compatible with the complex structure the following identity should hold for all $\varphi_1, \varphi_2 \in \mathcal{S}$:

$$\langle \mathrm{K}\,\varphi_1, \mathrm{J}\,\mathrm{K}\,\varphi_2 \rangle = \langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\mathrm{J}\,\varphi_2 \rangle = i \langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2 \rangle. \tag{5.1}$$

As the notation suggests, we are aiming at using $\mathrm{K}\,\mathcal{S}$ as the pre-Hilbert space, that the Fock space will ultimately be based on. In particular, we will show, that there are suitable complex structures, that allow a natural choice of $\mathrm{K}\,\mathcal{S}$.

[6] Kay and Wald [26], and Wald in [48] take a somewhat more general approach, based on bilinear forms satisfying a certain bound related to the symplectic form. In the case where the bound is saturated our approach is equivalent.

[7] We will chose inner products on complex spaces to be anti-linear in the first argument.

In order for this to be possible, the complex structure should have further properties, such that all the requirements in definition 5.9 can be fullfilled. The conditions (A1)-(A3) and (H1)-(H3) yield compatibility conditions for the complex structure and the canonical symplectic form on $\mathcal{S}$.

The creation and annihilation operator satisfy the commutation relations

$$[a^*(\mathrm{K}\,\varphi_1), a^*(\mathrm{K}\,\varphi_2)] = 0 = [a(\mathrm{K}\,\varphi_1), a(\mathrm{K}\,\varphi_2)] \quad \text{and}$$
$$[a(\mathrm{K}\,\varphi_1), a^*(\mathrm{K}\,\varphi_2)] = \langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2\rangle \mathbb{1}.$$

In combination with (A3) and (H3) one obtains

$$\Omega(\varphi_1, \varphi_2) = -2\,\mathfrak{Im}\,\langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2\rangle$$

by expressing the field operators in terms of creation and annihilation operators. From this identity and the properties of $\langle\cdot,\cdot\rangle$ it follows, that for all $\varphi_1, \varphi_2 \in \mathcal{S}$

$$\langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2\rangle = \tfrac{1}{2}\left(\gamma(\varphi_1, \varphi_2) - i\Omega(\varphi_1, \varphi_2)\right). \tag{5.2}$$

This implies in particular, that $\gamma(\cdot,\cdot) = -\Omega(\cdot, \mathrm{J}\,\cdot)$ is a positive real bilinear form on $\mathcal{S}$ and furthermore the symplectic form is conserved by J.

Given a complex structure on $\mathcal{S}$ possessing these properties, that is

- $-\Omega(\cdot, \mathrm{J}\,\cdot)$ is a real bilinear form on $\mathcal{S}$ and
- $\Omega(\mathrm{J}\,\cdot, \mathrm{J}\,\cdot) = \Omega(\cdot,\cdot)$,

one can construct a suitable subspace $\mathrm{K}\,\mathcal{S}$ of $\tilde{\mathfrak{H}}$ as follows.

Definition 5.10: Define the operators $P_\pm = \frac{1}{\sqrt{2}}\left(\mathbb{1} \mp i\,\mathrm{J}\right)$ on $\tilde{\mathfrak{H}}$.

Proposition 5.11: The operators $P_\pm$ are projectors on the eigenspaces $\tilde{\mathfrak{H}}_\pm$ of J to the eigenvalues $\pm i$, and $\mathfrak{H} = \tilde{\mathfrak{H}}_+ + \tilde{\mathfrak{H}}_-$, as well as $\overline{\tilde{\mathfrak{H}}_+} = \tilde{\mathfrak{H}}_-$.

Proof. Considering the squares one finds

$$(P_\pm)^2 = \tfrac{1}{2}\left(\mathbb{1} \mp 2i\,\mathrm{J} - \mathrm{J}^2\right) = 1 \mp i\,\mathrm{J} = P_\pm.$$

Note that $P_+ + P_- = \mathbb{1}$ and $P_\pm P_\mp = 0$. With $\tilde{\mathfrak{H}}_\pm = P_\pm \mathfrak{H}$ one gets

$$\begin{aligned}\mathrm{J}\,\tilde{\mathfrak{H}}_\pm &= \mathrm{J}\,P_\pm\mathfrak{H} = \sqrt{2}^{-1} J\,(\mathbb{1} \mp iJ)\,\mathfrak{H} = \sqrt{2}^{-1}\,(\mathrm{J} \pm i\mathbb{1})\,\mathfrak{H} = \pm i\sqrt{2}^{-1}\,(\mathbb{1} \mp i\,\mathrm{J})\,\mathfrak{H} \\ &= \pm i\tilde{\mathfrak{H}}_\pm.\end{aligned}$$

The final property follows as J commutes with complex conjugation on $\tilde{\mathfrak{H}}$. ■

Note, that $\varphi \in \mathcal{S} \subseteq \tilde{\mathfrak{H}}$ satisfy $\varphi = \overline{\varphi}$ and can therefore be written as[8] $\varphi_+ + \overline{\varphi_+}$. With this observation one finds $\mathcal{S}$ and $\tilde{\mathfrak{H}}_+$ to be isomorphic as real vector spaces via the real linear map

$$\begin{aligned} \mathrm{K} : \mathcal{S} &\to \tilde{\mathfrak{H}}_+ \\ \varphi &\mapsto P_+ \varphi \end{aligned}$$

with inverse $\tilde{\mathfrak{H}}_+ \ni \varphi_+ \mapsto \varphi_+ + \overline{\varphi_+} \in \mathcal{S}$. To turn $\tilde{\mathfrak{H}}_+$ into a suitable pre-Hilbert space, we need to endow it with an inner product, that is in addition compatible with the complex structure. To this avail, we show that the right hand side of equation (5.2) defines an hermitian inner product on $\tilde{\mathfrak{H}}_+$.

Lemma 5.12:

(i) For $\varphi_{1,+}, \varphi_{2,+} \in \tilde{\mathfrak{H}}_+$ the symplectic form vanishes, i.e. $\Omega(\varphi_{1,+}, \varphi_{2,+}) = 0$ and

(ii) if $\varphi_1, \varphi_2 \in \mathcal{S}$, the identities $\gamma(\varphi_1, \varphi_2) = 2\,\mathfrak{Re}\big(-i\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})\big)$ and $\Omega(\varphi_1, \varphi_2) = -2\,\mathfrak{Im}\big(-i\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})\big)$ hold.

(iii) Furthermore $\langle \cdot, \cdot \rangle = -i\Omega(\bar{\cdot}, \cdot)$ is an hermitian inner product on $\tilde{\mathfrak{H}}_+$, that is compatible with J.

Proof.

(i) As $\Omega(\mathrm{J}\,\varphi, \mathrm{J}\,\varphi') = \Omega(\varphi, \varphi')\ \forall \varphi, \varphi' \in \mathfrak{H}$ by the hermiticity of $\langle \cdot, \cdot \rangle$ one has

$$\Omega(\varphi_{1,+}, \varphi_{2,+}) = \Omega(\mathrm{J}\,\varphi_{1,+}, \mathrm{J}\,\varphi_{2,+}) = \Omega(i\varphi_{1,+}, i\varphi_{2,+}) = -\Omega(\varphi_{1,+}, \varphi_{2,+})$$

(ii) Using (i) one obtains

$$\begin{aligned} \gamma(\varphi_1, \varphi_2) = -\Omega(\varphi_1, \mathrm{J}\,\varphi_2) &= -\Omega\big(\varphi_{1,+} + \overline{\varphi_{1,+}}, \mathrm{J}(\varphi_{2,+} + \overline{\varphi_{2,+}})\big) \\ &= i\Omega(\varphi_{1,+}, \overline{\varphi_{2,+}}) - i\Omega(\overline{\varphi_{1,+}}, \varphi_2) \\ &= 2\,\mathfrak{Re}\big(-i\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})\big). \end{aligned}$$

The previous result and the bilinearity of Ω entail

$$\begin{aligned} \Omega(\varphi_1, \varphi_2) &= -\Omega(\varphi_1, \mathrm{J}^2\,\varphi_2) = 2\,\mathfrak{Re}\big(-i\Omega(\overline{\varphi_{1,+}}, J\varphi_{2,+})\big) \\ &= 2\,\mathfrak{Re}\big(\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})\big) = -2\,\mathfrak{Im}\big(-i\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})\big) \end{aligned}$$

(iii) By the preceeding two identities, one gets

$$\begin{aligned} \overline{\langle \varphi_{1,+}, \varphi_{2,+} \rangle} &= i\overline{\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+})} = i\Omega(\varphi_{1,+}, \overline{\varphi_{2,+}}) = -i\Omega(\overline{\varphi_{2,+}}, \varphi_{1,+}) \\ &= \langle \varphi_{2,+}, \varphi_{1,+} \rangle. \end{aligned}$$

[8] Here and in the following we denote the components in the subspaces $\tilde{\mathfrak{H}}_\pm$ by the respective subscript.

By the complex bilinearity of Ω this shows the sesquilinearity of $\langle\cdot,\cdot\rangle$. In addition our choice of K and J commute, which ensures the compatibility with the complex structure in the sense of equation 5.1:

$$\langle\varphi_1, \mathrm{J}\,\varphi_2\rangle = -i\Omega(\overline{\varphi_{1,+}}, \mathrm{J}\,\varphi_{2,+}) = -i^2\Omega(\overline{\varphi_{1,+}}, \varphi_{2,+}) = i\langle\varphi_1, \varphi_2\rangle.$$

Finally the positivity of $\langle\cdot,\cdot\rangle$ follows from the positivity of γ and the antisymmetry of Ω. ■

With the following definition the construction of the free quantum field theory is complete.

Definition 5.13: Define the one-particle Hilbert space $\mathfrak{H}$ to be the completion of $\tilde{\mathfrak{H}}_+$ in the norm induced by $\langle\cdot,\cdot\rangle$, and let $\mathfrak{F}$ the symmetric Fock space $\sum_{n=0}^{\infty}\mathfrak{H}^{\vee n}$.

5.1.3. The reverse construction

As announced earlier, we wil now reverse the construction of the Fock space in the sense, that we are not starting with a complex structure on the space of solutions to the classical theory, but with two functions on $\mathbb{R}^3$ that satisfy a certain positivity condition.

To begin with, we describe the construction of a linear complex structure on $\mathcal{S}$ that is compatible with the symplectic form starting from a subspace of $\tilde{\mathfrak{H}}$.[9]

Lemma 5.14: Assume $\tilde{\mathfrak{H}}_+$ is a subspace of $\tilde{\mathfrak{H}}$ such that $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_+ \oplus \overline{\tilde{\mathfrak{H}}_+}$ and for $\varphi_{1,+}, \varphi_{2,+} \in \tilde{\mathfrak{H}}_+$ the conditions

(I) $\Omega(\varphi_{1,+}, \varphi_{2,+}) = 0$,

(II) $-\mathfrak{Im}\,\Omega(\cdot, \bar{\cdot})$ is a real inner product on $\tilde{\mathfrak{H}}_+$

are satisfied. Then the linear operator $\mathrm{J}' : \tilde{\mathfrak{H}} \to \tilde{\mathfrak{H}}$ defined by

$$\mathrm{J}'\,\varphi = i\varphi_+ - i\varphi_-$$

reduces to a complex structure on $\mathcal{S}$. Furthermore $\langle\cdot,\cdot\rangle = -i\Omega(\bar{\cdot}, \cdot)$ defines an hermitian inner product on $\tilde{\mathfrak{H}}_+$, that can be expressed as the right hand side of (5.2) for two corresponding elements in $\mathcal{S}$ and J replaced by J$'$.

Proof. J$'$ is complex linear as $\tilde{\mathfrak{H}}_+$ is a linear subspace of $\tilde{\mathfrak{H}}$. Consequently

$$\mathrm{J}'^2\,\varphi = \mathrm{J}(i\varphi_+ - i\varphi_-) = -\varphi_+ - \varphi_- = -\varphi,$$

[9] In proposition 3.1 [26] the authors give the corresponding construction with the focus on bilinear forms on $\mathcal{S}$ satisfying a certain inequality with the symplectic form. The complex structures in the construction presented here always give rise to such bilinear form. These are given by $\frac{1}{2}\gamma$ in our notation, cf. page 82.

i.e. J' is a linear complex structure on $\tilde{\mathfrak{H}}$. In addition

$$\overline{\mathrm{J}'\,\varphi} = \overline{i\varphi_+ - i\varphi_-} = i\overline{\varphi_-} - i\overline{\varphi_+} = \mathrm{J}(\overline{\varphi_-} + \overline{\varphi_+}) = \mathrm{J}\,\overline{\varphi}.$$

As $\mathcal{S}$ embeds into $\tilde{\mathfrak{H}}$ as the set $\{\varphi \in \tilde{\mathfrak{H}} \mid \varphi = \overline{\varphi}\}$, one can restrict J' to a complex structure on $\mathcal{S}$.

The fact that $-i\Omega(\bar{\cdot},\cdot)$ is an inner product follows from conditions (I) and (II) and the calculations in the proof of lemma 5.14.

For $\varphi_{1,+}, \varphi_{2,+} \in \tilde{\mathfrak{H}}_+$ one can express the inner product as

$$-\tfrac{1}{2}\left[\Omega\left(\varphi_{1,+}\overline{\varphi_{1,+}}, \mathrm{J}'\left(\varphi_{2,+}\overline{\varphi_{2,+}}\right)\right) + i\Omega\left(\varphi_{1,+}\overline{\varphi_{1,+}}, \varphi_{2,+}\overline{\varphi_{2,+}}\right)\right].$$

where $\varphi_{1,+} + \overline{\varphi_{1,+}}$ and $\varphi_{2,+} + \overline{\varphi_{2,+}}$ are elements of $\mathcal{S}$. The calculation is the same as the proof of lemma 5.12. ■

Again a completion of $\tilde{\mathfrak{H}}_+$ in the norm induced by $\langle\cdot,\cdot\rangle$ yields the one-particle Hilbert space $\mathfrak{H}$.

Using the lemma, we are now going to construct a one-particle Hilbert space starting from the initial conditions.

We identify $\tilde{\mathfrak{H}}$ with the space of complex valued initial conditions $C_0^\infty(\mathbb{R}^3,\mathbb{C}) \times C_0^\infty(\mathbb{R}^3,\mathbb{C}) \ni (\alpha,\beta)$, cf. the discussion preceding definition 5.6 such that we absorb the volume element for the Cauchy surface in the functions β. The canonical symplectic form on this space is given by

$$\Omega\left((\alpha,\beta),(\alpha',\beta')\right) = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x}\ \ (\alpha'\beta - \alpha\beta').$$

Then we let $\tilde{\mathfrak{H}}_+$ be the graph $\Gamma(-i\,\mathrm{A})$ of an operator $-i\,\mathrm{A}$, i.e.

$$\tilde{\mathfrak{H}}_+ = \{(\alpha, -i\,\mathrm{A}\,\alpha) \mid \alpha \in \mathcal{D}(\mathrm{A})\}.$$

We require $\tilde{\mathfrak{H}}_+$ to satisfy $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_+ \oplus \overline{\tilde{\mathfrak{H}}_+}$. This excludes that $-i\,\mathrm{A}$ commutes with complex conjugation, as $-i\,\mathrm{A}\,\overline{\alpha} = \overline{-i\,\mathrm{A}\,\alpha}$ implies, that for $\alpha \in \mathcal{D}(a)$

$$\overline{\tilde{\mathfrak{H}}_+} \ni \overline{(\alpha, -i\,\mathrm{A}\,\alpha)} = \left(\overline{\alpha}, \overline{-i\,\mathrm{A}\,\alpha}\right) = (\overline{\alpha}, -i\,\mathrm{A}\,\overline{\alpha}) \in \Gamma(-i\,\mathrm{A}).$$

We assume furthermore, that $\mathrm{A} + \overline{\mathrm{A}}$ is invertible.[10]

Proposition 5.15: For an operator A being sufficiently well behaved in the sense of the previous discussion an element $(\gamma,\delta) \in \tilde{\mathfrak{H}}$ can be decomposed as

$$(\gamma,\delta) = (\alpha, -i\,\mathrm{A}\,\alpha) + \left(\overline{\beta}, i\overline{\mathrm{A}\,\beta}\right).$$

[10]By $\overline{\mathrm{A}}$ we denote the composition $\mathrm{C\,A\,C}$ with the complex conjugation C.

Proof. One can solve the equations

$$\gamma = \alpha + \overline{\beta}$$
$$\delta = i\overline{\mathrm{A}}(\gamma - \alpha) - i\,\mathrm{A}\,\alpha$$

for α and β by

$$\alpha = i(\mathrm{A} + \overline{\mathrm{A}})^{-1}(\delta - i\overline{\mathrm{A}}\gamma)$$
$$\beta = \overline{\gamma} + \overline{i(\overline{\mathrm{A}} + \mathrm{A})^{-1}(\delta - i\overline{\mathrm{A}}\gamma)}. \qquad \blacksquare$$

Imposing the conditions (I) and (II), cf. lemma 5.14, the operator A is required to satisfy $\overline{\mathrm{A}} = \mathrm{A}^*$, as for $\alpha, \alpha' \in \tilde{\mathfrak{H}}_+$, the first one reads

$$\Omega(\alpha, \alpha') = i \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \ \ (\alpha\,\mathrm{A}\,\alpha' - \alpha'\,\mathrm{A}\,\alpha) = i \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \ \ (\alpha\,\mathrm{A}\,\alpha' - \alpha\,\mathrm{C}\,\mathrm{A}^*\,\mathrm{C}\,\alpha') = 0$$

and $(\mathrm{A} + \mathrm{A}^*) < 0$, as by (II)

$$-\,\mathfrak{Re} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \ \ \overline{\alpha'}(\mathrm{A} + \mathrm{A}^*)\alpha$$

is required to be an inner product. Up to domain questions these conditions and proposition 5.15 yield $\tilde{\mathfrak{H}} = \tilde{\mathfrak{H}}_+ \oplus \overline{\tilde{\mathfrak{H}}_+}$.

The preceding discussions culminate in the next lemma, which is an essential ingredient to our following discussion of the stability.

Lemma 5.16: For $(\alpha, \beta) \in \tilde{\mathfrak{H}}$ satisfying $-\,\mathfrak{Im}\,\Omega\big((\alpha,\beta),(\overline{\alpha},\overline{\beta})\big) > 0$ there exists a linear complex structure J such that $(\alpha, \beta) \in \tilde{\mathfrak{H}}_+$.

Proof. Given a pair (α, β), we will construct an operator $-i\,\mathrm{A}$ such that $i\beta = \mathrm{A}\,\alpha$ and the operator satisfies the conditions of lemma 5.14 and proposition 5.15 .

Let $e = \{e_1, e_2, e_3, e_4\}$ be an orthonormal basis for the subspace $V \in \tilde{\mathfrak{H}}$ spanned by the real and imaginary parts of α and β. In the following, we shall construct the operator A on this subspace.

For some $a, b, c, d \in \mathbb{R}^4$, we can write $\alpha = a{\cdot}e + ib{\cdot}e$ and $\beta = c{\cdot}e + id{\cdot}e$. Decomposed into its real and imaginary part and written in block form, the operator A reads

$$\mathrm{A} = \begin{pmatrix} D & F \\ -F & D \end{pmatrix}.$$

To satisfy condition (I) from lemma 5.14 the real and imaginary part need to satisfy $D = D^t$ as and $F = F^t$ respectively, as by the proof of proposition 5.15 (I) is equivalent to $\mathrm{A}^* = \overline{\mathrm{A}}$, which translates to the symmetry of the block matrices. The second condition (II), which reads $\big(\mathrm{A} + \overline{\mathrm{A}}\big) > 0$ by the proof of the same proposition as before, expressed in terms of the block matrices is $D > 0$.

As we are aiming at using the construction in lemma 5.14 based on a subspace $\tilde{\mathfrak{H}}_+$ obtained via proposition 5.15, we want (α, β) to be an element in the graph of $-i\,\mathrm{A}$, i.e. we require $A\alpha = i\beta$, or in terms of the block matrices on V

$$\begin{aligned} Da - Fb &= -d, \\ Fa + Db &= c. \end{aligned}$$

The positivity condition in the statement of the lemma we are about to prove entails

$$b \cdot c - a \cdot d > 0. \tag{5.3}$$

First, consider the case $a = 0$, then (5.3) implies $b \cdot c > 0$. Let $\{\tilde{e}_1, \tilde{e}_2\}$ be a basis of the orthogonal complement of span $\{b\}$ in V. Note, that $\{c, \tilde{e}_1, \tilde{e}_2\}$ is a basis of V. We define

$$D = (b \cdot c)^{-1}\Big(|c\rangle\langle c| + |\tilde{e}_1\rangle\langle\tilde{e}_1| + |\tilde{e}_2\rangle\langle\tilde{e}_2| \Big),$$

then $Db = (b\cdot c)^{-1}(b\cdot c)c = c$. Furthermore, D is symmetric on V and positive definite, as for an arbitrary non-zero $v \in V$

$$v \cdot Dv = (b \cdot c)^{-1}\left((v \cdot c)^2 + (v \cdot \tilde{e}_1)^2 + (v \cdot \tilde{e}_2)^2\right) > 0.$$

Setting

$$F = \begin{cases} (b \cdot d)^{-1}\, |d\rangle\langle d| & \text{if } b \cdot d \neq 0, \\ (b)^{-2}\, (|d\rangle\langle b| + |b\rangle\langle d|) & \text{otherwise} \end{cases}$$

completes the construction of the A restricted to V in this case.

Second, let $b = 0$. By (5.3) one has $-a \cdot d > 0$. In analogy to the first case with $\{\tilde{e}_1, \tilde{e}_2\}$ being an orthonormal basis of $(\text{span}\,\{a\})^\perp \subset V$ one defines

$$D = -(a \cdot d)^{-1}\Big(|d\rangle\langle d| + |\tilde{e}_1\rangle\langle\tilde{e}_1| + |\tilde{e}_2\rangle\langle\tilde{e}_2| \Big),$$

and

$$F = \begin{cases} (a \cdot c)^{-1}\, |c\rangle\langle c| & \text{if } a \cdot c \neq 0, \\ (a)^{-2}\Big(|a\rangle\langle c| + |c\rangle\langle a| \Big) & \text{otherwise.} \end{cases}$$

Third, assume that a and b are linearly independent and set

$$D = \frac{b \cdot c - a \cdot d}{a^2 + b^2}\mathbb{1},$$

then define

$$\begin{aligned} b_1 &= Fb = Da + d = \frac{b \cdot c - a \cdot d}{a^2 + b^2}a + d, \\ a_1 &= Fa = c - Db = c - \frac{b \cdot c - a \cdot d}{a^2 + b^2}b. \end{aligned}$$

Note, that

$$a \cdot b_1 = b \cdot a_1 = \frac{a^2}{a^2 + b^2} b \cdot d + \frac{b^2}{a^2 + b^2} d \cdot a.$$

Now choose an arbitrary symmetric operator F_0, such that $F_0 a = a_1$ and decompose $F_0 = F + F_1$. Then it follows, that $F_1 a = 0$. Set $b_2 = F_1 b = b_1 - F_0 b$. It follows by the symmetry of F_0, that

$$a \cdot b_2 = a \cdot b_1 - a \cdot F_0 b = a_1 \cdot b - a \cdot F_0 b = b \cdot F_0 a - a \cdot F_0 b = 0.$$

Denote by b_3 the projection of b onto the orthogonal complement of a. Then

$$F_1 b = b_2 = F_1 b_3$$

is an equation for F_1 on the orthogonal complement of a and as $b_3 \neq 0$ it has a solution. If $b_2 = 0$, one has the trivial solution $F_1 = 0$. For non-vanishing b_2 and $b_2 \cdot b_3 = 0$, take $F_1 = b_3^{-2}\big(|b_2\rangle\langle b_3| + |b_3\rangle\langle b_2| \big)$. In the remaining case $b_2 \cdot b_3 \neq 0$ setting $F_1 = (b_3 \cdot b_2)^{-1} |b_2\rangle\langle b_2|$ solves the equation.

Finally, consider the case of linearly dependent a and b, i.e. $b = \lambda a$ with $\lambda \neq 0$. The equations to be satisfied read

$$(D - \lambda F)a = -d,$$
$$(F + \lambda D)a = c.$$

Combining these one gets

$$Da = \frac{\lambda c - d}{1 + \lambda^2}$$

and (5.3) implies $a \cdot (\lambda c - d) > 0$. As in the first two cases, let $\{\tilde{e}_1, \tilde{e}_2\}$ be a basis of $(\operatorname{span}\{a\})^\perp$. Define

$$D = (1 + \lambda^2)^{-1} \left(a \cdot (\lambda c - d)\right)^{-1} \Big(|\lambda c - d\rangle\langle \lambda c - d| + |\tilde{e}_1\rangle\langle \tilde{e}_1| + |\tilde{e}_2\rangle\langle \tilde{e}_2| \Big)$$

and F symmetric, such that

$$Fa = \frac{c + d}{1 + \lambda^2}.$$

Depending on the relations of c, d, and a, this can be done in a similar fashion as before.

Thereby the construction of the operators on V is complete. In all four possible cases the operator D is symmetric and positive definite and hence invertible by the finite dimensional spectral theorem. One can now extend the operator D to all of $\tilde{\mathfrak{H}}$ by defining it to be the identity operator on the orthogonal complement of V. This preserves all the required properties. Similarly we can extend F to $\tilde{\mathfrak{H}}$ by the zero operator.

By proposition 5.15 we can decompose $\tilde{\mathfrak{H}}$ into $\tilde{\mathfrak{H}}_+ \oplus \overline{\tilde{\mathfrak{H}}_+}$, where $\tilde{\mathfrak{H}}_+$ is the graph of $-i\,\mathrm{A}$, as the operator by construction satisfies conditions (I) and (II) in lemma 5.14. To complete the proof we use lemma 5.14 to construct the complex structure. ■

We collect the results of the discussion above in the following theorem:

Theorem 5.17: Given a pair of functions (α, β) in $C_0^\infty(\mathbb{R}^3, \mathbb{C}) \times C_0^\infty(\mathbb{R}^3, \mathbb{C})$ satisfying

$$0 < -\mathfrak{Im} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \ \left(\overline{\alpha}\beta - \alpha\overline{\beta}\right) = \mathfrak{Re}\left(-i\Omega\left(\overline{(\alpha,\beta)}, (\alpha,\beta)\right)\right),$$

there exists a Fock space representation of the field algebra $\mathfrak{A}$, such that (α, β) can be identified with the intial conditions of a classical solution corresponding to an element in the one-particle Hilbert space.

Proof. We start by identifying the initial conditions of complex solutions to the Klein-Gordon equation with $C_0^\infty(\mathbb{R}^3, \mathbb{C}) \times C_0^\infty(\mathbb{R}^3, \mathbb{C})$, cf. page 85. According to lemma 5.16, we can construct a complex structure J, such that (α, β) corresponds to an element of the corresponding pre-Hilbert space $\tilde{\mathfrak{H}}_+$. Completing $\tilde{\mathfrak{H}}_+$ in the norm induced by $-i\Omega(\bar{\cdot}, \cdot)$, we obtain the one-particle Hilbert space. The corresponding symmetric Fock space with its creation and anihilation operators completes the construction. ■

Proposition 5.18: The previously presented construction of a Fock space representation of the field algebra is naturally related to a pure quasi free state on the algebra, the vacuum in $\mathfrak{F}$.

Proof. Our construction gives a one-particle Hilbert space structure in the sense of proposition 3.1 in [26]. I.e. we associate to the symplectic space $(\mathcal{S}, \Omega)$ a complex Hilbert space $\mathfrak{H}$, an isomorphism of real vector spaces $\mathrm{K} : \mathcal{S} \to \mathrm{K}\,\mathcal{S} \subset \mathfrak{H}$, such that

1. the complexified range of K is dense in $\mathfrak{H}$,
2. $\gamma(\varphi_1, \varphi_2) = 2\,\mathfrak{Re}\,\langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2\rangle$ for all $\varphi_1, \varphi_2 \in \mathcal{S}$, and
3. $\Omega(\varphi_1, \varphi_2) = 2\,\mathfrak{Im}\,\langle \mathrm{K}\,\varphi_1, \mathrm{K}\,\varphi_2\rangle$ for all $\varphi_1, \varphi_2 \in \mathcal{S}$.

The Fock space representation obtained from this one-particle Hilbert space structure realizes the GNS construction for the algebraic state ω_γ defined by its smeared two point function

$$\omega_\gamma\left(F(\varphi_1)F(\varphi_2)\right) = \tfrac{1}{2}\left(\gamma(\varphi, \varphi) + i\Omega(\varphi_1, \varphi_2)\right),$$

together with the requirement, that all odd (smeared) n-point functions vanish by lemma A.2 in [26]. The lemma furthermore implies, that ω_γ is pure, as in our construction $\overline{\mathrm{K}\,\mathcal{S}} = \mathfrak{H}$, cf. page 83. ■

5.2. Stability and the relation to the classical problem

Consider the operator valued distributions ϕ, ϖ obtained from the canonical conjugate field operators, cf. definition 5.6. For (α, β) being the initial conditions of a real classical solution $\varphi \in \mathcal{S}$ to the Klein-Gordon equation one formally writes

$$\phi(\varphi) = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \phi(\mathbf{x})\,\mathrm{K}\,\beta(\mathbf{x}) \quad \text{and} \quad \varpi(\varphi) = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \varpi(\mathbf{x})\,\mathrm{K}\,\alpha(\mathbf{x}).$$

They satisfy the distributional commutation relations

$$[\phi(\mathbf{x}), \varpi(\mathbf{y})] = -i\delta(\mathbf{x} - \mathbf{y}).$$

5.2.1. Condensed states

Definition 5.19: Let $(\alpha, \beta) \in \mathfrak{H}$ be the normalized initial conditions for a complex solution to the Klein-Gordon equation and define the operators

$$a = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(\overline{\beta(\mathbf{x})}\phi(\mathbf{x}) + \overline{\alpha(\mathbf{x})}\varpi(\mathbf{x})\right),$$

$$a^* = \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(\beta(\mathbf{x})\phi(\mathbf{x}) + \alpha(\mathbf{x})\varpi(\mathbf{x})\right).$$

Proposition 5.20: The operators a and a^* satisfy the commutation relations of annihilation- and creation operators.

Proof. Considering the commutator, one finds

$$\begin{aligned}[a, a^*] &= i \int\limits_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(\beta(\mathbf{x})\overline{\alpha(\mathbf{x})} - \alpha(\mathbf{x})\overline{\beta(\mathbf{x})}\right) \\ &= -i\Omega\left(\overline{(\alpha, \beta)}, (\alpha, \beta)\right) = \|(\alpha, \beta)\|^2 = 1 \end{aligned}$$

■

Comparing the expression for the commutator with equation 4.32 we note that the commutator $[a, a^*]$ is corresponds to the classical charge of the classical complex field obtained from $\sqrt{2}\,(\overline{\alpha}, \beta)$. This will be relevant in the discussion of stability later on.

In the following we want to consider an N-particle state of the form

$$|\psi\rangle = (N!)^{-1}(a^*)^N\,|0\rangle$$

on the Fock space $\mathfrak{F}$. The following theorem establishes the connection between the complex classical field and the quantum field in the setting previously described.

Theorem 5.21: The expectation values of the normal ordered[11] operator valued distributions $:\phi^2(\mathbf{x}):$, $:\varpi^2(\mathbf{x}):$, and the contraction of $:(\partial_a\phi(\mathbf{x}))\partial_b\phi(\mathbf{x}):$ with any symmetric $(0,2)$-tensor in the state $|\psi\rangle$ are given by the corresponding values for the classical complex field obtained from $\sqrt{2N}(\alpha,\beta)$.[12]

Proof. To compute the first expectation value, note that formally $[a, :\phi^2(\mathbf{x}):] = [a,\phi^2(\mathbf{x})]$, as the field squared formally differs from the normal ordered one only by a constant.

Now, consider the commutator

$$\begin{aligned}\left[a^N,\phi(\mathbf{x})\right] &= a^{N-1}\left[a,\phi(\mathbf{x})\right] + \left[a^{N-1},\phi(\mathbf{x})\right]a\\ &= i\overline{\alpha(\mathbf{x})}a^{N-1} + \left[a^{N-1},\phi(\mathbf{x})\right]a.\end{aligned}$$

Using this recursion relation, one obtains

$$\left[a^N,\phi(\mathbf{x})\right] = iN\overline{\alpha(\mathbf{x})}a^{N-1}.$$

With this expression we compute the following commutator:

$$\begin{aligned}\left[a^N,\phi^2(\mathbf{x})\right] &= \phi(x)\left[a^N,\phi(\mathbf{x})\right] + \left[a^N,\phi(\mathbf{x})\right]\phi(\mathbf{x})\\ &= iN\overline{\alpha(\mathbf{x})}\left(\phi(\mathbf{x})a^{N-1} + a^{N-1}\phi(\mathbf{x})\right)\\ &= iN\overline{\alpha(\mathbf{x})}\left(2\phi(\mathbf{x})a^{N-1} + \left[a^{N-1},\phi(\mathbf{x})\right]\right)\\ &= iN\overline{\alpha(\mathbf{x})}\left(2\phi(\mathbf{x})a^{N-1} + iN\overline{\alpha(\mathbf{x})}a^{N-2}\right).\end{aligned}$$

Using this identity and the assumption of the vanishing expectation value of the normal ordered field square, one finds

$$\begin{aligned}(N!)\langle\psi|:\phi^2(\mathbf{x}):|\psi\rangle &= \langle 0|\,a^N :\phi^2(\mathbf{x}): a^{*N}\,|0\rangle = \langle 0|\left[a^N, :\phi^2(\mathbf{x}):\right]a^{*N}\,|0\rangle\\ &= \langle 0|\left[a^N,\phi^2(\mathbf{x})\right]a^{*N}\,|0\rangle\\ &= iN\overline{\alpha(\mathbf{x})}\,\langle 0|\left(2\phi(\mathbf{x})a^{N-1} + iN\overline{\alpha(\mathbf{x})}a^{N-2}\right)a^{*N}\,|0\rangle\,.\end{aligned}$$

[11] As we are working on a Fock space, the normal ordered (or Wick) products of the operator valued distribution $F(x) = a(x) + a^*(x)$ are defined by

$$:F(x_1)\dots F(x_n):= \sum_{I\subset\underline{n}}\prod_{i\in I}a^*(x_i)\prod_{j\in\underline{n}\setminus I}a(x_j).$$

These can be restricted to coinciding space-time points and the obtained Wick powers of the field are well defined as operator valued distributions again. Formally this amounts to subtracting the (infinite) vacuum expectation value. In this formal sense the normal ordered expression for $\phi^2(x)$ can be understood as

$$:\phi^2(x):= \phi^2(x) - \langle 0|\,\phi^2(x)\,|0\rangle\,\mathrm{Id}$$

,where the vacuum expectation value is in fact a constant, and analogously for $\varpi^2(x)$.

[12] One can similarly obtain the agreement of the expectation value of the normal ordered quantum in the state under consideration and the classical stress-energy tensor for the initial conditions by expressing the operator in terms of the field F and its derivatives.

With $\langle 0| a^{N-2}a^{*N} |0\rangle = 0$ and $a^{N-1}a^{*N} |0\rangle = (N!)a^* |0\rangle$ the above is equal to

$$\begin{aligned}&2N(N!)\overline{\alpha(\mathbf{x})} \langle 0| \phi(\mathbf{x})a^* |0\rangle \\ &= 2N(N!) |\alpha(\mathbf{x})|^2 .\end{aligned}$$

Along the lines of the previous calculation one obtains the second expectation value:

$$\begin{aligned}&\left[a^N, \varpi(\mathbf{x})\right] = -iN\overline{\beta(\mathbf{x})}a^{N-1}, \\ &\left[a^N, \varpi^2(\mathbf{x})\right] = -iN\overline{\beta(\mathbf{x})} \left(2\varpi(\mathbf{x})a^{N-1} - iN\overline{\beta(\mathbf{x})}a^{N-2}\right), \\ &\langle\psi| :\varpi^2(\mathbf{x}): |\psi\rangle = 2N |\beta(\mathbf{x})|^2 .\end{aligned}$$

To compute the last expectation value, we compute the following commutator:

$$\begin{aligned}\left[\partial_b\phi(\mathbf{x}), \textstyle\int \mathrm{d}\mathbf{y}\ \varpi(\mathbf{y})\overline{\alpha(\mathbf{y})}\right] &= \int \mathrm{d}\mathbf{y}\ \left[\partial_b\phi(\mathbf{x}), \varpi(\mathbf{y})\right] \overline{\alpha(\mathbf{y})} \\ &= \int \mathrm{d}\mathbf{y}\ \partial_{b,\mathbf{x}} \left[\phi(\mathbf{x}), \varpi(\mathbf{y})\right] \overline{\alpha(\mathbf{y})} \\ &= -i \int \mathrm{d}\mathbf{y}\ \partial_{b,\mathbf{x}}\delta(\mathbf{x}-\mathbf{y})\overline{\alpha(\mathbf{y})} = i\partial_b\overline{\alpha(\mathbf{x})}.\end{aligned}$$

Based on this one finds:

$$\begin{aligned}\left[a^N, \partial_a\phi(\mathbf{x})\right] &= iN\big(\partial_a\overline{\alpha(\mathbf{x})}\big)a^{N-1}, \\ \left[a^N, (\partial_a\phi(\mathbf{x}))\partial_b\phi(x)\right] &= iN\Big(\big(\partial_b\overline{\alpha(\mathbf{x})}\big)\partial_a\phi(\mathbf{x}) + \big(\partial_a\overline{\alpha(\mathbf{x})}\big)\partial_b\phi(\mathbf{x}) \\ &\qquad +i(N-1)\big(\partial_a\overline{\alpha(x)}\big)\big(\partial_b\overline{\alpha(\mathbf{x})}\big)a^{N-2}\Big).\end{aligned}$$

Using these on can compute the last vacuum expectation value

$$\begin{aligned}(N!)\,\langle\psi| :\big(\partial_a\phi(\mathbf{x})\big)\partial_b\phi(\mathbf{x}): |\psi\rangle &= \langle 0| \left[a^N, :\big(\partial_a\phi(\mathbf{x})\big)\partial_b\phi(\mathbf{x}):\right] a^{*N} |0\rangle \\ &= NN!\Big(\big(\partial_b\alpha(\mathbf{x})\big)\partial_a\overline{\alpha(\mathbf{x})} + \big(\partial_a\alpha(\mathbf{x})\big)\partial_b\overline{\alpha(\mathbf{x})}\Big).\end{aligned}$$

As the expression is symmetric in a and b, a contraction with any symmetric tensor yields the analogue for the classical complex field $\sqrt{2N}(\alpha, \beta)$. ■

5.2.2. More general N-particle states

In order to extend the statements of the previous section, we are required to generalize some parts of the underlying construction of the quantum theory based on given initial conditions.

To this avail we will generalize lemma 5.16 as follows:

Lemma 5.22: Suppose $\varphi_1, \ldots, \varphi_N \in \tilde{\mathfrak{H}}$ satisfy

(i) $\Omega\left(\varphi_i, \overline{\varphi}_j\right) = -i\delta_{i,j}$,

(ii) $\Omega(\varphi_i, \varphi_j) = 0$.

Then there exists a linear complex structure J, such that $\varphi_1, \ldots \varphi_N \in \tilde{\mathfrak{H}}_+$.

The proof of the previous lemma will be based on the following:

Lemma 5.23: Let $v_i,\ i \in \underline{N}$ be linearly independent vectors in a real pre-Hilbert space $\tilde{\mathfrak{H}}$. For $w_i \in \tilde{\mathfrak{H}},\ i \in \underline{N}$, satisfying

$$\langle v_i, w_j \rangle = \langle v_j, w_i \rangle$$

there exists a symmetric operator F such that $\mathrm{F}\, v_i = w_i\ i \in \underline{N}$.

Proof. Since the matrix M with elements $M_{ij} = \langle v_i, w_j \rangle$ is symmetric, there exists an orthogonal matrix O diagonalizing M, i.e.

$$OMO^t = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

Defining $v_i' = \sum_j O_{ij} v_j$, and $w_i' = \sum_j O_{ij} w_j$ one finds

$$\langle v_i', w_j' \rangle = \sum_{k,l} \langle O_{ik} v_k, O_{jl} w_l \rangle = \left(OMO^t\right)_{ij} = \lambda_i \delta_{i,j}. \tag{5.4}$$

Based on this observation, we will from now on assume that the v_i and w_i satisfy equation (5.4). We will furthermore order them, such that $\lambda_i \neq 0$ for $i \leq n$ and $\lambda_j = 0$ for $j \in \underline{N} \setminus \underline{n}$, and refer to their span as V.

We define the operator F_1 by

$$\mathrm{F}_1 = \sum_{i=1}^{n} \lambda_i^{-1} \left| w_i \right\rangle \left\langle w_i \right| .$$

It is symmetric and $\mathrm{F}_1\, v_i = w_i$ for $i \leq n$ and $\mathrm{F}_1\, v_j = 0$ for $j \in \underline{N} \setminus \underline{n}$.

By assumption $w_j \in \left(\mathrm{span}\, \{v_i\}_{i \in \underline{n}}\right)^\perp$ for for $j \in \underline{N} \setminus \underline{n}$. Let us denote the orthogonal projection of v_j on $\left(\mathrm{span}\, \{v_i\}_{i \in \underline{n}}\right)^\perp$ by $v_j^\perp$. Then $\left\{v_j^\perp\right\}_{j \in \underline{N} \setminus \underline{n}}$ is a basis for $\left(\mathrm{span}\, \{v_i\}_{i \in \underline{n}}\right)^\perp \cap V$ and thereby we can define an operator A on V by

$$\begin{cases} \mathrm{A}\, v_j^\perp = w_j & \text{for } j \in \underline{N} \setminus \underline{n} \\ \mathrm{A} \left(\mathrm{span}\, \left\{v_j^\perp\right\}_{j \in \underline{N} \setminus \underline{n}}\right)^\perp = 0 & \end{cases}$$

It follows from $\mathrm{A}\, V = \mathrm{span}\, \{w_j\}_{j \in \underline{N} \setminus \underline{n}} \subseteq \mathrm{span}\, \{v_i\}_{i \in \underline{n}}^\perp$, that $\mathrm{A}^t\, v_k = 0$ for $k \in \underline{N}$ as

$$\begin{aligned} \langle \mathrm{A}^t\, v_i, v_j \rangle &= \langle v_i, w_j \rangle = 0 && \text{for } i \in \underline{n} \text{ and } j \in \underline{N} \setminus \underline{n}, \\ \langle \mathrm{A}^t\, v_k, v_i \rangle &= \langle v_k, 0 \rangle = 0 && \text{for } k \in \underline{N} \text{ and } i \in \underline{n}, \\ \langle \mathrm{A}^t\, v_j, v_{j'} \rangle &= \lambda_j \delta_{j,j'} = 0 && \text{for } j, j' \in \underline{N} \setminus \underline{n}. \end{aligned}$$

Therefore

$$\begin{aligned}\left(\mathrm{A}+\mathrm{A}^t\right) v_i &= \mathrm{A}\, v_i = 0 && \text{for } i \in \underline{n}, \text{ and}\\ \left(\mathrm{A}+\mathrm{A}^t\right) v_j &= \mathrm{A}\, v_j = w_j && \text{for } j \in \underline{N} \setminus \underline{n}.\end{aligned}$$

Consequently the symmetric operator given by $\mathrm{F} = \mathrm{F}_1 + \left(\mathrm{A}+\mathrm{A}^t\right)$ satisfies the assertion of the lemma. ■

Proof of lemma 5.22. Identifying the φ_i with their initial condition (α_i, β_i) as functions on $\mathbb{R}^3$, the conditions (i) and (ii) read

$$\text{(i)} \quad \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(\overline{\alpha}_j \beta_i - \alpha_i \overline{\beta}_j\right) = -i\delta_{i,j}$$

$$\text{(ii)} \quad \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(\alpha_j \beta_i - \alpha_i \beta_j\right) = 0.$$

We decompose the initial conditions into their real and imaginary parts, which we denote as follows:

$$\begin{aligned}\alpha_j &= a_j + i b_j,\\ \beta_j &= c_j + i d_j.\end{aligned}$$

In terms of these on can reformulate the conditions as

$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; a_j c_i = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; a_i c_j \tag{5.5}$$

$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; b_j d_i = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; b_i d_j \tag{5.6}$$

$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{x} \;\; \left(a_j d_i - b_i c_j\right) = -\tfrac{1}{2}\delta_{i,j}. \tag{5.7}$$

As in the prove of lemma 5.16, our aim is to construct a positive definite operator D and a symmetric operator F on $L^2(\mathbb{R}^3)$, such that

$$\begin{cases}\mathrm{D}\, a_i - \mathrm{F}\, b_i = -d_i\\ \mathrm{F}\, a_i + \mathrm{D}\, b_i = c_i\end{cases} \quad \forall i \in \underline{N}. \tag{5.8}$$

To construct suitable operators, we will distinguish three different cases. For convenience we will denote the integrals over $\mathbb{R}^3$ by "$\cdot$".

First, we assume the $\{a_i, b_i\}_{i\in\underline{N}}$ to be linearly independent. This implies the existence of a bounded linear operator A on $V = \operatorname{span}\{a_i, b_i\}_{i\in\underline{N}}$, that maps $\{a_i, b_i\}_{i\in\underline{N}}$ to an orthogonal basis in V. We set $\mathrm{D} = \frac{1}{2}\,\mathrm{A}^t\,\mathrm{A}$ on V, then

$$\begin{aligned}&a_i \cdot \mathrm{D}\, a_j + b_i \cdot \mathrm{D}\, b_j = \tfrac{1}{2}\delta_{i,j}, \quad \text{and}\\ &a_i \cdot \mathrm{D}\, b_j = 0 = b_i \cdot \mathrm{D}\, a_j.\end{aligned}$$

With this choice for the operator D, the equations (5.8) that F needs to satisfy read

$$\mathrm{F}\,a_i = c_i - \mathrm{D}\,b_i = c_i',$$
$$\mathrm{F}\,b_i = d_i + Da_i = d_i'.$$

We note that, by the conditions (5.5-5.7),

$$a_j \cdot c_i' = a_j \cdot c_i - a_j \cdot \mathrm{D}\,c = a_j \cdot c_i = a_i \cdot c_j = a_i \cdot c_j', \tag{5.9}$$
$$b_j \cdot d_i' = b_i \cdot d_j'. \tag{5.10}$$

As we assume the $\{a_i, b_i\}_{i\in \underline{N}}$ to be linearly independent, to span a subspace of $L^2(\mathbb{R}^3)$, and to satisfy equations (5.9) and (5.10), we can apply lemma 5.23 to obtain a symmetric operator F, which concludes the proof of the assertion of the lemma in the case of linearly independent a_i and b_i.

Second, we furthermore assume $a_N = 0$. In this case the conditions (5.5)- (5.7) imply $\mathrm{D}\,b_N = c_N$, $\mathrm{F}\,b_N = c_N$, and $b_N \cdot c_N = \frac{1}{2}$. By lemma 5.23, there exists a symmetric operator F_N on $\tilde{V} = \mathrm{span}\,\{b_N, c_N, d_N\}$, that fulfills

$$\mathrm{F}_N\,b_N = d_N. \tag{5.11}$$

We set $D_N = 2\,|c_N\rangle\langle c_N|$ and

$$\mathrm{F} = F_{N-1} + F_N,$$
$$\mathrm{D} = D_{N-1} + D_N.$$

By (5.11), we require $\mathrm{D}_{N-1}\,b_N = 0$. As we need D to be symmetric and positive definite, D_{N_1} has to be positive definite on $b_N^\perp$. Likewise F_{N-1} has to be symmetric and satisfy $\mathrm{F}_{N_1}\,b_N = 0$. Now for $i \in \underline{N-1}$ the requirement (5.8) takes the form

$$\mathrm{D}_{N-1}\,a_i - \mathrm{F}_{N-1}\,b_i = -d_i + \mathrm{F}_N\,b_i = d_i',$$
$$\mathrm{D}_{N-1}\,b_i + \mathrm{F}_{N-1}\,a_i = c_i - \mathrm{F}_N\,a_i = c_i',$$

as

$$\mathrm{D}_N\,a_i = 2(c_N \cdot a_i)c_N = 2(c_i \cdot a_N)c_N = 0, \quad \text{and}$$
$$\mathrm{D}_N\,b_i = 2(c_N \cdot b_i)c_N = 2(\tfrac{1}{2}\delta_{i,N} - a_N \cdot b_i)c_N = 0.$$

Furthermore one finds, that for all $i, j \neq N$

$$a_i \cdot c_j' = a_i \cdot c_j - a_i \cdot \mathrm{F}_N\,a_j = a_j \cdot c_i - a_j \cdot \mathrm{F}_N\,a_i = a_j \cdot c_j',$$
$$b_i \cdot d_j' = b_i \cdot c_j - b_i \cdot \mathrm{F}_N\,b_j = d_j \cdot b_i - b_j \cdot \mathrm{F}_N\,b_i = b_j \cdot d_i', \quad \text{and}$$
$$a_i \cdot d_j' - b_j \cdot c_i' = a_i \cdot d_j - b_j \cdot c_i = -\tfrac{1}{2}\delta_{i,j}.$$

Moreover

$$b_N \cdot d_i' = b_N \cdot d_i - b_N \cdot F_N b_i = b_N \cdot d_i - d_N \cdot b_i = 0,$$
$$b_N \cdot c_i' = b_N \cdot c_i - b_N \cdot \mathrm{F}_N\,a_i = b_N \cdot c_i - d_N a_i = \tfrac{1}{2}\delta_{N,i} = 0.$$

In summary we have $\{d'_i, c'_i, a_i, b_i\}_{i\in\underline{N-1}} \subset b_N^{\perp}$, as well as $a_i \cdot c'_j = a_j \cdot c'_i$ and $b_i \cdot d'_j = b_j \cdot d'_i$ for all $i, j \in \underline{N-1}$. Therefore we can apply the assertion of the lemma proven for the first case to obtain D_{N-1} and F_{N-1} on $b_N^{\perp}$.

Finally, we consider the a_i and b_i to be linearly dependent. In the following we will show, that this case is equivalent to the previous one. For this purpose let us consider two real $N \times N$ matrices, A and B and set

$$\begin{aligned} a'_i &= \sum_{k=1}^{N} \left(A_{ik} a_k + B_{ik} b_k\right), \qquad & b'_i &= \sum_{k=1}^{N} \left(A_{ik} b_k - B_{ik} a_k\right), \\ c'_i &= \sum_{k=1}^{N} \left(A_{ik} c_k + B_{ik} d_k\right), \qquad & d'_i &= \sum_{k=1}^{N} \left(A_{ik} d_k - B_{ik} c_k\right). \end{aligned} \tag{5.12}$$

For $\{a_i, b_i, c_i, d_i\}_{i\in\underline{N}}$ satisfying the conditions (5.5)- (5.7), one finds

$$\begin{aligned} &a'_i \cdot c'_j = a'_j \cdot c'_i + \left(BA^t - AB^t\right)_{ij}, \\ &b'_i \cdot d'_j = b'_j \cdot d'_j - \tfrac{1}{2}\left(BA^t - AB^t\right)_{ij}, \quad \text{and} \\ &a'_i \cdot d'_j - c'_i \cdot b'_j = -\tfrac{1}{2}\left(AA^t + BB^t\right)_{ij}. \end{aligned}$$

Hence the $\{a'_i, b'_i, c'_i, d'_i\}$ defined above also satisfy the conditions (5.5)- (5.7), if the matrix

$$\mathcal{O} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

is orthogonal. The statement also holds the other way round. If $\{a'_i, b'_i, c'_i, d'_i\}$ satisfy the conditions, then

$$\begin{aligned} a_i &= \sum_{k=1}^{N} \left(A_{ki} a'_k - B_{ki} b'_k\right), \qquad & b_i &= \sum_{k=1}^{N} \left(A_{ki} b'_k + B_{ki} a'_k\right), \\ c_i &= \sum_{k=1}^{N} \left(A_{ki} c'_k - B_{ki} d'_k\right), \qquad & d_i &= \sum_{k=1}^{N} \left(A_{ki} d'_k + B_{ki} c'_k\right), \end{aligned}$$

also do, provided $\mathcal{O}$ is orthogonal. We will make use of this statement by constructing an orthogonal matrix mapping the $2N$ linear dependent a_i and b_i to $2N-1$ linear independent ones and the remaining one to zero, such that we can apply the assertion of the lemma in the second case proven before. We construct the matrix as follows:

- Pick coefficients $A_{11}, \ldots, A_{1N}, B_{11}, \ldots, B_{1N}$, such that $a'_N = \sum_{k=1}^{N}(A_{1k} a_k + B_{1k} b_k) = 0$.
- Choose a normalized vector $(A_{21}, \ldots, A_{2N}, B_{21}, \ldots, B_{2N}) \in \mathbb{R}^{2N}$ orthogonal to $(A_{11}, \ldots, A_{1N}, B_{11}, \ldots, B_{1N})$ and $(-B_{11}, \ldots, -B_{1N}, A_{11}, \ldots, A_{1N})$.

- Proceed by selecting a normalized $(A_{31},\dots,A_{3N},B_{31},\dots,B_{3N})$ orthogonal to $(A_{11},\dots,A_{1N},B_{11},\dots,B_{1N})$, $(-B_{11},\dots,-B_{1N},A_{11},\dots,A_{1N})$, $(A_{21},\dots,A_{2N},B_{21},\dots,B_{2N})$ and $(-B_{21},\dots,-B_{2N},A_{21},\dots,A_{2N})$.

- Continue analogously for the remaining $N-3$ vectors.

The matrix we obtain by the above construction is orthogonal. Furthermore the elements of $\{a'_i,b'_i,c'_i,d'_i\}_{i\in\underline{N}}\setminus\{a'_N\}$ defined by the equations (5.12) are linearly independent and satisfy the conditions (5.5)- (5.7). Consequently we are in the setting of the second case and can apply the statement of the lemma to obtain suitable operators D and F, such that the $\{a'_i,b'_i,c'_i,d'_i\}_{i\in\underline{N}}$ statisfy (5.8). It follows from the invertibility of the orthogonal matrix, that this also holds for the a_i, b_i, c_i and d_i.

To conclude the proof of the lemma, we note that the operator

$$\mathrm{A}=\begin{pmatrix}\mathrm{D} & \mathrm{F}\\ -\mathrm{F} & \mathrm{D}\end{pmatrix}$$

obtained individually for the three cases satisfies the assumptions of proposition 5.15, and its graph consequently is a suitable subspace to apply lemma 5.16, which proves the existence of a complex structure such that the $\varphi_i\in\tilde{\mathfrak{H}}_+$. ■

Definition 5.24: Let $\{\varphi_i\}_{i\in\underline{N}}\subset\tilde{\mathfrak{H}}_+$ be a set of normalized solutions to the Klein-Gordon equation satisfying

$$-i\Omega(\overline{\varphi_i},\varphi_j)=\delta_{i,j}$$

and denote the initial conditions corresponding to φ_i by (α_i,β_i). Then we define the operators

$$\begin{aligned}a_i&=\int_{\mathbb{R}^3}\mathrm{d}\mathbf{x}\ \ \left(\overline{\alpha_j(\mathbf{x})}\varpi(\mathbf{x})+\overline{\beta_i(\mathbf{x})}\phi(\mathbf{x})\right),\\ a_i^*&=\int_{\mathbb{R}^3}\mathrm{d}\mathbf{x}\ \ \left(\alpha_i(\mathbf{x})\varpi(\mathbf{x})+\beta_i(\mathbf{x})\phi(\mathbf{x})\right).\end{aligned}\tag{5.13}$$

Proposition 5.25: The operators a_i and a_i^* are pairs annihilation and creation operators for orthogonal one-particle states.

Proof.

$$\left[a_i,a_j^*\right]=i\int_{\mathbb{R}^3}\mathrm{d}\mathbf{x}\ \ \left(\overline{\alpha_i(\mathbf{x})}\beta_j(\mathbf{x})-\overline{\beta_i(\mathbf{x})}\alpha_j(\mathbf{y})\right)=-i\Omega(\overline{\varphi_i},\varphi_j)=\delta_{i,j}$$ ■

One obtains an analogous result to theorem 5.21 for the more general states, namely N-particle states built from N pairwise orthogonal one-particle states.

Theorem 5.26: Let $\{\varphi_i\}_{i\in\underline{N}}$ be a set of mutually orthogonal elements in the one-particle Hilbert space $\big(\mathfrak{H},\ -i\Omega(\bar{\cdot},\cdot)\big)$ and define the corresponding annihilation and creation operators, a_i and a_i^*, in terms of the respective initial conditions by (5.13). Then the expectation values of the normal ordered operator valued distributions $:\phi^2(\mathbf{x}):$, $:\varpi^2(\mathbf{x}):$, and $:(\partial_a\phi(\mathbf{x}))\partial_b\phi(\mathbf{x}):$ in a state $|\Upsilon\rangle = \prod_i a_i^*|0\rangle$ correspond to the multi-component classical scalar field given by $\sqrt{2}\sum_i(\alpha_i,\beta_i)$.

Proof. First, note that

$$\prod_i a_i \prod_j a_j^* |0\rangle = \prod_{i\neq k} a_i \left[a_k, \textstyle\prod_j a_j^*\right]|0\rangle = \prod_{i\neq k} a_i \prod_{j\neq k} a_j^* |0\rangle = |0\rangle\,.$$

This implies

$$\begin{aligned}\langle 0|\prod_i a_i :\phi^2(\mathbf{x}): \prod_j a_j^*|0\rangle &= \langle 0| :\phi^2(\mathbf{x}): |0\rangle + \langle 0|\,[\textstyle\prod_i a_i, :\phi^2(\mathbf{x}):]\displaystyle\prod_j a_j^*|0\rangle \\ &= \langle 0|\,[\textstyle\prod_i a_i, \phi^2(\mathbf{x})]\displaystyle\prod_j a_j^*|0\rangle\,.\end{aligned}$$

To compute the above expectation value, we use the following identities, which are direct consequences of the commutation relations:

$$\begin{aligned}[a_k,\phi(\mathbf{x})] &= \int_{\mathbb{R}^3}\mathrm{d}\mathbf{x}\ \left[\overline{\alpha_k(\mathbf{y})}\varpi(\mathbf{y}) + \overline{\beta_k(\mathbf{y})}\phi(\mathbf{y}), \phi(\mathbf{x})\right] = i\overline{\alpha_k(\mathbf{x})},\\ [a_k^*,\phi(\mathbf{x})] &= -i\alpha_k(\mathbf{x}),\\ \left[\textstyle\prod_j a_j,\phi(\mathbf{x})\right] &= a_k\left[\textstyle\prod_{j\neq k} a_j,\phi(\mathbf{x})\right] + [a_k,\phi(\mathbf{x})]\displaystyle\prod_{j\neq k} a_j\\ &= i\sum_{k=1}^N \overline{\alpha_k(\mathbf{x})}\prod_{j\neq k} a_j,\\ \left[\textstyle\prod_j a_j,\phi^2(\mathbf{x})\right] &= i\sum_{k=1}^N \overline{\alpha_k(\mathbf{x})}\left(2\phi(\mathbf{x})\prod_{j\neq k} a_j + i\sum_{l\neq k}\overline{\alpha_l(\mathbf{x})}\prod_{j\neq k,l} a_j\right).\end{aligned}$$

Then the expectation value can be rewritten as:

$$\begin{aligned}\langle\Upsilon| : \phi^2(\mathbf{x}) : |\Upsilon\rangle &= i\sum_{k=1}^N \alpha_k(\mathbf{x})\,\langle 0|\,2\phi(\mathbf{x})a_k^* + i\sum_{l\neq k}\alpha(\mathbf{x})a_k^* a_l^*\,|0\rangle\\ &= 2\sum_{k=1}^N |\alpha_k(\mathbf{x})|^2\,.\end{aligned}$$

Analogously,

$$\begin{aligned}
&[a_k, \varpi(\mathbf{x})] = -i\overline{\beta_k(\mathbf{x})},\\
&[a_k^*, \varpi(\mathbf{x})] = i\beta_k(\mathbf{x}),\\
&\Big[\prod_j a_j, \varpi(\mathbf{x})\Big] = -i\sum_{k=1}^{N}\overline{\beta_k(\mathbf{x})}\prod_{j\neq k} a_j,\\
&\Big[\prod_j a_j, \varpi^2(\mathbf{x})\Big] = -i\sum_{k=1}^{N}\overline{\beta_k(\mathbf{x})}\left(2\varpi(\mathbf{x})\prod_{j\neq k} a_j - i\sum_{l\neq k}\overline{\beta_l(\mathbf{x})}\prod_{j\neq k,l} a_j\right).
\end{aligned}$$

Hence the second expectation value is given as

$$\langle\Upsilon| :\varpi^2(\mathbf{x}): |\Upsilon\rangle = 2\sum_{k=1}^{N}|\beta_k(\mathbf{x})|^2 .$$

Finally one obtains

$$\begin{aligned}
&[a_k, \partial_a\phi(\mathbf{x})] = -i\partial_a\overline{\alpha_k(\mathbf{x})},\\
&[a_k^*, \partial_a\phi(\mathbf{x})] = i\partial_a\alpha_k(\mathbf{x}),\\
&\Big[\prod_j a_j, \partial_a\phi(\mathbf{x})\Big] = -i\sum_{k=1}^{N}\partial_a\overline{\alpha_k(\mathbf{x})}\prod_{j\neq k} a_j,\\
&\Big[\prod_j a_j, \big(\partial_a\phi(\mathbf{x})\big)\partial_b\phi(\mathbf{x})\Big]\\
&\qquad = i\sum_{k=1}^{N}\Bigg[\bigg(\Big(\partial_b\overline{\alpha_k(\mathbf{x})}\Big)\big(\partial_a\phi(\mathbf{x})\big) + \Big(\partial_a\overline{\alpha_k(\mathbf{x})}\Big)\big(\partial_b\phi(\mathbf{x})\big)\bigg)\prod_{j\neq k} a_j\\
&\qquad\qquad + i\sum_{l\neq k}\Big(\partial_a\overline{\alpha_k(\mathbf{x})}\Big)\Big(\partial_b\overline{\alpha_l(\mathbf{x})}\Big)\prod_{j\neq k,l} a_j\Bigg],
\end{aligned}$$

which yields the following for the final expectation value:

$$\langle\Upsilon| :\big(\partial_a\phi(\mathbf{x})\big)\partial_b\phi(\mathbf{x}): |\Upsilon\rangle = 2\sum_{k=1}^{N}\Big(\big(\partial_a\alpha_k(\mathbf{x})\big)\partial_b\overline{\alpha_k(\mathbf{x})} + \big(\partial_b\alpha_k(\mathbf{x})\big)\partial_a\overline{\alpha_k(\mathbf{x})}\Big).!$$

This concludes the proof of the theorem. ■

5.2.3. The semiclassical Einstein equation

Given a quantum theory of matter on a rather general curved space-time a question, that arises naturally, is that of the influence of the matter on the space-time.

In the classical, in the sense of non-quantum, general relativistic setting the situation is clear.

Consider, for example, a cloud of dust particles, subject to general relativity.

Then looking at Einstein's equations as a set of differential equations determining the evolution of space-time, the matter enters them as a source term, thus influencing solutions to the equations. In return the geometry of space-time is influencing the matter, for instance by compressing the cloud of dust particles.

It is this interaction of the matter and the geometry, that one would like to model with quantum matter, and that one refers to as back reaction.

In the case of matter, described by some quantum theory, the situation however becomes more complicated. As there is no full theory of quantum gravity available, one could give up right away considering all the fundamental differences of the left hand side, the geometric side, and the right hand side, the quantum theory side, of an Einstein equation that one would naively write down.

But instead, one can think about the situation carefully and try to modify the naive approach to make sense out of it. The way this is usually done, is by considering the so-called semi-classical Einstein equation, where one replaces the classical stress-energy tensor in Einstein's equations by the expectation value of the quantum stress-energy tensor operator in an appropriate state ω.

This idea poses two new problems, the first one being the quantum stress-energy tensor operator and the second one being the appropriate class of states. In the course of this section we are going to review the approaches taken to resolve these issues, starting with the former.

The classical expression is quadratic in the fields, and as the quantum field operators are defined as distributions, non-linear operations are not well defined in general. The way out is a regularization of the classical expression, where the classical fields are replaced by their operator-valued distribution analogoes of the quantum theory, we will frequently refer to this object as the stress-energy tensor $\hat{\mathcal{T}}_{\mu\nu}$. A study of the natural requirements on such a regularized stress-energy tensor $:\hat{\mathcal{T}}_{\mu\nu}:$ lead Wald [45] to the formulation of the following axioms, here presented in the version of Hack and Moretti in [24] adapted to the more recent developments in algebraic quantum field theory:

1. The commutator of the regularized stress-energy tensor $:\hat{\mathcal{T}}_{\mu\nu}(x):$ with any product of fields at different space-time points[13] $F(x_1)F(x_2)\dots F(x_n)$ equals the commutator of the non-regularized stress-energy tensor $\hat{\mathcal{T}}_{\mu\nu}(x)$ with $F(x_1)F(x_2)\dots F(x_n)$.

2. The regularizes stress-energy tensor $:\hat{\mathcal{T}}_{\mu\nu}(x):$ transforms covariantly under diffeomorphisms and does not depend on the metric and its derivatives at $y \neq x$.

3. Any expectation value of $\nabla^\mu :\hat{\mathcal{T}}_{\mu\nu}(x):$ vanishes.

4. In Minkowski space-time the vacuum expectation value of $:\hat{\mathcal{T}}_{\mu\nu}(x):$ vanishes.

[13] As long as $x_i \neq x_j \; \forall i,j \in \underline{n}$ the product $F(x_1)F(x_2)\dots F(x_n)$ is a well defined multivariate operator valued distribution.

5. No expectation value of : $\hat{\mathcal{T}}_{\mu\nu}(x)$: contains derivatives of the metric of order higher than two.

In a subsequent work Wald [46] proved, that the final axiom can never be satisfied. Detailed descriptions of the origin of these axioms can be found in the given original references, as well as in chapter 4 of [48]. The original reason for imposing this axiom had been to avoid a possible blow-up in finite time of the metric components, when solving the semi-classical equations, that this section leads up to. Even though this cannot be avoided one can systematically treat the regular solutions, cf. [15] and the references therein.

Regarding the stress-energy tensor, the remaining problem is to specify a constructition that produces an operator in accordance with the first four of the previously described axioms.

In order to construct well defined regularized stress-energy tensor operators one simultaneously needs to tackle the second problem, to find a suitable class of states. The class of quasi free states is, despite their property of having well defined n-point functions, too large, cf. [17], and [26], as well asreferences therein. The class of physically acceptable states, in the sense that they allow for well defined stress-energy tensor operators obeying the axioms 1.-4. above, are the so called Hadamard states. There are several ways of defining Hadamard states, may be the most elegant one is the definition in terms of the wave-front set of the two point function $\omega_2 : (\varphi_1, \varphi_2) \mapsto \omega(F(\varphi_1)F(\varphi_2))$ of a given algebraic state ω given by Radzikoswki in [40].

The characteristic feature of Hadamard states is the singular behaviour of the integral kernel $\omega_2(x, y)$ of their respective two-point function. For any Hadamard state $\omega_2(x, y)$ is given by the sum of a smooth function on $M \times M$ and the Hadamard parametrix $Z(x, y)$. The latter is completely determined by the metric g and the Klein-Gordon operator P.

On Hadamard states there is a procedure to construct regularized stress-energy tensor operators due to Moretti [33], which is a priori determined by the local geometry and the Klein-Gordon operator only. In particular it does not include any ad-hoc operations, which was a common drawback of earlier constructions, cf. [48] and references therein.

The basis for the construction, and for any so-called "point-splitting" procedure, is the observation, that $\omega_2(x, y)$ is regular, in fact even smooth, as long as x and y are neither identical nor connected by a light-like curve. To define the integral kernel of the expectation value of regularized square of the field $:\omega_2(x):$, one takes the integral kernel of another bi-distribution $\tilde{Z}(x, y)$ with the same singular behavior as $\omega_2(x, y)$. Then for split points x and y in the above sense, the difference $\omega_2(x, y) - \tilde{Z}(x, y)$ is regular, as both constituents are, and in particular has a limit[14] for $y \to x$. The latter is customarily referred to as the coincidence limit. One sets

$$:\omega_2(x):= \lim_{y \to x} \Big(\mathfrak{Re} \left(\omega_2(x, y) \right) - \tilde{Z}(x, y) \Big) .$$

[14] The notation $y \to x$ is in this case to be understood as y approaching x or any point that is connected to x by a light-like curve.

Prior to the aforementioned work of Moretti, the distribution $\tilde{Z}$ was chosen by hand to satisfy the axioms above, and turned out to be dependent on nothing but P and the local geometry, cf. [48] for the explicit construction.

In order to define the expectation value of the regularized stress-energy tensor in a Hadamard state according to Moretti, we denote by $\delta(z,x)$ the operator of geodesic transport from the tangent space to M at z to the one at x. One defines the linear operator $\mathrm{D}^{(\eta)}_{(z),\mu\nu}(x,y)$ by

$$\begin{aligned}\mathrm{D}^{(\eta)}_{(z),\mu\nu}(x,y) =& \tfrac{1}{2}\left(\delta^{\mu'}_{\mu}(z,x)\delta^{\nu'}_{\nu}(z,y)\nabla_{x,\mu'}\nabla_{y,\nu'} + \delta^{\mu'}_{\mu}(z,y)\delta^{\nu'}_{\nu}(z,x)\nabla_{y,\mu'}\nabla_{x,\nu'}\right)\\ &- \tfrac{1}{2}g_{\mu\nu}(z)\left(g^{\gamma\gamma'}(z)\delta^{\mu'}_{\gamma}(z,x)\delta^{\nu'}_{\gamma'}(z,y)\nabla_{x,\mu'}\nabla_{y,\mu'} + m^2\right)\\ &+ \tfrac{1}{2}\eta g_{\mu\nu}(z)\left(\mathrm{P}_x + \mathrm{P}_y\right).\end{aligned}$$

See equation (10) in the reference for the explicit expression for non-minimally coupled fields and non-zero potentials.

It is noteworthy, that the classical expression, one obtains from this operator by letting it act on the square of a function at split points $\varphi(x)\varphi(y)$ has an additional term, which in the coincidence limit reads

$$\eta g_{\mu\nu}\varphi(z)P\varphi(z),$$

in comparison to the usual classical stress-energy tensor. This additional term however vanishes, when considering classical solutions to the Klein-Gordon equation $P\varphi = 0$ and one recovers the usual classical theory.

For $Z_n(x,y)$ being the Hadamard parametrix truncated at nth order, the integral kernels of the expectation values of the regularized square of the field operator and the regularized stress-energy tensor are given by

$$\begin{aligned}z \mapsto \left\langle :F^2(z): \right\rangle_\omega &= \lim_{(x,y)\to(z,z)} \left(\mathfrak{Re}\left(\omega_2(x,y)\right) - Z_n(x,y)\right),\\ z \mapsto \left\langle :\hat{\mathcal{T}}^{(\eta)}_{\mu\nu}(z): \right\rangle_\omega &= \lim_{(x,y)\to(z,z)} \mathrm{D}^{(\eta)}_{z,\mu\nu}\left(\mathfrak{Re}\left(\omega_2(x,y)\right) - Z_n(x,y)\right).\end{aligned}$$

Both of these are smooth, independent of the order n of the truncation and if and only if[15] $\eta = \frac{1}{3}$, then $\left\langle :\hat{\mathcal{T}}^{(\eta)}_{\mu\nu}(z): \right\rangle_\omega$ is covariantly conserved. Furthermore one can define a corresponding stress-energy tensor that satisfies the remaining axioms 1.,2., and, 4.. For details see theorems 2.1 and 3.2 in [33].

In this setting one can unambiguously define the semi-classical Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G\left\langle :\hat{\mathcal{T}}^{(\eta)}_{\mu\nu}(z): \right\rangle_\omega, \tag{SCEE}$$

with solutions consisting of a state and a metric.

[15] We are only considering 4-dimensional space-times here.

5.2.4. On the stability

Let us for a moment relax the conditions discussed in the previous section – we will comment on the insufficiencies that this entails later – and look at the semi-classical Einstein equation for a larger class of states.

Theorem 5.27: The system of a classical static spherically symmetric space-time given in terms of two functions $u, v : \mathbb{R}_+ \to \mathbb{R}$ satisfying

(u) $\|u\|_\infty < \infty, \quad \|\partial_\rho u\|_1 < \infty$, and

(v) $\|v\|_\infty < \infty, \quad \|\partial_\rho(\rho v)\|_1 < \infty$,

by the line element

$$\mathrm{d}s^2 = -e^{2u}\,\mathrm{d}t^2 + e^{2v}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}a^2 + \sin^2(a)\,\mathrm{d}b^2\right). \tag{3.1}$$

and a minimally coupled free massive scalar quantum field, for which the expectation value of the regularized stress-energy tensor on a suitable Fock space is defined via normal ordering with respect to the pure quasifree vacuum state ω, interacting via the semiclassical Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G \left\langle\Psi\right| :\hat{\mathcal{T}}_{\mu\nu}: \left|\Psi\right\rangle,$$

is energetically unstable. I.e. there exists a pure quasi free state ω, and another state Ψ in the folium[16] of ω, such that the mass functional obtained from this state has no positive lower bound.

Proof. The proof of the theorem consists of a combination of the results of the previous section and the results on the classical complex scalar field. We pick a spherically symmetric manifold (M, g) and construct a state Ψ on the massive free scalar quantum field algebra on (M, g), whose expectation values for the normal ordered constituents of the stress-energy tensor mimic a classical field. In this setting the semiclassical Einstein equation turns into the classical Einstein's equations and we can apply the appropriate results to show the non-existence of a positive lower bound on the mass functional.

We start the construction with a pair of spherically symmetric initial conditions, i.e. functions $(\alpha, \beta) \in C_0^\infty(\mathbb{R}_+, \mathbb{C}) \times C_0^\infty(\mathbb{R}_+, \mathbb{C})$, such that

$$0 < -\mathfrak{Im}\int_0^\infty \left(\overline{\alpha}\beta - \alpha\overline{\beta}\right)\mathrm{d}r. \tag{5.14}$$

Then for any positive integer N, the pair $\sqrt{2N}(\alpha, \beta)$ is certainly an element in $\mathfrak{F}^m$, cf. definition 4.2 on page 44. Using the solutions (4.16) and (4.17) to the constraint

[16] The folium of an algebraic state is the set of states in the GNS Hilbert space associated to ω. In the explicit case here the folium is the Fock space.

equations (4.14) and (4.15) respectively, the pair $\sqrt{2N}(\alpha, \beta)$ defines a static spherically symmetric asymptotically flat globally hyperbolic space-time (M, g).

Given a pair of functions satisfying equation (5.14) on some static globally hyperbolic spacetime, we can apply theorem 5.17 and proposition 5.18 to obtain a Fock space representation of the algebra $\mathfrak{A}$ of massive free scalar quantum fields on (M, g), such that the pair (α, β) corresponds to an element of the one-particle Hilbert space when β is understood to be a density in the sense of page 85.

We proceed by constructing a state analogously to the ingredients of theorem 5.21. Then by the result of the theorem, the expectation values of the constituents of the normal ordered stress-energy tensor 5.18 correspond to their classical analogues for the classical complex scalar field $\sqrt{2N}(\alpha, \beta)$.

As in section 4.2.4, we define the total mass of the system. Note, that we are in the exact same situation as in the classical case, as we defined the space-time via the solutions of the constraint equations for the functions $\sqrt{2N}(\alpha, \beta)$, which is consistent with the expectation values on the quantum side of the equations.

The positivity condition (5.14) and the commutation relations in the proof of proposition 5.20 are all invariant under the scaling used in the proof of theorem 4.22. This assures that we can apply the same scaling argument as in the classical case. By scaling (α, β) appropriately, we can make the total mass of the system arbitrarily small while keeping the particle number and the normalization of the state fixed. ■

Insufficiencies of the construction

Despite its well-definedness in mathematical terms, the construction we just described has two major drawbacks when it comes to its applicability. The first one is the lack of control over the Hadamard condition. It is unclear, whether or not the pure quasi-free state ω related to the construction is Hadamard or not.

This implies in particular, that the existence of a well defined stress-energy tensor operator satisfying Wald's axioms is doubtful.

The second drawback is related to the regularization procedure in the definition of the expectation value of the stress-energy tensor. Opposed to the point-splitting procedure discussed in the case of Hadamard states, we are not only removing the singular part, but the whole two integral kernel of the two-point function. Thereby all a priori non-vanishing smooth contributions of the state ω are eliminated. In addition our procedure might violate the second axiom of Wald by including non-local terms.

Bibliography

[1] V. I. Arnold, *Ordinary differential equations*, 3. ed., Springer, 2006.

[2] R. Arnowitt, S. Deser, and C. W. Misner, *The dynamics of general relativity*, http://arxiv.org/abs/gr-qc/0405109v1, 2004.

[3] A. Ashtekar and A. Magnon, *Quantum fields in curved space-times*, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **346** (1975), no. 1646, 375–394.

[4] N. Bär, C. Ginoux and F. Pfäffle, *Wave equations on lorentzian manifolds and quantization*, European Mathematical Society, 2007.

[5] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, no. 7, Cambridge university press, 1984.

[6] P. Bizoń and A. Wasserman, *On existence of mini-boson stars*, Communications in Mathematical Physics **215** (2000), 357–373.

[7] R. Brunetti, K. Fredenhagen, and R. Verch, *The generally covariant locality principle – a new paradigm for local quantum field theory*, Communications in Mathematical Physics **237** (2003), no. 1-2, 31–68.

[8] S. Chandrasekhar, *The Maximum Mass of Ideal White Dwarfs*, Astrophysical Journal **74** (1931), 81.

[9] S. Chandrasekhar, *On stars, their evolution and their stability*, Reviews of modern physics **56** (1984), no. 2, 137–147.

[10] D. Christodoulou, *The problem of a self-gravitating scalar field*, Communications in Mathematical Physics **105** (1986), 337–361.

[11] C. Dappiaggi, K. Fredenhagen, and N. Pinamonti, *Stable cosmological models driven by a free quantum scalar field*, (2008).

[12] D. G. De Figueiredo, P. L. Lions, and R. D. Nussbaum, *A priori estimates and existence of positive solutions of semilinear elliptic equations*, Djairo G. de Figueiredo-Selected Papers, Springer, 2013, pp. 133–155.

[13] G. Fischer, *Lineare Algebra*, 15. ed., Vieweg, 2005.

[14] K. Fredenhagen, *Advanced quantum field theory*, Lecture Notes, 2013.

[15] K. Fredenhagen and T.-P. Hack, *Quantum field theory on curved spacetime and the standard cosmological model*, (2013).

[16] R. Friedberg, T. D. Lee, and Y. Pang, *Mini - Soliton Stars*, Phys.Rev. D **35** (1987), 3640.

[17] S. A. Fulling, M. Sweeny, and R. M. Wald, *Singularity structure of the two-point function in quantum field theory in curved spacetime*, Communications in Mathematical Physics **63** (1978), no. 3, 257–264 (English).

[18] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, Physical Review D **15** (1977), no. 10, 2752–2756.

[19] B. Gidas, W.-M. Ni, and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Communications in Mathematical Physics **68** (1979), no. 3, 209–243.

[20] S. R. Green, J. S. Schiffrin, and R. M. Wald, *Dynamic and thermodynamic stability of relativistic, perfect fluid stars*, Class. Quantum Grav. **31** (2014).

[21] G. Grubb, *Distributions and operators*, Graduate Texts in Mathematics, vol. 252, Springer, New York, 2009.

[22] R. Haag, *Local quantum physics: Fields, particles, algebras*, 2 ed., Springer Berlin, 1996.

[23] T.-P. Hack, *On the backreaction of scalar and spinor quantum fields in curved spacetimes*, Ph.D. thesis, Universität Hamburg, 2010.

[24] T.-P. Hack and V. Moretti, *On the stress-energy tensor of quantum fields in curved spacetimes - comparison of different regularization schemes and symmetry of the hadamard/seeley-dewitt coefficients*, Journal of Physics A: Mathematical and Theoretical **45** (2012).

[25] C. Hainzl, *On the static and dynamical collapse of white dwarfs*, 2010.

[26] B. S. Kay and R. M. Wald, *Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon*, Physics Reports **207** (1991), no. 2, 49 – 136.

[27] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. II, John Wiley and Sons, Inc., New York, 1969.

[28] L. D. Landau and E. M. Lifshitz, *Statistical physics, part 1*, Course of theoretical physics, vol. 5, Elsevier, 1980.

[29] T.D. Lee and Y. Pang, *Stability of mini-boson stars*, Nuclear Physics B **315** (1989), no. 2, 477 – 516.

[30] E. H. Lieb and R. Seiringer, *The stability of matter in quantum mechanics*, Cambridge University Press, 2010.

[31] E. H. Lieb and H.-T. Yau, *The chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Communications in Mathematical Physics **112** (1987), no. 1, 147–174.

[32] P. L. Lions, *On the existence of positive solutions of semilinear elliptic equations*, SIAM Review **24** (1982), no. 4, pp. 441–467 (English).

[33] V. Moretti, *Comments on the stress-energy tensor operator in curved spacetime*, Communications in Mathematical Physics **232** (2003), no. 2, 189–221 (English).

[34] Z. Nehari, *On a class of nonlinear second-order differential equations*, Transactions of the American Mathematical Society **95** (1960), no. 1, pp. 101–123 (English).

[35] W.-M. Ni, *Uniqueness of solutions of nonlinear Dirichlet problems*, J. Differential Equations **50** (1983), no. 2, 289–304. MR 719451 (85i:35060)

[36] W.-M. Ni and R. D. Nussbaum, *Uniqueness and nonuniqueness for positive radial solutions of* $\delta u + f(u, r) = 0$, Communications on Pure and Applied Mathematics **38** (1985), no. 1, 67–108.

[37] H. Olbermann, *States of low energy on robertson–walker spacetimes*, Classical and Quantum Gravity **24** (2007), no. 20, 5011.

[38] J. Oppenheimer, *On massive neutron cores*, Phys. Rev.; Physical Review **55** (1939), no. 4, 374–381.

[39] H. Pfister, *A new and quite general existence proof for static and spherically symmetric perfect fluid stars in general relativity*, Classical and Quantum Gravity **28** (2011), no. 7, 075006.

[40] M. J. Radzikowski, *Micro-local approach to the hadamard condition in quantum field theory on curved space-time*, Communications in Mathematical Physics **179** (1996), 529–553 (English).

[41] T. Regge and C. Teitelboim, *Role of surface integrals in the hamiltonian formulation of general relativity*, Annals of Physics **88** (1974), no. 1, 286 – 318.

[42] A. D. Rendall and B. G. Schmidt, *Existence and properties of spherically symmetric static fluid bodies with a given equation of state*, Classical and Quantum Gravity **8** (1991), no. 5, 985.

[43] U.M. Schaudt, *On static stars in newtonian gravity and lane-emden type equations*, Annales Henri Poincaré **1** (2000), no. 5, 945–976.

[44] R. Tolman, *Static solutions of einstein's field equations for spheres of fluid*, Phys. Rev.; Physical Review **55** (1939), no. 4, 364–373.

[45] R. M. Wald, *The back reaction effect in particle creation in curved spacetime*, Communications in Mathematical Physics **54** (1977), 1–19.

[46] ______, *Trace anomaly of a conformally invariant quantum field in curved spacetime*, Physical Review D **17** (1978), no. 6, 1477–1484.

[47] ______, *General Relativity*, The University of Chicago Press, 1984.

[48] ______, *Quantum field theory in curved spacetimes and black hole thermodynamics*, The University of Chicago Press, 1994.

[49] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, John Wiley and Sons, Inc., New York, 1972.

[50] E. Zeidler, *Applied functional analysis: Main principles and their applications*, Applied Mathematical Sciences Series, no. 109, Springer, 1995.

[51] E. Zeidler and L.F. Boron, *Variational methods and optimization*, Nonlinear functional analysis and its applications, vol. 3, Springer, 1984.

Supplements

A.1. Explicit curvature quantities

The following is a collection of explicit curvature quantities that are relevant for the various computations, where the spherically symmetric metric tensor is given by the line element (3.1).
Non-vanishing Christoffel Symbols:

$$\begin{aligned}
\Gamma^r_{rr} &= \tfrac{\partial v}{\partial r} & \Gamma^t_{rt} &= \tfrac{\partial u}{\partial r}\\
\Gamma^r_{tt} &= \tfrac{\partial u}{\partial r}e^{2(u-v)} & \Gamma^\alpha_{\alpha\beta} &= \tfrac{\cos\beta}{\sin\beta}\\
\Gamma^r_{\alpha\alpha} &= -re^{-2v}\sin^2\beta & \Gamma^\alpha_{\alpha r} &= r^{-1}\\
\Gamma^r_{\beta\beta} &= -re^{-2v} & \Gamma^\beta_{\alpha\alpha} &= -\cos\beta\sin\beta\\
\Gamma^r_{rt} &= \tfrac{\partial v}{\partial t} & \Gamma^\beta_{\beta r} &= r^{-1}\\
\Gamma^t_{rr} &= \tfrac{\partial v}{\partial t}e^{-2(u+v)} & &
\end{aligned}$$

Non-vanishing components of the Ricci curvature tensor:

$$R^t{}_t = \left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t}\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right)\right)e^{-2u} - \left(\frac{\partial^2 u}{\partial r^2} + \left(\frac{\partial u}{\partial r}\right)^2 - \frac{\partial v}{\partial r} + \frac{2}{r}\right)e^{-2v}$$

$$R^r{}_r = \left(\frac{\partial^2 v}{\partial t^2} - \left(\frac{\partial v}{\partial t}^2 - \frac{\partial v}{\partial t}\frac{\partial u}{\partial t}\right)\right)e^{-2u} + \left(\left(\frac{2}{r} + \frac{\partial u}{\partial r}\right)\frac{\partial v}{\partial r} - \frac{\partial^2 u}{\partial r^2} - \left(\frac{\partial u}{\partial r}\right)^2\right)e^{-2v}$$

$$R^t{}_r = -\frac{2}{r}\left(\frac{\partial v}{\partial t}\right)e^{-2u}$$

$$R^r{}_t = \frac{2}{r}\left(\frac{\partial v}{\partial t}\right)e^{-2v}$$

$$R^\alpha{}_\alpha = R^\beta{}_\beta = \frac{1}{r^2}\left(e^{2v} + r\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\right) - 1\right)e^{-2v}$$

Ricci scalar:

$$\begin{aligned}
R = 2r^{-2} &+ 2e^{-2u}\left[\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t}\frac{\partial u}{\partial t} + \left(\frac{\partial v}{\partial t}\right)^2\right]\\
&+ 2e^{-2v}\left[-\frac{\partial^2 u}{\partial r^2} - \left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{2}{r}\left(\frac{\partial v}{\partial r} - \frac{\partial u}{\partial r}\right) - r^{-2}\right]
\end{aligned}$$

A.2. Detailed calculations for the Fréchet differentiability of the Hamiltonian

To estimate $|\Delta H^g - \delta_u H^g - \delta_v H^g|$, expand $H^g(u+\delta u, v+\delta v)$ as follows

$$
\begin{aligned}
&H^g(u+\delta u, v+\delta v) \\
&= (2G)^{-1} \int_0^\infty \mathrm{d}r\, e^u \Big[2 - e^v - e^{-v} + 2\rho\left(1-e^{-v}\right)\partial_\rho u \\
&\qquad + \left(2 - e^v - e^{-v} + 2\rho\left(1-e^{-v}\right)\partial_\rho u\right)\delta u \\
&\qquad + \left(e^{-v} - e^v + 2\rho e^v \partial_\rho u\right)\delta v \\
&\qquad - e^v\left(e^{\delta u+\delta v} - (\delta u + \delta v + 1)\right) - e^{-v}\left(e^{\delta u - \delta v} - (\delta u - \delta v + 1)\right) \\
&\qquad + 2\rho(\partial_\rho u)\left[\left(e^{\delta u} - (\delta u + 1)\right) - e^{-v}\left(e^{\delta u-\delta v} - (\delta u - \delta v + 1)\right)\right] \\
&\qquad + 2\rho(\partial_\rho \delta u)\left(e^{\delta u} - e^{u-v-\delta v} - 2\right)\Big].
\end{aligned}
$$

With $\|\delta u\|_\infty \le 1$, $\|\delta v\|_\infty \le 1$, and $k \ge e^2$ one can bound $|\Delta H^g - \delta_u H^g - \delta_v H^g|$ by

$$
\begin{aligned}
&(2G)^{-1} e^{\|u\|_\infty}\Big[2k\left(\left\|2 - e^v - e^{-v}\right\|_1 + \|\partial_\rho u\|_1 \left\|\rho(1-e^{-v})\right\|_\infty\right)\left(\|\delta u\|_\infty^2 + \|\delta v\|_\infty^2\right) \\
&\qquad + 2k\left\|\rho(1-e^{-v})\right\|_\infty \|\partial_\rho \delta u\|_1 \left(\|\delta u\|_\infty + \|\delta v\|_\infty\right)\Big] \\
&\le\; C^g(\|u\|_\infty, \|v\|_\infty, \|\partial_\rho u\|_1, \|\partial_\rho(\rho v)\|_1)\left(\|\delta u\|_\infty^2 + \|\delta v\|_\infty^2 + \|\partial_\rho \delta u\|_1^2 \right. \\
&\qquad \left. + \|\partial_\rho(\rho\delta v)\|_1^2\right).
\end{aligned}
$$

In order to prove the Fréchet differentiability of the matter Hamiltonian, consider the expansion of $\mathcal{H}^m(u+\delta u, v+\delta v, \varphi+\delta\varphi, \Pi+\delta\Pi)$,

$$
\begin{aligned}
&4\pi \int_0^\infty \mathrm{d}r\, \rho^2 e^u \Big[m^2 e^v |\varphi|^2 \left(e^{\delta u+\delta v} - (1+\delta u+\delta v\right) \\
&\qquad + e^{-v}\left(|\partial_\rho\varphi|^2 + (16\pi^2\rho^4)^{-1}|\Pi|^2\right)\left(e^{\delta u-\delta v} - (1+\delta u - \delta v)\right) \\
&\qquad + m^2 e^v(\overline{\varphi}\delta\varphi + \varphi\overline{\delta\varphi})\left(e^{\delta u+\delta v} - 1\right) \\
&\qquad + e^{-v}\Big((\partial_\rho\overline{\varphi})\partial_\rho\delta\varphi + (\partial_\rho\varphi)\partial_\rho\overline{\delta\varphi} \\
&\qquad\qquad + (16\pi^2\rho^4)^{-1}(\overline{\Pi}\delta\Pi + \Pi\overline{\delta\Pi})\Big)\left(e^{\delta u-\delta v} - 1\right) \\
&\qquad + m^2 e^v |\delta\varphi|^2 e^{\delta u+\delta v} + e^{-v}\left(|\partial_\rho\delta\varphi|^2 + (16\pi^2\rho^4)^{-1}|\delta\varphi|^2\right)e^{\delta u+\delta v}\Big].
\end{aligned}
$$

For $\|\delta u\|_\infty + \|\delta v\|_\infty \leq 1$, $k_1 \geq e$, and $k_2 \geq (e-1)$ one can estimate $|\Delta H^m - \delta_u H^m - \delta_v H^m - \delta_\varphi H^m - \delta_\Pi H^m|$ by

$$\begin{aligned}
&4\pi e^{\|u\|_\infty + \|v\|_\infty} \Bigg[2k_1 \left(m^2 \|\rho\varphi\|_2^2 + \|\rho\partial_\rho\varphi\|_2^2 + \left\|\rho^{-1}\Pi\right\|_2^2 \right) \left(\|\delta u\|_\infty^{\;2} + \|\delta v\|_\infty^{\;2} \right) \\
&\qquad + 2k_1 \Big(m^2 \|\rho\varphi\|_2 \|\rho\delta\varphi\|_2 + \|\rho\partial_\rho\varphi\|_2 \|\rho\partial_\rho\delta\varphi\|_2 \\
&\qquad\qquad + \left\|\rho^{-1}\Pi\right\|_2 \left\|\rho^{-1}\delta\Pi\right\|_2 \Big) \left(\|\delta u\|_\infty + \|\delta v\|_\infty \right) \\
&\qquad + k_2 \left(m^2 \|\rho\delta\varphi\|_2^2 + \|\rho\partial_\rho\delta\varphi\|_2^2 + \left\|\rho^{-1}\delta\Pi\right\|_2^2 \right) \Bigg] \\
&\leq C^m(\|u\|_\infty, \|v\|_\infty, \|\rho\varphi\|_2, \|\rho\partial_\rho\varphi\|_2, \left\|\rho^{-1}\Pi\right\|_2) \Big(\|\delta u\|_\infty^{\;2} + \|\delta v\|_\infty^{\;2} + \|\rho\delta\varphi\|_2^2 \\
&\qquad + \|\rho\partial_\rho\delta\varphi\|_2^2 + \left\|\rho^{-1}\delta\Pi\right\|_2^2 \Big).
\end{aligned}$$

A.3. Additional results on the behavior of parameters of solutions to the TOV equation

In this section we provide some additional results from our numerical analysis of the solutions to the (TOV) equation.

On the particle number

In order to compute the particle number, one solves the equation of constant entropy per particle (3.11). For our exemplary equations of state $\varrho(p) = p^\alpha + 3p$ the number density solving equation (3.11) is given by

$$n(r) = p(r)^\alpha \exp\left(- \int_0^{p(r)} \frac{1}{s^\alpha + 4s} \, \mathrm{d}s \right).$$

Given the particle number density $n(r)$ one can compute the particle number $N(\varrho(p))$ corresponding to the solution p of the (TOV) equation by formula (3.10), see figure A.1 for the graphs.

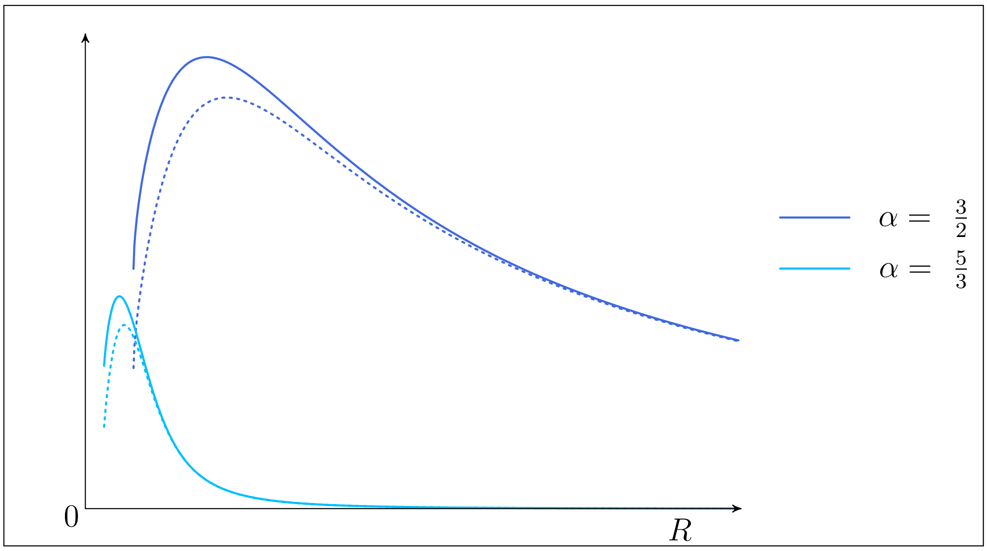

Figure A.1.: Plot of the total particle number $N(R)$ (dotted) and the mass $M(R)$ of solutions to the (TOV) equation vanishing at radius R for equations of states $\varrho(p) = p^\alpha + 3p$, against R.

An illustration of the instability

Plotting the respective total mass $M(R)$ and $N(R)$ against one another, one obtains an illustration of the instability by the cusps of the graphs, see figure A.2. Fixing either one of the two parameters, there might be two corresponding values of the other that lead to a solution of the (TOV) equation.

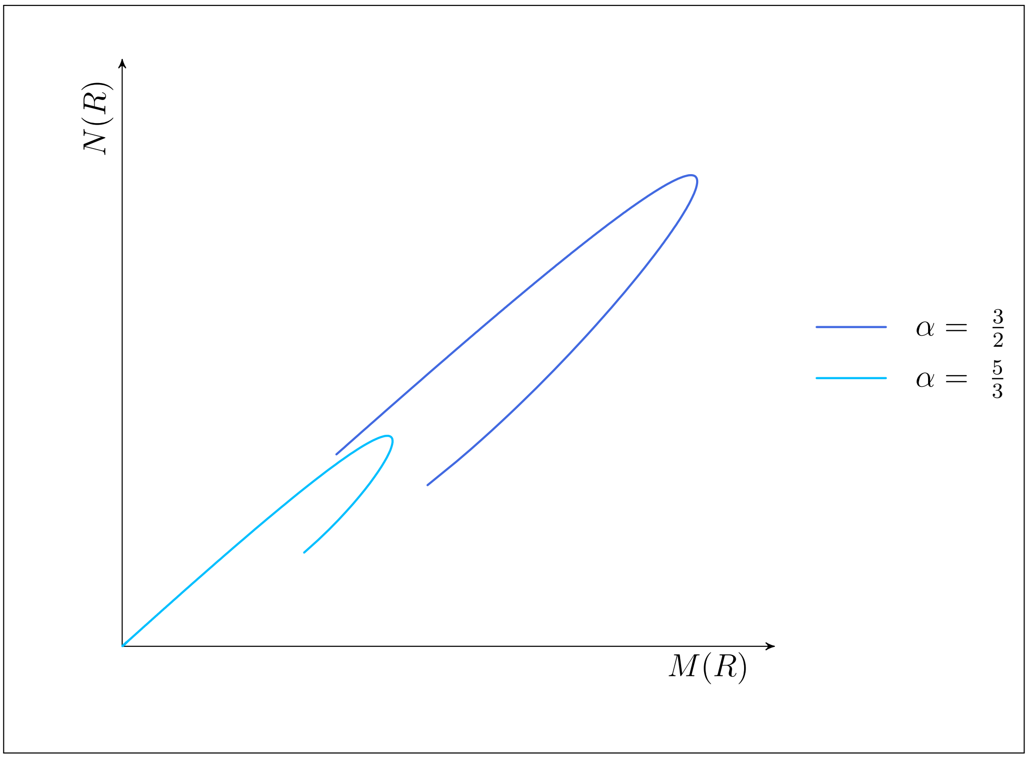

Figure A.2.: Plot of the total particle number $N(R)$ against the mass $M(R)$ of solutions to the (TOV) equation with equation of states $\varrho(p) = p^{\alpha} + 3p$.