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Groups, Actions, and C^*-Algebras

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To my family.

Abstract: In this thesis, we study C^*-simplicity of locally compact étale groupoids with compact unit space, and applications to a class of C^*-algebras constructed from uniformly recurrent subgroups.

Our novel methods for describing groupoid C^*-simplicity are largely enabled by a new induction procedure for actions of étale groupoids. This procedure allows us to construct injective objects in the category of operator systems with an action by a given groupoid $\mathcal{G}$, starting with injective C^*-algebras fibred over its unit space $\mathcal{G}^{(0)}$. Adapting work of Hamana on group-equivariant injective envelopes, we are thereby able to construct their groupoid-equivariant counterparts, and define the boundary groupoid $\tilde{\mathcal{G}}$ of $\mathcal{G}$ arising from the action on the equivariant envelope of $C(\mathcal{G}^{(0)})$. Using the boundary groupoid, we prove a new criterion for C^*-simplicity of $\mathcal{G}$ by relating the ideal structures of $\mathcal{G}$ and $\tilde{\mathcal{G}}$.

In the latter part of this thesis, we study a class of C^*-algebras associated with a uniformly recurrent subgroup Z of a finitely generated discrete group G via the Schreier graph, recently described by G. Elek. We reframe Elek's algebras as the reduced C^*-algebra of a groupoid designed to model the original construction. This groupoid turns out to be a quotient of (the opposite of) the transformation groupoid associated with the action of G on Z. Using well-known results about reduced groupoid C^*-algebras, our new construction provides simpler proofs of Elek's results that the C^*-algebra is nuclear if the Schreier graph has local property A and simple if the uniformly recurrent subgroup is generic. Furthermore, we show that local property A is in fact neccesary for the C^*-algebra to be nuclear, establishing it as an equivalent criterion. Finally, we apply our new criterion for C^*-simplicity from the first part of the thesis to give examples of simple Elek algebras whose uniformly recurrent subgroups are not generic.

To establish the necessary background, Chapters 2 and 3 of this thesis provide a self-contained discussion of the recent successful characterisations of C^*-simplicity for discrete groups by Kalantar–Kennedy, Breuillard–Kalantar–Kennedy–Ozawa, and Kennedy, as well as for crossed products of discrete groups with commutative unital C^*-algebras by Kawabe.

Résumé: Denne afhandling undersøger C^*-simplicitet af lokal kompakte étale gruppoider med kompakt enhedsrum gennem en ny randgruppoid som generaliserer metoder der for nylig blev brugt til at bresvare spørgsmålet om C^*-simplicitet for diskrete grupper og deres krydsprodukter. Derudover bruger vi vores nye resultater til at give nye eksempler på simple Elek algebraer.

Vores nye metoder for at beskrive gruppoid simplicitet er for det meste baseret på en ny induktionsfremgangsmåde af virkninger af étale gruppoider på operatorsystemer. Dette induktionsskema tillader os at konstruere injektive objekter i kategorien af operatorsystemer med virkning af en fast gruppoid $\mathcal{G}$, med injektive C^*-algebraer som er fibreret over enhedsrummet $\mathcal{G}^{(0)}$ som udgangspunkt. Baseret på Hamanas ækvivariante injektive hylstre for grupper, konstruerer vi ækvivariante injektive hylstre for gruppoider og definerer randgruppoiden $\tilde{\mathcal{G}}$, som er tilknyttet til virkningen af $\mathcal{G}$ på det gruppoid-ækvivariante hylster af $C(\mathcal{G}^{(0)})$. Vi relaterer idealstrukturen af $\tilde{\mathcal{G}}$ og $\mathcal{G}$ og beviser derved et nyt tilstrækkeligt kriterium til C^*-simplicitet.

I den anden del af afhandlingen undersøger vi en ny klasse af C^*-algebraer tilknyttet til "uniformly recurrent" delgrupper Z af endelig genererede diskrete grupper G, som for nyligt blev beskrevet af G. Elek gennem deres "Schreier" grafer. Inspireret af Eleks konstruktion beskriver vi en konkret étale gruppoid, en kvotient af transformationsgruppoiden for virkningen af G på Z, og vi viser, at dens reducerede C^*-algebra er Eleks algebra tilknyttet til Z. Sådan giver vi kortere beviser for Eleks resultater at C^*-algebraen er nukleær hvis Schreier grafen har local property A og er simpel hvis Z er generisk, og viser derudover at local property A er faktisk ækvivalent med nuklearitet. Til sidst giver vi nye exempler af simpel Elek algebraer tilknyttet til uniformly recurrent delgrupper ikke er generisk.

Kapitler 2 og 3 rekapitulerer baggrunden for Kalantar–Kennedys, Breuillard–Kalantar–Kennedy–Ozawas and Kennedys resultater om C^*-simplicitet af diskrete grupper, såvel som Kawabes arbejde vedrørende simplicitet af reducerede krydsprodukter af diskrete grupper med kommutative unitale C^*-algebraer.

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Introduction

The field of Operator Algebras is well-known for its tendency to borrow objects from different branches of mathematics, relate them to its own language, and build a rich theory on top of the already existing structure. A prime example of this modus operandi is the theory of group C^*-algebras, which provide a handy way to generate concrete examples of C^*-algebras with desired properties - as long as you understand how to translate back and forth between group-theoretic and operator-algebraic statements. A recent success story in making these translations concerns the question of when the reduced group C^*-algebra $C_r^*(G)$ is simple, the group G consequently being called *C^*-simple*, which was completely resolved for discrete groups by Kennedy [40]. In this thesis, we generalise Kennedy's characterisation of C^*-simplicity to Hausdorff étale groupoids with compact unit space, providing a new sufficient criterion for simplicity of their associated C^*-algebras. We then apply this criterion to a class of C^*-algebras recently defined by G. Elek, for which we provide a new groupoid model.

The earliest C^*-simplicity result goes back to 1975, when Powers [52] proved that the non-abelian free group on two generators $\mathbb{F}_2$ is C^*-simple. His proof used a variant of the *Dixmier property*, showing that conjugates of every element of $C_r^*(\mathbb{F}_2)$ can be made to approximate a scalar. For several decades, Powers' techniques formed the de-facto only method to prove C^*-simplicity of (discrete, countable) groups and many more groups to which his arguments applied could be identified. These are now called *Powers groups*, see for example de la Harpe's 2007 survey [19].

A group G satisfies another operator algebraic property, the *unique trace* property, if the canonical trace on $C_r^*(G)$ is its only tracial state. Since Power's methods for proving C^*-simplicity also forced uniqueness of the trace, all examples of C^*-simple groups known at the time of de la Harpe's survey also had unique trace. Both properties had been shown to force the group to be not only non-amenable, but also to rule out the presence of non-trivial normal amenable subgroups. This is often easier stated as the largest normal amenable subgroup, the *amenable radical*, being trivial. However, no other relation between C^*-simplicity, the unique trace property, and triviality of the amenable radical was known: It remained open whether there were any groups satisfying either C^*-simplicity or unique trace, but not the other, and de la Harpe [19] raised the question of whether there were any non-C^*-simple countable groups with trivial amenable radical, besides the trivial group.

Progress on these questions was only made in 2014, when Kalantar and Kennedy [36], for the first time, gave a dynamic characterisation of C^*-simplicity. Shortly after, in a joined effort with Breuillard and Ozawa [9], this led to a group-theoretic characterisation of the unique trace property, which indeed turned out to be equivalent to the amenable radical being trivial. When Le Boudec [41] answered de la Harpe's question in 2016 by giving an example of a non-C^*-simple group with unique trace, all implications between the three properties were known: C^*-simplicity implies unique trace, which is equivalent to having trivial amenable radical, but the converse does not hold.

Kalantar–Kennedy's dynamical description was largely enabled by their new description of an old tool: The *injective envelope* $I(A)$ of a C^*-algebra A, defined and proved to exist by Hamana [29] in 1979, is the smallest injective C^*-algebra into which A embeds. Hamana developed his theory further in the following years [30, 34], providing among other things a group-equivariant version $I_G(A)$ of the injective envelope for a C^*-algebra A with an action of a group G. On the other hand, Furstenberg [25] coined a notion of *boundary* of a discrete group G, a strongly proximal, minimal, compact space with G-action, and proved in 1973 the existence of a universal boundary $\partial_F G$, of which every other boundary is a factor. This boundary $\partial_F G$ is now called the *Furstenberg boundary*. Kalantar and Kennedy realised that the Furstenberg boundary is G-equivariantly homeomorphic to the spectrum of the G-equivariant injective envelope $I_G(\mathbb{C})$ of the complex numbers $\mathbb{C}$ equipped with the trivial action of G. Leveraging the new connection, they proved that G is C^*-simple, if and only if the action of G on $I_G(\mathbb{C})$ is topologically free, providing the first non-operator theoretic description of C^*-simplicity.

Shortly after, Kennedy [40] developed these ideas further and arrived at a purely group-theoretic characterisation of C^*-simplicity. A subgroup H of a discrete group G is called *recurrent*, if there is a finite subset F of $G \setminus \{e\}$ such that every subgroup conjugate to H intersects F. Kennedy's characterisation, in its simplest form, states that G is C^*-simple if and only if G contains no non-trivial amenable recurrent subgroups. Since any non-trivial normal subgroup is invariant under conjugation, it is clearly recurrent and we obtain an easy proof that every C^*-simple group has unique trace, as not having recurrent amenable subgroups implies not having normal amenable subgroups.

Kennedy's work has since been generalised in several directions [4, 13, 43, 44]. In particular, Kawabe [38] has derived an analogous characterisations for simplicity of the crossed product of a compact space by a discrete group in his PhD thesis under the supervision of Ozawa. Kawabe's techniques are remarkably straightforward: Considering the crossed product $C(X) \rtimes_r G$, the role of the injective envelope $I_G(\mathbb{C})$ with respect to C^*-simplicity of G is replaced by the injective envelope $I_G(C(X))$ of the algebra on which G acts, which specialises to Kennedy's proof since $C^*_{\mathrm{r}}(G)$ is isomorphic to the crossed product $\mathbb{C} \rtimes_r G$ for the trivial action of G on $\mathbb{C}$. Proving simplicity of a crossed product $C(X) \rtimes_r G$ is usually done in two steps. First, the action of G on X has to be *minimal*, that is, every orbit is dense in X, to make sure that $C(X)$ does not have G-invariant ideals. Then, one should prove the *intersection property*, introduced by Svensson and Tomiyama [58], demanding that every non-trivial ideal of $C(X) \rtimes_r G$ intersects $C(X)$ non-trivially. Kawabe proves that $C(X) \rtimes_r G$ has the intersection property if and only if $I_G(C(X)) \rtimes_r G$ does, and that this is furthermore the case if and only if the action of G on the spectrum of $I_G(C(X))$ is topologically free. From this he derives a simplicity criterion for minimal actions of G, which we restate in slightly simpler language: The crossed product is simple if and only if no stabiliser contains a recurrent amenable subgroup of G.

The central piece of original research presented in this thesis is a further generalisation of Kennedy's and Kawabe's work to Hausdorff étale groupoids with compact unit space and their associated reduced C^*-algebras. These generalise group C^*-algebras and crossed products, but instead of a single group acting on a space, the set of available elements to act may vary from point to point. Consequently, group-equivariant injective envelopes cannot meaningfully be applied to the theory of groupoid C^*-algebras. Instead, we supply a new framework of *induction of groupoid actions*, allowing us to construct injective objects in the category of operator systems with an action by a fixed groupoid $\mathcal{G}$ from an injective C^*-algebra fibred over its unit space $\mathcal{G}^{(0)}$. Following Hamana's scheme for the construction of group-equivariant injective envelopes, we are able to adapt the theory to provide the groupoid-equivariant analogues. For a suitable groupoid $\mathcal{G}$ we obtain a boundary groupoid $\tilde{\mathcal{G}}$ from the action of $\mathcal{G}$ on the spectrum of the $\mathcal{G}$-equivariant injective envelope of $C(\mathcal{G}^{(0)})$, the *boundary groupoid* of $\mathcal{G}$. As it was the case for groups and crossed products, C^*-simplicity and the intersection property are closely linked to the dynamics of the boundary groupoid and we prove that $\mathcal{G}$ has the intersection property if $\tilde{\mathcal{G}}$ does. The boundary groupoid furthermore shares the properties of the Furstenberg boundary for groups that make its dynamics more tractable, like extremal disconnectedness of its unit space and amenability of its stabilisers. We leverage these into a sufficient criterion for C^*-simplicity of $\mathcal{G}$ solely in terms of certain subgroups of the isotropy groups of $\mathcal{G}$ we call *dynamically recurrent*.

We then apply our results to a class of C^*-algebras constructed from uniformly recurrent subgroups and recently devised by G. Elek. These algebras arise as a completion of the algebra of local kernels on a Schreier graph associated with the uniformly recurrent subgroups. We develop a groupoid picture that closely models Elek's construction. This groupoid turns out to be a quotient of (the opposite of) the transformation groupoid associated with the action of the surrounding group on its uniformly recurrent subgroup. As the groupoid model enables us to use the powerful tools available to the theory of ample étale groupoids, we are able to simplify and strengthen several of Elek's proofs. In particular, we show that Elek's sufficient condition for nuclearity, the so-called local property A of the Schreier graph, is in fact an equivalent condition. Finally, we apply our new results on groupoid simplicity to obtain new examples of Elek algebras that are not covered by Elek's simplicity criteria.

In addition to this introduction there are four Chapters. Chapter 2 provides a self-contained review of Kalantar–Kennedy's work, including Hamana's theory of injective envelopes. For almost all statements in this chapter, we provide proofs with very little or no modification from the original papers. In Chapter 3, we review Kawabe's generalisations of these techniques to crossed products, and the main results regarding their simplicity. As Kawabe's work remains unpublished, we have taken the liberty to modify its statements and language slightly, providing alternative or streamlined proofs. We introduce our new notion of *dynamically recurrent* subgroup of a stabiliser which generalises well to our own results for groupoids and add an equivalent condition to his theorem on simplicity of minimal crossed products (see Proposition 3.3.12). Our own original research is presented in Chapter 4. Section 4.2 contains the main results establishing new simplicity criterion for groupoid C^*-algebras along with the construction of the groupoid Furstenberg boundary, based on a pre-print [8] currently under review for publication. In Section 4.3 we provide a new groupoid model for the Elek algebras associated with uniformly recurrent subgroups. We then provide new examples of simple Elek algebras using one of our simplicity conditions. This section is based on a paper [7] to appear in Mathematica Scandinavica. Finally, Chapter 5 collects some open questions for further research on the groupoid Furstenberg boundary and simplicity of reduced groupoid C^*-algebras.

Groups

We start out by reviewing the recent breakthrough results on C^*-simplicity of discrete groups, which the later chapters will generalise. After providing the necessary background in Section 2.1, the fundamental insights regarding the connection between injective envelopes and the Furstenberg boundary are presented in Section 2.2. This enables us to prove Kalantar–Kennedy's characterisations in Section 2.3.

2.1 Group Basics

C^*-algebras associated with groups and their many generalisations give a particularly rich class of examples in operator algebras and are essential in tying the field to many other areas of mathematics.

In order to fix notation and be self-contained, we introduce some basic terminology.

2.1.1 Group C^*-algebras

In its essence, the group C^*-algebra of a group G models the group structure as unitaries acting on some Hilbert space $\mathcal{H}$ via a representation of G, that is, a group homomorphism from G into the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$. Immediately, there is a choice of representation with one of the most important candidates being the so-called *left-regular representation*. Throughout this thesis we will consider this representation and its generalisations, giving rise to the so-called *reduced* C^*-algebras. In general, we will assume discrete groups to be countable, unless explicitly stated otherwise. If the countability assumption is particularly important, we will repeat it in the affected statements.

The left-regular representation λ represents G on the Hilbert space $\mathcal{H} = \ell^2(G)$. It sends a group element $g \in G$ to the unitary $\lambda_g \in \mathcal{U}(\ell^2(G))$ acting as

$$\lambda_g \delta_h = \delta_{gh}.$$

Here, $\delta_h \in \ell^2(G)$ denotes the canonical basis vector which takes value one at $h \in G$ and zero everywhere else. For ease of notation, we will refer to the left-regular representation simply as the *regular* representation. The reduced group C^*-algebra of G is then isomorphic to the C^*-algebra generated by the image $\lambda(G)$ inside $\mathcal{B}(\ell^2(G))$. Slightly more abstractly, we consider the group algebra $\mathbb{C}[G]$ consisting of all formal linear combinations of finitely many group elements over the field $\mathbb{C}$ equipped with multiplication

$$\Big(\sum_{g\in G} a_g\, g\Big)\Big(\sum_{g\in G} b_g\, g\Big) = \sum_{g,h\in G} a_{gh^{-1}} b_h\, g$$

and involution

$$\Big(\sum_{g\in G} a_g\, g\Big)^* = \sum_{g\in G} \overline{a_{g^{-1}}}\, g.$$

This multiplication and involution make the representation of G on $\mathbb{C}[G]$ by sending $g \in G$ to $g \in \mathbb{C}[G]$ a unitary representation. In fact, asking for this map to be a unitary representation determines the multiplication and involution on $\mathbb{C}[G]$ uniquely. Furthermore, we equip $\mathbb{C}[G]$ with the norm induced by the representation λ, that is,

$$\Big\| \sum_{g\in G} a_g\, g \Big\|_\lambda = \Big\| \sum_{g\in G} a_g \lambda_g \Big\|_{\mathcal{B}(\ell^2(G))},$$

and define the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$ of G to be the completion of $\mathbb{C}[G]$ in that norm.

For a more thorough introduction including proofs that $\mathbb{C}[G]$ and $C^*_{\mathrm{r}}(G)$ as described are indeed a *-algebra and a C^*-algebra, we refer to the excellent exposition of Brown–Ozawa [12, Chapter 4.1].

Another important C^*-algebra associated with the discrete group G is the *universal* C^*-*algebra* $C^*_u(G)$. It is given by the completion of $\mathbb{C}[G]$ in the universal norm $\|\cdot\|_u$ taking into account "all representations" of G, but there are some subtle size issues as the collection of "all representations" of the group is not a set. However, the collection of *unitary equivalence* classes of *cyclic* representations of G is a set, where a representation π of G on a Hilbert space $\mathcal{H}$ is called *cyclic* if there is a vector $\xi \in \mathcal{H}$ such that $\pi(G)\xi$ spans $\mathcal{H}$ densely and two representations π and π' on $\mathcal{H}$ and $\mathcal{H}'$ are unitarily equivalent, if there is a unitary operator $T\colon \mathcal{H} \to \mathcal{H}'$ such that $T\pi(g) = \pi'(g)T$ for all $g \in G$. We say that such T intertwines π and π'. We denote the set of cyclic representations of G up to unitary equivalence by $\mathrm{cRep}(G)/{\sim}$. Then we define the *universal norm* for $f \in \mathbb{C}[G]$ as

$$\|f\|_u := \sup\{\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \mid [\pi] \in \mathrm{cRep}(G)/{\sim}\}. \tag{2.1}$$

Clearly, if π and π' are unitarily equivalent representations, then $\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} = \|\pi'(f)\|_{\mathcal{B}(\mathcal{H}')}$ for all $f \in \mathbb{C}[G]$, which makes the choice of representing element π for $[\pi]$ in Equation (2.1) irrelevant. Note that for $g \in G \subseteq \mathbb{C}[G]$ we have $\|\pi(g)\|_{\mathcal{B}(\mathcal{H})} = 1$ for every representation π, and hence

$$\Big\| \sum_{g\in G} a_g\, g \Big\|_u \le \sum_{g\in G} |a_g|\, \|g\|_u = \sum_{g\in G} |a_g|,$$

which is finite as the sum is finite, and therefore the supremum in Equation (2.1) is finite. Furthermore, $\|\cdot\|$ is easily seen to indeed be a norm on $\mathbb{C}[G]$. We define the universal C^*-algebra $C^*_u(G)$ associated with G to be the completion of $\mathbb{C}[G]$ in the universal norm.

If $\xi \in \mathcal{H}$ and π is a (not necessarily cyclic) representation of G on $\mathcal{H}$, then $\pi(G)$ leaves $\mathcal{H}_\xi := \overline{\operatorname{span}}\, \pi(G)\xi$ invariant and we obtain a representation π_ξ of G on $\mathcal{H}_\xi$ by restricting. It is not hard to see that $\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} = \sup\{\|\pi_\xi(f)\|_{\mathcal{B}(\mathcal{H}_\xi)} \mid \xi \in \mathcal{H}\}$. Hence $\|\pi(f)\|_{\mathcal{B}(\mathcal{H})}$ may be approximated by $\|\pi'(f)\|_{\mathcal{B}(\mathcal{H}')}$ for cyclic representations π'. Combining this with the observation that

$\|\pi(f)\|_{\mathcal{B}(\mathcal{H})}$ depends on π only up to unitary equivalence, we find that $\|\pi(f)\|_{\mathcal{B}(\mathcal{H})} \leq \|f\|_u$ for any representation π and $f \in \mathbb{C}[G]$. Consequently, $C^*_u(G)$ has the universal property that for every representation π of G on $\mathcal{H}$ there is a unique *-homomorphism $C^*_u(G) \to \mathcal{B}(\mathcal{H})$ with $g \mapsto \pi(g)$ for $g \in G \subseteq \mathbb{C}[G] \subseteq C^*_u(G)$. In particular, there is a surjective *-homomorphism $\rho\colon C^*_u(G) \to C^*_{\mathrm{r}}(G)$.

In this thesis we explore the fundamental question of when a reduced C^*-algebra is simple.

Definition 2.1.1: A *two-sided ideal* in a C^*-algebra A is a linear subspace closed under multiplication with arbitrary elements of A from the left and right. The C^*-algebra A is called *simple*, if it contains no proper closed two-sided ideals; that is, if $I \lhd A$ is a closed two-sided ideal then $I = \{0\}$ or $I = A$.

A discrete group G is called *C^*-simple* if its reduced algebra $C^*_{\mathrm{r}}(G)$ is simple.

Due to the canonical surjective *-homomorphism $\rho\colon C^*_u(G) \to C^*_{\mathrm{r}}(G)$ obtained from the universal property, the universal C^*-algebra $C^*_u(G)$ is never simple, unless $C^*_{\mathrm{r}}(G)$ happens to be simple and ρ is an isomorphism. Hence simplicity of $C^*_u(G)$ is completely characterised by simplicity of $C^*_{\mathrm{r}}(G)$, assuming that we can describe when ρ is an isomorphism, which we will do in the next section.

2.1.2 Amenability

Recall that the famous Banach-Tarski paradox states that the ball in three dimensions can be decomposed into finitely many disjoint subsets which may then be moved by only translations and rotations such that they assemble again into two translated copies of the original ball. This is a property of the action of the group G of translations and rotations on $\mathbb{R}^3$ and we say that the action admits a *paradoxical decomposition*. That is, the unit ball of $\mathbb{R}^3$ contains disjoint subsets A_i and B_j and group elements $g_i, h_j \in G$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ such that such that the unit ball equals

$$\bigcup_{i=1}^{n} g_i.A_i = \bigcup_{j=1}^{m} h_j.B_j.$$

On the contrary, *amenability* has been conceived as a property of groups that prevents its actions from having such paradoxical compositions.

Let G be a discrete group and let G act on $\ell^\infty(G)$ by shifting the argument; that is,

$$(g.f)(h) = f(g^{-1}h)$$

for $g, h \in G$ and $f \in \ell^\infty(G)$.

Definition 2.1.2: A *mean* on G is a positive linear functional $m\colon \ell^\infty(G) \to \mathbb{C}$ of norm one. If

$$m(g.f) = m(f)$$

for all $g \in G$ and $f \in \ell^\infty(G)$ the mean is called *invariant*, and G is *amenable* if it admits an invariant mean.

Any finite or commutative group is amenable, while the easiest example of a non-amenable group is given by the free group in two generators [52]. In fact it was a long-standing conjecture by Day and von Neumann that every non-amenable group contains a copy of the free group on two generators, until proven false by Olšanskiĭ [45].

There is a plethora of equivalent criteria for amenability. For reference we collect some of them from [12, Theorem 2.6.8] without explaining the additional terminology not required elsewhere:

Proposition 2.1.3: Let G be a discrete group. The following are equivalent:

1. G is amenable, i.e. it admits an invariant mean.
2. The universal and the reduced norm coincide on $\mathbb{C}[G]$; in other words, the canonical *-homomorphism $\rho\colon C^*_u(G) \to C^*_\mathrm{r}(G)$ is an isomorphism.
3. $C^*_\mathrm{r}(G)$ is nuclear.
4. $C^*_\mathrm{r}(G)$ has a character.
5. G satisfies the Følner condition.
6. The trivial representation τ_G of G is weakly contained in the regular representation λ_G of G, i.e. $\|\tau_G(f)\| \leq \|\lambda_G(f)\|$ for all $f \in \mathbb{C}[G]$.
7. There is a net φ_λ of finitely supported positive definite functions on G such that $\lim_\lambda \varphi_\lambda(g) = 1$ for all $g \in G$.

Note that, as in the previous proposition, we will often denote nets simply as elements indexed by some index λ without making the underlying directed set explicit.

Amenability questions are deeply intertwined with the question of C^*-simplicity. One of the obstructions to C^*-simplicity discovered first is the existence of a non-trivial normal amenable subgroup, see for example the exposition of de la Harpe [19, Proposition 3]. Note that the trivial subgroup $\{e\}$ is always normal and amenable. In fact, every group G contains a unique maximal normal amenable subgroup, called the *amenable radical* of G, that conains all other normal amenable subgroups by a result of Day [18, Section 4, Lemma 1]. We will go into more details on these results in Section 2.3.

2.1.3 Completely Positive Maps

In many places, we will deal not with C^*-algebras, but operator systems:

Definition 2.1.4: An *operator system* S is linear subspace of a unital C^*-algebra A that contains the unit and is closed under the adjoint operation of A.

To avoid some subtlety later on, we keep the surrounding C^*-algebra A explicit in the definition of S.

Definition 2.1.5: An element a of a C^*-algebra A is *positive* if it is of the form $a = b^*b$ for some $b \in A$. A linear map φ between two operator systems $S \subseteq A$ and $T \subseteq B$ is *positive* if it sends every element of S that is positive in A to an element of T that is positive in B.

The matrix algebra $M_n(A)$ over a C^*-algebra A is the C^*-algebra of all n-by-n matrices $[a_{ij}]$ with entries in A equipped with the usual matrix multiplication

$$[a_{ij}][b_{ij}] = [c_{ij}] \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

and adjoint operation

$$[a_{ij}]^* = [a_{ji}^*].$$

If A is represented on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we may represent $M_n(A)$ on $\mathcal{B}(\mathcal{H}^{\oplus n})$ and equip it with the operator norm of $\mathcal{B}(\mathcal{H}^{\oplus n})$. Likewise, the matrix algebra $M_n(S) \subseteq M_n(A)$ is an operator system inside $M_n(A)$ for any operator system $S \subseteq A$. A linear map $\varphi\colon S \to T$ between two operator system S, T has *amplification* $\varphi^{(n)}\colon M_n(S) \to M_n(T)$ defined by

$$\varphi^{(n)}([a_{ij}]) = [\varphi(a_{ij})].$$

This yields a stronger notion of positivity:

Definition 2.1.6: Let $\varphi\colon S \to T$ be a linear map between two operator systems S and T. Then φ is called *n-positive* if $\varphi^{(n)}$ is positive and *completely positive* if it is n-positive for every $n \in \mathbb{N}$.

If φ is n-positive, then it is clearly m-positive for every $m \leq n$. For a simple example of a positive map which is not completely positive, consider the transposition $t\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$, which is not even 2-positive.

Any positive map φ is bounded and if the domain is unital with unit $\mathbb{1}$, then in fact $\|\varphi\| = \|\varphi(\mathbb{1})\|_B$ by [50, Corollary 2.9]. A completely positive map φ that satisfies $\|\varphi^{(n)}\| \leq 1$ for all $n \in \mathbb{N}$ is furthermore called *completely contractive*, or *ccp* for short. A completely positive map that is unital is called *unital completely positive* or *ucp*, and any ucp map is ccp. A ucp map with ucp inverse is called a *complete order isomorphism*. If every amplification $\varphi^{(n)}$ is an isometry, φ is called a complete isometry.

As an operator system is not closed under multiplication, ucp maps offer the "next best" replacement for *-homomorphisms and in fact exhibit several useful properties that make them almost as tractable as if they were multiplicative. We therefore form a category with operator systems as objects and unital completely positive maps between these as morphisms, called the category of operator systems.

Note that the order on (the self-adjoint elements of) a C^*-algebra A is determined by the cone of positive elements, that is $a \leq b$ for $a, b \in A$ self-adjoint if and only if $b - a$ is a positive element of A.

Proposition 2.1.7: If $\varphi\colon S \to T$ is a ucp map between two operator systems S and T, it satisfies the Schwarz inequality

$$\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$$

for all $a \in S$.

This is a simple consequence of the Stinespring dilation for ucp maps, see [12, Proposition 1.5.7].

Definition 2.1.8: Let $\varphi\colon A \to B$ be a ucp map between two C^*-algebras A and B. The multiplicative domain $\mathrm{mult}(\varphi)$ is given by

$$\mathrm{mult}(\varphi) = \{a \in A \mid \varphi(a)^*\varphi(a) = \varphi(a^*a) \text{ and } \varphi(a)\varphi(a)^* = \varphi(aa^*)\}.$$

The multiplicative domain lends its name from the fact that φ is multiplicative whenever at least one of the factors in the multiplication comes from the multiplicative domain. That is, whenever $x \in \mathrm{mult}(\varphi)$ and $y \in A$, we have $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(yx) = \varphi(y)\varphi(x)$. If the domain of φ is a C^*-algebra A then $\mathrm{mult}(\varphi)$ is a sub-C^*-algebra of A and in fact the largest sub-C^*-algebra on which φ restricts to a *-homomorphism. We will sometimes use the following description of the multiplicative domain:

Lemma 2.1.9: *Let $\varphi\colon A \to B$ be a ucp map between two C^*-algebras A and B. Then*

$$\mathrm{mult}(\varphi) = \overline{\mathrm{span}}\{u \in A \mid \|u\|_A = 1 \text{ and } \varphi(u) \text{ unitary}\}.$$

Proof. If $u \in A$ with $\|u\| = 1$ is such that $\varphi(u)$ is unitary, then $\mathbb{1}_B = \varphi(u)^*\varphi(u) \leq \varphi(u^*u)$ by the Schwarz inequality, but $\|\varphi(u^*u)\| \leq \|u\|^2 = 1$ and hence $\varphi(u^*u) \leq \mathbb{1}_B$. We conclude $\varphi(u)^*\varphi(u) = \varphi(u^*u)$. Similarly, $\varphi(u)\varphi(u)^* = \varphi(uu^*)$, so u is contained in the multiplicative domain of φ.

Conversely, $\mathrm{mult}(\varphi)$ is a unital *-subalgebra of A and as such spanned by the unitaries that it contains. But if $u \in \mathrm{mult}(\varphi)$ is a unitary, then $\varphi(u)^*\varphi(u) = \varphi(u^*u) = \varphi(\mathbb{1}_A) = \mathbb{1}_B$ and likewise $\varphi(u)\varphi(u)^* = \mathbb{1}_B$, so $\varphi(u)$ is a unitary in B. We conclude that

$$\mathrm{mult}(\varphi) \subseteq \overline{\mathrm{span}}\{u \in A \mid \|u\|_A = 1 \text{ and } \varphi(u) \text{ unitary}\}$$

and thereby the statement. □

A particularly useful class of completely positive maps is formed by the so-called conditional expectations.

Definition 2.1.10: Let A be a C^*-algebra and $B \subseteq A$ a sub-C^*-algebra. A *conditional expectation $E\colon A \to B$* from A onto B is a ccp map that restricts to the identity on B.

For example, consider a discrete group G with normal subgroup N. Then $\mathbb{C}[N] \subseteq \mathbb{C}[G]$ and a quick calculation detailed in [12, Proposition 2.5.8] shows that we may extend this to an inclusion of C^*-algebras $C^*_{\mathrm{r}}(N) \subseteq C^*_{\mathrm{r}}(G)$. The map $E_N\colon C^*_{\mathrm{r}}(G) \to C^*_{\mathrm{r}}(N)$ given by

$$E_N(\lambda_g) = \begin{cases} \lambda_g & \text{if } g \in N, \\ 0 & \text{else,} \end{cases} \tag{2.2}$$

is a conditional expectation by [12, Proposition 2.5.9, 2.5.12]. Further examples include the canoncial conditional expectation E_X from a crossed product $C(X) \rtimes_r G$ onto $C(X)$ introduced later in Equation (3.2) and the analogue expectation onto the diagonal for groupoids.

2.2 Injective Envelopes and the Furstenberg Boundary

In this section we review the results leading to Kalantar–Kennedy's identification of the Hamana and Furstenberg boundaries, paving the way for the simplicity results of Section 2.3.

2.2.1 The Furstenberg Boundary

Recall that an *action* of a discrete group G on a Hausdorff topological space X is simply a group homomorphism from G into the group of homeomorphisms of X. For a discrete group G a *boundary* of G is a non-empty compact topological space X with a G-action, such that the action is *minimal* and *strongly proximal*. Minimality is simply the assertion that the G-orbit of every point in X is dense in X. To define strong proximality, note that G also acts via shifting of the argument on the space $\mathcal{M}(X)$ of Radon probability measures on X equipped with the weak-*-topology of $C(X)$ and that X embeds into $\mathcal{M}(X)$ as the point measures. The action on X is then called strongly proximal if any measure μ contains a point measure in the closure of its G-orbit $G.\mu$ in $\mathcal{M}(X)$.

The Furstenberg Boundary $\partial_F G$ is the universal G-boundary in the sense of the following proposition. We follow Ozawa [48, Section 1] to prove existence and uniqueness:

Proposition 2.2.1 (Furstenberg): Let G be a discrete group. Up to G-equivariant homeomorphism, there is a unique G-boundary $\partial_F G$ such that every other G-boundary is G-equivariantly homeomorphic to a quotient of $\partial_F G$ with the quotient relationship respecting the G-action. Then $\partial_F G$ is called the *Furstenberg boundary* of G.

Proof. Existence: Let $\{X_i\}$ be the representing elements of all G-homeomorphism classes of G-boundaries and consider $\prod_i X_i$, equipped with the coordinate-wise action of G. Then the action of G on $\prod_i X_i$ is strongly proximal. Indeed, if X and Y are two strongly proximal G-spaces, then $X \times Y$ is strongly proximal: given a measure $\mu \in \mathcal{M}(X \times Y)$, we obtain a measure $q(\mu)$ on X by $q(\mu)(A) := \mu(A \times Y)$ for $A \subseteq X$ measurable. As the action on X is strongly proximal, the orbit closure of $q(\mu)$ contains a point measure δ_x for some $x \in X$ and as the action of G is coordinate-wise, q is G-equivariant, there is a net $g_\lambda \in G$ such that $q(g_\lambda.\mu) = g_\lambda.q(\mu)$ approximates δ_x. Since $\mathcal{M}(X \times Y)$ is compact and q continuous, we may pass to a convergent subnet and obtain a measure μ' in the orbit closure of μ such that $q(\mu') = \delta_x$. Hence $\mu'(B) = \mu'(\pi_1^{-1}(\{x\}) \cap B)$ for π_1 the projection on the first coordinate and therefore clearly $\mu' = \delta_x \otimes \nu$ for a measure $\nu \in \mathcal{M}(Y)$ defined by $\nu(A) = \mu'(\{x\} \times A)$ for $A \subseteq Y$ measurable. As the action on Y is strongly proximal, there is a net $g'_\lambda \in G$ such that $g'_\lambda.\nu$ converges to δ_y for some $y \in Y$ and since X is compact we may again pass to a subnet such that $g'_\lambda.x$ converges to some $x' \in X$. Then $g'_\lambda.\mu' = \delta_{g'_\lambda.x} \otimes g'_\lambda.\nu$ converges to the point measure $\delta_{x'} \otimes \delta_y$ and we conclude that the action on $X \times Y$ is strongly proximal. By induction, finite products of strongly-proximal G-spaces are strongly proximal and hence so are infinite products, as the topology is defined by convergence in every finite selection of coordinates.

Therefore the product of all G-boundaries $\prod_i X_i$ is compact and strongly proximal, and as every compact space contains a minimal closed subspace, we may choose a minimal closed subspace K of $\prod_i X_i$, which is a G-boundary. Since the image of K under the projection π_i onto the i-th coordinate is non-empty and X_i is minimal, $\pi_i(K)$ is all of X_i, so any G-boundary X_i is a quotient of K, up to G-equivariant homeomorphism.

Uniqueness: Let X and Y be two universal G-boundaries in the sense of the proposition. Then there are G-equivariant quotient maps $\varphi\colon X \to Y$ and $\psi\colon Y \to X$. We show that $\psi \circ \varphi$ is the identity on X and uniqueness follows by symmetry. Let ι be the embedding of X into $\mathcal{M}(X)$ and consider the map $\eta\colon X \to \mathcal{M}(X)$ defined by $\eta(x) = (\iota(x) + \iota(\psi \circ \varphi(x)))/2$. Then η is a G-equivariant map hence $\eta(X)$ is a G-invariant compact subset of $\mathcal{M}(X)$. As the action on X is strongly proximal, $\eta(X)$ contains $\iota(X)$. But as X is minimal, $\eta^{-1}(\iota(X))$ contains all of X, so η

only takes values in point measures. Since $\eta(x) = (\delta_x + \delta_{\psi\circ\varphi(x)})/2$ is only a point measure if $\psi \circ \varphi(x) = x$, we conclude that $\psi \circ \varphi$ is the identity on X. □

An extensive list of examples and constructions of group boundaries has been compiled by Bryder [13, Section 2.2].

2.2.2 Equivariant Injective Envelopes

On the other side of the story sits another old and well-known tool, Hamana's equivariant injective envelopes.

Despite being C^*-algebras, injective envelopes are described in the larger category of *operator systems* with unital completely positive maps, as presented in Section 2.1.3. A discrete group G can act on an operator system S by invertible morphisms in the category of operator systems, that is, ucp maps with ucp inverse. These are called the *complete order automorphisms* of S and the group action is simply a group homomorphism from G into the group of complete order automorphisms of S.

Injective operator systems, as intensely studied by Choi and Effros [17], are exactly the injective objects in the category of operator systems. In more detail:

Definition 2.2.2: An operator system I is called *injective*, if for any pair of operator systems V, W with a unital completely isometric map $\kappa\colon V \hookrightarrow W$ and a ucp map $\varphi\colon V \to I$ there is a ucp map $\tilde{\varphi}\colon W \to I$ that extends φ, that is, $\tilde{\varphi} \circ \kappa = \varphi$ as in Diagram 2.3 below.

$$\begin{tikzcd} W \arrow[r, dashed, "\tilde{\varphi}"] & I \\ V \arrow[u, hook, "\kappa"] \arrow[ur, "\varphi"] & \end{tikzcd} \tag{2.3}$$

Note that the extension $\tilde{\varphi}$ does not need to be unique. We sometimes refer to a unital completely isometric map between operator systems as an *embedding*.

Hamana developed a rich theory of so-called *injective envelopes* for operator systems [30] as well as C^*-algebras [29], with and without a group action [34]. The following definitions are taken from [30, Section 2]:

Definition 2.2.3: Let V be an operator system. An *extension* of V is an operator system W with a unital completely isometric map $\kappa\colon V \hookrightarrow W$.

Definition 2.2.4: An extension $\kappa\colon V \hookrightarrow W$ of an operator system V is called...

- *injective* if W is injective as an operator system.
- *essential* if whether a map out of W is a complete isometry can be decided on V; that is, if any ucp map $\varphi\colon W \to Z$ into a third operator system Z is completely isometric if and only if $\varphi \circ \kappa$ is.
- *rigid* if any ucp map that fixes V inside W pointwise also fixes all of W pointwise; that is, the only ucp map $\varphi\colon W \to W$ such that $\varphi \circ \kappa = \kappa$ is the identity on W.

- the *injective envelope* of V if it is injective and essential.

When working in the category of operator systems with G-action, there are analogous definitions for a G-extension being G-injective, G-essential, G-rigid, or a G-equivariant injective envelope, in which all operator systems carry an action of a discrete group G and all ucp maps are G-equivariant.

One might wonder what the role of rigidity is, as it is seemingly not involved in the definition of an injective envelope. However, rigidity provides a convenient stepping stone to prove essentiality:

Proposition 2.2.5 (Hamana [30, Lemma 3.7]): Let $\kappa\colon V \to W$ be an extension of the operator system V. If the extension is rigid and injective, then it is essential:

Proof. Let $\varphi\colon W \to Z$ be a ucp map into a third operator system Z, such that $\varphi \circ \kappa$ is an embedding of V into Z. As W is injective, we may therefore extend the ucp map $\kappa\colon V \to W$ to a ucp map $\psi\colon Z \to W$ such that $\psi \circ (\varphi \circ \kappa) = \kappa$.

$$\begin{tikzcd}
W \arrow[rr, bend left, "\varphi"] & & Z \arrow[ll, dashed, "\psi" description] \\
& V \arrow[ul, hook, "\kappa"] \arrow[ur, hook, "\varphi\circ\kappa"'] &
\end{tikzcd}$$

Then $\psi \circ \varphi$ is a ucp map $W \to W$ fixing $\kappa(V)$ pointwise, so by rigidity of the extension it is the identity of W. We conclude that φ is a unital complete isometry since it has a ucp left inverse. □

The analogous statement and proof work for G-operator systems. Even more generally, essentiality implies rigidity even if the extension is not injective, but the proof will have to wait until after we have established the existence of an injective envelope.

For the rest of this section we follow Hamana's proof of the existence of an equivariant injective envelope from [34]. As it contains the case of a classical injective envelope by letting the trivial group act, we formulate all results for the G-equivariant injective envelope of operator systems with G-action.

Hamana's procedure works in two steps, first identifying any G-injective G-extension, and then cutting it down with a Zorn's argument to obtain the actual envelope.

For plain operator systems without a group action, the existence of an injective extension is guaranteed by Arveson extension [12, Theorem 1.6.1], which simply states that $\mathcal{B}(\mathcal{H})$ is an injective operator system for any Hilbert space $\mathcal{H}$, and the GNS construction [35, Chapter 4.5], ensuring that any C^*-algebra embeds into $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. For G-operator systems, however, we need to find a G-extension. Luckily, any injective extension can be made G-equivariant via induction, a process we will generalise to groupoids in Section 4.2.1. Given any operator system S and a discrete group G, note that $\ell^\infty(G, S)$, the algebra of bounded S-valued functions on G, carries a G-action by acting on the argument. That is,

$$(h.f)(g) = f(h^{-1}.g)$$

for $f \in \ell^\infty(G, S)$ and $g, h \in G$. If S sits inside the C^*-algebra A represented on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, then $\ell^\infty(G, S)$ sits as an operator system inside of $\mathcal{B}(\mathcal{H} \otimes \ell^2(G))$. Thus, we construct G-injective operator systems from injective ones:

Proposition 2.2.6 (Hamana [34, Lemma 2.2]): Let I be an injective operator system and G a discrete group. Then $\ell^\infty(G, I)$ is a G-injective G-operator system.

Proof. Let $\iota\colon V \hookrightarrow W$ be a G-extension of G-operator systems and $\varphi\colon V \to \ell^\infty(G, I)$ a G-equivariant ucp map. We have to show that it extends to some G-equivariant ucp map $\tilde{\varphi}\colon W \to \ell^\infty(G, I)$. First note that $\mathrm{eval}_e\colon \ell^\infty(G, I) \to I$, which maps $f \in \ell^\infty(G, I)$ to its evaluation $f(e)$ at the neutral element of G is a unital *-homomorphism. As I is injective as an operator system, we may extend $\mathrm{eval}_e \circ \varphi$ to a ucp map $\psi\colon W \to I$ as in Diagram (2.4).

$$\begin{tikzcd}
W \arrow[rrd, dashed, "\tilde{\varphi}"] \arrow[rd, dashed, "\psi"] & & \\
& I & \ell^\infty(G, I) \arrow[l, "\mathrm{eval}_e"'] \\
V \arrow[uu, hook, "\iota"] \arrow[ru, dashed, "\mathrm{eval}_e \circ \varphi"] \arrow[rru, bend right, "\varphi"'] & &
\end{tikzcd} \tag{2.4}$$

This, however, induces a G-equivariant ucp map $\tilde{\varphi}\colon W \to \ell^\infty(G, I)$ by

$$(\tilde{\varphi}(w))(g) := \psi(g^{-1}.w)$$

for $w \in W$ and $g \in G$. Indeed, it is easily verified that $\tilde{\varphi}$ is linear and unital and since every amplification $(\tilde{\varphi})^{(n)}$ is simply a shift of $\psi^{(n)}$, it is ucp. For $v \in V$ we calculate

$$(\tilde{\varphi}(\iota(v)))(g) = \psi(\iota(g^{-1}.v)) = \mathrm{eval}_e(\varphi(\iota(g^{-1}.v))) = (g^{-1}.\varphi(\iota(v)))(e) = \varphi(\iota(v))(g).$$

Hence, $\tilde{\varphi}$ restricts to φ on $\iota(V)$ as desired and we conclude that $\ell^\infty(G, I)$ is a G-injective G-operator system. □

As promised, the next step is to cut this G-injective G-extension down by the appropriate family of seminorms, defined as follows:

Definition 2.2.7: Let $\iota\colon V \hookrightarrow W$ be a G-extension of G-operator systems. A G-equivariant ucp map $\varphi\colon W \to W$ is called a *V-projection* if it is idempotent and restricts to the identity on $\iota(V)$. Furthermore, a *V-seminorm* on W is defined as the seminorm given by $\|w\|_\psi := \|\psi(w)\|_W$ for a G-equivariant ucp self-map on W. The set of V-projections on W is partially ordered by $\prec$ with $\|\bullet\|_\psi \prec \|\bullet\|_{\psi'}$ if and only if $\|w\|_\psi \leq \|w\|_{\psi'}$ for all $w \in W$.

Clearly, every V-projection has an associated V-seminorm, and the partial order between V-projections φ and ψ, can be restated as $\varphi \prec \psi \Leftrightarrow \varphi \circ \psi = \varphi$: If the latter holds, then $\|\varphi(w)\| = \|\varphi \circ \psi(w)\| \leq \|\psi(w)\|$. On the other hand, if $\|\varphi(w)\| \leq \|\psi(w)\|$ for all $w \in W$ then

$$\|\varphi(w - \psi(w))\| \leq \|\psi(w - \psi(w))\| = \|\psi(w) - \psi^2(w)\| = 0$$

since ψ is idempotent and so $\varphi = \varphi \circ \psi$.

Hamana's main argument is the existence of a *minimal* such V-seminorm:

Proposition 2.2.8 (Hamana [34, Lemma 2.4]): Let $\iota\colon V \hookrightarrow W$ be a G-extension of G-operator systems. Any decreasing net of V-seminorms on W has a lower bound and hence there is a minimal V-seminorm on W with respect to the partial order $\prec$.

For the proof recall that for an operator system W and a von-Neumann algebra M the *point-weak* topology* on $\mathcal{B}(W, M)$ is characterised by bounded nets φ_λ converging if and only if they converge pointwise ultraweakly in M and that the unit ball of $\mathcal{B}(W, M)$ is point-weak*-compact by Banach–Alaoglu [54, Theorem 3.15].

Proof. Let φ_λ be the ucp maps $W \to W$ defining a decreasing net of V-seminorms $\|\bullet\|_{\varphi_\lambda}$. Assume that $W \subseteq \mathcal{B}(H)$ is represented on some Hilbert space $\mathcal{H}$ and embed W G-equivariantly into $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ as in the proof of Proposition 2.2.6 by sending $w \in W$ to $g \mapsto g^{-1}.w$. Then we may regard the φ_λ as ucp maps $W \to \ell^\infty(G, \mathcal{B}(\mathcal{H}))$ and as they are contractions, we obtain a net inside the unit ball of $\mathcal{B}(W, \ell^\infty(G, \mathcal{B}(\mathcal{H})))$. By compactness, we may pick a point-weak* limit $\tilde{\varphi} \in \mathcal{B}(W, \ell^\infty(G, \mathcal{B}(\mathcal{H})))$ of φ_λ, after passing to a subnet. It remains to verify that $\tilde{\varphi}$ is the desired lower bound. As the necessary conditions are pointwise and preserved under weak*-convergence, $\tilde{\varphi}$ is again a ucp map that on $\iota(V)$ restricts to the embedding into $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ and since the G-action on $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ is weak*-continuous, $\tilde{\varphi}$ is again G-equivariant. Composing with the evaluation eval_e at the identity of G we obtain a G-equivariant ucp map $\varphi\colon W \to W$ that restricts to the identity on $\iota(V)$, giving rise to the V-seminorm $\|\bullet\|_\varphi$. Then for $w \in W$

$$\|w\|_\varphi = \|\mathrm{eval}_e \circ \tilde{\varphi}(w)\| \leq \|\tilde{\varphi}(w)\| \leq \limsup_\lambda \|\varphi_\lambda(w)\|,$$

and since $\|\varphi_\lambda(w)\|$ is decreasing, $\|\bullet\|_\varphi$ is a lower bound for $\|\bullet\|_{\varphi_\lambda}$ as desired. □

This minimal V-seminorm must indeed come from a minimal V-projection, as we will see in the proof of the following proposition:

Proposition 2.2.9: Let $\iota\colon V \hookrightarrow W$ be a G-extension of G-operator systems. There exists a minimal V-projection with respect to the partial order $\prec$.

Proof. From Proposition 2.2.8 we know of the existence of a ucp map $\varphi\colon W \to W$ that restricts to the identity on $\iota(V)$ and which gives rise to a minimal V-seminorm. For $n \in \mathbb{N}$ consider

$$\bar{\varphi}_{(n)} := \frac{1}{n}(\varphi + \varphi^2 + \ldots + \varphi^n),$$

the average over the first n powers of φ. As in the proof of Proposition 2.2.8, we may understand $\bar{\varphi}_{(n)}$ as a sequence in the unit ball of $\mathcal{B}(W, \ell^\infty(G, \mathcal{B}(\mathcal{H})))$ for an appropriate Hilbert space $\mathcal{H}$ on which W is represented. Continuing in the spirit of said proof we may pick a point-weak* convergent subnet $\bar{\varphi}_{(n_\lambda)}$ of $\bar{\varphi}_{(n)}$ and argue that after composition with eval_e its limit $\bar{\varphi}$ is a G-equivariant ucp self-map of W which restricts to the identity on $\iota(V)$. Then note that for $w \in W$

$$\begin{aligned}\|\mathrm{eval}_e \circ \bar{\varphi}(w)\| \leq \|\bar{\varphi}(w)\| &\leq \limsup_\lambda \|\bar{\varphi}_{(n_\lambda)}(w)\| \\ &= \limsup_\lambda \frac{1}{n_\lambda}\|\varphi(w) + \varphi^2(w) + \ldots \varphi^{n_\lambda}(w)\| \\ &\leq \|\varphi(w)\|\end{aligned} \tag{2.5}$$

and since the latter defines a minimal V-seminorm, all of these are in fact equal so φ and $\mathrm{eval}_e \circ \bar{\varphi}$ define the same V-seminorm. In particular, the above shows that $\|\varphi(w)\| = \limsup_\lambda \|\bar{\varphi}_{(n_\lambda)}(w)\|$ whence also

$$\|\varphi(w - \varphi(w))\| = \limsup_\lambda \|\bar{\varphi}_{(n_\lambda)}(w) - \bar{\varphi}_{(n_\lambda)}(\varphi(w))\| = \limsup_\lambda \frac{1}{n_\lambda}\|\varphi(w) - \varphi^{n_\lambda+1}(w)\| = 0$$

and we may conclude that the original ucp self-map φ defining the minimal V-seminorm was in fact idempotent and therefore a V-projection. □

This minimal V-projection gives rise to a G-rigid G-extension as follows:

Proposition 2.2.10: Let $\iota\colon V \hookrightarrow W$ be a G-extension of G-operator systems and let φ be a minimal V-projection on W. Then $\varphi(W)$ is an operator system containing $\iota(V)$ and $\iota\colon V \hookrightarrow \varphi(W)$ is a G-rigid G-extension.

Proof. Clearly, $\varphi(W)$ contains $\iota(V)$, as φ restricts to the identity there. Since φ is G-equivariant, $\varphi(W)$ is G-invariant and forms a G-operator system.

We proceed to show that it is a G-rigid G-extension. Let $\psi\colon \varphi(W) \to \varphi(W)$ be a G-equivariant ucp self-map of $\varphi(W)$ that restricts to the identity on $\iota(V)$. Consider again

$$\overline{(\psi \circ \varphi)}_{(n)} := \frac{1}{n}\Big((\psi \circ \varphi) + (\psi \circ \varphi)^2 + \ldots + (\psi \circ \varphi)^n\Big).$$

As in the proof of Proposition 2.2.9 we may pick a point-weak* limit of a subnet $\overline{(\psi \circ \varphi)}_{(n_\lambda)}$ to obtain $\chi \in \mathcal{B}(W, \ell^\infty(G, \mathcal{B}(\mathcal{H})))$ and in analogy to Equation (2.5) we have

$$\|\mathrm{eval}_e \circ \chi(w)\| \leq \|\chi(w)\| \leq \limsup_\lambda \|\overline{(\psi \circ \varphi)}_{(n_\lambda)}(w)\| \leq \|\psi \circ \varphi(w)\| \leq \|\varphi(w)\|$$

where consequently all of these terms have to be equal by minimality of the V-projection φ. Then

$$\begin{aligned}\|w - \psi(w)\| = \|\varphi(w - \psi \circ \varphi(w))\| &= \limsup_\lambda \|\overline{(\psi \circ \varphi)}_{(n_\lambda)}(w - \psi \circ \varphi(w))\| \\ &= \limsup_\lambda \frac{1}{n_\lambda}\|(\psi \circ \varphi)(w) - (\psi \circ \varphi)^{n_\lambda+1}(w)\| \\ &= 0\end{aligned}$$

as $\varphi(w) = w$ for $w \in \varphi(W)$ since it is a projection and we conclude that ψ is the identity on all of $\varphi(W)$. □

We collect these results into Hamana's existence and uniqueness theorem for G-equivariant injective envelopes:

Theorem 2.2.11 (Hamana [30, Theorem 4.1], [34, Theorem 2.5]): *Let V be a G-operator system. There exists an operator system $I_G(V)$ and a ucp embedding $\iota\colon V \to I_G(V)$ that forms a G-injective G-essential G-extension. Then $I_G(V)$ is called the G-equivariant injective envelope of V. Given any other G-injective G-essential G-extension $\kappa\colon V \to W$, there is a G-equivariant ucp map $\psi\colon I_G(V) \to W$ with G-equivariant ucp inverse that preserves the embedding of V as $\psi \circ \iota = \kappa$. In other words, the G-equivariant injective envelope is unique up to G-equivariant complete order isomorphism.*

Proof. Assume that V is represented on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and recall that V embeds G-equivariantly into $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$. Let φ be a minimal V-projection on $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$. Define $I_G(V) := \varphi(\ell^\infty(G, \mathcal{B}(\mathcal{H})))$, then by Proposition 2.2.10 $I_G(V)$ is a G-rigid G-extension of V and we denote the embedding of V by ι. Since φ is idempotent and $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ is G-injective, $\varphi(\ell^\infty(G, \mathcal{B}(\mathcal{H})))$ is also G-injective: Given an embedding $X \hookrightarrow Y$ of G-operator systems, any G-equivariant ucp map $\chi \colon X \to \varphi(\ell^\infty(G, \mathcal{B}(\mathcal{H})))$ is a map into $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ and can therefore be extended to a G-equivariant ucp map $\tilde{\chi} \colon Y \to \varphi(\ell^\infty(G, \mathcal{B}(\mathcal{H})))$. Then $\varphi \circ \tilde{\chi}$ is a G-equivariant ucp map from Y into $I_G(V)$ that restricts to χ on X.

Hence $\iota \colon V \to I_G(V)$ is a G-injective G-rigid G-extension of V and by Proposition 2.2.5 it is G-essential as claimed.

Given any other G-injective G-essential G-extension $\kappa \colon V \to W$, consider Diagram (2.6).

$$\begin{tikzcd}
I_G(V) \arrow[rr, dashed, bend left, "\psi"] & & W \arrow[ll, dashed, "\hat{\psi}"] \\
& V \arrow[ul, hook, "\iota"] \arrow[ur, hook', "\kappa"'] &
\end{tikzcd} \tag{2.6}$$

By G-injectivity of W we may extend κ to a G-equivariant ucp map $\psi \colon I_G(V) \to W$ and by G-injectivity of $I_G(V)$ we may extend ι to a G-equivariant ucp map $\hat{\psi} \colon W \to I_G(V)$ as in the diagram. Then $\hat{\psi} \circ \psi$ is a self-map of $I_G(V)$ which restricts to the identity on $\iota(V)$ and therefore the identity by G-rigidity of $I_G(V)$. On the other hand, $\hat{\psi}$ is injective by G-essentiality of W and therefore $\psi \circ \hat{\psi} = \mathrm{id}_W$ since

$$\hat{\psi}(w - \psi \circ \hat{\psi}(w)) = \hat{\psi}(w) - (\hat{\psi} \circ \psi)(\hat{\psi}(w)) = \hat{\psi}(w) - \hat{\psi}(w) = 0.$$

We conclude that ψ is a G-equivariant complete order isomorphism with inverse $\hat{\psi}$. □

Having proven the existence of a G-injective envelope, we get the promised implication of G-rigidity as a bonus in adaptation of the remark after [30, Lemma 3.7]:

Proposition 2.2.12 (Hamana [30]): Every G-essential G-extension is G-rigid.

Proof. Let $\iota \colon V \to W$ be a G-essential G-extension of the G-operator system V and let $\varphi \colon W \to W$ be a G-equivariant ucp map such that it restricts to the identity on $\iota(V)$. Let $\kappa \colon V \to I_G(V)$ be the embedding into the G-equivariant injective envelope. Then we may extend κ to a G-equivariant ucp map $\tilde{\kappa} \colon W \to I_G(V)$ by G-injectivity of $I_G(V)$.

$$\begin{tikzcd}
& & I_G(V) \\
W \arrow[r, hook, "\varphi"] & W \arrow[ur, "\tilde{\kappa}\circ\varphi"] \arrow[r, hook, dashed, "\tilde{\kappa}"] & I_G(V) \arrow[u, dashed, "\psi"] \\
& V \arrow[ul, hook, "\iota"] \arrow[u, hook, "\iota"] \arrow[ur, hook', "\kappa"] &
\end{tikzcd} \tag{2.7}$$

By G-essentiality of W, we know that $\tilde{\kappa}$ is a complete isometry and therefore $\tilde{\kappa} \circ \varphi$ can be extended from W along the embedding $\tilde{\kappa}$ to a G-equivariant ucp map $\psi\colon I_G(V) \to I_G(V)$ as in the upper right corner of Diagram (2.7). Consequently, $\tilde{\kappa} \circ \varphi = \psi \circ \tilde{\kappa}$. Since φ restricts to the identity on $\iota(V)$, a simple diagram chase yields

$$\psi \circ \kappa = \psi \circ \tilde{\kappa} \circ \iota = \tilde{\kappa} \circ \varphi \circ \iota = \tilde{\kappa} \circ \iota = \kappa$$

and we conclude that ψ restricts to the identity on $\kappa(V)$. By G-rigidity of $I_G(V)$, however, this means that ψ is the identity on all of $I_G(V)$ and consequently $\tilde{\kappa} \circ \varphi = \tilde{\kappa}$. Now, as $\tilde{\kappa}$ is injective, φ is the identity on all of W as desired. □

While we have alluded to it, we have not yet argued that injective envelopes are in fact C^*-algebras, not merely operator systems. This is in fact due to an older and more general result of Choi and Effros [17, Theorem 3.1], equipping any injective operator system with a C^*-algebra-structure. Recall that a G-operator system $V \subseteq \mathcal{B}(\mathcal{H})$ embeds G-equivariantly into $\ell^\infty(G, \mathcal{B}(H))$ by sending $v \in V$ to the function $g \mapsto g^{-1}.v$. The following observation makes Choi–Effros' results apply to G-equivariant envelopes:

Proposition 2.2.13: Let $I \subseteq \mathcal{B}(\mathcal{H})$ be a G-operator system represented on some Hilbert space $\mathcal{H}$. Then I is a G-injective G-operator system, if and only if it is injective as an operator system and there is a G-equivariant ucp map $\ell^\infty(G, I) \to I$ that restricts to the identity on I.

Proof. First assume that I is G-injective. Then the identity on I extends along the embedding $I \hookrightarrow \ell^\infty(G, \mathcal{B}(H))$ to a G-equivariant ucp map $\psi\colon \ell^\infty(G, \mathcal{B}(H)) \to I$. Let $V \hookrightarrow W$ be an extension of operator systems without G-action and let $\varphi\colon V \to I$ be a ucp map. As $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$ is an injective C^*-algebra since it is a direct sum of copies of the injective C^*-algebra $\mathcal{B}(\mathcal{H})$, we may extend φ to a ucp map $\tilde{\varphi}\colon W \to \ell^\infty(G, \mathcal{B}(\mathcal{H}))$ and consequently $\psi \circ \tilde{\varphi}\colon W \to I$ is the desired extension of φ showing that I is an injective operator system. The restriction of ψ to $\ell^\infty(G, I)$ is furthermore the desired G-equivariant ucp map as claimed in the proposition.

Conversely, let I be an injective operator system and $\psi\colon \ell^\infty(G, I) \to I$ a G-equivariant ucp map that restricts to the identity on I. Since I is an injective operator system, $\ell^\infty(G, I)$ is a G-injective G-operator system by Proposition 2.2.6. Consequently, every G-equivariant ucp map $\varphi\colon V \to I$ for an embedding $V \hookrightarrow W$ of G-operator systems has a G-equivariant extension $\tilde{\varphi}\colon W \to \ell^\infty(G, I)$ and $\psi \circ \tilde{\varphi}\colon W \to I$ once again shows that I is G-injective. □

The C^*-algebra-structure on an injective operator system I is given by the Choi–Effros product associated with an idempotent on I:

Definition 2.2.14: Let A be a C^*-algebra and $\varphi\colon A \to A$ a unital idempotent contraction satisfying the *Schwarz inequality*

$$\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$$

for all $a \in A$. The *Choi–Effros product* $\circ$ on $\varphi(A)$ is defined by

$$a \circ b := \varphi(ab)$$

for $a, b \in A$ with ab their multiplication in A.

Indeed, this gives rise to a C^*-algebra-structure on $\varphi(A)$:

Theorem 2.2.15 (Hamana [29, Theorem 2.3], Choi–Effros [17, Section 3]): *Let A be a C^*-algebra and $\varphi\colon A \to A$ a ucp idempotent. Then $\varphi(A)$ equipped with the Choi–Effros product $\circ$ associated with φ and the norm and adjoint operation of A is a unital C^*-algebra that is completely order isomorphic to $\varphi(A)$ as an operator system.*

By [17, Theorem 3.1], this C^*-algebra-structure is even unique up to *-isomorphism respecting the canonical complete order isomorphisms to $\varphi(A)$, but we omit the proof.

Proof. Since φ is a continuous idempotent, its range is closed and therefore complete in the norm of A. We first prove the C^*-identity for $a \in \varphi(A)$. Then $\varphi(a) = a$ as φ is idempotent and by the Schwarz-inequality, which holds for ucp maps, we have $\varphi(a)^*\varphi(a) \leq \varphi(a^*a) = a^* \circ a$ whence

$$||a||^2 = ||a^*a|| \leq ||a^* \circ a||.$$

However, φ is ucp and therefore of norm one, so

$$||a^* \circ a|| = ||\varphi(a^*a)|| \leq ||a^*a|| = ||a||^2$$

and we conclude that all of these inequalities are in fact equalities. Furthermore, it clearly holds for $b \in \varphi(A)$ that

$$(a \circ b)^* = \varphi(ab)^* = \varphi(b^*a^*) = b^* \circ a^*.$$

To see that $\circ$ is associative, first note that

$$\varphi(a\varphi(x)) = \varphi(ax) \quad \text{and} \quad \varphi(\varphi(x)a) = \varphi(xa) \tag{2.8}$$

for $a \in \varphi(A)$ and arbitrary $x \in A$, which we will justify in a separate proof below. Then for $a, b, c \in \varphi(A)$

$$(a \circ b) \circ c = \varphi(\varphi(ab)c) = \varphi(abc) = \varphi(a\varphi(bc)) = a \circ (b \circ c).$$

All other requirements for $\varphi(A)$ to be a C^*-algebra do not involve the multiplication and therefore hold trivially.

We next show that $\varphi(A)$ is complete order isomorphic to $(\varphi(A), \circ)$ equipped with the product $\circ$. Denote by $(M_n(\varphi(A)), \circ)$ the algebra of n-by-n matrices over $(\varphi(A), \circ)$. Then for $[a_{ij}], [b_{ij}] \in M_n(\varphi(A))$ we have

$$[a_{ij}] \circ [b_{ij}] = \sum_{k=1}^{n} a_{ik} \circ b_{kj} = \left[\varphi\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)\right] = \varphi^{(n)}([a_{ij}][b_{ij}]).$$

Hence there is no difference between passing from A to $\varphi(A)$ via φ and then taking the matrix algebras $M_n(\varphi(A))$ or taking matrix algebras first and passing from $M_n(A)$ to $\varphi^{(n)}(M_n(A))$ via the amplification $\varphi^{(n)}$. Consequently, the norms on the two coincide as the norm of a C^*-algebra is determined algebraically. Likewise, positivity is determined by the norm, since a self-adjoint element x in any C^*-algebra is positive if and only if

$$||(||x|| \cdot \mathbb{1}) - x|| \leq ||x||$$

and therefore $[a_{ij}] \in M_n(\varphi(A))$ is positive if and only if it is positive in $M_n(A)$. Consequently, the identity map from $\varphi(A)$ as an operator system inside A to $\varphi(A)$ as a C^*-algebra equipped with $\circ$ sends exactly the positive elements to the positive elements in every amplification and we conclude that it is a complete order isomorphism. □

Proof of Equation (2.8). Let $\varphi\colon A \to A$ be a ucp idempotent as above and let both $a \in \varphi(A)$ and $x \in A$ be self-adjoint. Then the matrix

$$\begin{pmatrix} 0 & a \\ a & x \end{pmatrix}$$

is self-adjoint and by the Schwarz inequality we find that

$$\begin{aligned}
& \varphi^{(2)}\begin{pmatrix} 0 & a \\ a & x \end{pmatrix}\varphi^{(2)}\begin{pmatrix} 0 & a \\ a & x \end{pmatrix} \leq \varphi^{(2)}\left(\begin{pmatrix} 0 & a \\ a & x \end{pmatrix}\begin{pmatrix} 0 & a \\ a & x \end{pmatrix}\right) \\
\Leftrightarrow \quad & \begin{pmatrix} a^2 & a\varphi(x) \\ \varphi(x)a & a^2 + \varphi(x)^2 \end{pmatrix} \leq \begin{pmatrix} \varphi(a)^2 & \varphi(ax) \\ \varphi(xa) & \varphi(a^2 + x^2) \end{pmatrix},
\end{aligned}$$

and applying $\varphi^{(2)}$ to both sides yields

$$\begin{aligned}
\Rightarrow \quad & \begin{pmatrix} \varphi(a^2) & \varphi(a\varphi(x)) \\ \varphi(\varphi(x)a)) & \varphi(a^2 + \varphi(x)^2) \end{pmatrix} \leq \begin{pmatrix} \varphi(a)^2 & \varphi(ax) \\ \varphi(xa) & \varphi(a^2 + x^2) \end{pmatrix} \\
\Leftrightarrow \quad & 0 \leq \begin{pmatrix} 0 & \varphi(ax) - \varphi(a\varphi(x)) \\ \varphi(xa) - \varphi(\varphi(x)a) & * \end{pmatrix},
\end{aligned}$$

disregarding the entry marked $*$. Hence, there is a self-adjoint matrix $[y_{ij}] \in M_2(A)$ such that

$$\begin{aligned}
\begin{pmatrix} 0 & \varphi(ax) - \varphi(a\varphi(x)) \\ \varphi(xa) - \varphi(\varphi(x)a) & * \end{pmatrix} &= \begin{pmatrix} y_{11} & y_{12} \\ y_{12}^* & y_{22} \end{pmatrix}\begin{pmatrix} y_{11} & y_{12} \\ y_{12}^* & y_{22} \end{pmatrix} \\
&= \begin{pmatrix} y_{11}^2 + y_{12}y_{12}^* & y_{11}y_{12} + y_{12}y_{22} \\ y_{12}^*y_{11} + y_{22}y_{12}^* & * \end{pmatrix}.
\end{aligned}$$

But then $y_{11} = 0 = y_{12}$ from the upper left entry and we conclude that $\varphi(ax) - \varphi(a\varphi(x)) = 0$ and $\varphi(xa) - \varphi(\varphi(x)a) = 0$, proving Equation (2.8). □

We can easily apply Theorem 2.2.15 to equip G-equivariant injective envelopes with a C^*-algebra-structure:

Proposition 2.2.16: Let I be a G-injective operator system. Then I is completely order isomorphic to a unital C^*-algebra with G-action.

Proof. Let I be represented as a subset of a G-C^*-algebra A, for example by representing on some $\mathcal{B}(\mathcal{H})$ and embedding in $\ell^\infty(G, \mathcal{B}(\mathcal{H}))$. Then, since I is G-injective, there is a G-equivariant ucp map $\varphi\colon A \to I$ that extends the identity on I along the embedding of I into A. Consequently, $\varphi(A) = I$ and we may equip I with the Choi–Effros product associated with φ. Since I is G-invariant and by G-equivariance of φ

$$g.a \circ g.b = \varphi((g.a)(g.b)) = \varphi(g.(ab)) = g.\varphi(ab) = g.(a \circ b)$$

for $a, b \in I$, the C^*-algebra-structure on I carries the same G-action as I. □

Employing the uniqueness statement of [17, Theorem 3.1], this C^*-algebra-structure is furthermore unique up to *-isomorphism.

Finally, note that the Choi–Effros product is commutative if the surrounding C^*-algebra was commutative. As a consequence, the the G-equivariant injective envelope of a commutative G-C^*-algebra C is again commutative, since $I_G(C)$ sits inside the commutative C^*-algebra $\ell^\infty(G, C)$ and using $A = \ell^\infty(G, C)$ in the proof above shows that the Choi–Effros product is commutative.

2.2.3 Kalantar–Kennedy's Identification

The novel approach of Kalantar and Kennedy [36] towards simplicity of reduced group C^*-algebras largely rests on their identification of the Furstenberg boundary with the spectrum of the appropriate G-equivariant injective envelope, sometimes called *Hamana boundary*:

Definition 2.2.17: Let G be a discrete group and let G act trivially on the C^*-algebra $\mathbb{C}$ of complex numbers. Then $I_G(\mathbb{C})$ is a unital, commutative C^*-algebra and hence isomorphic to $C(\partial_H G)$ for some compact Hausdorff space $\partial_H G$ with a G-action and $\partial_H G$ is called the *Hamana boundary* of G.

Indeed, the Hamana boundary is a boundary in the sense of Furstenberg:

Proposition 2.2.18 (Kalantar–Kennedy [36, Proposition 3.4, Proposition 3.7]): The action of G on its Hamana boundary $\partial_H G$ is minimal and strongly proximal; that is, $\partial_H G$ is a boundary in the sense of Furstenberg.

Proof. We first show that the action is minimal. Let $x \in \partial_H G$ and consider its orbit closure $\overline{G.x}$. This is a closed, G-invariant subspace and the restriction map $\mathrm{res}_{\overline{G.x}} \colon C(\partial_H G) \to C(\overline{G.x}) \subseteq C(\partial_H G)$ is unital and completely positive. Note that $C(\partial_H G) = I_G(\mathbb{C})$ is a G-essential G-extension of $\mathbb{C}$, with the embedding sending $z \in \mathbb{C}$ to the scalar $z\mathbb{1} \in C(\partial_H G)$ for the unit $\mathbb{1} \in C(\partial_H G)$. Hence $\mathrm{res}_{\overline{G.x}}$ is a ucp map on $I_G(\mathbb{C})$ that restricts to the identity on the embedding of $\mathbb{C}$ and therefore it is a complete isometry by G-essentiality. We conclude that $\overline{G.x}$ is all of $\partial_H G$, as the restriction to $\overline{G.x}$ is injective.

We furthermore show that the action is strongly proximal. Let $\mu \in \mathcal{M}(\partial_H G)$ be a Radon probability measure. The *Poisson map* $P_\mu \colon C(\partial_H G) \to \ell^\infty(G)$ associated with μ is defined by

$$(P_\mu(f))(g) := \int_{\partial_H G} f \,\mathrm{d}g.\mu$$

for $f \in C(\partial_H G)$ and $g \in G$ and is clearly linear, G-equivariant, positive, and unital. As a positive map into a commutative C^*-algebra, it is furthermore completely positive and therefore a complete isometry as it restricts to a complete isometry on the scalars and $C(\partial_H G)$ is a G-essential G-extension. Assume for contradiction that the point measure δ_x of some $x \in \partial_H G$ was not contained in the weak*-closed convex hull K of the G-orbit of μ inside $\mathcal{M}(\partial_H G)$. Then, using the Hahn–Banach separation theorem [54, Theorem 3.4], δ_x can be separated from K by a linear functional described by some positive function $f \in C(\partial_H G)$ with $f(x) = \int_{\partial_H G} f \,\mathrm{d}\delta_x = 1$ such that

$$(P_\mu(f))(g) = \int_{\partial_H G} f \,\mathrm{d}g.\mu < 1 - \epsilon$$

for all $g \in \mathcal{G}$ and some $\epsilon > 0$. Hence $P_\mu(f)$, which is real-valued as f is positive, only takes values between 0 and $1 - \epsilon$, although $f(x) = 1$ in contradiction to P_μ being an isometry. Therefore, the closed convex hull of $G.\mu$ inside $\mathcal{M}(\partial_H G)$ contains all point measures of $\mathcal{M}(\partial_H G)$ and as these are exactly the extreme points of the compact, convex space $\mathcal{M}(\partial_H G)$, we conclude that $K = \mathcal{M}(\partial_H G)$ by Krein–Milman [54, Theorem 3.23]. By Milman's partial converse [54, Theorem 3.25], the orbit $G.\mu$ approximates all extreme points δ_x for $x \in \partial_H G$ and the action is strongly proximal. □

Remark. Note that the proof works not only for the injective envelope, but for every commutative G-essential G-extension of the trivial action on the complex numbers, hence the spectrum of every commutative G-essential G-extension is a boundary in the sense of Furstenberg.

2.2.4 An Incomplete Digression on Monotone Completeness

Some of Hamana's original arguments require a few results and constructions involving monotone complete C^*-algebras.

Definition 2.2.19: A C^*-algebra A is called *monotone complete* if every norm-bounded, increasing net of self-adjoint elements of A has a least upper bound in A.

By work of Takesaki [59, Proposition III.1.7], a commutative C^*-algebra $C_0(X)$ is monotone complete if and only if its spectrum X is *extremally disconnected*, that is, the closure of every open set in X is again open. If X is compact and extremally disconnected, it is called *Stonean*.

Proposition 2.2.20: Any unital, commutative, injective C^*-algebra is monotone complete.

Proof. Let X be a compact Hausdorff space such that $C(X)$ is an injective C^*-algebra. By injectivity we may extend the identity on $C(X)$ along the embedding $C(X) \hookrightarrow \ell^\infty(X)$ to a conditional expectation E from $\ell^\infty(X)$ onto $C(X)$. Note that $\ell^\infty(X)$ is monotone complete since we may form a least upper bound simply by taking the pointwise supremum. Let a_λ be an increasing, norm-bounded net of self-adjoint functions in $C(X)$ and a a least upper bound for a_λ in $\ell^\infty(X)$. Then $E(a) \geq E(a_\lambda) = a_\lambda$ and given any upper bound $b \in C(X)$ such that $b \geq a_\lambda$ for all λ we have $b \geq a$ inside $\ell^\infty(X)$ since a is a least upper bound and therefore $b = E(b) \geq E(a)$. We conclude that $E(a)$ is a least upper bound for a_λ in $C(X)$, so $C(X)$ is monotone complete. □

Consequently, the G-equivariant injective envelope of a commutative C^*-algebra with G-action is monotone complete, since it is an injective C^*-algebra by Proposition 2.2.13 and the Choi–Effros product of Definition 2.2.14 is commutative.

The remainder of this section is dedicated to Hamana's *monotone complete crossed products* as defined in [33, Chapter 3].

Let $A \subseteq \mathcal{B}(\mathcal{H})$ be an operator system and G a discrete group. Define the operator system

$$A \overline{\otimes} \mathcal{B}(\ell^2(G)) := \left\{ [x_{g,h}] \in \mathcal{B}(\mathcal{H} \otimes \ell^2(G)) \;\middle|\; x_{g,h} \in A \text{ for all } g, h \in G \right\} \subseteq \mathcal{B}(\mathcal{H} \otimes \ell^2(G)),$$

that is, understand the operators in $\mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ as $\mathcal{B}(\mathcal{H})$-valued matrices and consider only those operators x whose matrix elements

$$x_{g,h} := (\mathrm{id}_{\mathcal{H}} \otimes \delta_g)^* x (\mathrm{id}_{\mathcal{H}} \otimes \delta_h)$$

lie in A for all $g, h \in G$. If A is a monotone complete C^*-algebra, $A \overline{\otimes} \mathcal{B}(\ell^2(G))$ is Hamana's *monotone complete tensor product* from [32, Theorem 3.12], which again carries the structure of a monotone complete C^*-algebra.

Definition 2.2.21: Let A be a monotone complete C^*-algebra with an action by a discrete group G. The *monotone complete crossed product* $M(A, G)$ is given by

$$M(A,G) := \left\{ [x_{g,h}] \in A \overline{\otimes} \mathcal{B}(\ell^2(G)) \;\middle|\; s^{-1}.x_{g,h} = x_{gs,hs} \text{ for all } g, h, s \in G \right\} \subseteq A \overline{\otimes} \mathcal{B}(\ell^2(G)).$$

This is an operator system which, by [33, Chapter 3], may be equipped with the structure of a monotone complete C^*-algebra with a G-action.

We cite Hamana's results that we will need in our applications of the montone complete crossed product, but refer the reader to the original papers, as they require significant background and we have nothing to add to the proofs.

Proposition 2.2.22 (Hamana [33, Lemma 3.1]): Let A be a monotone complete C^*-algebra with an action by a discrete group G. Then A is G-injective if and only if $M(A, G)$ is injective.

The following statement is contained in the proof of [33, Lemma 3.1] with added details from [13, before Lemma 5.3.2]:

Proposition 2.2.23 (Hamana): The map $E\colon M(A, G) \to A$ given by evaluation at the matrix entry

$$E(x) = x_{e,e}$$

for $x \in M(A, G)$ is a faithful G-equivariant conditional expectation from $M(A, G)$ onto A.

2.3 Simplicity of Reduced Group C^*-algebras

The identification of the Hamana and Furstenberg boundaries immediately yielded a dynamical characterisation of C^*-simplicity by Kalantar–Kennedy. After presenting streamlined proofs of their results, we briefly discuss implications regarding the unique trace property as discovered by Breuillard–Kalantar–Kennedy–Ozawa and then finally turn to Kennedy's group-theoretic characterisation of C^*-simplicity.

2.3.1 Kalantar–Kennedy's Dynamical Charcterisation

The main application of the Furstenberg boundary and its reinterpretation as the spectrum of a G-equivariant injective envelope we are considering is simplicity of reduced group C^*-algebras. Indeed, the reduced C^*-algebra $C^*_{\mathrm{r}}(G)$ associated with a discrete group G is simple, if and only if the reduced crossed product C^*-algebra associated with the action of G on $\partial_F G$ is simple, as established in Kalantar–Kennedy's original work [36, Theorem 6.2]. We offer a slightly more in-depth treatment of reduced crossed product algebras in Section 3.1. For now let it be said that the reduced crossed product $C(X) \rtimes_r G$ associated with a discrete group G acting on a commutative C^*-algebra $C(X)$ arises much like $C^*_{\mathrm{r}}(G)$ from completing the group-algebra $C(X)[G]$ with coefficients in $C(X)$. The completion is done with respect to the norm of a left-regular representation $\pi \rtimes \lambda_G$ on $\mathcal{B}(\mathcal{H}) \otimes \ell^2(G)$ built from the left-regular representation λ_G of G and a faithful representation of $C(X)$ on some Hilbert space $\mathcal{H}$.

Theorem 2.3.1 (Kalantar–Kennedy [36, Theorem 6.2]): *Let G be a discrete group and $\partial_F G$ its Furstenberg boundary. Then $C^*_r(G)$ is simple if and only if $C(\partial_F G) \rtimes_r G$ is simple.*

We will follow the original proof from [36, Theorem 6.2], relying on technical arguments from [34] and [33]. Although there is a more straightforward method we will later present in Theorem 3.3.2, we chose to present this approach since we feel that both methods offer different perspectives which will be useful when attempting generalizations.

Before we prove Theorem 2.3.1, we collect a couple of auxiliary results.

Lemma 2.3.2 (Hamana [34, Theorem 3.4]): *Let A be a G-C^*-algebra and let $I(A \rtimes_r G)$ denote the non-equivariant injective envelope of the crossed product $A \rtimes_r G$. Then $I_G(A) \rtimes_r G$ embeds into $I(A \rtimes_r G)$, respecting the embedding of $A \rtimes_r G$ as in the following diagram:*

$$\begin{tikzcd}
I_G(A) \rtimes_r G \arrow[r, hook, "\kappa"] & I(A \rtimes_r G) \\
A \rtimes_r G \arrow[u, hook, "\iota_G"] \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

Proof. We want to find an embedding κ as in the diagram in the statement. Since $I(A \rtimes_r G)$ is injective, there is such a ucp map κ extending the embedding of $\iota\colon A \rtimes_r G \hookrightarrow I(A \rtimes_r G)$ along the embedding $\iota_G\colon A \rtimes_r G \hookrightarrow I_G(A) \rtimes_r G$ coming from $A \hookrightarrow I_G(A)$. We have to show that it is a complete order isomorphism.

Consider the monotone complete crossed product $M(I_G(A), G)$ as in Definition 2.2.21. Since $I_G(A)$ is G-injective, $M(I_G(A), G)$ is injective by Proposition 2.2.22. Both $A \rtimes_r G \subseteq I_G(A) \rtimes_r G$ and $M(I_G(A), G)$ are defined as subalgebras of $\mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ up to a representation of $A \subseteq I_G(A)$ on some Hilbert space $\mathcal{H}$. After that identification $M(I_G(A), G)$ clearly contains $A \rtimes_r G$. Hence by injectivity of $M(I_G(A), G)$ we may extend this inclusion along ι to a ucp map $\eta\colon I(A \rtimes_r G) \to M(I_G(A), G)$ which is furthermore a complete isometry by essentiality of $I(A \rtimes_r G)$. We may therefore consider all algebras involved as subsets of $M(I_G(A), G)$ and κ a ucp map between operator systems inside $M(I_G(A), G)$. We want to show that κ is the identity on $I_G(A) \rtimes_r G$ and to do so adapt arguments from [34, Lemma 3.3].

First consider $\kappa|_{I_G(A)}$ restricted to $I_G(A)$ as a subset of $M(I_G(A), G)$ and let $E\colon M(I_G(A), G) \to I_G(A)$ be the conditional expectation from Proposition 2.2.23. Then $E \circ \kappa|_{I_G(A)}\colon I_G(A) \to I_G(A)$ is a G-equivariant ucp map that restricts to the identity on A and therefore the identity by G-rigidity. Furthermore, for $x \in I_G(A)$ the Schwarz inequality for κ yields

$$E(\kappa(x)^*\kappa(x)) \leq E(\kappa(x^*x)) = x^*x$$

while on the other hand the Schwarz inequality for E yields

$$x^*x = E(\kappa(x))^*E(\kappa(x)) \leq E(\kappa(x)^*\kappa(x))$$

and we conclude that

$$E(\kappa(x)^*\kappa(x)) = x^*x.$$

Therefore, both $\kappa(I_G(A))$ and $I_G(A)$ are contained in the multiplicative domain of E and so $x - \kappa(x) \in \operatorname{mult}(E)$ for every $x \in I_G(A)$. Consequently,

$$\|E((x-\kappa(x))^*(x-\kappa(x)))\| = \|(E(x-\kappa(x)))^*(E(x-\kappa(x)))\| = \|E(x) - E\circ\kappa(x)\|^2 = \|x - x\| = 0.$$

Since E is faithful, we conclude that $x - \kappa(x) = 0$ for $x \in I_G(A)$ so $\kappa|_{I_G(A)}$ is indeed the identity on $I_G(A)$.

However, κ also restricts to the identity on $A \rtimes_r G$ and hence fixes $\lambda_g \in \mathcal{B}(\mathcal{H} \otimes \ell^2(G))$ for all $g \in G$ so that it also restricts to the identity on $C^*_{\mathrm{r}}(G) \subseteq I_G(A) \rtimes_r G$. But $I_G(A)$ and $C^*_{\mathrm{r}}(G)$ span $I_G(A) \rtimes_r G$ densely, whence κ is simply the identity. Stated differently, κ is the inclusion of one subset of $M(I_G(A), G)$ into another, larger subset. We conclude that κ is a complete isometry. □

The next lemma is particularly useful to kill off ideals of $I_G(A) \rtimes_r G$:

Lemma 2.3.3 (Hamana [31, Lemma 1.2]): *Let A be a unital C^*-algebra and $I(A)$ its injective envelope. For sake of notation consider $A \subseteq I(A)$ to be a subalgebra of $I(A)$. Let B be a C^*-subalgebra of $I(A)$ such that $A \cap B = \{0\}$ and $xB + By \subseteq B$ for all $x, y \in A$. Then $B = \{0\}$.*

Proof. By the assumptions, $A + B$ is closed under multiplication and therefore a unital C^*-subalgebra of $I(A)$ and B is a closed, two-sided ideal of $A + B$. Let π denote the quotient *-homomorphism $\pi\colon A+B \to (A+B)/B$. Then its restriction to A is injective, since $A \cap B = \{0\}$. We want to show that the inclusion $A \subseteq A+B$ is an essential extension, so that we can conclude that π is injective by essentiality because $\pi|_A$ is. Then the conclusion $B = \{0\}$ follows.

Indeed, consider Diagram (2.9), where ι denotes the embedding of A into its injective envelope.

$$\begin{tikzcd}
I(A) \arrow[drr, dashed, bend left, "\psi"] & & \\
A + B \arrow[u, hook] \arrow[r, "\pi"] & (A+B)/B \arrow[r, dashed, "\varphi"] & I(A) \\
& A \arrow[ul, hook'] \arrow[u, hook, "\pi|_A"'] \arrow[ur, hook, "\iota"'] &
\end{tikzcd} \tag{2.9}$$

By injectivity of $I(A)$ we may extend ι along the embedding $\pi|_A$ to a ucp map $\varphi\colon (A+B)/B \to I(A)$. Then, using injectivity of $I(A)$ once more we may extend $\varphi \circ \pi$ from $A + B$ along the inclusion $A + B \subseteq I(A)$ to another ucp map $\psi\colon I(A) \to I(A)$. But ψ restricts to the identity on A and as $I(A)$ is rigid, it is therefore the identity on all of $I(A)$. We conclude that $\varphi \circ \pi$ and therefore π are injective. □

These are enough to make a simple argument for the first implication of Theorem 2.3.1, as known already by Hamana [34]. For the converse, we streamline the proof by Kennedy–Kalantar [36, Theorem 6.2] as inspired by Ozawa [48, Lemma 16] in a way that will later be easier to generalize for groupoids in Lemma 4.2.22 and 4.2.23.

Proof of Theorem 2.3.1: First assume that G is C^*-simple and note that $C^*_{\mathrm{r}}(G)$ is canonically isomorphic to $\mathbb{C} \rtimes_r G$. Then by Lemma 2.3.2 we have that $C(\partial_F G) \rtimes_r G \subseteq I(\mathbb{C} \rtimes_r G) \cong I(C^*_{\mathrm{r}}(G))$ since $C(\partial_F G) \cong I_G(\mathbb{C})$. Now if $J \lhd C(\partial_F G) \rtimes_r G$ is a non-trivial, closed, two-sided ideal, then $J \cap C^*_{\mathrm{r}}(G)$ is a closed, two-sided ideal of $C^*_{\mathrm{r}}(G)$ and is not all of $C^*_{\mathrm{r}}(G)$, since it would otherwise contain the unit of $C^*_{\mathrm{r}}(G)$ and therefore be trivial as an ideal of $C(\partial_F G) \rtimes_r G$. By simplicity of $C^*_{\mathrm{r}}(G)$, it follows that $J \cap C^*_{\mathrm{r}}(G) = \{0\}$. Now since $JC^*_{\mathrm{r}}(G) + C^*_{\mathrm{r}}(G)J \subseteq J$ inside $I(C^*_{\mathrm{r}}(G))$, we may conclude from Lemma 2.3.3 that $J = \{0\}$ and have proven simplicity of $C(\partial_F G) \rtimes_r G$.

Conversely, assume that G is not C^*-simple and let J be a non-trivial, closed, invariant ideal of $C^*_{\mathrm{r}}(G)$. Assume that $C^*_{\mathrm{r}}(G)/J \subseteq \mathcal{B}(\mathcal{H})$ is represented on some Hilbert space $\mathcal{H}$ and let $\pi\colon C^*_{\mathrm{r}}(G) \to C^*_{\mathrm{r}}(G)/J$ be the associated quotient map. By Arveson's injectivity of $\mathcal{B}(\mathcal{H})$, we may extend π along the embedding $C^*_{\mathrm{r}}(G) \subseteq C(\partial_F G) \rtimes_r G$ to a ucp map $\tilde{\pi}\colon C(\partial_F G) \rtimes_r G \to \mathcal{B}(\mathcal{H})$, which includes $C^*_{\mathrm{r}}(G)$ in its multiplicative domain as it restricts to a *-homomorphism on $C^*_{\mathrm{r}}(G)$.

Let E denote the C^*-algebra generated by $\tilde{\pi}(C(\partial_F G))$ inside $\mathcal{B}(\mathcal{H})$ and note that G acts on E with $g \in G$ acting by adjoining $\tilde{\pi}(\lambda_g)$. As $C(\partial_F G)$ is G-injective and both $C(\partial_F G)$ and E are unital, we may extend the embedding of $\mathbb{C}$ into $C(\partial_F G)$ along the embedding of $\mathbb{C}$ into E to a G-equivariant ucp map $\varphi\colon E \to C(\partial_F G)$. Then $\varphi \circ \tilde{\pi}$ is a G-equivariant ucp map $C(\partial_F G) \to C(\partial_F G)$ and therefore the identity by G-rigidity of $C(\partial_F G) \cong I_G(\mathbb{C})$ as it fixes $\mathbb{C}$ pointwise. Consequently, $\tilde{\pi}(C(\partial_F G))$ is contained in the multiplicative domain of φ as $\varphi \circ \tilde{\pi}|_{C(\partial_F G)}$ is a *-homomorphism and therefore φ itself is a *-homomorphism since $\tilde{\pi}(C(\partial_F G))$ generates its domain E. Hence $\ker(\varphi)$ is a closed, two-sided ideal of E and as φ is G-equivariant, its kernel is G-invariant. Let $F := \overline{\ker(\varphi) \cdot \pi(C^*_{\mathrm{r}}(G))}$ which sits inside the sub-C^*-algebra D of $\mathcal{B}(\mathcal{H})$ generated by $\tilde{\varphi}(C(\partial_F G) \rtimes_r G)$. It is clearly a closed, right-sided ideal by construction and since $\ker(\psi)$ is G-invariant, it is a two-sided ideal. If f_λ is an approximate identity of $\ker(\varphi)$ then $d \in D$ belongs to F if and only if $\lim_\lambda f_\lambda d = d$ and therefore $F \cap E = \ker(\varphi)$. Let Φ denote the quotient *-homomorphism $D \to D/F$ and note that φ by design factors through Φ to an injective *-homomorphism $\bar{\varphi}$. Since $\varphi \circ \tilde{\pi}|_{C(\partial_F G)}$ was the identity, so is $\bar{\varphi} \circ (\Phi \circ \tilde{\pi})|_{C(\partial_F G)}$, but now $\bar{\varphi}$ is an *injective* left inverse *-homomorphism to $\Phi \circ \tilde{\pi}$ as depicted in Diagram (2.10).

$$\begin{tikzcd}
C(\partial_F G) \arrow[r, "\tilde{\pi}"] \arrow[rr, bend left, "\mathrm{id}_{C(\partial_F G)}"] & D \arrow[r, "\varphi"] \arrow[d, "\Phi"] & C(\partial_F G) \\
& D/F \arrow[ur, hook, "\bar{\varphi}"'] &
\end{tikzcd} \tag{2.10}$$

This means that the a-priori ucp map $(\Phi \circ \tilde{\pi})|_{C(\partial_F G)}$ is a *-homomorphism, since for $f, f' \in C(\partial_F G)$ we have

$$\varphi \circ (\Phi \circ \tilde{\pi})(ff') = ff' = (\varphi \circ (\Phi \circ \tilde{\pi})(f))(\varphi \circ (\Phi \circ \tilde{\pi})(f')) = \varphi((\Phi \circ \tilde{\pi})(f)(\Phi \circ \tilde{\pi})(f'))$$

and φ is injective. On the other hand, $(\Phi \circ \tilde{\pi})|_{C^*_{\mathrm{r}}(G)} = \Phi \circ \pi$ is also a *-homomorphism and since $C(\partial_F G)$ and $C^*_{\mathrm{r}}(G)$ generate $C(\partial_F G) \rtimes_r G$, indeed $\Phi \circ \tilde{\pi}$ is a *-homomorphism. However, $\Phi \circ \tilde{\pi}$ can neither be zero, since it is injective on $C(\partial_F G)$, nor injective, since π is not injective on $C^*_{\mathrm{r}}(G)$, so we conclude that $\ker(\Phi \circ \tilde{\pi})$ is a non-trivial, closed, two-sided ideal of $C(\partial_F G) \rtimes_r G$. □

One advantage of the description of C^*-simplicity through simplicity of $C(\partial_F G) \rtimes_r G$ is that the action on the envelope has many special properties. The following is one of them: Let G_x denote the stabiliser subgroup that consists of exactly these elements $g \in G$ that fix x as $g.x = x$ for some $x \in \partial_F G$.

Lemma 2.3.4: *Let G be a discrete group and $\partial_F G$ its Furstenberg boundary. Then the stabiliser group G_x is amenable for every $x \in \partial_F G$.*

Proof. Note that the C^*-algebra $\ell^\infty(G)$ carries a G-action by acting on the argument. Hence we may extend the embedding $\mathbb{C} \hookrightarrow C(\partial_F G)$ along the embedding $\mathbb{C} \hookrightarrow \ell^\infty(G)$ to a G-equivariant ucp map $\varphi\colon \ell^\infty(G) \to C(\partial_F G)$ by G-injectivity of $C(\partial_F G)$. For $x \in \partial_F G$, the map $\mathrm{eval}_x \circ \varphi$ is a positive linear functional on $\ell^\infty(G)$ that is invariant under the action of G_x. This shows so-called *relative amenability* of $G_x \le G$ and by [16, Theorem 2] G_x is amenable, but we give a more direct proof of amenability of G_x by embedding $\ell^\infty(G_x)$ unitally and G_x-equivariantly into $\ell^\infty(G)$. To do so let $K \subseteq G$ be a choice of representing elements, one for each right coset in $G_x \backslash G = \{G_x g \mid g \in G\}$. Then every group element $g \in G$ can uniquely be written as $g = hk$ for $h \in G_x$ and $k \in K$ the representing element of $G_x g$ in K. The embedding $\iota\colon \ell^\infty(G_x) \hookrightarrow \ell^\infty(G)$ takes a function $f \in \ell^\infty(G_x)$ and copies it equivariantly on each of the disjoint G_x orbits of G, with the neutral element of G_x matched with the chosen representing element in K. That is, ι is given by $\iota(f)(g) = f(h)$ for $g \in G$ with $g = hk$ for $h \in G_x$ and $k \in K$ as above. Consequently, $\mathrm{eval}_x \circ \varphi \circ \iota$ is a left-invariant mean on $\ell^\infty(G_x)$ and we conclude that G_x is amenable by Definition 2.1.2. □

Lemma 2.3.5 (Kalantar–Kennedy [36, Remark 3.16], BKKO [9, Proposition 2.4]): *Let G be a discrete group and $\partial_F G$ its Furstenberg boundary. Then $\partial_F G$ is extremally disconnected, that is, the closure of every open subset of $\partial_F G$ is open.*

Proof. By Proposition 2.2.13, $C(\partial_F G)$ is an injective C^*-algebra and therefore monotone complete by Proposition 2.2.20. As remarked after Definition 2.2.19, it follows that $\partial_F G$ is extremally disconnected. □

In the following theorem we see how both extremal disconnectedness of $\partial_F G$ and amenability of the stabilisers of the G-action on $\partial_F G$ are helpful in characterising simplicity of $C(\partial_F G) \rtimes_r G$. The theorem and its proof use the concepts of *topological freeness* and *topological stabiliser G_x°*. We provide detailed definitions of these in the chapter on crossed products, see Definition 3.2.1 and Proposition 3.2.2, and therefore postpone most of the proof of Theorem 2.3.6 to Lemma 3.2.5.

Theorem 2.3.6 (KK [36, Theorem 6.2], BKKO [9, Theorem 3.1], O [48, Theorem 14]): *Let G be a discrete group and $\partial_F G$ its Furstenberg boundary. Then $C(\partial_F G) \rtimes_r G$ is simple if and only if the action of G on $\partial_F G$ is topologically free if and only if the action is free.*

Proof. By Lemma 2.3.4 all (topological) stabilisers of the action of G on $\partial_F G$ are amenable, hence Lemma 3.2.5 applies and we conclude that $C(\partial_F G) \rtimes_r G$ is simple if and only if the action is topologically free.

Furthermore, $\partial_F G$ is extremally disconnected by Lemma 2.3.5 and hence the fixed point set of any homeomorphism of ∂G is open by Frolík's theorem [24], [51, Proposition 2.7]. Consequently, if some $g \in G$ fixes a point $x \in \partial_F G$ then $y \mapsto g.y$ and hence g fix a whole neighbourhood of x pointwise. Therefore the action of G on $\partial_F G$ is free if and only if it is topologically free. □

2.3.2 Unique Trace

Further building on the description of the Furstenberg boundary as the spectrum of the equivariant injective envelope of the trivial action, Breuillard–Kalantar–Kennedy–Ozawa characterised another property of discrete groups closely related to C^*-simplicity: the unique trace property.

Definition 2.3.7: For a discrete group G the *canonical trace* τ on $C^*_{\mathrm{r}}(G)$ is

$$\tau\colon C^*_{\mathrm{r}}(G) \to \mathbb{C} \qquad \text{given by} \qquad \tau(x) = \langle x\delta_e, \delta_e\rangle_{\ell^2(G)}$$

for $x \in C^*_{\mathrm{r}}(G)$ with δ_e the canonical basis vector of $\ell^2(G)$ supported at the neutral element $e \in G$. A discrete group G is said to have the *unique trace property*, if the canonical trace is the only tracial state on $C^*_{\mathrm{r}}(G)$, that is, τ is the only positive linear functional of norm one on $C^*_{\mathrm{r}}(G)$ that satisfies $\tau(ab) = \tau(ba)$ for $a, b \in C^*_{\mathrm{r}}(G)$.

A non-trivial discrete amenable group N never has the unique trace property since the trivial representation of N on $\mathbb{C}$, in which every group element is sent to the unit, extends continuously to a character τ' on $C^*_{\mathrm{r}}(N)$. As $\tau'(\lambda_n) = 1$ for every $n \in N$, τ is not the canonical trace as long as N is non-trivial. Likewise, the existence of a non-trivial normal amenable subgroup $N \leq G$ of a discrete group G always gives rise to a non-canonical trace on $C^*_{\mathrm{r}}(G)$. Let τ' be the trace obtained from the trivial representation as above and consider the composition $\tau' \circ E_N$ with the conditional expectation E_N of $C^*_{\mathrm{r}}(G)$ onto $C^*_{\mathrm{r}}(N)$ sending $\lambda_g \in C^*_{\mathrm{r}}(G)$ to $\lambda_g \in C^*_{\mathrm{r}}(N)$ if $g \in N \subseteq G$ and to zero otherwise. For $g, h \in G$ we have $gh \in N$ if and only if $hg \in N$ as N is normal. Hence, $\tau' \circ E_N(\lambda_g\lambda_h) = 0 = \tau' \circ E_N(\lambda_h\lambda_g)$ if $gh \notin N$. On the other hand, if $gh \in N$ then

$$\tau' \circ E_N(\lambda_g\lambda_h) = \tau'(\lambda_{gh}) = 1 = \tau'(\lambda_{hg}) = \tau' \circ E_N(\lambda_h\lambda_g)$$

and we conclude that $\tau' \circ E_N$ is a trace on $C^*_{\mathrm{r}}(G)$ by linearity. Since $\tau' \circ E_N(\lambda_g) = 1$ for $g \in N \setminus \{e\}$, it is not the canonical trace.

Breuillard–Kalantar–Kennedy–Ozawa showed that absence of normal amenable subgroups is indeed sufficient to satisfy the unique trace property. The same statement was independently obtained by Haagerup [28], who gave a simpler proof without the need invoke the theory of injective envelopes.

Theorem 2.3.8 (BKKO [9, Corollary 4.3], Haagerup [28, Theorem 3.3]): *A discrete group G has the unique trace property if and only if its amenable radical is the trivial subgroup $\{e\}$.*

As a consequence, C^*-simplicity of a discrete group G implies the unique trace property with a proof due to de la Harpe [19, Proposition 3] which we sketch briefly: If G does *not* have the unique trace property, then by Theorem 2.3.8 we know that it contains a non-trivial normal amenable subgroup N. The trivial representation τ' of N induces a representation $\mathrm{Ind}_N^G \tau'$ of G which is weakly contained in the left regular representation of G, since τ' is weakly contained in the left regular representation of N (see [23, Chapter 6] and [22]). Working out the definition of $\mathrm{Ind}_N^G \pi$, it is the unitary representation of G on $\ell^2(G/N)$, sending $g \in \mathcal{G}$ to λ_{gN}. Since τ' is weakly contained in the left regular representation of G it extends continuously and linearly to a *-homomorphism $\pi\colon C^*_{\mathrm{r}}(G) \to C^*_{\mathrm{r}}(G/N) \subseteq \ell^2(G/N)$. The kernel of π is then a closed, two-sided ideal of $C^*_{\mathrm{r}}(G)$, which is not $\{0\}$ since $\lambda_n - \lambda_e \in \ker(\pi)$ for $n \in N \setminus \{e\}$ and not all of $C^*_{\mathrm{r}}(G)$ since $\pi(\lambda_e) = \lambda_{eN}$.

The converse, however, does not hold, with the first example of a non-C^*-simple discrete group with the unique trace property given by Le Boudec [41]. In the next section, we shed some light on exactly why .

2.3.3 Kennedy's Intrinsic Characterisation

While C^*-simplicity of a discrete group G is defined as an operator-algebraic property attached to $C^*_{\mathrm{r}}(G)$, Kalantar–Kennedy first gave a *dynamical* description when characterising it as freeness of the action on the Furstenberg boundary. In a further break-through result, Kennedy gave several *intrinsic* characterisations of C^*-simplicity in purely group-theoretic terms.

Let $\mathrm{Sub}(G)$ be the space of subgroups of G equipped with the topology of pointwise convergence, understanding a subgroup of G as an indicator function on G. Alternatively, but equivalently, think of $\mathrm{Sub}(G)$ as a subset of the power set $\{0, 1\}^G$ and equip it with the subspace topology of the product topology of $\{0, 1\}^G$. As $\mathrm{Sub}(G) \subseteq \{0, 1\}^G$ is closed, this shows that it is compact Hausdorff. This topology is otherwise known as the *Chabauty topology* or sometimes the *Fell topology*. We equip $\mathrm{Sub}(G)$ with the G-action by conjugation.

Definition 2.3.9: A *uniformly recurrent subgroup* of G is a minimal closed G-invariant subspace $Z \subseteq \mathrm{Sub}(G)$. It is called *amenable* if all subgroups contained in Z are amenable and *non-trivial* if $Z \neq \{\{e\}\}$.

Note that for a *discrete* group G amenability is a closed condition in $\mathrm{Sub}(G)$ with the Chabauty topology and therefore, by minimality, a uniformly recurrent subgroup Z is amenable if and only if all contained subgroups are amenable if and only if any contained subgroup is amenable. We prove this fact for later reference:

Proposition 2.3.10: Let G be a discrete group and $H_\lambda \in \mathrm{Sub}(G)$ a net of subgroups converging to some group $H \in \mathrm{Sub}(G)$ in the Chabauty topology. If all H_λ are amenable, then so is H.

Proof. Assume that the net H_λ is indexed by $\lambda \in \Lambda$ for a directed set Λ. Define

$$H'_\lambda := \bigcap_{\mu \in \Lambda : \lambda \leq \mu} H_\mu.$$

Then $H'_{\lambda_1} \leq H'_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$ and we consider the direct limit and claim that

$$H = \bigcup_{\lambda \in \Lambda} H'_\lambda.$$

Indeed, if $g \in G$ is not contained in H, then $g \notin H_\lambda$ eventually and therefore $g \notin H'_\lambda$ for all $\lambda \in \Lambda$. On the other hand, if $g \in H$, then g is eventually contained in H_λ, so there exists a

$\lambda \in \Lambda$ such that $g \in H_\mu$ for all $\mu \geq \lambda$ and therefore $g \in H'_\lambda$. Since amenability is closed under taking subgroups, all $H'_\lambda \leq H_\lambda$ are amenable and since it is closed under taking direct limits so is H. □

Theorem 2.3.11 (Kennedy [40, Theorem 4.1]): *A discrete group G is C^*-simple if and only if* $\mathrm{Sub}(G)$ *contains no non-trivial amenable uniformly recurrent subgroup.*

Proof. First suppose that Z is a non-trivial amenable uniformly recurrent subgroup of G and let $H \in Z$. Let $\mathcal{M}(\partial_F G)$ be the space of probability measures on the Furstenberg boundary $\partial_F G$ of G. As H is amenable, it acts amenably on $\partial_F G$ and so there is a measure $\mu \in \mathcal{M}(\partial_F G)$ which is invariant under the action of H. As $\partial_F G$ is a boundary, there is a net $g_\lambda \in G$ such that $\lim_\lambda g_\lambda.\mu = \delta_x$ for some $x \in \partial_F G$. Since $\mathrm{Sub}(G)$ is compact, we may assume that $g_\lambda.H$ converges to some subgroup $K \in Z$ after possibly passing to a subnet. In particular, we may assume that $\bigcap_\lambda g_\lambda.H \neq \{e\}$ after possibly passing to a second subnet. Indeed, if $\bigcap_\eta g_{\lambda_\eta}.H = \{e\}$ for all subnets g_{λ_η}, then $K = \{e\}$ which would make Z trivial by minimality. Now pick a non-trivial $h \in \bigcap_\lambda g_\lambda.H$. Then $g_\lambda^{-1} h g_\lambda.\mu = \mu$ for all λ, since $g_\lambda^{-1} h g_\lambda \in H$ and μ is H-invariant. Consequently, $hg_\lambda.\mu = g_\lambda.\mu$ and hence $h.\delta_x = \delta_x$ by taking limits. Therefore, $h \in G \setminus \{e\}$ fixes $x \in \partial_F G$, so the action of G on $\partial_F G$ is not free. We conclude that G is not C^*-simple by Theorem 2.3.6.

For the converse assume that G is not C^*-simple. We will show that the collection of stabilisers of the action on the Furstenberg boundary forms the desired uniformly recurrent subgroup. Indeed let $Z := \{G_x \mid x \in \partial_F G\}$. By Lemma 2.3.4, all subgroups contained in Z are amenable and since G is not C^*-simple, at least one subgroup contained in Z is not trivial. Since $g.G_x = G_{g.x}$, the subset Z is furthermore G-invariant and it remains to show that it is minimal and closed. Consider the map $\partial_F G \to \mathrm{Sub}(G)$ that assigns every point to its stabiliser $x \mapsto G_x$. Recall from the proof of Theorem 2.3.6 that Lemma 2.3.5 implies that every homeomorphism on $\partial_F G$ has open fixed point set. Hence for a convergent net $x_\lambda \to x$ in $\partial_F G$ we find that if $g \in G_x$ then $g \in G_{x_\lambda}$ eventually, as x_λ eventually enters the neighbourhood of x that g fixes pointwise. On the other hand, if $g \notin G_x$, then there is no subnet x_{λ_η} of x_λ such that $g \in G_{x_{\lambda_\eta}}$ for all η, since otherwise $g.x_{\lambda_\eta} = x_{\lambda_\eta}$ implies $g.x = x$. Hence $g \notin G_{x_\lambda}$ eventually, and we conclude that G_{x_λ} converges to G_x in the Chabauty topology. Consequently, the map $x \mapsto G_x$ is a continuous, G-equivariant map $\partial_F G \to \mathrm{Sub}(G)$ with image Z and we conclude that Z is compact and minimal. □

To translate Theorem 2.3.11 into a characterisation that is truly only in elementary group-theoretic terms, consider the following definition:

Definition 2.3.12: Let G be a discrete group and $H \leq G$ a subgroup. Then H is called *recurrent* if and only if there is a finite set $F \subseteq G \setminus \{e\}$ such that F intersects every subgroup gHg^{-1} conjugate to H.

Remark. Kennedy has since changed his terminology to call these subgroups *residually normal*, but we prefer the previous term and continue to call them *recurrent*.

Lemma 2.3.13: *Let H be a subgroup of a discrete group G. Then H is recurrent if and only if the trivial subgroup $\{e\}$ is not contained in the orbit closure of H inside* $\mathrm{Sub}(G)$.

Proof. Assume that H is recurrent, so there is a finite set $F \subseteq G \setminus \{e\}$ that intersects every conjugate of H. If there was a net of conjugates of $g_\lambda H g_\lambda^{-1}$ approximating $\{e\}$, then $f \notin g_\lambda H g_\lambda^{-1}$ eventually for all $f \in F$, contradicting the assumption on F.

Conversely, assume that H is not recurrent, so for every finite set $F \subseteq G \setminus \{e\}$ there exists some $g_F \in G$ such that $g_F H g_F^{-1}$ does not intersect F. Note that the set $\mathcal{F}$ of finite subsets of $G \setminus \{e\}$ is directed with respect to the inclusion $\subseteq$ since two sets $F, F' \in \mathcal{F}$ have upper bound $F \cup F' \in \mathcal{F}$. Then $\lim_{F \in \mathcal{F}} g_F H g_F^{-1} = \{e\}$, since every group element $g \neq e$ is eventually not contained in $g_F H g_F^{-1}$ and we conclude that $\{e\}$ is contained in the orbit closure of H. □

Lemma 2.3.14: *Let X be a compact space with G-action. There is a non-empty, closed, G-invariant subset $Y \subseteq X$ such that the G-action on Y is minimal.*

We are not aware of the origin of Lemma 2.3.14 and consider it folklore.

Proof. Conider the collection $\mathcal{K}$ of all non-empty, closed, G-invariant subsets of X, ordered by inclusion. Let $\mathcal{S} \subseteq \mathcal{K}$ be a totally ordered family of such subsets. As it is totally ordered, every finite choice of subsets $S_1, \ldots, S_n$ in S has a lower bound; that is, one of the subsets S_i is contained in all other subsets S_j for $j = 1, \ldots, n$. Hence $\bigcap_{j=1}^n S_j = S_i$ is non-empty, so $\mathcal{S}$ has the finite intersection property and since X is compact we conclude that

$$Z := \bigcap_{S \in \mathcal{S}} S$$

is non-empty. Clearly, Z is closed and G-invariant as an intersection of closed, G-invariant sets and hence it forms a lower bound of $\mathcal{S}$ in $\mathcal{K}$.

As every totally ordered family in $\mathcal{K}$ has a lower bound, by Zorn's lemma there is a minimal element of $\mathcal{K}$, say $Y \in \mathcal{K}$. Then Y is a non-empty, closed, G-invariant subset of X whose only closed, G-invariant subsets are $\emptyset$ and Y itself. As for arbitrary $x \in Y$ the orbit closure of x is a non-empty, closed, G-invariant subset of Y, it must be all of Y and so every $x \in Y$ has dense orbit in Y. Hence Y is minimal as desired. □

Lemma 2.3.15 (Kennedy [40, Section 5]): *A discrete group G has a recurrent amenable subgroup H if and only if it has a non-trivial amenable uniformly recurrent subgroup Z.*

Proof. Let H be a recurrent amenable subgroup of G and consider the orbit closure $\overline{G.H}$ of H in $\mathrm{Sub}(G)$. Then the trivial subgroup $\{e\}$ is not contained in $\overline{G.H}$ by Lemma 2.3.13. As H is amenable and amenability is a closed condition in $\mathrm{Sub}(G)$ by Proposition 2.3.10, $\overline{G.H}$ is a compact G-invariant subspace of $\mathrm{Sub}(G)$ containing only amenable groups. By Lemma 2.3.14, $\overline{G.H}$ contains a (non-empty) minimal closed G-invariant subspace and since $\{e\} \notin \overline{G.H}$ this subspace forms a non-trivial amenable uniformly recurrent subgroup of G.

On the other hand, assume that $Z \subseteq \mathrm{Sub}(G)$ is a non-trivial amenable uniformly recurrent subgroup of G and pick any $H \in Z$. Then H is amenable and $\{e\}$ is not contained in the orbit closure $\overline{G.H}$ of H since otherwise $\{e\} \in Z$ but hence $Z = \{\{e\}\}$ by minimality contradicting non-triviality. So H is recurrent by Lemma 2.3.13. □

Theorem 2.3.16 (Kennedy [40, Theorem 5.3]): *A discrete group G is C^*-simple if and only if it contains no amenable recurrent subgroups.*

Proof. This is a simple combination of Theorem 2.3.11 and Lemma 2.3.15. □

Crossed Products

We turn our focus to C^*-simplicity of crossed products. While group C^*-algebras already provide a wide variety of examples, an even larger class of examples is provided by the crossed product construction, with only a mild increase in complexity. After recalling the basic theory in Section 3.1 we turn to simplicity of reduced crossed products in Sections 3.2 and 3.3.

3.1 The Reduced Crossed Product

Consider a compact Hausdorff topological space X and a discrete group G. An action of G on X is a group homomorphism φ from G into the group $\mathrm{Homeo}(X)$ of homeomorphisms from X to X, that is, the group of continuous bijections $X \to X$ (with continuous inverse). Via precomposition with the inverse, φ can be understood as an action of G on $C(X)$, the C^*-algebra of complex-valued continuous functions on X; that is, a group homomorphism from G into the automorphism group $\mathrm{Aut}(C(X))$ consisting of automorphisms $C(X) \to C(X)$. In more precise terms, for a group element $g \in G$ and a function $f \in C(X)$ the action φ^* of G on $C(X)$ is given by

$$\varphi^*(g)(f)(x) = f(\varphi(g^{-1})(x)) \tag{3.1}$$

for $x \in X$. We will often drop φ and φ^* from our notation in favour of the abbreviated $g.x := \varphi(g)(x)$. In this notation, Equation (3.1) becomes

$$(g.f)(x) := f(g^{-1}.x).$$

Recall that the left-regular representation λ_G of G on $\ell^2(G)$ gives rise to the reduced group C^*-algebra by completion of the group ring $\mathbb{C}[G]$ of G with coefficients in $\mathbb{C}$ in the associated norm. To construct a C^*-algebra associated with the action of G on $C(X)$, consider the group ring $C(X)[G]$ of G with coefficients in $C(X)$ instead, that is, the ring of finitely supported functions on G taking values in $C(X)$ with pointwise addition and multiplication derived from

$$(f\lambda_g)(f'\lambda_{g'}) = f(g.f')\lambda_{gg'},$$

where for $f, f' \in C(X)$ and $g, g' \in G$ we denote by $f\lambda_g$ the function from G to $C(X)$ supported only on $g \in G$ where it takes the value f. An involution on $C(X)[G]$ is defined by

$$(f\lambda_g)^* = (g^{-1}.f^*)\lambda_{g^{-1}}$$

with f^* the involution in $C(X)$ given by pointwise complex conjugation. Note that $C(X)[G]$ contains a copy of $C(X)$ as a sub-*-algebra by identifying a function f with $f\lambda_e$ for e the neutral element of G. As we restricted ourselves to X being compact so that $C(X)$ is unital with unit $\mathbb{1}$, we may also identify a copy of $\mathbb{C}[G]$ by sending λ_g to $\mathbb{1}\lambda_g$ and $\mathbb{1}\lambda_e$ is the identity of $C(X)[G]$. In that case, the elements λ_g are unitaries implementing the action of G on $C(X)$ as inner automorphism of $C(X)[G]$, since

$$\lambda_g f \lambda_{g^{-1}} = (g.f)\lambda_g\lambda_{g^{-1}} = (g.f)\lambda_e.$$

Let π be a faithful representation of $C(X)$ on some Hilbert space $\mathcal{H}$. We denote by $\pi \rtimes \lambda_G$ the (left-)regular representation of $C(X)[G]$ on $\mathcal{B}(\mathcal{H}) \otimes \ell^2(G)$ associated with π. It is given by

$$(\pi \rtimes \lambda_G)(f\lambda_g)(\xi \otimes \delta_h) = (\pi(h^{-1}g^{-1}.f)\xi) \otimes \delta_{gh}$$

for $\xi \in \mathcal{H}$ and δ_h the canonical basis vector of $\ell^2(G)$ supported in $h \in G$.

Definition 3.1.1: Let X be a compact Hausdorff topological space and G a discrete group. Choose a faithful representation π of $C(X)$ on some Hilbert space. The *reduced crossed product* $C(X) \rtimes_r G$ is the completion of $C(X)[G]$ with respect to the norm induced by the regular representation $\pi \rtimes \lambda_G$.

Indeed, $C(X) \rtimes_r G$ forms a C^*-algebra and the construction is independent of the choice of faithful representation π, see e.g. [12, Proposition 4.1.5] or [61, Chapter 7.2]. As in the construction of group C^*-algebras, there is a notion of universal crossed product $C(X) \rtimes G$, which we will only rarely use in this thesis.

The reduced crossed product construction generalises the construction of the reduced group C^*-algebra: In the case of X consisting of a single point fixed by the action of G, $C(X)$ specialises to $\mathbb{C}$ with the trivial action of G in which every group element acts neutrally. Hence $C(X)[G]$ matches the group ring $\mathbb{C}[G]$ and the regular representation $\pi \rtimes \lambda_G$ will be equivalent to λ_G, giving rise to $C^*_{\mathrm{r}}(G)$ as $\mathbb{C} \rtimes_r G$.

On the other hand, there are several further generalisations of reduced crossed products. While we have notationally restricted ourselves to the case of a group acting on a unital commutative C^*-algebra $C(X)$, there is an obvious generalisation to actions on a not necessarily commutative, not necessarily unital C^*-algebra A replacing $C(X)$. One possible generalisation of the results in this chapter to such crossed products has been investigated by Bryder [13]. Keeping the commutativity of the algebra, but relaxing the notion of a "group" acting, the reduced C^*-algebras of locally compact Hausdorff étale groupoids are a generalisation in a different direction, which we will describe in detail in Section 4.1.

An important tool in the study of reduced crossed products are conditional expectations as defined in Definition 2.1.10. In particular, the *canonical conditional expectation* E_X from $C(X) \rtimes_r G$ to $C(X)$ is obtained as in the group case by extending evaluation at the neutral element of the group, that is,

$$E_X(f\lambda_g) = \begin{cases} f & \text{if } g = e, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

and extended by linearity. Identifying $C(X) \rtimes_r G$ with the closure of the image of $C(X)[G]$ in $\mathcal{B}(\mathcal{H}) \otimes \ell^2(G)$ under the regular representation $\pi \rtimes \lambda_G$, E_X is given by compression with the projection $\mathbb{1} \otimes |\delta_e\rangle\langle\delta_e|$ and this shows that it is continuous on $C(X) \rtimes_r G$ and ucp. Noting furthermore that $C(X) \rtimes_r G$ is even a subalgebra of $\mathcal{B}(\mathcal{H}) \otimes C^*_{\mathrm{r}}(G) \subseteq \mathcal{B}(\mathcal{H}) \otimes \ell^2(G)$, where E_X is given by $\mathbb{1} \otimes \tau$ for τ the faithful canonical expectation of $C^*_{\mathrm{r}}(G)$ shows that E_X is faithful, see further [12, Lemma 4.1.9]. In addition, we will make use of conditional expectations onto the algebras of the stabiliser subgroups: For a discrete group G acting on a compact Hausdorff space X, the *stabiliser* G_x of a point $x \in X$ is the subgroup of G consisting of all group elements fixing x. As for any subgroup, there is a conditional expectation E_{G_x} from $C^*_{\mathrm{r}}(G)$ onto $C^*_{\mathrm{r}}(G_x)$ given by cutting the representation on $\ell^2(G)$ down with the projection onto $\ell^2(G_x)$, that is

$$E_{G_x}(\lambda_g) = \begin{cases} \lambda_g & \text{if } g \in G_x, \\ 0 & \text{else,} \end{cases}$$

and extended by linearity. Pre-composed with evaluation at x, this gives rise to a conditional expectation E_x of the crossed product onto $C^*_{\mathrm{r}}(G_x)$ given by

$$E_x(f\lambda_g) = f(x)E_{G_x}(\lambda_g) \tag{3.3}$$

and extended by linearity. For a more detailed treatment of crossed products, the reader is advised to consult the books by Brown–Ozawa [12] or Williams [61].

3.2 Early Work on Crossed Product Simplicity

The first observation concerning simplicity of reduced crossed products is that ideals of the algebra acted upon that are invariant under the group action give rise to ideals in the crossed product. In the case of a commutative, unital C^*-algebra $C(X)$, closed, two-sided ideals correspond to closed subsets K of the compact space X, with the ideal $I_K \lhd C(X)$ given by $I_K := \{f \in C(X) \mid f|_K \equiv 0\}$. The ideal I_K is invariant under the G-action on $C(X)$ if and only if K is invariant under the G-action on X. Then $I_K \rtimes_r G$, seen as a subset of $C(X) \rtimes_r G$, is an ideal in the reduced crossed product, and clearly proper if and only if I_K is a proper ideal of $C(X)$. If X has no closed G-invariant subsets, or equivalently if $C(X)$ has no proper G-invariant closed ideals, the action of G on X is called *minimal*. While the crossed product associated with a minimal action does not have any proper ideals coming from $C(X)$ in this way, it might still have other proper ideals. The easiest example of a minimal non-simple crossed product is any non-C^*-simple discrete group acting trivially on a one-point space.

Given a closed, two-sided ideal J of $C(X) \rtimes_r G$, its intersection with $C(X)$ is again a closed, two-sided ideal of $C(X)$ and is furthermore invariant, as the G-action on $C(X)$ can be implemented by conjugation with the associated unitaries λ_g if X is compact. The action of G on $C(X)$ is said to have the *intersection property*, coined by Svensson and Tomiyama [58], if this intersection $J \cap C(X)$ is non-zero for every non-zero ideal J of $C(X) \rtimes_r G$. It immediately follows that the crossed product of a minimal action is simple if and only if the action has the intersection property, and as minimality is well-understood, we focus our further efforts on describing the intersection property.

As a first experiment, note that in the case of the reduced C^*-algebra of a group G, thought of as the crossed product of G acting trivially on a single point, the intersection property is simply stating that every non-zero ideal of $C^*_{\mathrm{r}}(G)$ contains a non-zero scalar and is therefore all of $C^*_{\mathrm{r}}(G)$.

Next, we make the case that every action which is "free enough" has the intersection property. We introduce some common notation:

Definition 3.2.1: Let G be a discrete group acting on a space X. The *topological stabiliser* G°_x of a point $x \in X$ is the subgroup of G consisting of all group elements for which there is a neighbourhood of x that they are fixing pointwise. The *fixed point set* X^g of a group element $g \in G$ is the set of all points in X which g fixes. In other words,

$$\begin{aligned} G_x &:= \{g \in G \mid g.x = x\}, \\ G^\circ_x &:= \{g \in G \mid \exists U \subseteq X \text{ s.th. } x \in U^\circ \text{ and } g.y = y \,\forall y \in U\}, \\ X^g &:= \{x \in X \mid g.x = x\}. \end{aligned}$$

The action is called *topologically free* if all topological stabilisers G°_x for $x \in X$ are trivial.

It is important to note that both the stabiliser and the topological stabiliser translate nicely under the group action:

$$G_{g.x} = gG_xg^{-1} \qquad \text{and} \qquad G^\circ_{g.x} = gG^\circ_xg^{-1}$$

for $g \in G$ and $x \in X$, since the action of G is by homeomorphism and therefore transports open sets to open sets. There are several different definitions of topological freeness in the literature, but in the case of a countable discrete group acting on a space, most of them are equivalent:

Proposition 3.2.2: Let G be a countable discrete group acting on a compact Hausdorff space X. The following are equivalent:

1) The set of points $x \in X$ with trivial stabiliser G_x is dense in X.

2) The action is topologically free in the sense of Definition 3.2.1.

3) For any $g \in G \setminus \{e\}$ the set of points X^g fixed by g has empty interior.

Proof. 1) $\Rightarrow$ 2): If a dense set of points has trivial stabiliser, every open subset of X will contain a point with trivial stabiliser, and so every topological stabiliser has to be trivial.

2) $\Rightarrow$ 3): If X^g does not have empty interior for some nontrivial $g \in G$, then it contains a neighbourhood of some x and clearly $e \neq g \in G^\circ_x$.

3) $\Rightarrow$ 1): As X^g is closed and has empty interior, $X \setminus X^g$ is an open, dense subset of X. If G is countable, then by the Baire category theorem [54, Theorem 2.2], $\bigcap_{g \in G \setminus \{e\}} X \setminus X^g$ will again be dense in X and is exactly the set of points with trivial stabiliser. □

Note that 2) and 3) are equivalent even without the countability assumption on G, since condition 3) prevents any non-trivial element of G from fixing a non-empty open set pointwise.

By an early result of Archbold and Spielberg [3], every topologically free action of a discrete group G on a compact Hausdorff space X has the intersection property. Instead of giving their proof, we provide a stronger result given by Ozawa [48, Theorem 14 (1)] in the minimal and Kawabe [38, Lemma 2.4 (i)] in the general setting.

Lemma 3.2.3 (Kawabe–Ozawa): *Let G be a discrete group acting on a compact Hausdorff space X such that the set of points x with C^*-simple stabiliser G_x is dense. Then the action has the intersection property. In particular, if the action is minimal and has at least one C^*-simple stabiliser then the associated crossed product $C(X) \rtimes_r G$ is simple.*

Note that the statement also follows as a special case from our later results of Corollary 4.2.29 in the étale groupoid setting.

We elaborate on the proofs from [38, Lemma 2.4 (i)] and [48, Theorem 14 (1)]:

Proof. Assume that the set of points with C^*-simple stabiliser is dense in X. Let I be an ideal of $C(X) \rtimes_r G$ such that $I \cap C(X) = \{0\}$. We want to show that $I = \{0\}$. Let us momentarily fix an $x \in X$ such that $C^*_{\mathrm{r}}(G_x)$ is C^*-simple. As the conditional expectation E_x is the identity on $C^*_{\mathrm{r}}(G_x)$ seen as a subalgebra of $C(X) \rtimes_r G$, this subalgebra is contained in the multiplicative domain of E_x and therefore $E_x(I)$ is again a two-sided ideal of $C^*_{\mathrm{r}}(G_x)$, although a-priori not necessarily closed. As $C^*_{\mathrm{r}}(G_x)$ is simple and unital, it is algebraically simple and therefore $E_x(I)$ has to be either zero or all of $C^*_{\mathrm{r}}(G_x)$.

Consider the sub-C^*-algebra $C(X) + I$ of $C(X) \rtimes_r G$. As $C(X) \cap I = \{0\}$, the quotient $(C(X) + I)/I \cong C(X)/(C(X) \cap I)$ is isomorphic to $C(X)$ and by evaluating at x we obtain a *-homomorphism

$$C(X) + I \xrightarrow{[\cdot]_I} (C(X) + I)/I \cong C(X) \xrightarrow{\mathrm{eval}_x} \mathbb{C}$$

which extends to a state $\varphi \colon C(X) \rtimes_r G \to \mathbb{C}$ by Hahn-Banach [54, Theorem 3.3]. Then φ vanishes on I, while its multiplicative domain contains $C(X)$, where it is given by evaluation at x. Hence, for $f \in C(X)$ and $g \in G$ we obtain

$$f(x)\varphi(\lambda_g) = \varphi(f\lambda_g) = \varphi(\lambda_g(g^{-1}.f)) = \varphi(\lambda_g)(g^{-1}.f)(x) = f(g.x)\varphi(\lambda_g).$$

If $g \notin G_x$, then we may choose $f' \in C(X)$ such that it separates x and $g.x$. From the above statement $(f'(x) - f'(g.x))\varphi(\lambda_g) = 0$ we can therefore conclude that $\varphi(\lambda_g) = 0$ if $g \notin G_x$ and hence $\varphi(E_x(f\lambda_g)) = 0 = \varphi(f\lambda_g)$ for arbitrary $f \in C(X)$. On the other hand, if $g \in G_x$ then

$$\varphi(E_x(f\lambda_g)) = \varphi(f(x)\lambda_g) = f(x)\varphi(\lambda_g) = \varphi(f)\varphi(\lambda_g) = \varphi(f\lambda_g)$$

again using that $C(X)$ is contained in the multiplicative domain of φ. As elements of the form $f\lambda_g$ span $C(X) \rtimes_r G$ densely, we conclude that $\varphi = \varphi \circ E_x$ and hence $\varphi(E_x(I)) = \varphi(I) = \{0\}$. Since $\varphi(\lambda_e) = 1$, λ_e cannot be contained in $E_x(I) \subseteq C^*_{\mathrm{r}}(G_x)$ and hence $E_x(I) \neq C^*_{\mathrm{r}}(G_x)$ so that it must have been zero.

Let $S = \{x \in X \mid C^*_{\mathrm{r}}(G_x) \text{ is } C^*\text{-simple}\}$ be the set of points with C^*-simple stabiliser, which is dense in X by assumption. We have just shown that $E_x(I) = \{0\}$ for $x \in S$. Let a be any element of I and note that a^*a is a positive element contained in I. Since $\mathrm{eval}_x \circ E_X = \tau_{G_x} \circ E_x$ with τ_{G_x} the canonical trace of $C^*_{\mathrm{r}}(G_x)$ and E_X the canonical conditional expectation of $C(X) \rtimes_r G$ onto $C(X)$, we see that $E_X(a^*a) \in C(X)$ vanishes on the dense set $S \subseteq X$ and therefore $E_X(a^*a) = 0$ by continuity so that $a^*a = 0$ and therefore $a = 0$ by faithfulness of E_X. Hence $I = \{0\}$.

If the action is minimal and has a single C^*-simple stabiliser, then all stabilisers on the same orbit are also C^*-simple and as the orbit is dense, the crossed product is simple. □

Corollary 3.2.4: *Let G be a (not necessarily countable) discrete group acting on a compact Hausdorff space X. If the action is topologically free, it has the intersection property. If it is furthermore minimal, the associated crossed product is simple.*

Proof. Assume that the action of G on X is topologically free and let I be an ideal of $C(X) \rtimes_r G$ such that $I \cap C(X) = \{0\}$. Let a be any element of I and choose a *countable* subgroup H of G such that a is contained in $C(X) \rtimes_r H$, which is possible since $C(X) \rtimes_r G$ is the closure of $C(X)[G]$ whose elements are finitely supported. As the action of H on X is likewise topologically free, characterisation 1) of Proposition 3.2.2 tells us that there is a dense set of points $x \in X$ whose stabiliser G_x intersects the countable subgroup H trivially. Therefore, the action of H on X has a dense set of points with trivial, hence C^*-simple, stabiliser. Applying the above Lemma 3.2.3 to $I \cap C(X) \rtimes_r H \lhd C(X) \rtimes_r H$ shows that $a \in I \cap C(X) \rtimes_r H$ vanishes and since a was arbitrary, we once again conclude that $I = \{0\}$. □

Topological freeness is, as Lemma 3.2.3 suggests, a rather strong condition. In the special case of group C^*-algebras where the group is acting on a single point, the action is topologically free if and only if the group is trivial and the lemma above degenerates to the trivial statement that $C^*_{\mathrm{r}}(G)$ is simple if G is C^*-simple. Notably, however, Kawamura and Tomiyama proved in [39, Theorem 4.1] a partial converse: If the group acting is *amenable* then the action has the intersection property if and only if it is topologically free. We provide their statement in the slightly more general version given by Kawabe [38, Lemma 2.4 (ii)] and Ozawa [48, Theorem 14 (2)]:

Lemma 3.2.5 (Kawabe–Ozawa): *Let G be a discrete group acting on a compact Hausdorff space X such that the set of points x with amenable topological stabiliser G_x° is dense. The action has the intersection property, if and only it is topologically free.*

In particular, if the action is minimal and has at least one amenable topological stabiliser then the associated crossed product $C(X) \rtimes_r G$ is simple if and only if the action is topologically free.

We elaborate on the proofs from [38, Lemma 2.4 (ii)] and [48, Theorem 14 (2)]:

Proof. Using Corollary 3.2.4, we only need to show that the action is topologically free if it has the intersection property. Let X be as in the statement and assume that the action has the intersection property. For $x \in X$ such that G_x° is amenable consider the representation π_x of the universal crossed product $C(X) \rtimes G$ on $\ell^2(G/G_x^\circ)$ given by $\pi_x(f\lambda_g)\delta_{hG_x^\circ} = f(gh.x)\delta_{ghG_x^\circ}$ and extended linearly and continuously to all of $C(X) \rtimes G$. Let $\varphi\colon C(X) \rtimes G \to C(X) \rtimes_r G$ be the canonical surjective *-homomorphism. The amenability of G_x° ensures that π_x factors through φ to a representation of the reduced crossed product $C(X) \rtimes_r G$: First, a quick calculation shows that

$$\langle \pi(a)\delta_{eG_x^\circ}, \delta_{eG_x^\circ}\rangle = \tau_{G_x^\circ} \circ E_{G_x^\circ} \circ E_x \circ \varphi$$

for $a \in C(X) \rtimes G$ with $\tau_{G_x^\circ}$ the trivial representation of G_x°, which extends continuously to $C^*_{\mathrm{r}}(G_x^\circ)$ since the topological stabiliser is amenable, $E_{G_x^\circ}$ the conditional expectation of $C^*_{\mathrm{r}}(G_x)$ onto $C^*_{\mathrm{r}}(G_x^\circ)$ defined in Equation 2.2, and E_x the conditional expectation of $C(X) \rtimes_r G$ onto $C^*_{\mathrm{r}}(G_x)$ defined in Equation 3.3. Hence, if $a \in \ker(\varphi)$ then

$$\langle \pi(a)\delta_{gG_x^\circ}, \delta_{hG_x^\circ}\rangle = \langle \pi(\lambda_{h^{-1}} a \lambda_g)\delta_{eG_x^\circ}, \delta_{eG_x^\circ}\rangle = 0$$

since $\lambda_{h^{-1}} a \lambda_g \in \ker(\varphi)$ and consequently $\pi(a) \in \mathcal{B}(\ell^2(G/G_x^\circ))$ vanishes. We conclude that π_x factors through φ to a continuous representation of $C(X) \rtimes_r G$ on $\ell^2(G/G_x^\circ)$, which we again denote by π_x.

We denote the dense set of $x \in X$ with amenable topological stabiliser G_x° by $S \subset X$ and consider the representation $\pi = \bigoplus_{x \in S} \pi_x$. Note that $\pi_x(f\lambda_e)\delta_{eG_x^\circ} = f(x)\delta_{eG_x^\circ}$, so $\pi(f\lambda_e) \neq 0$ for $f \neq 0$ and hence $\ker(\pi) \cap C(X) = \{0\}$. By the intersection property, π is therefore faithful. Assume that X^g has non-empty interior for some $g \in G$ and pick a non-empty open set U contained in X^g. Clearly, $g.x = x$ for all $x \in U$. Pick a non-zero $f \in C(X)$ supported on U. Note that since g fixes U pointwise, so does g^{-1} and therefore $g.x$ is contained in U if and only if x already is. Hence $g.f = f$ since either $g.x = x$ if $x \in U$ or $f(g.x) = 0 = f(x)$ if $x \notin U$. Then for $x \in S$ we find

$$\begin{aligned} \pi_x(f(\lambda_g - \lambda_e))\delta_{hG_x^\circ} &= f(gh.x)\delta_{ghG_x^\circ} - f(h.x)\delta_{hG_x^\circ} \\ &= f(h.x)\Big(\delta_{ghG_x^\circ} - \delta_{hG_x^\circ}\Big) \end{aligned}$$

For h such that $h.x$ is not contained in U, the term vanishes as f vanishes. If on the other hand $h.x \in U$, then note that $g \in G_{h.x}^\circ = hG_x^\circ h^{-1}$ since U is a neighbourhood of $h.x$ and therefore $gh \in hG_x^\circ$, so $\delta_{ghG_x^\circ} = \delta_{hG_x^\circ}$ and the term above vanishes again. Hence $\pi_x(f(\lambda_g - \lambda_e)) = 0$ for any $x \in S$ and by faithfulness of π, we conclude that $f(\lambda_g - \lambda_e)$ vanishes. As f was non-zero, g must have been the neutral element e. Since $g \in G$ was chosen arbitrarily such that X^g has non-empty interior, we conclude topological freeness by condition 3) of Proposition 3.2.2.

If the action has a single amenable topological stabiliser, then all topological stabilisers on the same orbit are likewise amenable, since they are conjugate to each other. Hence our lemma applies to a minimal action with at least one amenable topological stabiliser, as every orbit is dense. As, in the minimal case, the intersection property is furthermore equivalent to simplicity of the crossed product, the second claim follows. □

3.3 Kawabe's Simplicity Characterisation

We take this opportunity to present Kawabe's results in our own language of Section 4.2.4. Recall that the compact *Chabauty topology* on the space of subgroups Sub(G) of a given discrete group G is given by the topology of pointwise convergence of the characteristic functions associated with the subgroups. For Kennedy's characterisation of C^*-simplicity of discrete groups it was essential in defining the notion of a *recurrent* subgroup. To describe the intersection property of crossed products coming from discrete groups acting on compact spaces, we will provide a generalised notion of recurrent subgroup: For a discrete group G acting on a compact space X, let $\mathrm{Sub}(X, G)$ denote the space $\{(x, H) \in X \times \mathrm{Sub}(G) \mid H \leq G_x\}$ of subgroups of the stabiliser groups together with a base point, seen as a subspace of the product space $X \times \mathrm{Sub}(G)$. Clearly, G acts on $X \times \mathrm{Sub}(G)$ component-wise via its action on X and conjugation of subgroups of G. Since the stabiliser subgroups are closed under conjugation as $gG_xg^{-1} = G_{g.x}$ for any $g \in G$ and $x \in X$, this action restricts to $\mathrm{Sub}(X, G)$.

Definition 3.3.1: Let G be a discrete group acting on a compact space X. Let H be a subgroup of the stabiliser G_x at some $x \in X$. We call H a *dynamically recurrent* subgroup of G_x, if $(x, \{e\})$ is not contained in the orbit closure of (x, H) in $\mathrm{Sub}(X, G)$. That is, if there is no net g_λ of group elements such that $g_\lambda.x \to x$ in X, while $g_\lambda H g_\lambda^{-1} \to \{e\}$ in $\mathrm{Sub}(G)$.

Furthermore, choosing a subgroup H_x of G_x at every $x \in X$, we call $\Lambda = \{(x, H_x) \mid x \in X\}$ a *section of stabiliser subgroups*. We call such a section *recurrent*, if its orbit closure does not contain all trivial subgroups, that is, if there is $x_0 \in X$ such that $(x_0, \{e\})$ cannot be approximated by any net $g_\lambda.(x_\lambda, H_{x_\lambda})$ for $(x_\lambda, H_{x_\lambda}) \in \Lambda$ and $g_\lambda \in G$.

Our terminology for *dynamically recurrent* was inspired by the webcomic of Figure 3.1.

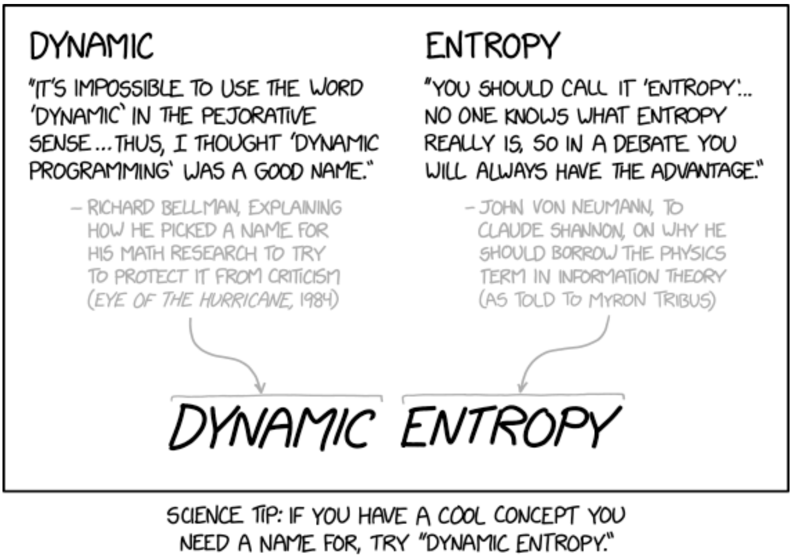

Figure 3.1: "Despite years of effort by my physics professors to normalize it, deep down I remain convinced that 'dynamical' is not really a word". Available at `xkcd.com/2318`.

Just as Kalantar–Kennedy's characterizations of C^*-simplicity of discrete groups, Kawabe's results rely heavily on the use of Hamana's injective G-equivariant envelopes. Recall that the Furstenberg boundary of a group G was G-equivariantly homeomorphic to the spectrum of the injective G-equivariant envelope of the trivial action of G on a single point.

Let X be a compact Hausdorff space with an action of a discrete group G. The G-injective envelope of the commutative unital C^*-algebra $C(X)$ with G-action is again a commutative unital C^*-algebra with a G-action. We denote its spectrum by $\tilde{X}$. As $C(X)$ is identified with a unital G-invariant C^*-subalgebra of $C(\tilde{X})$, every pure state on $C(\tilde{X})$ restricts to a unique pure state on $C(X)$ and we obtain a continuous surjection $q\colon \tilde{X} \to X$ determined by $\mathrm{eval}_{\tilde{x}}|_{C(X)} = \mathrm{eval}_{q(\tilde{x})}$ for $\tilde{x} \in \tilde{X}$. Since the inclusion is G-equivariant and $\mathrm{eval}_{g.x}(f) = \mathrm{eval}_x(g^{-1}.f)$, the surjection q is likewise G-equivariant.

The key observation in Kawabe's work is a generalisation of Theorem 2.3.1, namely that the action of G on X has the intersection property, if and only if the action of G on $\tilde{X}$ does:

Theorem 3.3.2 (Kawabe [38, Theorem 3.4]): *Given a compact Hausdorff space X with an action of a discrete group G, let $\tilde{X}$ be the spectrum of the G-equivariant injective envelope of $C(X)$, equipped with its associated G-action. The action of G on $C(X)$ has the intersection property if and only if the action of G on $C(\tilde{X})$ does.*

The proof is surprisingly elementary, leveraging only the definitions of the intersection property and the G-equivariant injective envelope themselves. Below, we elaborate on the proof given in [38, Theorem 3.4], while preparing the language we will later use in the proofs of Section 4.2.4. The second half of the proof is not remarkably different from the proof of the same implication for Theorem 2.3.1.

Proof. We first show that the action of G on the envelope $C(\tilde{X})$ has the intersection property, if the action on $C(X)$ does.

Suppose that the action of G on $C(X)$ has the intersection property and that I is a closed, two-sided ideal of $C(\tilde{X}) \rtimes_r G$ such that $I \cap C(\tilde{X}) = \{0\}$. Let $\pi\colon C(\tilde{X}) \rtimes_r G \to (C(\tilde{X}) \rtimes_r G)/I$ be the associated quotient *-homomorphism with kernel I. Treating $C(X) \rtimes_r G$ as a subalgebra of $C(\tilde{X}) \rtimes_r G$, the restriction of π to $C(X) \rtimes_r G$ is faithful, since the action of G on X has the intersection property and $I \cap C(X) = \{0\}$. Note that the canonical conditional expectation $E_X\colon C(X) \rtimes_r G \to C(X)$ is a ucp map into $C(\tilde{X})$ via the inclusion of $C(X)$. Furthermore, $C(X) \rtimes_r G$ carries a natural action of G by conjugation with the associated unitaries:

$$g.(f\lambda_h) = \lambda_g(f\lambda_h)\lambda_{g^{-1}} = (g.f)\lambda_{ghg^{-1}}.$$

Consequently, as $ghg^{-1} = e$ if and only if $h = e$, the conditional expectation E_X is G-equivariant. As $C(\tilde{X})$ is the G-injective envelope of $C(X)$, we obtain a G-equivariant ucp map $\varphi\colon (C(\tilde{X}) \rtimes_r G)/I \to C(\tilde{X})$ that restricts to the conditional expectation on the subalgebra $C(X) \rtimes_r G$ as in the following diagram:

$$\begin{tikzcd}
C(\tilde{X}) \rtimes_r G \arrow[r, "\pi"] & (C(\tilde{X}) \rtimes_r G)/I \arrow[r, dashed, "\varphi"] & C(\tilde{X}) \\
C(X) \arrow[u, hook] \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"] & C(X) \rtimes_r G \arrow[u, hook] \arrow[r, "E_X"] & C(X) \arrow[u, hook]
\end{tikzcd}$$

(3.4)

As visible from Diagram (3.4), the G-equivariant ucp map $\varphi \circ \pi$ restricts to the identity on $C(X)$ and by G-rigidity of $C(X)$ inside $C(\tilde{X})$ it therefore also restricts to the identity on $C(\tilde{X})$, whence $C(\tilde{X})$ is contained in the multiplicative domain of $\varphi \circ \pi$. Therefore,

$$\varphi \circ \pi(f\lambda_g) = \varphi \circ \pi(f)\varphi \circ \pi(\lambda_g) = fE_X(\lambda_g) = \begin{cases} f & \text{if} g = e \\ 0 & \text{else} \end{cases}$$

and hence $\varphi \circ \pi = E_{\tilde{X}}$ is the faithful canonical conditional expectation of $C(\tilde{X}) \rtimes_r G$. As π is faithful we conclude that $I = \{0\}$.

Conversely, we show that the action of G on $C(X)$ has the intersection property, if the action on the envelope $C(\tilde{X})$ does.

Suppose that the action of G on $C(\tilde{X})$ has the intersection property and that I is a closed, two-sided ideal of $C(X) \rtimes_r G$ such that $I \cap C(X) = \{0\}$. Let $\pi\colon C(X) \rtimes_r G \to (C(X) \rtimes_r G)/I$ be the associated quotient *-homomorphism and let ρ be a faithful representation of $(C(X) \rtimes_r G)/I$ on some Hilbert space $\mathcal{B}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is injective, $\rho \circ \pi$ extends to a ucp map $\tilde{\pi}\colon C(\tilde{X}) \rtimes_r G \to \mathcal{B}(\mathcal{H})$. Let D denote the C^*-algebra generated by $\tilde{\pi}(C(\tilde{X}) \rtimes_r G)$ inside $\mathcal{B}(\mathcal{H})$. Once again, G

acts on D by conjugation with the associated unitaries, for $d \in D$ and $g \in G$ defined by

$$g.d := \big(\rho \circ \pi(\lambda_g)\big)\, d\, \big(\rho \circ \pi(\lambda_{g^{-1}})\big).$$

Since $\tilde{\pi}$ restricts to the *-homomorphism $\rho \circ \pi$ on $C^*_{\mathrm{r}}(G) \subseteq C(X) \rtimes_r G \subseteq C(\tilde{X}) \rtimes_r G$, it is G-equivariant if we equip $C(\tilde{X}) \rtimes_r G$ with the G-action by conjugation with the unitaries λ_g. Let E denote the C^*-algebra generated by $\tilde{\pi}(C(\tilde{X}))$ inside $\mathcal{B}(\mathcal{H})$, which contains $C(X)$ as a sub-C^*-algebra, since $\rho \circ \pi$ and therefore $\tilde{\pi}$ are faithful on $C(X)$. Extending the embedding of $C(X)$ into $C(\tilde{X})$ from this subalgebra to E by G-injectivity, we obtain a G-equivariant ucp map $\varphi\colon E \to C(\tilde{X})$ as in the diagram below:

$$\begin{tikzcd}
C(\tilde{X}) \arrow[r, "\tilde{\pi}"] & E \arrow[r, dashed, "\varphi"] & C(\tilde{X}) \\
C(X) \arrow[u, hook] \arrow[r, equal] & C(X) \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd} \tag{3.5}$$

Since $\varphi \circ \tilde{\pi}$ restricts to the identity on $C(X)$, $\varphi \circ \tilde{\pi}|_{C(\tilde{X})}$ is the identity on $C(\tilde{X})$ by G-rigidity of $C(X)$ inside $C(\tilde{X})$. In such a composition of ucp maps yielding the identity, the left inverse can readily seen to be a *-homomorphism: Recall from Definition 2.1.8 that the multiplicative domain of φ can be characterised as the closed span of all $u \in E$ of norm one such that $\varphi(u)$ is a unitary in $C(\tilde{X})$. As $\varphi \circ \tilde{\pi}|_{C(\tilde{X})} = \mathrm{id}_{C(\tilde{X})}$, $\tilde{\pi}|_{C(\tilde{X})}$ is an isometry, and so $\tilde{\pi}(u')$ is contained in the multiplicative domain of φ for every unitary u' in $C(\tilde{X})$. As such unitaries span $C(\tilde{X})$ densely, their images under $\tilde{\pi}$ span E densely, and we conclude that the multiplicative domain of φ is all of E and φ is therefore a *-homomorphism.

Like in the proof of Theorem 2.3.1, we want to show that $\tilde{\pi}|_{C(\tilde{X})}$ is a *-homomorphism which we could readily conclude if φ were injective. As before we work around this issue by passing to the appropriate quotient: Let F denote the closure of $\ker(\varphi) \cdot \tilde{\pi}(C(\tilde{X}) \rtimes_r G)$ inside $\mathcal{B}(\mathcal{H})$. Since φ is G-equivariant, so is its kernel and as

$$\tilde{\pi}(f)\rho \circ \pi(\lambda_g) = \rho \circ \pi(\lambda_{g^{-1}} g.(\tilde{\pi}(f))$$

for $\tilde{\pi}(f) \in \ker(\varphi)$ we see that $\ker(\varphi) \cdot \tilde{\pi}(C(\tilde{X}) \rtimes_r G) = \tilde{\pi}(C(\tilde{X}) \rtimes_r G) \cdot \ker(\varphi)$. Therefore, F is a closed, two-sided ideal of D, namely all elements of D fixed by an approximate unit of $\ker(\varphi)$. Let Φ denote the quotient *-homomorphism $D \to D/F$ and consider the ucp map $\Phi \circ \tilde{\pi}\colon C(\tilde{X}) \rtimes_r G \to D/F$. By definition of F, φ factors through the restriction of Φ to E as $\bar{\varphi}$ and $\bar{\varphi} \circ (\Phi \circ \tilde{\pi})|_{C(\tilde{X})} = \mathrm{id}_{C(\tilde{X})}$. As $\bar{\varphi}$ is injective on E/F, we may now conclude that $(\Phi \circ \tilde{\pi})|_{C(\tilde{X})}$ is in fact a faithful *-homomorphism. Since $\tilde{\pi}$ restricts to the *-homomorphism π on $C^*_{\mathrm{r}}(G) \subset C(\tilde{X}) \rtimes_r G$, both $C^*_{\mathrm{r}}(G)$ and $C(\tilde{X})$ are contained in the multiplicative domain of $\Phi \circ \tilde{\pi}$ and as their product is dense in $C(\tilde{X}) \rtimes_r G$, $\Phi \circ \tilde{\pi}$ itself is a *-homomorphism. Applying the intersection property of the action on $\tilde{X}$, we may conclude that $\Phi \circ \tilde{\pi}$ itself is faithful, as it is injective on $C(\tilde{X})$. In particular, π is faithful as a restriction of $\tilde{\pi}$ as claimed. □

We continue to denote the spectrum of the G-equivariant injective envelope of a unital commutative C^*-algebra $C(X)$ with G-action by $\tilde{X}$. Recall from Proposition 2.2.13 that $C(\tilde{X})$ is an injective C^*-algebra and hence $\tilde{X}$ is Stonean by Proposition 2.2.20. Recall furthermore that all stabilisers of the action on $\tilde{X}$ are amenable by Lemma 2.3.4. Both of these circumstances make the action on $\tilde{X}$ significantly more tractable than the action on X. The main benefit of passing the intersection property on to the action on the injective envelope is the drastic simplification provided by the following proposition:

Proposition 3.3.3 (Kawabe [38, Theorem 3.4]): Let $\tilde{X}$ be the spectrum of the G-equivariant injective envelope of $C(X)$ for X a compact space with G-action. The following are equivalent:

1) The action of G on X has the intersection property.

2) The action of G on $\tilde{X}$ has the intersection property.

3) The action of G on $\tilde{X}$ is topologically free.

4) The action of G on $\tilde{X}$ is free.

This generalises Theorem 2.3.6.

Proof. The equivalence of 1) and 2) is given in Theorem 3.3.2 and only repeated for completeness, while the implications 4) ⇒ 3) ⇒ 2) are clear, using Corollary 3.2.4.

Noting that the topological stabiliser groups are subgroups of the stabiliser groups at the same point, the intersection property of $\tilde{X}$ implies topological freeness by Lemma 3.2.5, as all stabiliser groups of the G-action on $\tilde{X}$ are amenable by Lemma 2.3.4. This proves the implication 2) ⇒ 3).

Finally, since $C(\tilde{X})$ is injective by Proposition 2.2.13, $\tilde{X}$ is Stonean and therefore free if the action is topologically free: By Frolík's theorem [24] the fixed point set of every homeomorphism is clopen, in particular the set $\tilde{X}^g$ of points fixed by a given element $g \in G$ is open. In other words, whenever a group element g fixes a single point $x \in X$, it fixes a whole neighbourhood U of x pointwise. Therefore, $g \in G_y^\circ$ for all $y \in U$. Assuming topological freeness, the points with trivial topological stabiliser are dense, so there is some $y \in U$ for which $g \in G_y^\circ = \{e\}$, hence g is trivial and the action is free. □

We rephrase Kawabe's characterisations in a language that will generalise well to our results for groupoid algebras in Section 4:

Recall the definition of sections of stabiliser subgroups from Definition 3.3.1. We call such a section *amenable*, if all its constituent groups are amenable. Just like absence of recurrent amenable subgroups describes C^*-simplicity of discrete groups, the absence of recurrent amenable sections of stabiliser subgroups begets the intersection property of the action:

Theorem 3.3.4 (Kawabe [38, Theorem 5.2]): *Let G be a discrete group acting on a compact space X. If the action has no recurrent amenable sections of stabiliser subgroups it has the intersection property.*

To prove the above theorem, we provide a brief lemma:

Lemma 3.3.5 (Kawabe [38, Proposition 3.3]): *Let G be a discrete group acting on a compact space X and let $\tilde{X}$ be the spectrum of the G-equivariant injective envelope of $C(X)$ with surjection $q\colon \tilde{X} \to X$ associated with the embedding of $C(X)$ into $C(\tilde{X})$.*

If $Y \subset \tilde{X}$ is a closed, G-invariant subset of $\tilde{X}$ such that $q(Y) = X$, then $Y = \tilde{X}$.

Proof. Let Y be as in the statement of the lemma. As Y is G-invariant, G acts on Y and both $q|_Y \colon Y \to X$, as well as the inclusion $Y \hookrightarrow \tilde{X}$ are G-equivariant. Since $q(Y) = X$, we may embed $C(X)$ into $C(Y)$ by precomposition with q, that is

$$\begin{aligned} C(X) &\hookrightarrow C(Y) \\ f &\mapsto f \circ q. \end{aligned}$$

By G-injectivity of $C(\tilde{X})$, we can extend the embedding $C(X) \hookrightarrow C(\tilde{X})$ along the embedding $C(X) \hookrightarrow C(Y)$ to a G-equivariant ucp map $\varphi \colon C(Y) \to C(\tilde{X})$ as in Diagram 3.6 below. On the other hand, restriction to Y is a G-equivariant quotient *-homomorphism $\mathrm{res}_Y \colon C(\tilde{X}) \to C(Y)$ which preserves the embedding of $C(X)$ as in the same diagram:

$$\begin{tikzcd} C(Y) \arrow[r, dashed, "\varphi"] & C(\tilde{X}) \arrow[l, bend right, "\mathrm{res}|_Y"'] \\ C(X) \arrow[u, hook] \arrow[ur, hook] & \end{tikzcd} \tag{3.6}$$

Clearly, $\varphi \circ \mathrm{res}|_Y$ is a G-equivariant ucp map $C(\tilde{X}) \to C(\tilde{X})$ fixing $C(X)$ pointwise and is therefore the identity on $C(\tilde{X})$ by G-rigidity. Consequently, the restriction $\mathrm{res}|_Y$ to Y is injective and since $\tilde{X}$ is Hausdorff, Y must be all of $\tilde{X}$. □

Theorem 3.3.4 is the first half of the statement [38, Theorem 5.2]. We give a simpler proof than is provided there.

Proof. Let G act on the compact space X such that there are no amenable recurrent sections of stabiliser subgroups. Note that if a group element $g \in G$ stabilises a point $\tilde{x} \in \tilde{X}$ then $g.q(\tilde{x}) = q(g.\tilde{x}) = q(\tilde{x})$ as q is G-equivariant and so the stabiliser $G_{\tilde{x}}$ of the action of G on $\tilde{X}$ is a subgroup of the stabiliser $G_{q(\tilde{x})}$ of the action of G on X. Let $\Phi \colon \tilde{X} \to \mathrm{Sub}(G)$ be the map sending a point $\tilde{x}$ in the boundary $\tilde{X}$ to its stabiliser subgroup $G_{\tilde{x}}$. Using Frolík's theorem [24] again, we see that the stabiliser $G_{\tilde{x}}$ and the topological stabiliser $G^{\circ}_{\tilde{x}}$ coincide for every $\tilde{x} \in \tilde{X}$, as the fixed point set of the homeomorphism of $\tilde{X}$ associated with g is open and therefore $\tilde{x}$ is fixed by g if and only if a neighbourhood of $\tilde{x}$ is fixed pointwise. This shows continuity of Φ: Recall that a neighbourhood subbasis of $H \in \mathrm{Sub}(G)$ is given by the sets of subgroups $\mathcal{U}_{g,H} = \{H' \leq G \mid g \in H' \Leftrightarrow g \in H\}$. Let $\Phi(\tilde{x}) \in \mathcal{U}_{g,H}$ and assume that $g \in H$. Then $g \in G_{\tilde{x}} = G^{\circ}_{\tilde{x}}$ and so $g \in G_{\tilde{y}}$ for y in some neighbourhood $O \subseteq \tilde{X}$ of $\tilde{x}$. Therefore, $\tilde{x}$ is contained in the interior of $\Phi^{-1}(\mathcal{U}_{g,H})$. The case $g \notin H$ is analogous, and this proves continuity at $\tilde{x}$. Hence, $\Phi(\tilde{X})$ is a compact G-invariant subset of $\mathrm{Sub}(G)$, containing at least one amenable subgroup of every stabiliser G_x for $x \in X$ since all stabilisers of the boundary $\tilde{X}$ are amenable. Pick a section of stabiliser subgroups consisting of groups in $\Phi(\tilde{X})$, which is not recurrent by assumption. Then for every $x \in X$ we find nets g_λ and $\tilde{x}_\lambda$ such that $(g_\lambda.q(\tilde{x}_\lambda), g_\lambda.G_{\tilde{x}_\lambda}) \to (x, \{e\})$. As $q \times \Phi \colon \tilde{X} \to X \times \mathrm{Sub}(G)$ is continuous, $q \times \Phi(\tilde{X})$ is compact and therefore closed, so it contains $(x, \{e\})$. In other words, for every $x \in X$ there is $\tilde{x} \in \tilde{X}$ with $q(\tilde{x}) = x$ such that $G_{\tilde{x}} = \{e\}$ and so $Y := \Phi^{-1}(\{e\})$ is a closed, G-invariant subspace of $\tilde{X}$ satisfying $q(Y) = X$. By Lemma 3.3.5 we conclude that $Y = \tilde{X}$ and hence the action of G on $\tilde{X}$ is free. By Proposition 3.3.3, we conclude that the action of G on X has the intersection property. □

As for discrete groups, absence of recurrent amenable sections of stabiliser subgroups is actually equivalent to the action having the intersection property.

Theorem 3.3.6 (Kawabe [38, Theorem 5.2]): *Let G be a discrete group acting on a compact space X. If the action has the intersection property, then no amenable section of stabiliser subgroups is recurrent.*

By Theorem 3.3.4, having the intersection property is therefore equivalent to not having recurrent amenable sections of stabiliser subgroups.

To prove Theorem 3.3.6, we need to establish what Kawabe calls the analogue of the unique trace property for group actions.

Lemma 3.3.7 (Kawabe [38, Lemma 4.1]): *Let G be a discrete group acting on a compact space X. If the action is topologically free, the only conditional expectation from $C(X) \rtimes_r G$ onto $C(X)$ is the canonical conditional expectation E_X. Conversely, if X is Stonean, all stabilisers of the action of G on X are amenable, and the conditional expectation onto $C(X)$ is unique, then the action is topologically free.*

Seen in a wider context, uniqueness of the conditional expectation stems from the fact that $C(X)$ is a Cartan subalgebra of $C(X) \rtimes_r G$ if the action is topologically free (see e.g [53, Theorem 5.2]) and that conditional expectations onto Cartan subalgebras are unique (see [53, Corollary 5.10]). The theory of Cartan subalgebras is already well-developed in a groupoid contex, in fact, every pair of C^*-algebras (A, B) where B is a Cartan subalgebra of A can be modeled by a (twisted) groupoid C^*-algebra of a topologically free groupoid and its diagonal subalgebra (see [53, Theorem 5.9]).

We repeat the proof given by Kawabe in [38].

Proof. Let E be a conditional expectation from $C(X) \rtimes_r G$ onto $C(X)$. Let $x \in X$ and $g \in G$ be such that $g.x \neq x$. This forces $E(\lambda_g)$ to vanish at x: Since $g.x \neq x$ and the action is by homeomorphism, we may choose a neighbourhood $\mathcal{U}$ of x such that $\mathcal{U}$ and $g.\mathcal{U}$ are disjoint. Let $f \in C(X)$ be supported in $\mathcal{U}$ with $f(x) = 1$ and note that $f \cdot g.f = 0$. Then

$$E(\lambda_g)(x) = (f \cdot E(\lambda_g) \cdot f)(x),$$

and as $f \in C(X)$ is in the multiplicative domain of E,

$$E(\lambda_g)(x) = E(f\lambda_g f)(x) = (f \cdot g.f)(x) \cdot E(\lambda_g)(x) = 0.$$

As the action of G on X is topologically free, the set of points x in X with trivial stabiliser G_x is dense by Proposition 3.2.2. That is, for every $g \in G \backslash \{e\}$ we can find a dense set of points x such that $g \notin G_x$ and therefore $E(\lambda_g) = 0 = E_X(\lambda_g)$ by the above. Clearly $E(\lambda_e) = \mathbb{1}_X = E_X(\lambda_e)$, as this is the inclusion of $C(X)$ into $C(X) \rtimes_r G$. Hence E is the canonical conditional expectation E_X.

Conversely, we show that if X is Stonean, all stabilisers G_x are amenable, and the action is not topologically free, there is a non-canonical conditional expectation E given by

$$E(f\lambda_g) = f \cdot \chi_{X^g}$$

for $g \in G$ with X^g the subspace of points fixed by g. Since X is Stonean, X^g is clopen by Frolík's theorem [24] and therefore the indicator function χ_{X^g} is continuous for any $g \in G$. The definition of E extends from $C(X)[G]$ to all of $C(X) \rtimes_r G$: For $x \in X$ consider $\mathrm{eval}_x \circ E$ and note that

$$\mathrm{eval}_x \circ E(\sum_{g \in G} f_g \lambda_g) = \sum_{g \in G_x} f_g(x) = \tau(\sum_{g \in G_x} f_g(x)\lambda_g) = \tau \circ E_x\Big(\sum_{g \in G} f_g \lambda_g\Big)$$

for E_x the conditional expectation onto G_x and τ the trivial representation of G_x which extends to $C^*_{\mathrm{r}}(G_x)$ because the stabiliser is amenable. Hence $\mathrm{eval}_x \circ E = \tau \circ E_x$ is a positive linear functional on $C(X) \rtimes_r G$ and therefore $\|\mathrm{eval}_x \circ E\| = \mathrm{eval}_x \circ E(\mathbb{1}) = 1$ by [62, Theorem 13.5]. Consequently, E as defined on $C(X)[G]$ is a contraction in the norm of $C(X) \rtimes_r G$ and therefore extends to all of $C(X) \rtimes_r G$.

If the action is not topologically free, there exists a non-trivial $g \in G$ such that X^g is non-empty and hence E is distinct from E_X since $E_X(\lambda_g) = 0 \neq \chi_{X^g} = E(\lambda_g)$. □

We use uniqueness of the conditional expectation in the following version:

Lemma 3.3.8 (Kawabe [38, Theorem 4.2]): *Let G be a discrete group acting on a compact space X and let $\tilde{X}$ be the spectrum of the G-equivariant injective envelope of X. The action has the intersection property if and only if the only G-equivariant ucp map $C(X) \rtimes_r G \to C(\tilde{X})$ that fixes $C(X)$ pointwise is the canonical conditional expectation E_X.*

We elaborate on the proof given by Kawabe in [38].

Proof. Note again that $C(X) \rtimes_r G$ embeds into $C(\tilde{X}) \rtimes_r G$ by using the embedding of $C(X)$ into $C(\tilde{X})$ and that this is G-equivariant if we equip the crossed products with an action of G where $g \in G$ acts by conjugation with the associated unitary λ_g. Let $\varphi\colon C(X) \rtimes_r G \to C(\tilde{X})$ be a G-equivariant ucp map fixing $C(X)$ pointwise. By G-injectivity of $C(\tilde{X})$ we may, as in Diagram (3.7), extend φ to a G-equivariant ucp map $\Phi\colon C(\tilde{X}) \rtimes_r G \to C(\tilde{X})$ along the embedding described above.

$$\begin{tikzcd} C(\tilde{X}) \rtimes_r G \arrow[r, dashed, "\Phi"] & C(\tilde{X}) \\ C(X) \rtimes_r G \arrow[u, hook] \arrow[ur, "\varphi"] & C(X) \arrow[l, hook'] \arrow[u, hook] \end{tikzcd} \tag{3.7}$$

Restricting Φ to $C(\tilde{X})$ yields a G-equivariant ucp map $C(\tilde{X}) \to C(\tilde{X})$ that fixes $C(X)$ pointwise, hence $\Phi|_{C(\tilde{X})} = \mathrm{id}_{C(\tilde{X})}$ is the identity by G-rigidity of $C(\tilde{X})$. Therefore, $C(\tilde{X})$ is contained in the multiplicative domain of Φ and hence

$$\Phi(f\lambda_g) = \Phi(f)\Phi(\lambda_g) = f\varphi(\lambda_g).$$

Consequently, if φ is the canonical conditional expectation onto $C(X)$, then Φ is the canonical conditional expectation onto $C(\tilde{X})$. Conversely, given any G-equivariant ucp map Φ that fixes $C(X)$ pointwise, it has to be a conditional expectation onto $C(\tilde{X})$ by G-rigidity and restricts to the canonical conditional expectation onto $C(X)$ if it is the canonical conditional expectation onto $C(\tilde{X})$. Existence of a G-equivariant ucp map $C(X) \rtimes_r G \to C(\tilde{X})$ that fixes $C(X)$ pointwise but is not the canonical conditional expectation onto $C(X)$ is therefore equivalent to the existence of a non-canonical conditional expectation $C(\tilde{X}) \rtimes_r G \to C(\tilde{X})$. As $\tilde{X}$ is Stonean by Proposition 2.2.13 and all stabilisers of the G-action on $\tilde{X}$ are amenable by Lemma 2.3.4, this is furthermore equivalent to the intersection property of the G-action on $\tilde{X}$ by the previous Lemma 3.3.7 and finally equivalent to the intersection property of the G-action on X by Proposition 3.3.3, proving the claim. □

This enables us to prove Theorem 3.3.6, as in the second half of [38, Theorem 5.2]:

Proof. Given an amenable section of stabiliser subgroups, consider its orbit closure Y in $X \times \mathrm{Sub}(G)$. This is a closed, G-invariant set. We define a G-equivariant ucp map θ from $C(X) \rtimes_r G$ to $C(Y)$ by

$$\theta(f\lambda_g)(x, H) \coloneqq f(x)\chi_H(g)$$

for $f \in C(X)$, $g \in G$, $H \leq G$ a subgroup, and χ_H the indicator function of H on G. To see that $\theta(f\lambda_g)(x, H)$ is indeed continuous on Y, note that f is continuous on X and $H \mapsto \chi_H(g)$ is continuous on $\mathrm{Sub}(G)$ for fixed g, since H contains g if and only if all subgroups in a neighbourhood of H do. Extending linearly, θ is a positive unital map $C(X)[G] \to C(Y)$: Consider

$$\begin{tikzcd}
C(\tilde{X}) \rtimes_r G \arrow[r, "E_x"] & C^*_{\mathrm{r}}(G_x) \arrow[r, "E_H"] & C^*_{\mathrm{r}}(H) \arrow[r, "\tau"] & \mathbb{C} \\
f\lambda_g \arrow[r, mapsto] & f(x)E_{G_x}(\lambda_g) \arrow[r, mapsto] & f(x)E_H(\lambda_g) \arrow[r, mapsto] & f(x)\chi_H(g)
\end{tikzcd}$$

With E_x the conditional expectation from $C(\tilde{X}) \rtimes_r G$ onto the algebra $C^*_{\mathrm{r}}(G_x)$ of the stabiliser G_x at x, E_H the conditional expectation from $C^*_{\mathrm{r}}(G_x)$ onto the algebra $C^*_{\mathrm{r}}(G)$ of its subgroup H and τ the trivial representation of H sending every group element to the unit of the unitary group $\mathcal{U}(\mathbb{C})$ of $\mathbb{C}$. As H is amenable, τ is continuous and since all involved maps are positive, $\mathrm{eval}_{(x,H)} \circ \theta$ is positive on $C(X)[G]$. Since $\mathrm{eval}_{(x,H)} \circ \theta$ is a positive linear functional, we have that $\|\mathrm{eval}_{(x,H)} \circ \theta\| = \theta(\mathbb{1})(x, H) = 1$ by [62, Theorem 13.5]. Consequently, θ is bounded in norm. Positivity on the commutative C^*-algebra $C(Y)$ is decided precisely by positivity after applying all these extremal states and therefore θ extends by linearity and continuity to a unital positive map on all of $C(X) \rtimes_r G$. As $C(Y)$ is commutative, θ is furthermore completely positive. Finally, θ is G-equivariant as

$$\begin{aligned}
\theta(g.(f\lambda_h))(x, H) &= \theta(\lambda_g(f\lambda_h)\lambda_g^{-1})(x, H) = \theta((g.f)\lambda_{ghg^{-1}})(x, H) = (g.f)(x)\chi_H(ghg^{-1}) \\
&= f(g^{-1}.x)\chi_{g^{-1}Hg}(h) = \theta(f\lambda_h)((g^{-1}).x, (g^{-1}).H) = g.(\theta(f\lambda_h))(x, H)
\end{aligned}$$

for $g, h \in G$, $f \in C(X)$, $x \in X$, and $H \leq G$.

Since Y contains at least one subgroup in every stabiliser, $C(X)$ embeds into $C(Y)$ by precomposition with projection onto the first coordinate. By G-injectivity of $C(\tilde{X})$, we may extend the embedding $C(X) \hookrightarrow C(\tilde{X})$ along the embedding $C(X) \hookrightarrow C(Y)$ to a ucp G-map $\varphi\colon C(Y) \to C(\tilde{X})$. Now $\varphi \circ \theta$ is a G-equivariant ucp map $C(X) \rtimes_r G \to C(X)$ that fixes $C(X)$ pointwise. As the action of G on X has the intersection property by assumption, $\tilde{X}$ is topologically free by Proposition 3.3.3 and therefore $\varphi \circ \theta$ is the canonical conditional expectation onto $C(X)$ by Lemma 3.3.8.

Let q again denote the quotient map $\tilde{X} \to X$ coming from $C(X) \hookrightarrow C(\tilde{X})$ and let $\tilde{x} \in \tilde{X}$. Consider the functional $\mathrm{eval}_{\tilde{x}} \circ \varphi \colon C(Y) \to \mathbb{C}$. We show that it is supported on $\{q(\tilde{x})\} \times \mathrm{Sub}(G)$: Let $f \in C(Y)$ be such that it vanishes on $\{q(\tilde{x})\} \times \mathrm{Sub}(G)$ and let $\mathcal{U}$ be an open neighbourhood of $q(\tilde{x})$. Let $h_{\mathcal{U}} \in C(X)$ be supported on $\mathcal{U}$ with $0 \leq h_{\mathcal{U}} \leq 1$ and $h_{\mathcal{U}}(q(\tilde{x})) = 1$. Then

$$|\mathrm{eval}_{\tilde{x}} \circ \varphi(f)| = |\mathrm{eval}_{\tilde{x}} \circ ((h_{\mathcal{U}} \circ q) \cdot \varphi(f))|.$$

Noting that $h_{\mathcal{U}} \circ q$ is the image of $h_{\mathcal{U}}$ under the embedding of $C(X)$ into $C(\tilde{X})$, we have $h_{\mathcal{U}} \circ q = \varphi(h_{\mathcal{U}} \circ \pi_1)$ with π_1 the coordinate projection from $X \times \mathrm{Sub}(G)$ onto X. Hence $h_{\mathcal{U}} \circ \pi_1$ is contained in the multiplicative domain of φ so that

$$\begin{aligned} |\mathrm{eval}_{\tilde{x}} \circ \varphi(f)| &= |\mathrm{eval}_{\tilde{x}} \circ ((h_{\mathcal{U}} \circ q) \cdot \varphi(f))| \\ &= |\mathrm{eval}_{\tilde{x}} \circ \varphi((h_{\mathcal{U}} \circ \pi_1) \cdot f)| \\ &\leq \|(h_{\mathcal{U}} \circ \pi_1) \cdot f\|_{C(Y)}. \end{aligned}$$

As f vanishes on $\{q(\tilde{x})\} \times \mathrm{Sub}(G)$ and $\mathrm{Sub}(G)$ is compact, for every $\epsilon > 0$ we may find a neighbourhood $\mathcal{U}$ of $q(\tilde{x})$ such that $\|f|_{\mathcal{U} \times \mathrm{Sub}(G)}\|_{C(Y)} < \epsilon$. Then $\|(h_{\mathcal{U}} \circ \pi_1) \cdot f\|_{C(Y)} < \epsilon$ and the above shows that $|\mathrm{eval}_{\tilde{x}} \circ \varphi(f)| < \epsilon$. We conculde that $\mathrm{eval}_{\tilde{x}} \circ \varphi(f) = 0$ if f vanishes on $\{q(\tilde{x})\} \times \mathrm{Sub}(G)$. Let $\mu_{\tilde{x}}$ denote the Radon probability measure on Y associated with the functional $\mathrm{eval}_{\tilde{x}} \circ \varphi$. The above shows that $\mu_{\tilde{x}}$ is supported on $\{q(\tilde{x})\} \times \mathrm{Sub}(G)$, or in other words, $\mu_{\tilde{x}}(\{q(\tilde{x})\} \times \mathrm{Sub}(G)) = 1$. However, for $e \neq g \in G$ we calculate that

$$\mu_{\tilde{x}}(\{(x, H) \mid g \in H\}) = \int_Y \chi_H(g) \, \mathrm{d}\mu_{\tilde{x}}(x, H) = \int_Y \theta(\lambda_g) \, \mathrm{d}\mu_{\tilde{x}}(x, H) = \mathrm{eval}_{\tilde{x}} \circ \varphi \circ \theta(\lambda_g) = 0,$$

the last equality following from the previous observation that $\varphi \circ \theta$ has to be the canonoical conditional expectation from $C(X) \rtimes_r G$ onto $C(X)$. If the given amenable section of stabiliser subgroups was recurrent, we could pick $x_0 \in X$ such that $(x_0, \{e\}) \notin Y$. Hence for $\tilde{x}_0 \in \tilde{X}$ such that $q(\tilde{x}_0) = x_0$ we would find that

$$1 = \mu_{\tilde{x}_0}(Y) \leq \mu_{\tilde{x}_0}\left(\bigcup_{e \neq g \in G} \{(x, H) \mid g \in H\} \right) \leq \sum_{e \neq g \in G} \mu_{\tilde{x}_o}(\{(x, H) \mid g \in H\}) = 0,$$

which is clearly a contradiction. □

While absence of recurrent amenable sections of stabiliser subgroups characterises the intersection property, the more easily stated absence of dynamically recurrent subgroups of the stabilisers is instead equivalent to the *residual* intersection property. Recall from Definition 3.3.1 that if G is acting on X, we call $H \leq G_x$ a dynamically recurrent subgroup if $(x, \{e\}) \in X \times \mathrm{Sub}(G)$ cannot be approximated by G-conjugates of (x, H). Note that H might be recurrent in G_x but not dynamically recurrent, as the former notion is only concerned with conjugation by elements of G_x, while the latter uses all elements of G. For example, the whole stabiliser G_x is normal and therefore recurrent in G_x, but not dynamically recurrent, unless G_x was a recurrent subgroup of G.

Definition 3.3.9: Let G be a discrete group acting on a compact space X. We say that the action of G on X has the *residual intersection property*, if the action of G on every closed, G-invariant subspace of X has the intersection property.

Theorem 3.3.10 (Kawabe [38, Theorem 5.3]): *Let G be a discrete group acting on a compact space X. The action has the residual intersection property, if and only if no stabiliser has a dynamically recurrent amenable subgroup.*

Proof. First assume that the action has no dynamically recurrent amenable subgroups in its stabilisers and let $Y \subset X$ be a closed, G-invariant subspace. Any amenable section of stabiliser subgroups for the action of G on Y contains an amenable subgroup (y, H) of G_y for every $y \in Y$. As (y, H) is by assumption not dynamically recurrent, its orbit closure contains $(y, \{e\})$ and since this is the case for all $y \in Y$, the orbit closure of the section of stabiliser subgroups contains $Y \times \{\{e\}\}$. By Theorem 3.3.4, the action of G on Y has the intersection property.

Conversely, suppose that the action on X has the residual intersection property and that H is an amenable subgroup of G_x for some $x \in X$. Let Y be the orbit closure of x in X, then the action of G on Y has the intersection property by assumption. For every point y in the orbit of x pick a group element g_y such that $g_y.x = y$. Then $\left\{(g_y.x, g_yHg_y^{-1}) \mid y \in G.\{x\}\right\}$ is a collection of amenable subgroups of stabilisers and can be extended to an amenable section of stabiliser subgroups by choosing a limit point $(y, H_y) \in Y \times \mathrm{Sub}(G)$ for every $y \in Y \setminus G.\{x\}$, which exists as $\mathrm{Sub}(G)$ is compact. As the action is continuous, H_y is a subgroup of G_y and since amenability is a closed condition in $\mathrm{Sub}(G)$ by Proposition 2.3.10, it is amenable. The resulting amenable section of stabiliser subgroups consists only of subgroups which can be expressed as limits of conjugate subgroups of (x, H) and its orbit closure contains $(x, \{e\})$ by Theorem 3.3.6, so $(x, \{e\})$ is likewise a limit of conjugate subgroups of (x, H). Hence H is not dynamically recurrent by Definition 3.3.1. □

We finish our review of Kawabe's work by summing up the above criteria in the minimal case, where the (residual) intersection property is equivalent to simplicity of the reduced C^*-algebra. Note that there are no non-trivial closed G-invariant subspaces in the minimal case, hence the intersection property coincides with the residual intersection property.

Proposition 3.3.11 (Kawabe [38, Theorem 6.1]): Let G be a discrete group acting minimally on a compact space X. The following are equivalent:

1) The reduced crossed product $C(X) \rtimes_r G$ is simple.

2) The action has the (residual) intersection property.

3) The action has no dynamically recurrent amenable subgroups of stabilisers.

4) There is $x_0 \in X$ such that G_{x_0} has no dynamically recurrent amenable subgroups.

We choose to give a different proof for the equivalence of 3) and 4) than in [38, Theorem 6.1].

Proof. We have already remarked upon the well-known equivalence of 1) and 2) and the equivalence of 2) and 3) is given in Theorems 3.3.4 and 3.3.6. We show that 4) implies 3), as the converse clearly holds.

Let $x, x_0 \in X$ and let H be an amenable subgroup of G_x. As the action is minimal, there is a net $g_\lambda \in G$ such that $g_\lambda.x \to x_0$. Since $\mathrm{Sub}(G)$ is compact we may assume that $g_\lambda.H \to H' \leq G$ for some amenable subgroup H' of G after possibly passing to a subnet. Using that $g_\lambda.H \leq G_{g_\lambda.x}$ it follows that $H' \leq G_{x_0}$. By assumption of 4), (x_0, H') is not dynamically recurrent, so $(x_0, \{e\})$ is contained in its orbit closure and therefore in the orbit closure of (x, H). Finally, since $g_\lambda^{-1}.(x_0, \{e\}) \to (x, \{e\})$, we conclude that H is not a dynamically recurrent subgroup of G_x and 3) follows. □

As we feel it is more insightful, we add the following two equivalent conditions to Proposition 3.3.11:

Proposition 3.3.12: Let G be a discrete group acting minimally on a compact space X. The following are equivalent:

1) The reduced crossed product $C(X) \rtimes_r G$ is simple.

5) For all $x \in X$, the stabiliser G_x contains no recurrent amenable subgroups of G.

6) There is $x_0 \in X$ such that G_{x_0} contains no recurrent amenable subgroups of G.

Proof. 1) ⇒ 5) If any G_{x_0} contains a recurrent amenable subgroup H of G, then H is a dynamically recurrent subgroup of G_{x_0} and the reuced crossed product is not simple by the previous proposition.

5) ⇒ 6) is clear.

6) ⇒ 4) ⇔ 1) Let $x_0 \in X$ be such that G_{x_0} contains no recurrent amenable subgroup of G. Then given any amenable subgroup $H \leq G_{x_0}$, there is a net $g_\lambda \in G$ such that $g_\lambda.H \to \{e\}$. As X is compact, we may pass to a convergent subnet of $g_\lambda.x$, say $g_\lambda.x \to y \in X$, so $(y, \{e\})$ is contained in the orbit closure of (x, H). As the action is minimal, x is in the orbit closure of y and hence conjugates of $(y, \{e\})$ approximate $(x, \{e\})$, showing that H is not dynamically recurrent in G_x. This shows condition 4) of the previous Propsition 3.3.11, so the reduced crossed product is simple. □

Groupoids

We finally get to the heart of this thesis, generalising Kennedy's results of Section 2.3.3 and Kawabe's results of Section 3.3 to étale groupoids. While Bryder–Kennedy [14] and Bryder [13] investigated C^*-simplicity in the case of crossed products where the commutative algebra $C(X)$ is replaced by a not-necessarily commutative C^*-algebra, we are taking a different approach with groupoids, keeping the commutative C^*-algebra over what will be called the "unit space" of the groupoid, but weakening the assumptions of a "group" acting. We recall the basic theory of groupoids in Section 4.1 and present our results on C^*-simplicity in Section 4.2. Afterwards, we present a groupoid model for a recent C^*-algebraic construction by G. Elek in Section 4.3 and apply our simplicity results.

4.1 Groupoid Basics

For decades, C^*-algebras associated with (topological) groups have been studied intensely as they provide both interesting links between Operator Algebras and other fields of mathematics like group theory and geometry, as well as an easily manipulated model for examples of C^*-algebras. The crossed products of Section 3 take this perspective further, allowing to enter all commutative C^*-algebras into the mix. One possible next step in this process of generalisation is to investigate the C^*-algebras modelled by *groupoids*, which mimic the action of a group on a space, but allow the group acting to vary from point to point. These models are immensely powerful. For example, every simple classifiable C^*-algebra arises as the reduced C^*-algebra of a groupoid [46, 42], and although not every C^*-algebra is modelled by a groupoid [15], we are not aware of an example that is known not to arise from a groupoid and a *twist*, a slightly more involved construction which is out of scope here.

A groupoid is often quickly defined as a "small category with inverses", but we collect the details in the following definition from [49, p.7]:

Definition 4.1.1: A groupoid $\mathcal{G}$ is a set together with a *composition map* m from a subset $\mathcal{G}^{(2)}$ of the product set $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$ and an *inverse map* $i\colon \mathcal{G} \to \mathcal{G}$ such that

1) $(\gamma, \eta), (\eta, \kappa) \in \mathcal{G}^{(2)} \Rightarrow (m(\gamma, \eta), \kappa), (\gamma, m(\eta, \kappa)) \in \mathcal{G}^{(2)}$ and $m(m(\gamma, \eta), \kappa) = m(\gamma, m(\eta, \kappa))$.

2) $i \circ i = \mathrm{id}_{\mathcal{G}}$

3) For all $\gamma \in \mathcal{G}$ we have $(\gamma, i(\gamma)) \in \mathcal{G}^{(2)}$

4) If $(\gamma, \eta) \in \mathcal{G}^{(2)}$, then $m(i(\gamma), m(\gamma, \eta)) = \eta$ and $m(m(\gamma, \eta), i(\eta)) = \gamma$.

We refer to elements of $\mathcal{G}$ as “arrows” and pairs of arrows that are contained in $\mathcal{G}^{(2)}$ as “composable”. For simplicity, we write the composition $m(\gamma, \eta)$ of a pair of composable arrows as $\gamma\eta$. Condition 1) above then simply states that composition is associative, which includes the correct pairs of arrows to be composable for the statement to make sense. The next three conditions establish the inverse map, which we for convenience write as $i(\gamma) = \gamma^{-1}$. By conditions 2) and 3), arrows are composable with their inverses in either order and by condition 4) they “revert” the effects of composition. In other words, the map $m(i(\gamma), \bullet)$ is inverse to the map $m(\gamma, \bullet)$.

While the above statement sounds intuitive, we may ask ourselves where the map $m(\gamma, \bullet)$ is actually defined. Clearly, it maps the set of arrows η such that (γ, η) is composable bijectively onto the set of ρ such that (γ^{-1}, ρ) is composable. Upon further inspection, we see that if η_1 and $\eta_2 \in \mathcal{G}$ are such that (γ, η_i) is composable for $i = 1, 2$, then $\eta_1\eta_1^{-1} = \eta_2\eta_2^{-1}$, since

$$\eta_i\eta_i^{-1} = (\gamma^{-1}(\gamma(\eta_i\eta_i^{-1}))) = ((\gamma^{-1}\gamma)\eta_i)\eta_i^{-1} = \gamma^{-1}\gamma \text{ for } i = 1, 2.$$

Consequently, we call $r(\eta) := \eta\eta^{-1}$ the range of η, defining the range map $r\colon \mathcal{G} \to \mathcal{G}$ and $s(\gamma) := \gamma^{-1}\gamma$ the source of γ, defining the source map $s\colon \mathcal{G} \to \mathcal{G}$. Indeed, a pair of arrows (γ, η) is composable if and only if $s(\gamma) = r(\eta)$, since

$$\gamma^{-1}(\gamma\eta)\eta^{-1} = (\gamma^{-1}\gamma)(\eta\eta^{-1}),$$

which by associativity coincides with both $s(\gamma)$ and $r(\eta)$. We call the image of the range and source maps the unit space $\mathcal{G}^{(0)} := r(\mathcal{G}) = s(\mathcal{G})$, whose elements are special arrows called *units*. Note that the two definitions using range or source coincide, since $r(\gamma) = s(\gamma^{-1})$. As the unit space will play the role of the space X in our ongoing analogy of a groupoid $\mathcal{G}$ generalising the action of a group G on a space X, we often prefer to use latin letters like $x \in \mathcal{G}^{(0)}$ for units, instead of the greek letters for general arrows. We denote by

$$\mathcal{G}^x := \{\gamma \in \mathcal{G} \mid r(\gamma) = x\} = r^{-1}(x) \qquad \text{and} \qquad \mathcal{G}_x := \{\gamma \in \mathcal{G} \mid s(\gamma) = x\} = s^{-1}(x)$$

the range and source fibres of a unit $x \in \mathcal{G}^{(0)}$. To answer our previous question, composition with a fixed element γ is a map $m(\gamma, \bullet)\colon \mathcal{G}^{s(\gamma)} \to \mathcal{G}^{r(\gamma)}$. As a special case, composition with a unit $x \in \mathcal{G}^{(0)}$ is the identity map $m(x, \bullet)\colon \mathcal{G}^x \to \mathcal{G}^x$.

Example 4.1.2. *Any group G is a groupoid $\mathcal{G} = G$ in which the composition and inversion are given by the group law. The unit space $\mathcal{G}^{(0)}$ then consists of the neutral element only and any pair of arrows is composable.*

Example 4.1.3. *Let G be a group acting on a set X. The* transformation groupoid *$G \ltimes X$ is given by the set $\mathcal{G} = G \times X$. Intuitively, we imagine an arrow (g, x) to be pointing from x to $g.x$, that is, $r(g, x) = g.x$ and $s(g, x) = x$. The set of composable pairs of arrows is hence defined as*

$$\mathcal{G}^{(2)} := \{((g, x), (g', x')) \in \mathcal{G} \mid g'.x' = x\}$$

and the composition of such pairs given by $(g, x)(g', x') = (gg', x')$ with inverses by $(g, x)^{-1} = (g^{-1}, g.x)$. The unit space is simply the copy of X in $\mathcal{G}$ given by $\{e\} \times X$ for the neutral element $e \in G$.

4.1.1 Topological, Étale, and Ample Groupoids

We wish to employ a groupoid as a model for a C^*-algebra, where the unit space $\mathcal{G}^{(0)}$ gives rise to a commutative C^*-algebra and the groupoid generalises the action of a group on $\mathcal{G}^{(0)}$. Hence we introduce a topology on $\mathcal{G}$ that turns $\mathcal{G}^{(0)}$ into a topological space and explain what it means for the "action" to be continuous:

Definition 4.1.4: A *topological groupoid* is a groupoid $\mathcal{G}$ with a locally compact Hausdorff topology such that the composition, inverse, range, and source maps are continuous.

The unit space $\mathcal{G}^{(0)}$ forms a closed subspace of $\mathcal{G}$, since any net of units η_λ converging to some arrow $\eta \in \mathcal{G}$ satisfies $\eta_\lambda = r(\eta_\lambda)$ and hence $\eta = r(\eta)$ if G is Hausdorff. In fact, the unit space being closed is equivalent to $\mathcal{G}$ being Hausdorff, if it is otherwise a topological groupoid as defined above, see [56, Lemma 2.3.2].

As we will be generalising results characterising the simplicity of C^*-algebras arising from actions of *discrete* groups, we may ask what properties of a topological groupoid would make it an appropriate generalisation of such actions. While it initially seems sensible that these would be groupoids carrying the discrete topology, note that this would mean we also endow the unit space with the discrete topology. Building the transformation groupoid as in Example 4.1.3 for the action of a discrete group G on a locally compact Hausdorff topological space X by automorphisms, we would then lose all information about the topology of X. The correct, or rather the *interesting*, analogue of a discrete group action in the groupoid world is therefore a different one:

Definition 4.1.5: We call a topological groupoid $\mathcal{G}$ *étale*, if its range map $r\colon \mathcal{G} \to \mathcal{G}$ is a local homeomorphism. That is, if for every arrow $\gamma \in \mathcal{G}$ there is an open neighbourhood U of γ such that $r(U)$ is open in $\mathcal{G}$ and $r|_U\colon U \to r(U)$ is a homeomorphism.

As the inverse map on $\mathcal{G}$ is a homeomorphism, we could of course equivalently ask the source map s to be a local homeomorphism. To understand why these generalise actions of discrete groups, note that in a transformation groupoid of G acting on X, we can identify every range fibre $\mathcal{G}^x$ or source fibre $\mathcal{G}_x$ with G. In an étale groupoid, these fibres will turn out to be discrete, but the proof is best done using another important concept, bisections:

Definition 4.1.6 ([56, Definition 2.4.8]): Let $\mathcal{G}$ be an étale groupoid. We call a subset $B \subseteq \mathcal{G}$ a *bisection*, if there is an open set U containing B such that both $s(U)$ and $r(U)$ are open in $\mathcal{G}^{(0)}$ and the source and range map restrict to homeomorphisms on U.

Note in particular that any bisection intersects a given range or source fibre either in a unique arrow or not at all. There are plenty of bisections in an étale groupoid. Enough, in fact to determine the topology:

Proposition 4.1.7: Let $\mathcal{G}$ be an étale groupoid. The topology of $\mathcal{G}$ has a basis consisting of open bisections. If the topology is second-countable, a countable basis can be chosen. Consequently, the range and source fibres are discrete subsets of $\mathcal{G}$.

Proof. As $\mathcal{G}$ is étale, every arrow γ is contained in an open set U_γ on which the range map restricts to a homeomorphism onto an open set. Likewise, there is an open set $V_\gamma = U_{\gamma^{-1}}$ on which the source map restricts to a homeomorphism and $U_\gamma \cap V_\gamma$ is an open bisection containing γ. Hence each arrow is contained in an open bisection and noting that the intersection of an open bisection with any open set is again an open bisection, we see that the topology of $\mathcal{G}$ has a basis containing of open bisections. If $\mathcal{G}$ is second countable, the basis can be made countable by picking γ above from a dense sequence and intersecting with countable neighbourhood bases for each γ as in [56, Lemma 2.4.9].

If B is an open bisection containing $\gamma \in \mathcal{G}$, then $\{\gamma\} = B \cap \mathcal{G}^{r(\gamma)}$ is open in $\mathcal{G}^{r(\gamma)}$ and hence the range and source fibres are discrete subspaces by the above. □

A class of étale groupoids that is particularly easy to deal with is that of *ample groupoids*:

Definition 4.1.8: Let $\mathcal{G}$ be an étale groupoid. If its topology has a basis of *compact* open bisections, $\mathcal{G}$ is called *ample*.

What makes these particularly tractable is the abundance of continuous indicator functions that come with these compact, open bisections, as we will see in Section 4.2.4 on groupoid C^*-simplicity.

4.1.2 C^*-algebras of Étale Groupoids

Just as with crossed products or group C^*-algebras, there is a reduced C^*-algebra associated with a topological groupoid, which is built from the appropriate algebra of functions completed with respect to the appropriate representation.

Let $\mathcal{G}$ be an étale groupoid and consider $C_c(\mathcal{G})$, the algebra of continuous, compactly supported, complex valued functions on $\mathcal{G}$. We endow $C_c(\mathcal{G})$ with the multiplication $*$ and involution * defined by

$$(f * g)(\gamma) = \sum_{\eta \in \mathcal{G}^{r(\gamma)}} f(\eta) g(\eta^{-1}\gamma),$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})},$$

for functions $f, g \in C_c(\mathcal{G})$ and an arrow $\gamma \in \mathcal{G}$. Often, $f * g$ is called the "convolution" of f and g. To see that $f * g$ is again continuous and compactly supported, note that all sums involved are finite, since f and g are compactly supported and the range fibres are discrete.

Constructing the regular representation, consider the Hilbert space $\ell^2(\mathcal{G}_x)$ for a unit $x \in \mathcal{G}^{(0)}$ an let π_x be the representation of $C_c(\mathcal{G})$ on $\ell^2(\mathcal{G}_x)$ by

$$\pi_x(f)\delta_\gamma = \sum_{\eta \in \mathcal{G}_{r(\gamma)}} f(\eta)\delta_{\eta\gamma}$$

for a function $f \in C_c(\mathcal{G})$ and δ_γ the basis vector of $\ell^2(\mathcal{G}_x)$ supported on $\gamma \in \mathcal{G}_x$. To obtain the reduced C^*-algebra $C^*_{\mathrm{r}}(\mathcal{G})$ of $\mathcal{G}$, we complete $C_c(\mathcal{G})$ in the reduced norm $\| \bullet \|_\lambda$ given by

$$\|f\|_\lambda = \sup_{x \in \mathcal{G}^{(0)}} \|\pi_x(f)\|_{\ell^2(\mathcal{G}_x)}$$

for $f \in C_c(\mathcal{G})$ or equivalently we can take the closure of the image of the representation $\bigoplus_{x \in \mathcal{G}^{(0)}} \pi_x$ in $\bigoplus_{x \in \mathcal{G}^{(0)}} \ell^2(\mathcal{G}_x)$. This representation is injective on $C_c(\mathcal{G})$ (see [56, Corollary 3.3.4]) and the norm restricts to $\|f\|_\lambda = \|f\|_\infty$ for $f \in C_c(\mathcal{G}^{(0)})$. The resulting copy of $C_0(\mathcal{G}^{(0)})$ in $C^*_{\mathrm{r}}(\mathcal{G})$ is often referred to as the *diagonal*.

As for groups, there is also a notion of the *maximal* C^*-algebra associated with a groupoid $\mathcal{G}$, given by a universal representation. In the simplest formulation, it is given by completing $C_c(\mathcal{G})$ with the *-algebra-structure above in the norm given by the supremum over all norms of *-representations on Hilbert spaces. As we will mainly be concerned with the reduced algebra, we only mention the maximal algebra in passing.

In analogy to crossed products, there is a conditional expectation $E\colon C^*_{\mathrm{r}}(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$ onto the commutative "diagonal" by keeping only the parts supported at units:

$$E(f) = f|_{\mathcal{G}^{(0)}}$$

for $f \in C^*_{\mathrm{r}}(G)$.

Example 4.1.9. *If a group G is understood as a groupoid $\mathcal{G}$ as in Example 4.1.2, then the associated C^*-algebras $C^*_r(G)$ and $C^*_r(\mathcal{G})$ are isomorphic with the isomorphism sending $\lambda_g \in \mathbb{C}[G]$ to the characteristic function $\chi_{\{g\}} \in C_c(\mathcal{G})$. Analogously, if the group G acts on a space X, then the crossed product $C_0(X) \rtimes_r G$ is isomorphic to the reduced C^*-algebra $C^*_r(G \ltimes X)$ of the associated transformation groupoid with the isomorphism sending a $f\lambda_h \in C_c(X)[G]$ to the function $(g, x) \mapsto f(x) \cdot \chi_{\{h\}}(g)$ in $C_c(G \ltimes X)$.*

4.1.3 Isotropy and Topological Principality

Point stabilisers G_x of a point $x \in X$ for a group G acting on a space X are generalised by the so-called *isotropy groups* in the groupoid setting. The isotropy group $\mathcal{G}^x_x = \mathcal{G}_x \cap \mathcal{G}^x$ at $x \in \mathcal{G}^{(0)}$ is given by the intersection of the range and source fibres at x, containing exactly those arrows of $\mathcal{G}$ that start and end at x. Consequently, all pairs of arrows in $\mathcal{G}^x_x$ are composable in either order and the isotropy group is indeed a group with unit x. The union of all isotropy groups in $\mathcal{G}$ is simply called the *isotropy* $\mathrm{Iso}(\mathcal{G})$ of $\mathcal{G}$ and contains at the minimum $\mathcal{G}^{(0)}$. In a Hausdorff groupoid the isotropy is closed, since for any net $\gamma_\lambda \to \gamma$ of arrows γ_λ in the isotropy converging to $\gamma \in \mathcal{G}$ we have $r(\gamma_\lambda) \to r(\gamma)$ and $s(\gamma_\lambda) \to s(\gamma)$ whence $r(\gamma) = s(\gamma)$. In general, the isotropy might not be open, not even in a Hausdorff étale groupoid. If the isotropy is trivial, that is, if $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$, then $\mathcal{G}$ is called *principal* which corresponds to freeness of the action of G on X in our analogy to group actions. Finding an analogue of topological stabilisers is more subtle. Recall that a group element $g \in G$ is in the topological stabiliser G°_x of $x \in X$ if and only if it fixes a neighbourhood U of x pointwise. In other words, $\{g\} \times U \subseteq G \ltimes X$ is contained in the isotropy of the transformation groupoid $G \ltimes X$. As the topology on $G \ltimes X$ is given by the product topology of $G \times X$, this is an open set if G is discrete and hence (g, x) is contained in the interior of the isotropy. On the other hand, if (g, x) is contained in the interior of the isotropy then there exists a neighbourhood U as above and g is contained in the topological stabiliser of x. The correct generalisation of a topological stabiliser at x is therefore the intersection of the isotropy group at x with the interior of the isotropy.

Definition 4.1.10: A groupoid $\mathcal{G}$ is called *topologically principal*, if the interior of the isotropy is the unit space $\mathcal{G}^{(0)}$.

As for group actions in Proposition 3.2.2, this simplifies in case the groupoid is étale:

Proposition 4.1.11: Let $\mathcal{G}$ be a second-countable étale groupoid. The groupoid $\mathcal{G}$ is topologically principal if and only if the set of points $x \in \mathcal{G}^{(0)}$ with trivial isotropy group $\mathcal{G}_x^x = \{x\}$ is dense in $\mathcal{G}^{(0)}$.

Proof. Assume that $\mathcal{G}$ is topologically principal, so $\mathrm{Iso}(\mathcal{G})^\circ = \mathcal{G}^{(0)}$. Recall that if $\mathcal{G}$ is étale, its topology has a basis of open bisections. Let B be an open bisection that does not intersect the unit space $\mathcal{G}^{(0)}$. As the isotropy is closed, $B \setminus \mathrm{Iso}(\mathcal{G})$ is open. Hence $s(B \setminus \mathrm{Iso}(\mathcal{G}))$ is open in $\mathcal{G}^{(0)}$ and since $B \cap \mathrm{Iso}(\mathcal{G})$ has empty interior by assumption, $B \setminus \mathrm{Iso}(\mathcal{G})$ is dense in B and $s(B \setminus \mathrm{Iso}(\mathcal{G}))$ is dense in $s(B)$. Aiming to apply the Baire category theorem as in the proof of Proposition 3.2.2, we add the interior of the complement of $s(B)$ and obtain the set $s(B \setminus \mathrm{Iso}(\mathcal{G})) \cup \mathcal{G}^{(0)} \setminus \overline{s(B)}$, which is open and dense in $\mathcal{G}^{(0)}$. Taking B from a *countable* basis of open bisections $\mathcal{B}$, the Baire category theorem yields that

$$V := \bigcap_{B \in \mathcal{B}} s(B \setminus \mathrm{Iso}(\mathcal{G})) \cup \mathcal{G}^{(0)} \setminus \overline{s(B \setminus \mathcal{G}^{(0)})} \subseteq \mathcal{G}^{(0)},$$

is dense in $\mathcal{G}^{(0)}$. If $\mathcal{G}_x^x \neq \{x\}$ is non-trivial, and $x \neq \gamma \in \mathcal{G}_x^x$ then there exists an open bisection $B \in \mathcal{B}$ such that $\gamma \in B$ and therefore $x \in s(B \setminus \mathcal{G}^{(0)})$ but $x \notin s(B \setminus \mathrm{Iso}(\mathcal{G}))$. Consequently, $x \notin V$ and V is therefore a dense subset of points in $\mathcal{G}^{(0)}$ whose isotropy groups are trivial.

On the other hand, as $\mathcal{G}^{(0)}$ is closed, $\mathrm{Iso}(\mathcal{G})^\circ \setminus \mathcal{G}^{(0)}$ is open in $\mathcal{G}$ and as the range map is open in an étale groupoid, $r(\mathrm{Iso}(\mathcal{G})^\circ \setminus \mathcal{G}^{(0)})$ is likewise an open subset of $\mathcal{G}^{(0)}$. Hence, if $\mathrm{Iso}(\mathcal{G})^\circ \setminus \mathcal{G}^{(0)}$ is non-empty $r(\mathrm{Iso}(\mathcal{G})^\circ \setminus \mathcal{G}^{(0)})$ is non-empty and the units with trivial isotropy are not dense in $\mathcal{G}^{(0)}$. □

The preceding proof is a good example of how concepts from the world of discrete group actions carry over to the world of étale groupoids, with “global” statements using single group elements replaced by “local” statements using bisections instead. Indeed, the French word “étale” means “spread out” and we believe that the nomenclature is motivated by single arrows of an étale groupoid being able to “spread out” to an open bisection and act locally instead of on a single point. In the case of a transformation groupoid $G \ltimes X$, an arrow (g, x) may simply spread out globally to the open bisection $\{g\} \times X$, but in a general étale groupoid more care is necessary. An important drawback, which was not an issue in the above proof, is that this choice of bisection is far from unique and we have to take care to make all arguments independent of the choice.

A groupoid is *minimal*, if the orbit of every unit is dense; that is, if $r(\mathcal{G}_x)$ is dense in $\mathcal{G}^{(0)}$ for every unit $x \in \mathcal{G}^{(0)}$. As for crossed products in Lemma 3.2.3, topological principality and minimality yield a sufficient criterion for simplicity of the reduced groupoid C^*-algebra:

Proposition 4.1.12 (Brown–Clark–Farthing–Sims [11, Theorem 5.1]): Let $\mathcal{G}$ be a topologically principal, minimal, Hausdorff étale groupoid. Then $C^*_{\mathrm{r}}(\mathcal{G})$ is simple.

We will not give a complete proof of Proposition 4.1.12, but point out the intermediate results of [11] that it relies on. First, topological principality ensures the analogue of the intersection property for crossed products by a result of Exel:

Lemma 4.1.13 (Exel [21, Theorem 4.4]): *Let $\mathcal{G}$ be a topologically principal, Hausdorff étale groupoid and I a non-trivial ideal of $C^*_r(\mathcal{G})$. Then $I \cap C_0(\mathcal{G}^{(0)}) \neq \{0\}$.*

On the other hand, minimality ensures that such ideals cannot be proper:

Lemma 4.1.14 (Brown–Clark–Farthing–Sims [11, Proposition 5.7]): *Let $\mathcal{G}$ be a Hausdorff étale groupoid. Then $\mathcal{G}$ is minimal if and only if for every non-zero $f \in C_0(\mathcal{G}^{(0)})$ the ideal of $C_r^*(\mathcal{G})$ generated by f is all of $C_r^*(\mathcal{G})$.*

Together, these imply Proposition 4.1.12:

Proof of Proposition 4.1.12: Let I be a non-zero ideal of $C_{\mathrm{r}}^*(\mathcal{G})$. By Lemma 4.1.13 I contains a non-zero $f \in C_0(\mathcal{G}^{(0)})$ and hence by Lemma 4.1.14 it is all of $C_{\mathrm{r}}^*(\mathcal{G})$. □

Prior to our results of Section 4.2.4, this is the only general criterion for simplicity of étale groupoids we were aware of, and for second-countable groupoids it follows easily from our result by Lemma 4.2.30.

4.1.4 Amenability

For discrete or even topological groups, amenability is a relatively clear-cut concept. While there famously roughly $10^{10^{10}}$ different characterisations (see [12, Chapter 2.6]), all of these are equivalent and maybe best described as the maximal and the reduced C^*-algebra of the group in question coinciding.

For an étale topological groupoid $\mathcal{G}$ the situation is quite different and, as far as we are aware, still under development. Once again the simplest condition imaginable is asking for the maximal and reduced norms on $C_c(\mathcal{G})$ to coincide, which Sims and Williams call “metric amenability” in [57]. However, metric amenability is not equivalent to groupoid-generalisations of the other, more powerful characterisations of amenability for discrete groups. In particular, the first notion of “amenability” for groupoids by Renault, now often called “topological amenability”, asks for the existence of a net $f_\lambda \in C_c(\mathcal{G})$ such that $f_\lambda * f_\lambda^*|_{\mathcal{G}^{(0)}}$ is uniformly bounded on the unit space $\mathcal{G}^{(0)}$ while $f_\lambda * f_\lambda^*$ converges to the constant function $\mathbb{1}$ uniformly on compact subsets of $\mathcal{G}$. This notion was later developed further by Anantharaman-Delaroche and Renault [2] in what is maybe the most extensive work on groupoid amenabilty. Among others, they provide the following characterisations:

Definition 4.1.15 [2, Proposition 2.2.13]: Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. We call $\mathcal{G}$ (topologically) amenable, if one of the following equivalent definitions holds:

1. There is a net of positive functions $f_\lambda \in C_c(\mathcal{G})^+$ such that
   (a) $\sum_{\gamma \in \mathcal{G}^x} f_\lambda(\gamma) \leq 1$ for all $x \in \mathcal{G}^{(0)}$.
   (b) $x \mapsto \sum_{\gamma \in \mathcal{G}^x} f_\lambda(\gamma)$ converges to 1 uniformly on compact subsets of $\mathcal{G}^{(0)}$.
   (c) $\gamma \mapsto \sum_{\eta \in \mathcal{G}^{r(\gamma)}} \left|f_\lambda(\gamma^{-1}\eta) - f(\eta)\right|$ converges to 0 uniformly on compact subsets of $\mathcal{G}$.
2. There is a net of functions $f_\lambda \in C_c(\mathcal{G})$ such that
   (a) $\sum_{\gamma \in \mathcal{G}^x} |f_\lambda(\gamma)|^2 \leq 1$ for all $x \in \mathcal{G}^{(0)}$.

   (b) $x \mapsto \sum\limits_{\gamma \in \mathcal{G}^x} |f_\lambda(\gamma)|^2$ converges to 1 uniformly on compact subsets of $\mathcal{G}^{(0)}$.

   (c) $\gamma \mapsto \sum\limits_{\eta \in \mathcal{G}^{r(\gamma)}} \left| f_\lambda(\gamma^{-1}\eta) - f(\eta) \right|^2$ converges to 0 uniformly on compact subsets of $\mathcal{G}$.

3. There is a net of functions $f_\lambda \in C_c(\mathcal{G})$ such that
   (a) $x \mapsto f_\lambda * f_\lambda^*(x)$ is uniformly bounded on $\mathcal{G}^{(0)}$.
   (b) $\gamma \mapsto f_\lambda * f_\lambda^*(\gamma)$ converges to 1 uniformly on compact subsets of $\mathcal{G}$.

Characterisations of amenability in terms of an invariant mean are more subtle, starting at the question of what measure space that invariant mean would be defined on. Indeed, there is a notion of "quasi-invariant measure" μ on the unit space, which induces a measure ν on $\mathcal{G}$ and a groupoid $\mathcal{G}$ is called "measurewise amenable", if $L^\infty(\mathcal{G}, \nu)$ has an invariant mean for every quasi-invariant measure μ. In general, topological amenability is stronger than measurewise amenability, but by [2, Theorem 3.3.7] they coincide for étale groupoids. The following easily defined characterisation of étale groupoid amenability in terms of invariant means is given by Sims in [56, Definition 4.1.2]:

Proposition 4.1.16: Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. Then $\mathcal{G}$ is amenable if and only if there is an *approximate invariant continuous mean* for $\mathcal{G}$. That is, if there are nets of Radon probability measures $(\mu_\lambda^x)_{\lambda \in \Lambda}$ for every $x \in \mathcal{G}^{(0)}$ such that

- the support of μ_λ^x is contained in $\mathcal{G}^x$
- $x \mapsto \int_{\mathcal{G}} f \, \mathrm{d}\mu_\lambda^x$ is continuous for every $\lambda \in \Lambda$ and $f \in C_c(\mathcal{G})$
- the net of functions M_λ on $\mathcal{G}$ given by
$$M_\lambda(\gamma) = \left\| \mu_\lambda^{r(\gamma)}(\gamma \cdot) - \mu_\lambda^{s(\gamma)}(\cdot) \right\|_1$$
converges to 0 uniformly on compact subsets of $\mathcal{G}$.

For étale groupoids (topological) amenability of $\mathcal{G}$ is indeed equivalent to nuclearity of the reduced C^*-algebra $C_r^*(\mathcal{G})$ by [2, Theorem 6.2.14], but while it implies that the maximal and reduced groupoid C^*-algebras are canonically isomorphic, it is not equivalent to the latter. A counterexample was famously given by Willett [60] based on a construction by Higson, Lafforgue, and Skandalis. His example is a group bundle, that is, a groupoid which consists of isotropy only. It is formed from a single group G by choosing an approximating sequence K_n of normal, finite-index subgroups of G and placing the groups G/K_n at $n \in \mathbb{N}$ over the one-point compactification $\mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ as unit space, with G at ∞. It turns out that amenability of the groupoid thus constructed, if equipped with the appropriate quotient topology, is equivalent to amenability of the group G, while the reduced and maximal C^*-algebras of the groupoid can be forced to coincide if the quasi-regular representations of G on the quotients G/K_n approximate the maximal norm on $\mathbb{C}[G]$. Willett gives an example of such an approximating sequence for G the free group in two generators.

Amenability still provides the partial converse to Proposition 4.1.12 given by Kawamura and Tomiyama in the context of crossed products. This is again a result by Brown, Clark, Farthing, and Sims:

Proposition 4.1.17 (Brown–Clark–Farthing–Sims [11, Theorem 5.1]): Let $\mathcal{G}$ be an amenable Hausdorff étale groupoid. Then $C^*_r(\mathcal{G})$ is simple if and only if it is minimal and topologically principal.

Our results in Section 4.2.4 generalise Proposition 4.1.17 by Lemma 4.2.31.

4.1.5 Groupoid Actions

While the action of a group on a topological space or C^*-algebra can be readily defined as a homomorphism into the appropriate automorphism group, the fibred nature of a groupoid makes its actions much more delicate and the objects acted upon will in general have to be fibred over its unit space. Below we introduce bundles of C^*-algebras and actions of groupoids on such bundles following the exposition [27] of Goehle.

Definition 4.1.18: For a locally compact Hausdorff space X, a *$C_0(X)$-algebra* is a C^*-algebra A equipped with a *-homomorphism Φ from $C_0(X)$ into the centre of the multiplier algebra of A such that Φ is non-degenerate in the sense that $\Phi(C_0(X))A$ is dense in A. A ucp map between two $C_0(X)$-algebras that respects the multiplication with elements of $C_0(X)$ will be called a *$C_0(X)$-map*.

For $f \in C_0(X)$ and $a \in A$ we will often suppress Φ in the notation and write fa instead of $\Phi(f)a$. There is a one-to-one correspondence between $C_0(X)$-algebras and upper-semicontinuous bundles of C^*-algebras over X: see for example Williams [61, Appendix C]. Therefore, we will not define such bundles separately, but instead explain how to interpret a $C_0(X)$-algebra as a bundle:

Let A be a $C_0(X)$-algebra and take any $x \in X$. Then $C_0(X \setminus x)$ is an ideal in $C_0(X)$ and consequently $I_x := \overline{C_0(X \setminus x)A}$ is a closed, two-sided ideal in A. We denote by $A_x := A/I_x$ the quotient of A by the ideal I_x and assemble these as fibres into the bundle $\mathcal{A} := \bigsqcup_{x \in X} A_x$. This bundle then carries a unique topology for which the bundle map $p \colon \mathcal{A} \to X$ sending any element of A_x to x is continuous and open and the C^*-algebra

$$\Gamma_0(X, \mathcal{A}) = \left\{ f \colon X \to \mathcal{A} \text{ continuous} \,\middle|\, f(x) \in A_x, \|f(x)\|_{A_x} \xrightarrow{x \to \infty} 0 \right\}$$

of continuous sections in $\mathcal{A}$ vanishing at infinity is isomorphic to A by the isomorphism $A \to \Gamma_0(X, \mathcal{A})$ sending $a \in A$ to the function $x \mapsto a + I_x \in A_x$. In this way, we may pass from bundles $\mathcal{A}$ to $C_0(X)$-algebras A and back by forming the algebra of continuous sections and by assembling the bundle fibrewise.

Definition 4.1.19: Let $\tau \colon Y \to X$ be a continuous map between two locally compact Hausdorff spaces and A a $C_0(X)$-algebra. The *pullback bundle $Y *_\tau \mathcal{A}$* is given by

$$Y *_\tau \mathcal{A} := \left\{ (y, a) \in Y \times \mathcal{A} \,\middle|\, \tau(y) = p(a) \right\}$$

with the relative topology of $Y \times \mathcal{A}$. It is an upper-semicontinuous bundle over Y with bundle map $q(y, a) = y$.

We are now ready to define the action of a groupoid on a C^*-algebra fibred over its unit space. Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid and A a $C_0(\mathcal{G}^{(0)})$-algebra. An *action* α of $\mathcal{G}$ on A is a family of *-isomorphisms $\alpha_\gamma\colon A_{s(\gamma)} \to A_{r(\gamma)}$ for $\gamma \in \mathcal{G}$ such that $\alpha_{\eta\gamma} = \alpha_\eta \circ \alpha_\gamma$ whenever η and γ are composable and such that the map

$$\mathcal{G} *_s \mathcal{A} \to \mathcal{A} \qquad \text{given by} \qquad (\gamma, a) \mapsto \alpha_\gamma(a)$$

is continuous. We then call A a $\mathcal{G}$-algebra. If there is no confusion about the action, we will abbreviate $\alpha_\gamma(a)$ to $\gamma.a$.

A $C_0(\mathcal{G}^{(0)})$-map $\varphi\colon A \to B$ between two $\mathcal{G}$-algebras with $\mathcal{G}$-actions denoted by α and β is called *$\mathcal{G}$-equivariant* if the induced maps on the fibres satisfy

$$\varphi_{r(\gamma)}\big(\alpha_\gamma(a)\big) = \beta_\gamma\big(\varphi_{s(\gamma)}(a)\big)$$

for all $\gamma \in \mathcal{G}$ and $a \in A_{s(\gamma)}$.

Accordingly, we provide a definition for an operator system with groupoid action:

Definition 4.1.20: A $\mathcal{G}$-operator system S is an operator system in a unital $\mathcal{G}$-algebra A that is closed under both the action of $C_0(\mathcal{G}^{(0)})$ and the action of $\mathcal{G}$, the latter meaning that $\alpha_\gamma(S/I_{s(\gamma)}) \subseteq A_{r(\gamma)}$ is contained in $S/I_{r(\gamma)}$.

Note that this is a *concrete* operator system, meaning that it is defined explicitly as a subset of a $\mathcal{G}$-algebra. While a general operator system may be defined abstractly without an enveloping C^*-algebra, defining a groupoid action requires a notion of ideals which only makes sense in the surrounding C^*-algebra and hence we deal only with concrete operator systems.

4.2 The Groupoid Furstenberg Boundary

This section is based on [8] and contains the main results of this thesis. Building on the work of Hamana [29, 30, 34], we define the Furstenberg boundary of a groupoid, generalising the Furstenberg boundary for discrete groups. This is achieved by providing a construction of a groupoid-equivariant injective envelope through a new induction procedure. Using this injective envelope, we establish the absence of dynamically recurrent amenable subgroups in the isotropy as a sufficient criterion for the intersection property of a locally compact Hausdorff étale groupoid with compact unit space and no fixed points. In turn, this yields a criterion for C^*-simplicity of minimal groupoids.

Following the recent successes of Kalantar–Kennedy [36] and Kennedy [40] several authors set forth to generalise Hamana's construction to obtain Furstenberg-type boundaries in different settings in the hope of providing similarly powerful tools, compare for example Bearden–Kalantar [4] and Monod [43]. While these works remain in the realm of groups, we provide an injective envelope construction that generalises the Furstenberg boundary to groupoids.

We introduce our new induction functor in Section 4.2.1, allowing us to transport injective operator systems to the category of operator systems with groupoid action, and enabling us to construct the boundary of a given groupoid in said category in Section 4.2.2. Consequently, we examine some properties of the boundaries thus defined in Section 4.2.3 and apply them to obtain a sufficient condition for C^*-simplicity of an étale groupoid in Section 4.2.4.

4.2.1 Induction of Groupoid Actions

The first step towards constructing an injective envelope among the $\mathcal{G}$-operator systems is to find a suitable, but possibly too large, injective object in this category. As injectivity of operator systems is well-understood in the absence of an action, we aim to obtain a $\mathcal{G}$-injective object by inducing the trivial action of $\mathcal{G}^{(0)}$ to an action of $\mathcal{G}$, and transporting morphisms in a natural fashion. We are grateful to R. Meyer for pointing out that this is the correct notion behind the construction of group-equivariant injective envelopes in Proposition 2.2.6.

Let $\mathcal{G}$ be a Hausdorff étale groupoid that acts on a $\mathcal{G}$-algebra W, and $\mathcal{H} \subseteq \mathcal{G}$ a closed subgroupoid such that $\mathcal{H}^{(0)} = \mathcal{G}^{(0)}$. Then W is also an $\mathcal{H}$-algebra and we denote the resulting restriction functor from $\mathcal{G}$-algebras to $\mathcal{H}$-algebras by $\operatorname{res}^{\mathcal{G}}_{\mathcal{H}}$. As alluded to above, we set out to find a right adjoint to $\operatorname{res}^{\mathcal{G}}_{\mathcal{H}}$ between the categories of $\mathcal{G}$- and $\mathcal{H}$-operator systems with equivariant unital completely positive maps as morphisms. More precisely, we seek to assign a $\mathcal{G}$-algebra $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}}(A)$ to every $\mathcal{H}$-algebra A such that for every $\mathcal{G}$-algebra W there is a natural bijection

$$\operatorname{Hom}_{\mathcal{G}}\Big(W, \operatorname{Ind}^{\mathcal{G}}_{\mathcal{H}}(A)\Big) \cong \operatorname{Hom}_{\mathcal{H}}\Big(\operatorname{res}^{\mathcal{G}}_{\mathcal{H}} W, A\Big). \tag{4.1}$$

To construct injective envelopes, we may restrict to the case where $\mathcal{H} = \mathcal{G}^{(0)}$.

In his recent PhD-thesis [5, Chapter 3], Bönicke provides a method for inducing groupoid-C^*-algebras from subgroupoids, but this construction does not provide a right adjoint to restriction. In this section, we modify his construction to obtain an induction functor satisfying Equation (4.1).

As $\mathcal{H} = \mathcal{G}^{(0)}$ consists only of units, there is no additional data in the $\mathcal{G}^{(0)}$-action and any $\mathcal{G}^{(0)}$-algebra that we want to induce to a $\mathcal{G}$-algebra will have no further structure than that of a $C(\mathcal{G}^{(0)})$-algebra. Let A be such a $\mathcal{G}^{(0)}$-algebra. As before we let $\mathscr{A}$ denote the bundle over $\mathcal{G}^{(0)}$ associated with A. We may then form its pullback $\mathcal{G} *_s \mathscr{A}$ along the source map $s\colon \mathcal{G} \to \mathcal{G}^{(0)}$, which is a bundle over $\mathcal{G}$. We define the induced C^*-algebra

$$\operatorname{Ind} A := \Gamma_{\mathrm{b}}(\mathcal{G}, \mathcal{G} *_s \mathscr{A}) \tag{4.2}$$

to be the *bounded* continuous sections from $\mathcal{G}$ into that pullback.

Remark. It is here that we deviate from Bönicke's construction [5, page 15], where only sections *vanishing at infinity* in the appropriate sense are considered: compare condition (2) on page 15 of [5]. Note that condition (1) on the same page of [5] is trivial for $\mathcal{H} = \mathcal{G}^{(0)}$, but could well be added to obtain a more general induction functor. Contrary to [5], our definition of $\operatorname{Ind} A$ does not yield a $C_0(\mathcal{G})$-algebra, as the action of $C_0(\mathcal{G})$ is degenerate unless $\mathcal{G}$ is compact.

Pushforward along the range map gives an action of $C(\mathcal{G}^{(0)})$ on $\operatorname{Ind} A$ by central multipliers: For $f \in \Gamma_b(\mathcal{G}, \mathcal{G} *_s \mathscr{A})$ and $g \in C(\mathcal{G}^{(0)})$, we set

$$gf := (g \circ r)f.$$

This action is non-degenerate, as $C(\mathcal{G}^{(0)})$ is unital. It is worth pointing out that the fibre of $\operatorname{Ind} A$ at a unit u is *not* given by $\Gamma_b(\mathcal{G}^u, \mathcal{G}^u *_s, \mathscr{A})$ as in [5], which makes it more difficult to formulate a $\mathcal{G}$-action on the induced bundle. This is alleviated by the fibre projection only depending on the restriction to a neighbourhood of $\mathcal{G}^u$, as we will see in the following lemma.

Recall that for a unit $u \in \mathcal{G}^{(0)}$ the fibre $(\operatorname{Ind} A)_u$ at u of the bundle $\operatorname{Ind} A$ is given by $(\operatorname{Ind} A)/I_u$, where $I_u = \overline{C_0(\mathcal{G}^{(0)} \setminus u) \operatorname{Ind} A}$.

Lemma 4.2.1: *Let $f, g \in \operatorname{Ind} A$ be such that their restrictions to $r^{-1}(V)$ coincide for a neighbourhood $V \subseteq \mathcal{G}^{(0)}$ of a unit $u \in \mathcal{G}^{(0)}$. Then $f + I_u = g + I_u$.*

Proof. Take f and g as above and pick $h \in C(\mathcal{G}^{(0)})$ such that $h(u) = 0$ and $h \equiv 1$ outside of V. Such h since $\mathcal{G}^{(0)}$ is normal. Then $g = f + h(g - f)$ while $h(g - f) \in C_0(\mathcal{G}^{(0)} \setminus u) \operatorname{Ind} A$, so $f - g \in I_u$. □

For $f \in \operatorname{Ind} A$ and $u \in \mathcal{G}^{(0)}$ we will write $[f]_u$ for the fibre projection $f + I_u$.

Proposition 4.2.2: If A is a $\mathcal{G}^{(0)}$-algebra, then $\operatorname{Ind} A$ is a $\mathcal{G}$-algebra.

Remark. In the spirit of Bönicke's construction [5], where fibre are associated with restrictions to $\mathcal{G}^u$, the action should be by composition on the argument. However, as the fibre projection of a section $f \in \operatorname{Ind} A$ at a unit $u \in \mathcal{G}^{(0)}$ is not determined by the values on $\mathcal{G}^u$ alone, we cannot act by the single element $\gamma \in \mathcal{G}$. Given that values on a *neighbourhood* of $\mathcal{G}^u$ determine the fibres, we may instead choose a bisection around γ and use this to act on the argument. Using the locality of Lemma 4.2.1, this will turn out to be well-defined.

Proof. We have already seen that $\operatorname{Ind} A$ is a $C_0(\mathcal{G}^{(0)})$-algebra, so it remains to describe the $\mathcal{G}$-action.

Let $f \in \operatorname{Ind} A$ be a section of the induced bundle, $u, v \in \mathcal{G}^{(0)}$ be units, and $\gamma \in \mathcal{G}_u^v$ be an arrow with source u and range v. Considering $[f]_u \in (\operatorname{Ind} A)_u$ we want to define $\gamma.[f]_u \in (\operatorname{Ind} A)_v$. As $\mathcal{G}$ is étale, we may pick an open neighbourhood B of γ that is a bisection. Then $U = s(B)$ and $V = r(B)$ are open neighbourhoods of u and v. We write $B.f$ for the section

$$B.f(\eta) := f(B^{-1}\eta) \quad \text{in} \quad \Gamma_b(r^{-1}(V), r^{-1}(V) *_s \mathcal{A}),$$

where $\eta \in r^{-1}(V)$ and $B^{-1}\eta = \xi^{-1}\eta$ for the unique $\xi \in B$ with $r(\xi) = r(\eta)$. In order to extend $B.f$ to a section in $\operatorname{Ind} A$, we choose a function $h \in C(\mathcal{G}^{(0)})$ which is identically one on a neighbourhood of v and vanishes outside of V. Such an h exists by normality of $\mathcal{G}^{(0)}$. Then $h(B.f)$ extends to a section on all of $\mathcal{G}$ that vanishes outside of $r^{-1}(V)$, and we set $\gamma.[f]_u = [h \cdot B.f]_v$.

The resulting class is independent of the choice of h, as two different choices h_1 and h_2 coincide on a neighbourhood V' of v and therefore the extensions $h_1(B.f)$ and $h_2(B.f)$ coincide on $r^{-1}(V')$. Hence the fibre projections coincide by Lemma 4.2.1.

Similarly, this is independent of the choice of bisection B: For two open bisections B_1 and B_2 containing γ, the intersection $B_1 \cap B_2$ is still an open bisection containing γ, so we may assume $B_2 \subseteq B_1$. But then $r(B_2) \subseteq r(B_1)$ and we can choose the same h for both B_2 and B_1. With this choice, $h(B_1.f)$ and $h(B_2.f)$ are equal as sections, even before passing to the fibre projection.

For simplicity, we may from now on drop h in the construction above and assume that $B.f$ can be extended to a section on all of $\mathcal{G}$ without prior modification, as we can just intersect B with the r-preimage of the neighbourhood of v on which h is constant.

To finish off the argument that $\gamma.[f]_u := [B.f]_v$ is well-defined, we show that it only depends on the u-fibre projection of f. Assuming that $[f]_u = [g]_u$, we may for every $\epsilon > 0$

find $h \in C_0(\mathcal{G}^{(0)} \setminus u) \cdot \operatorname{Ind} A$ such that g and $f + h$ are ϵ-close. Writing $h = h_1 \cdot h_2$ with $h_1 \in C_0(\mathcal{G}^{(0)} \setminus u)$ and $h_2 \in \operatorname{Ind} A$, we consider $B.h = B.h_1 \cdot B.h_2$, where the action of B on $\mathcal{G}^{(0)}$ is given by the local homeomorphism $s(B) \to r(B)$ via $r \circ (s|_B)^{-1}$. By the arguments above, we can assume that $B.h_1$ and $B.h_2$ can be extended to functions on all of $\mathcal{G}^{(0)}$ and $\mathcal{G}$, respectively. As then $B.h_1 \in C_0(\mathcal{G}^{(0)} \setminus v)$, we find that $[B.f]_v = [B.f + B.h]_v$. But $B.f + B.h$ and $B.g$ are ϵ-close on $r^{-1}(V)$ since the action of B is pointwise, and by extending as above with a cut-off function that is bounded by one, we may modify these to two ϵ-close functions on all of $\mathcal{G}$ without changing their classes in the v-fibre. Therefore $[B.f]_v$ and $[B.g]_v$ are ϵ-close for all $\epsilon > 0$, hence equal.

The $\mathcal{G}$-action satisfies $(\gamma\eta).[f]_u = \gamma.(\eta.[f]_u)$, as we can compose open bisections where defined to obtain another open bisection. As the defined maps act pointwise, they are *-homomorphisms, and as they are invertible we obtain *-isomorphisms.

Finally, we check that the $\mathcal{G}$-action is continuous. That is, if for $f, f_\lambda \in \operatorname{Ind} A$ and $\gamma, \gamma_\lambda \in \mathcal{G}$ we have $[f_\lambda]_{s(\gamma_\lambda)} \to [f]_{s(\gamma)}$ and $\gamma_\lambda \to \gamma$, then we need to show that $\gamma_\lambda.[f_\lambda]_{s(\gamma_\lambda)} \to \gamma.[f]_{s(\gamma)}$. If B is an open bisection containing γ, then we will eventually have $\gamma_\lambda \in B$. Hence, we may use the same bisection to construct $\gamma_\lambda.[f_\lambda]_{s(\gamma_\lambda)}$ and $\gamma.[f]_{s(\gamma)}$. Consequently, we have to show that $[B.f_\lambda]_{r(\gamma_\lambda)} \to [B.f]_{r(\gamma)}$, which is equivalent to $\inf_r \|B.f_\lambda - B.f + h'\|_\infty \to 0$ as $\lambda \to \infty$, where $\inf_r$ denotes the infimum taken over all $h' \in C_0(\mathcal{G}^{(0)} \setminus r(g_\lambda)) \cdot \operatorname{Ind} A$. We know, however, that $\inf_s \|f_\lambda - f + h\|_\infty \to 0$, where $\inf_s$ is the infimum taken over all $h \in C_0(\mathcal{G}^{(0)} \setminus s(\gamma_\lambda)) \cdot \operatorname{Ind} A$. By choosing h' as $B.h$, we may bound

$$\inf_r \|B.f_\lambda - B.f + h'\|_\infty \leq \inf_s \|f_\lambda - f + h\|_\infty \to 0 \tag{4.3}$$

which yields the claim. □

Remark. Note that we did not require f to be continuous in the proof of the previous lemma. We will later turn the bounded, *not necessarily continuous* sections $l^\infty(\mathcal{G}, \mathcal{G} *_s \mathcal{A})$ into a $\mathcal{G}$-algebra in the same way.

The following notational remark is essential to avoid confusion:

Remark. For a section $a \in A = \Gamma(\mathcal{G}^{(0)}, \mathcal{A})$ we will denote by a_u the value $a(u)$ of a at the unit $u \in \mathcal{G}^{(0)}$, which is the same as the projection $[a]_u$ of a onto the appropriate fibre. This is straightforward enough for A, but note that we defined elements $f \in \operatorname{Ind} B$ in the induced C^*-algebra as functions on $\mathcal{G}$, not $\mathcal{G}^{(0)}$. Nevertheless, the associated bundle is of course still fibred over $\mathcal{G}^{(0)}$, taking values in the appropriate quotients of $\operatorname{Ind} B$. As such, while it makes sense to speak of $f(\gamma) \in B_{s(\gamma)}$, this is *not* the value of the section associated with f at any given fibre, even if γ were a unit. We will therefore refrain from writing f_u for the projection $[f]_u \in (\operatorname{Ind} B)_u$ of f in the fibre at a unit u, as this class should not be confused with $f(u) \in B_u$.

Let A be a $\mathcal{G}$-algebra and B a $\mathcal{G}^{(0)}$-algebra. Below, we explain how a $\mathcal{G}^{(0)}$-*-homomorphism $\varphi\colon A \to B$ lifts to a $\mathcal{G}$-*-homomorphism $\psi\colon A \to \operatorname{Ind} B$. First, we note that A embeds into $\operatorname{Ind} A$ as $\mathcal{G}$-algebras, where we drop spelling out the restriction and denote A as a $\mathcal{G}$- and a $\mathcal{G}^{(0)}$-algebra simultaneously: Let

$$\iota\colon A = \Gamma(\mathcal{G}^{(0)}, \mathcal{A}) \to \Gamma_b(\mathcal{G}, \mathcal{G} *_s \mathcal{A}) = \operatorname{Ind} A$$

be given by sending a section a to the section $\gamma \mapsto \alpha_\gamma^{-1}(a_{r(\gamma)})$, where α is the $\mathcal{G}$-action on $\mathcal{A}$. This is obviously a $C(\mathcal{G}^{(0)})$-linear map, since multiplication is pointwise after the pushforward via the range map. The section thus defined is continuous, since it is given by a composition of continuous maps, namely

$$\begin{aligned}\mathcal{G} &\to \mathcal{G} *_r \mathcal{G}^{(0)} \to \mathcal{G} *_r \mathcal{A} \to \mathcal{G} *_s \mathcal{A}\\ \gamma &\mapsto (\gamma, r(\gamma)) \mapsto (\gamma, a_{r(\gamma)}) \mapsto (\gamma, \alpha_\gamma^{-1}(a_{r(\gamma)})),\end{aligned}$$

where the last map is continuous by continuity of the $\mathcal{G}$-action on $\mathcal{A}$. Furthermore, ι is $\mathcal{G}$-equivariant, as $\mathcal{G}$ acts on the argument on $\operatorname{Ind} A$. Note that all sections $\iota(a)$ have constant norm on range-fibres and that this implies that $[\iota(a)]_u$ is determined by its values on $\mathcal{G}^u$, in contrast to general sections in $\operatorname{Ind} A$.

Second, from a $\mathcal{G}^{(0)}$-*-homomorphism $\varphi\colon A \to B$, we obtain a $\mathcal{G}$-*-homomorphism $\operatorname{Ind}\varphi$ mapping $\operatorname{Ind} A \to \operatorname{Ind} B$ as follows:

$$\begin{aligned}\operatorname{Ind}\varphi\colon \operatorname{Ind} A = \Gamma_b(\mathcal{G}, \mathcal{G} *_s \mathcal{A}) &\to \Gamma_b(\mathcal{G}, \mathcal{G} *_s \mathcal{B}) = \operatorname{Ind} B\\ f &\mapsto \big(\gamma \mapsto \varphi_{s(\gamma)}(f(\gamma))\big).\end{aligned}$$

This is $C(\mathcal{G}^{(0)})$-linear, since the appropriate fibre of φ is applied pointwise; it is also $\mathcal{G}$-equivariant, since the map is pointwise and $\mathcal{G}$ acts on the argument. The assigned section is obviously bounded, so it only remains to show that it is continuous. Given a net $\gamma_\lambda \to \gamma$ in $\mathcal{G}$ and $f \in \operatorname{Ind} A$, we have $f(\gamma_\lambda) \to f(\gamma)$ and so for all sections $a \in A$ with $a_{s(\gamma)} = f(\gamma)$ we find that $\|a_{s(\gamma_\lambda)} - f(\gamma_\lambda)\|_{A_{s(\gamma_\lambda)}} \to 0$. Therefore

$$\left\|(\varphi(a))_{s(\gamma_\lambda)} - \operatorname{Ind}\varphi(f)(\gamma_\lambda)\right\|_{B_{s(\gamma_\lambda)}} = \left\|\varphi_{s(\gamma_\lambda)}\big(a_{s(\gamma_\lambda)}\big) - \varphi_{s(\gamma_\lambda)}(f(\gamma_\lambda))\right\|_{B_{s(\gamma_\lambda)}} \le \left\|a_{s(\gamma_\lambda)} - f(\gamma_\lambda)\right\|_{A_{s(\gamma_\lambda)}}$$

vanishes in the limit. As φ is continuous, we have $\varphi(a) \in B$ with $(\varphi(a))_{s(\gamma)} = \varphi_{s(\gamma)}(f(\gamma))$. We may conclude that $(\operatorname{Ind}\varphi(f))(\gamma_\lambda)$ converges to $(\operatorname{Ind}\varphi(f))(\gamma)$.

Altogether, the adjoint isomorphism $\operatorname{rInd}\colon \operatorname{Hom}_{\mathcal{G}^{(0)}}(\operatorname{res} A, B) \to \operatorname{Hom}_{\mathcal{G}}(A, \operatorname{Ind} B)$ is given by

$$\operatorname{rInd}\varphi := \operatorname{Ind}\varphi \circ \iota$$

for $\varphi \in \operatorname{Hom}_{\mathcal{G}^{(0)}}(A, B)$. Explicitly, a section $a \in A$ is mapped to the section given by

$$((\operatorname{rInd}\varphi)(a))_\gamma = \varphi_{s(\gamma)}\Big(\alpha_\gamma^{-1}\big(a_{r(\gamma)}\big)\Big). \tag{4.4}$$

We now see why we had to modify the induction procedure of [5]: The sections defined in Equation (4.4) do not vanish at infinity.

Conversely, any $\mathcal{G}$-*-homomorphism $\psi\colon A \to \operatorname{Ind} B$ restricts to a $\mathcal{G}^{(0)}$-*-homomorphism, which we denote as $\operatorname{res}\psi\colon A \to B$, by restricting from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ as

$$(\operatorname{res}\psi(a))_u := \psi(a)(u).$$

Extending φ and then restricting to $\operatorname{res}\operatorname{rInd}\varphi$ obviously gives back φ, since $\varphi_u(\alpha_u^{-1}(a_u)) = \varphi_u(a_u)$. On the other hand, we will see that restricting ψ and then extending the result to $\operatorname{rInd}\operatorname{res}\psi$ also gives back ψ.

For every $\gamma \in \mathcal{G}$ we obtain a *-homomorphism $\operatorname{eval}_\gamma\colon \operatorname{Ind} B \to B_{s(\gamma)}$ by evaluating a section in $\operatorname{Ind} B$ at γ. As $\gamma \in \mathcal{G}^{r(\gamma)}$, this *-homomorphism factors through the quotient map to the fibre at $r(\gamma)$ and satisfies a straightforward equivariance condition:

Lemma 4.2.3: *For* $b_{r(\gamma)} \in B_{r(\gamma)}$ *with* $\gamma, \eta \in \mathcal{G}$ *where* $s(\gamma) = r(\eta)$, *we have*

$$\mathrm{eval}_\eta\Big(\gamma^{-1}.\big(b_{r(\gamma)}\big)\Big) = \mathrm{eval}_{\gamma\eta}\big(b_{r(\gamma)}\big).$$

Proof. This comes down to γ acting on the argument. If $b \in B$ is a section with value $b_{r(\gamma)}$ in the appropriate fibre and S is an open bisection containing γ^{-1}, then any lift of $\gamma^{-1} b_{r(\gamma)}$ to a section Sb in $\mathrm{Ind}\, B$ is given by $(Sb)(\eta) = b(S^{-1}\eta)$ on a neighbourhood of $\mathcal{G}^{s(\gamma)}$. Hence for $\eta \in \mathcal{G}^{s(\gamma)}$ we find

$$\mathrm{eval}_\eta\big(\gamma^{-1} b_{r(\gamma)}\big) = b(S^{-1}\eta) = b(\gamma\eta) = \mathrm{eval}_{\gamma\eta}\big(b_{r(\gamma)}\big).$$

□

Using the previous lemma, restricting $\psi \in \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Ind}\, B)$ and then lifting the restriction gives

$$\begin{aligned}(\mathrm{rInd}\,\mathrm{res}\,\psi)(a)(\gamma) &= (\mathrm{res}\,\psi)_{s(\gamma)}\Big(\gamma^{-1}\big(a_{r(\gamma)}\big)\Big)\\ &= \mathrm{eval}_{s(\gamma)}\bigg(\psi_{s(\gamma)}\Big(\gamma^{-1}\big(a_{r(\gamma)}\big)\Big)\bigg)\\ &= \mathrm{eval}_{s(\gamma)}\bigg(\gamma^{-1}\Big(\psi_{r(\gamma)}\big(a_{r(\gamma)}\big)\Big)\bigg)\\ &= \mathrm{eval}_{\gamma}\Big(\psi_{r(\gamma)}\big(a_{r(\gamma)}\big)\Big)\\ &= \psi(a)(\gamma).\end{aligned}$$

Therefore, rInd : $Hom_{\mathcal{G}^{(0)}}(A, B) \cong \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Ind}\, B)$ is a bijection, as the two constructions are inverse to each other.

We proceed to show that this isomorphism is natural. That is, given a $\mathcal{G}$-*-homomorphism $j \colon A \to B$ between $\mathcal{G}$-algebras A and B, as well as a $\mathcal{G}^{(0)}$-algebra C, the following diagram commutes:

$$\begin{tikzcd}\mathrm{Hom}_{\mathcal{G}^{(0)}}(B, C) \arrow[r, "\mathrm{rInd}", "\cong"'] \arrow[d, "j^*"] & \mathrm{Hom}_{\mathcal{G}}(B, \mathrm{Ind}\, C) \arrow[d, "j^*"] \\ \mathrm{Hom}_{\mathcal{G}^{(0)}}(A, C) \arrow[r, "\mathrm{rInd}", "\cong"'] & \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Ind}\, C)\end{tikzcd} \tag{4.5}$$

Given $\varphi \in \mathrm{Hom}_{\mathcal{G}^{(0)}}(B, C)$ we have to show that $\mathrm{rInd}\,(\varphi \circ j)$ equals $(\mathrm{rInd}\,\varphi) \circ j$. Indeed we find, using equivariance of j, that

$$\begin{aligned}(\mathrm{rInd}\,(\varphi \circ j))(a)(\gamma) &= (\varphi \circ j)_{s(\gamma)}\Big(\gamma^{-1}\big(a_{r(\gamma)}\big)\Big)\\ &= \varphi_{s(\gamma)}\bigg(\gamma^{-1}\Big(j_{r(\gamma)}\big(a_{r(\gamma)}\big)\Big)\bigg)\\ &= (\mathrm{rInd}\,\varphi)(j(a))(\gamma),\end{aligned}$$

where the action of γ^{-1} on $a_{r(\gamma)}$ and $b_{r(\gamma)}$ is the $\mathcal{G}$-action on A or B, respectively.

So far, we have only considered *-homomorphisms as morphisms between $\mathcal{G}$-algebras. To apply our induction procedure in the construction of a $\mathcal{G}$-equivariant injective envelope, we will need to broaden our scope to also include unital, completely positive, $\mathcal{G}$-equivariant $C(\mathcal{G}^{(0)})$-maps between $\mathcal{G}$-algebras.

First we note that being positive is a fibre-wise condition:

Lemma 4.2.4: *Let $\varphi\colon A \to B$ be a $C_0(X)$-map between two $C_0(X)$-algebras A and B. Then φ is positive if and only if $\varphi_x\colon A_x \to B_x$ is positive for every x in X. Likewise, $\varphi\colon A \to B$ is completely positive if and only if all $\varphi_x\colon A_x \to B_x$ are completely positive.*

Proof. We may embed A into $\bigoplus_{x\in X} A_x$, and as injective *-homomorphisms are order embeddings, the first statement follows.

For the second half observe that $M_n(A)$ is a $C_0(X)$-algebra whose fibre can be understood as $(M_n(A))_x \cong M_n(A_x)$ and that under this identification $\left(\varphi^{(n)}\right)_x = (\varphi_x)^{(n)}$. □

Now, for a ucp $\mathcal{G}^{(0)}$-map $\varphi\colon A \to B$ from a $\mathcal{G}$-algebra A to a $\mathcal{G}^{(0)}$-algebra B, we may as above form

$$\operatorname{Ind}\varphi(f)(\gamma) := \varphi_{s(\gamma)}(f(\gamma)) \qquad \text{and} \qquad \operatorname{rInd}\varphi := \operatorname{Ind}\varphi \circ \iota.$$

Without any modification to the arguments above, rInd still is natural and an inverse to the restriction, and $\operatorname{rInd}\varphi$ is ucp, since a section $f \in \Gamma_b(\mathcal{G}, \mathcal{G} *_s \mathcal{A})$ is positive if and only if $f(\gamma)$ is positive for all $\gamma \in \mathcal{G}$. Hence, if $\varphi\colon A \to B$ is completely positive then so are $\operatorname{Ind}\varphi\colon \operatorname{Ind}A \to \operatorname{Ind}B$ and $\operatorname{rInd}\varphi\colon A \to \operatorname{Ind}B$.

4.2.2 The Groupoid Furstenberg Boundary

Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid with compact unit space $X := \mathcal{G}^{(0)}$. Using the construction of $\mathcal{G}$-equivariant injective envelopes above, we may now find a $\mathcal{G}$-injective $\mathcal{G}$-algebra enveloping $C(X)$ in the category of $\mathcal{G}$-operator systems. Note that this generalises the Furstenberg boundary of a discrete group G, where the unit space X is a single point and the boundary coincides with the spectrum of the G-equivariant injective envelope of $C(X) = \mathbb{C}$. The methods below may however be used more generally to construct groupoid-equivariant injective envelopes.

Consider the (non-dynamic) injective envelope $I := I(C(X))$. Let $\tilde{X}$ denote its spectrum, so that $I = C(\tilde{X})$. As $C(X)$ embeds into $C(\tilde{X})$, the latter is a $C(X)$-algebra and it is furthermore injective among such: For any two (unital) $C(X)$-algebras $V \subseteq W$, we may lift a ucp $C(X)$-map $\psi\colon V \to C(\tilde{X})$ to a ucp map $W \to C(\tilde{X})$ by disregarding the $C(X)$-structure, and the result will necessarily be a $C(X)$-map if all maps and algebras are unital, as in that case the action of $C(X)$ on V and W is determined by a subalgebra of V which lies in the multiplicative domain of ψ.

Therefore, the induced C^*-algebra, $\operatorname{Ind}C(\tilde{X})$, is a $\mathcal{G}$-injective $\mathcal{G}$-algebra: Given two $\mathcal{G}$-algebras $V \subseteq W$ and a $\mathcal{G}$-equivariant ucp map $\psi\colon V \to \operatorname{Ind}C(\tilde{X})$, we may first restrict to $\operatorname{res}\psi\colon V \to C(\tilde{X})$, then use the $\mathcal{G}^{(0)}$-injectivity to extend the restriction to $\varphi\colon W \to C(\tilde{X})$ and consequently lift to a $\mathcal{G}$-*-homomorphism $\operatorname{rInd}\varphi\colon W \to \operatorname{Ind}C(\tilde{X})$. By naturality as in Diagram (4.5), this is the desired extension, as lifting $\operatorname{res}\psi$ results in $\operatorname{rInd}\operatorname{res}\psi = \psi$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\mathrm{rInd}\,\varphi"] & & \\
& C(\tilde{X}) & \mathrm{Ind}\,C(\tilde{X}) \arrow[l, "\mathrm{res}"'] \\
V \arrow[uu, hook] \arrow[ur, "\mathrm{res}\,\psi"] \arrow[urr, bend right, "\psi=\mathrm{rInd}\,\mathrm{res}\,\psi"'] & &
\end{tikzcd} \tag{4.6}$$

Having found a $\mathcal{G}$-injective $\mathcal{G}$-extension of the $\mathcal{G}$-C^*-algebra $C(X)$, we follow Hamana's scheme [34] from Section 2.2.2 to construct an injective envelope, adapted to a $\mathcal{G}$-equivariant setting. The following definitions are $\mathcal{G}$-equivariant adaptations of the corresponding notions from Definitions 2.2.7 and 2.2.4.

Definition 4.2.5: Let A and B be $\mathcal{G}$-C^*-algebras with A a sub-$\mathcal{G}$-C^*-algebra of B. An *A-seminorm* on B is a seminorm p on B, such that $p(b) = \|\varphi(b)\|_B$ for some $\mathcal{G}$-equivariant $C(\mathcal{G}^{(0)})$-linear ucp map $\varphi\colon B \to B$ that restricts to the identity on A.

The A-seminorms are partially ordered by the pointwise order where an A-seminorm p is dominated by an A-seminorm q if $p(b) \leq q(b)$ for all $b \in B$. We write $p \prec q$.

Definition 4.2.6: Let A and B be $\mathcal{G}$-C^*-algebras with A a sub-$\mathcal{G}$-C^*-algebra of B. A map φ as above which is furthermore idempotent is called an *A-projection* on B, that is, a $\mathcal{G}$-equivariant $C(\mathcal{G}^{(0)})$-linear idempotent ucp map $\varphi\colon B \to B$ that restricts to the identity on A.

Note that despite the name the range of an A-projection is in general not equal to A, but merely contains it. To every A-projection φ on B there is an associated A-seminorm p defined by $p(b) = \|\varphi(b)\|_B$ for $b \in B$. The partial order on seminorms translates to a partial order on A-projections as $\varphi \leq \psi$ if $\varphi \circ \psi = \varphi$.

Definition 4.2.7: Let A be a $\mathcal{G}$-algebra and ι be a $\mathcal{G}$-equivariant embedding of A into another $\mathcal{G}$-algebra B. We then call (B, ι) a *$\mathcal{G}$-extension* of A. The extension is said to be *$\mathcal{G}$-injective* if B is a $\mathcal{G}$-injective $\mathcal{G}$-algebra.

Definition 4.2.8: A $\mathcal{G}$-extension (B, ι) of A is said to be *$\mathcal{G}$-essential* if any ucp $\mathcal{G}$-map $\varphi\colon B \to C$ into a third $\mathcal{G}$-algebra C is injective if $\varphi \circ \iota$ is injective on A.

The $\mathcal{G}$-extension is said to be *$\mathcal{G}$-rigid* if the identity is the unique ucp $\mathcal{G}$-map $\psi\colon B \to B$ that satisfies $\psi \circ \iota = \iota$.

We will set out to show the existence of a minimal A-seminorm from which we will construct a minimal A-projection. The minimal A-projection then yields a $\mathcal{G}$-rigid $\mathcal{G}$-injective $\mathcal{G}$-extension by equipping its image with the Choi–Effros multiplication from Definition 2.2.14, which turns its range into a C^*-algebra without changing the complete order isomorphism class. As any rigid injective extension is essential, this will be the desired $\mathcal{G}$-injective envelope:

Definition 4.2.9: A *$\mathcal{G}$-injective envelope* of a $\mathcal{G}$-algebra is a $\mathcal{G}$-extension which is both $\mathcal{G}$-injective and $\mathcal{G}$-essential.

Aiming to apply Zorn's lemma, we show that every decreasing net of A-seminorms has a lower bound in analogy to Proposition 2.2.8. Again we denote by $X := \mathcal{G}^{(0)}$ the unit space of a Hausdorff étale groupoid $\mathcal{G}$ and by $I = C(\tilde{X})$ the non-dynamic injective envelope of $C(X)$, such that $\operatorname{Ind} I$ is a $\mathcal{G}$-injective $\mathcal{G}$-algebra.

Lemma 4.2.10: *Every decreasing net p_i of $C(X)$-seminorms on* $\operatorname{Ind} I$ *has a lower bound.*

Proof. In order to take weak*-limits we embed $\operatorname{Ind} I$ into a von-Neumann algebra. Observe that the embedding $C(X) \hookrightarrow I = C(\tilde{X})$ yields a surjective continuous map $q\colon \tilde{X} \to X$ and that the fibres of I as a $C(X)$-algebra are of the form $C(q^{-1}(x))$. We consider $\ell^\infty(\tilde{X})$ seen as a $C(X)$-algebra with the obvious map of $C(X)$ into its central multipliers, yielding a decomposition of the associated bundle into fibres as $\bigsqcup_{x\in X}\ell^\infty(q^{-1}(x))$. We then denote by

$$M := \ell^\infty\left(\mathcal{G}, \mathcal{G} *_s \bigsqcup_{x\in X} \ell^\infty\Big(q^{-1}(x)\Big)\right) \cong \ell^\infty\Big(\Big\{(\gamma,\tilde{x}) \in \mathcal{G}\times\tilde{X} \;\Big|\; s(\gamma) = q(\tilde{x})\Big\}\Big),$$

the bounded sections in the pullback along s that are not necessarily continuous. As noted before, M can be equipped with a $\mathcal{G}$-action in the same way that $\operatorname{Ind} I$ can, making it a $\mathcal{G}$-algebra. We can embed $\operatorname{Ind} I$ into M as a $\mathcal{G}$-algebra by utilizing the inclusion of $I = C(\tilde{X})$ into $\ell^\infty(\tilde{X})$. We denote this inclusion $\operatorname{Ind} I \to M$ by κ. By $\mathcal{G}$-injectivity of $\operatorname{Ind} I$, we may lift the identity on $\operatorname{Ind} I$ along the inclusion $\kappa\colon \operatorname{Ind} I \hookrightarrow M$ to a ucp $\mathcal{G}$-map $E\colon M \to \operatorname{Ind} I$, which necessarily contains the image of $\operatorname{Ind} I$ in its multiplicative domain. Using this, we may take weak limits in M and project them back down to $\operatorname{Ind} I$ as follows.

Let p_i be the decreasing net of $C(X)$-seminorms on $\operatorname{Ind} I$, and let $\varphi_i\colon \operatorname{Ind} I \to \operatorname{Ind} I$ be the ucp $\mathcal{G}$-maps fixing $C(X)$ associated with the seminorms defined by $p_i(x) = \|\varphi_i(x)\|_{\operatorname{Ind} I}$. As the maps $\kappa\circ\varphi_i\colon \operatorname{Ind} I \to M$ are bounded, there is a point-weak* convergent subnet of $\kappa\circ\varphi_i$. For ease of notation we drop κ and consider φ_i as maps $\operatorname{Ind} I \to M$, so that passing to a subnet we may assume that $\varphi_i(f)$ converges to $\varphi(f)$ in the weak*-topology for some map $\varphi\colon \operatorname{Ind} I \to M$ and every $f \in \operatorname{Ind} I$. The limit φ will still be a $C(X)$-linear ucp map fixing $C(X)$, as all of these are pointwise conditions, but we need to check that it is a $\mathcal{G}$-map. That is, for every $f, g \in \operatorname{Ind} I$ and $\gamma \in \mathcal{G}_u^v$ with $\gamma.[f]_u = [g]_v$, we need to show that

$$\gamma.[\varphi(f)]_u = [\varphi(g)]_v. \tag{4.7}$$

Fixing an open bisection B containing γ, we may assume that f is supported in $r^{-1}(s(B))$ and that $g = B.f$.

We claim that $\varphi(B.f) = B.\varphi(f)$: The predual of M,

$$M_* \cong \ell^1\Big(\Big\{(\gamma,\tilde{x}) \in \mathcal{G}\times\tilde{X} \;\Big|\; s(\gamma) = q(\tilde{x})\Big\}\Big) \cong \ell^1\left(\mathcal{G}, \mathcal{G} *_s \bigsqcup_{x\in X} \ell^1\Big(q^{-1}(x)\Big)\right),$$

carries an analogous $\mathcal{G}$-action. For $f \in \operatorname{Ind} I \subseteq M$ and $\chi \in M_*$, the evaluation $f(\chi)$ only depends on the values of χ on the support of f, seen as a function on $\mathcal{G}^{(0)}$. Therefore, when f is supported in $s(B)$, $B.f$ is supported in $r(B)$. To evaluate $(B.f)(\chi)$ we may assume that χ is supported on $r(B)$ and find $(B.f)(\chi) = f\Big(B^{-1}(\chi)\Big)$. Now for $\operatorname{supp}(f) \subseteq s(B)$ and $\operatorname{supp}(\chi) \subseteq r(B)$ we calculate

$$\varphi_i(B.f)(\chi) = (B.\varphi_i(f))(\chi) = \varphi_i(f)\Big(B^{-1}.\chi\Big) \to \varphi(f)\Big(B^{-1}.\chi\Big) = (B.(\varphi(f)))(\chi).$$

As the left hand expression converges to $\varphi(B.f)(\chi)$, we can conclude that $\varphi(B.f) = B.\varphi(f)$. Since $B.f$ is a valid choice of γ in Equation (4.7), we conclude that

$$\gamma.(\varphi_u([f]_u)) = \gamma.[\varphi(f)]_u = [B.\varphi(f)]_v = [\varphi(B.f)]_v = \varphi_v([B.f]_v) = \varphi_v(\gamma.[f]_u).$$

As every element of $(\operatorname{Ind} I)_u$ is of the form $[f]_u$ for some such f, this proves $\mathcal{G}$-equivariance of the limit φ.

We may thus define a $C(X)$-seminorm p on $\operatorname{Ind} I$ by $p(f) = \|E \circ \varphi(f)\|_{\operatorname{Ind} I}$. As in [30, Lemma 3.4], we then find that

$$p(f) = \|E \circ \varphi(f)\| \leq \|\varphi(f)\| \leq \limsup \|\varphi_j(f)\| = \lim p_i(f),$$

with j indexing the convergent subnet chosen before. Now p is the desired lower bound. □

By Zorn's lemma we know of the existence of a minimal $C(X)$-seminorm on $\operatorname{Ind} I$, from which we obtain a minimal $C(X)$-projection in analogy to Proposition 2.2.9:

Lemma 4.2.11: *There is a minimal $C(X)$-projection on* $\operatorname{Ind} I$.

Proof. We follow Hamana [30, Thm 3.5], which originally followed Kaufman [37, Thm 1]. Let p be a minimal $C(X)$-seminorm and φ the ucp map implementing it. We show that φ is a projection.

Define the net

$$\varphi_n \colon \operatorname{Ind} I \to \operatorname{Ind} I \subseteq M \qquad \text{by} \qquad \varphi_n := \left(\varphi + \varphi^2 + \cdots + \varphi^n\right)/n,$$

and pass to a point-weak* convergent subnet as in the proof of Lemma 4.2.10, and denote the limit by $\varphi_\infty \colon \operatorname{Ind} I \to M$. Using the conditional expectation $E \colon M \to \operatorname{Ind} I$ as before, we note that $E \circ \varphi_\infty$ induces a $C(X)$-seminorm and for $f \in \operatorname{Ind} I$ we obtain that

$$\|E \circ \varphi_\infty(f)\| \leq \|\varphi_\infty(f)\| \leq \limsup \|\varphi_n(f)\| \leq \|\varphi(f)\| = p(f),$$

which implies $\|\varphi(f)\| = \limsup \|\varphi_n(f)\|$ by minimality of p. Hence

$$\|\varphi(f) - \varphi^2(f)\| = \|\varphi(f - \varphi(f))\| = \limsup \|\varphi_n(f - \varphi(f))\| = 0,$$

so φ is idempotent and therefore a $C(X)$-projection.

Among $C(X)$-projections, φ is also minimal: Given any other $C(X)$-projection ψ with $\psi \leq \varphi$, or equivalently $\psi \circ \varphi = \psi = \varphi \circ \psi$, we find that

$$\|\psi(f)\| = \|\psi \circ \varphi(f)\| \leq \|\varphi(f)\| = p(f),$$

and so ψ and φ define the same seminorm by minimality of p. In particular, $\ker \psi = \ker \varphi$. As ψ is idempotent, we have $\psi(f) - f \in \ker \psi = \ker \varphi$, for every $f \in \operatorname{Ind} I$, and $\psi(f) = \varphi \circ \psi(f) = \varphi(f)$, so the two projections coincide on all of $\operatorname{Ind} I$. □

From a minimal $C(X)$-projection φ, we build the Choi–Effros algebra $C^*(\varphi)$, as originally constructed in [17, Theorem 3.1] and described in the context of injective envelopes in [29, Theorem 2.3]. This is done by equipping its range with the Choi–Effros product from Definition 2.2.14, turning the range of φ into a C^*-algebra that is completely order isomorphic to the range of φ as an operator system. In our setting, we have to verify that $C^*(\varphi)$ is indeed a $\mathcal{G}$-algebra, and that the ucp map $\operatorname{Ind} I \to C^*(\varphi)$ induced by φ is $\mathcal{G}$-equivariant.

Proposition 4.2.12: For a $C(X)$-projection φ on $\operatorname{Ind} I$, the Choi–Effros algebra $C^*(\varphi)$ can be given the structure of a $\mathcal{G}$-algebra such that the ucp map $\operatorname{Ind} I \to C^*(\varphi)$ induced by φ is $\mathcal{G}$-equivariant.

Proof. First we check that $C(X)$ still acts by central multipliers on $C^*(\varphi)$, giving it the structure of a $C(X)$-algebra. The algebra $C^*(\varphi)$ is the range of φ inside $\operatorname{Ind} I$ as underlying set, equipped with the norm of $\operatorname{Ind} I$ and multiplication $\circ$ given by $x \circ y = \varphi(xy)$. As φ is a $C(X)$-map, its range is closed under multiplication by $g \in C(X)$. Inheriting this action still gives an adjointable map that is its own adjoint, as

$$x \circ (f.y) = \varphi(x(f.y)) = \varphi((f.x)y) = (f.x) \circ y,$$

for $x, y \in \operatorname{Ind} I$ and $f \in C(\tilde{X})$. We conclude that the action of $C(\tilde{X})$ on $\operatorname{Ind} I$ via $y \mapsto f.y$ is by central multipliers. Hence, the surjective ucp map $j\colon \operatorname{Ind} I \to C^*(\varphi)$ induced by φ as $j(x) = \varphi(x)$ factors through the fibres to obtain maps $j_u\colon (\operatorname{Ind} I)_u \to (C^*(\varphi))_u$. For j to be $\mathcal{G}$-equivariant, the $\mathcal{G}$-action on $C^*(\varphi)$ has to be given by

$$\gamma.(j_u([f]_u)) = j_u(\gamma.[f]_u) \qquad \text{for } f \in \operatorname{Ind} I,$$

where the action of γ is the $\mathcal{G}$-action associated with the appropriate C^*-algebra. Picking an open bisection B around γ and cutting down the section f to be supported in $s(B)$ as above yields

$$\gamma.(j_u([f]_u)) = [j(B.f)]_u = [B.j(f)]_v,$$

where the action of B on appropriately supported functions in $C^*(\varphi)$ is as a subspace of $\operatorname{Ind} I$. Continuity of the action is shown just as in Equation (4.3). □

The following arguments need almost no modification to those given in the work of Hamana. First, we show that $C^*(\varphi)$ is a $\mathcal{G}$-rigid extension by noting that it carries a unique $C(X)$-seminorm and repeating the arguments of Lemma 4.2.11. Then we show that every rigid injective extension is also essential, making it an injective envelope.

Lemma 4.2.13: *For φ a minimal $C(X)$-projection on $\operatorname{Ind} I$, the extension $C(X) \hookrightarrow C^*(\varphi)$ is a $\mathcal{G}$-rigid, $\mathcal{G}$-injective $\mathcal{G}$-extension.*

Proof. By Proposition 2.2.13, $C^*(\varphi)$ is injective and even $\mathcal{G}$-injective, as φ is a $\mathcal{G}$-map.

From any $C(X)$-seminorm on $C^*(\varphi)$ given by a ucp $\mathcal{G}$-map φ' we obtain a $C(X)$-seminorm on $\operatorname{Ind} I$ given by $\varphi' \circ \varphi$. As it is then dominated by the minimal $C(X)$-seminorm of φ, the C^*-norm on $C^*(\varphi)$ is its unique $C(X)$-seminorm. Let $\psi\colon C^*(\varphi) \to C^*(\varphi)$ be a ucp $\mathcal{G}$-map restricting to the identity on $C(X)$. Then, analogously to the proof of Lemma 4.2.11, for $a \in C^*(\varphi)$ we obtain that

$$\limsup \left\| \left(\psi(a) + \psi^2(a) + \ldots + \psi^n(a) \right) / n \right\| = \limsup \|\psi_n(a)\| = \|a\|$$

and hence

$$\|\psi(a) - a\| = \limsup \|\psi_n(\psi(a) - a)\| = 0.$$

Therefore ψ is the identity. □

Lemma 4.2.14: *Every $\mathcal{G}$-injective $\mathcal{G}$-rigid extension is also $\mathcal{G}$-essential.*

Proof. The proof is almost abstract nonsense as in [30, Lemma 3.7].

Let (I, ι) be a $\mathcal{G}$-injective $\mathcal{G}$-extension of A and $\varphi\colon I \to B$ a ucp $\mathcal{G}$-map such that $\varphi \circ \iota\colon A \to B$ is injective. We have to show that φ itself is already injective. By $\mathcal{G}$-injectivity of I we find a ucp $\mathcal{G}$-map $\psi\colon B \to I$ that restricts to ι when A is seen as a subalgebra of B via the embedding $\varphi \circ \iota$, as in Diagram (4.8):

$$\begin{tikzcd}
B \arrow[r, dashed, "\psi"] & I \\
A \arrow[u, hook, "\varphi\circ\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd} \tag{4.8}$$

Now $\psi \circ \varphi$ is a ucp $\mathcal{G}$-selfmap of I that restricts to the identity on $\iota(A)$. By $\mathcal{G}$-rigidity it is therefore the identity, implying that φ is injective. □

Finally, we may remark that this injective envelope is unique up to $\mathcal{G}$-equivariant isomorphism preserving the embedding:

Theorem 4.2.15: *For an étale groupoid $\mathcal{G}$ with compact unit space X the $\mathcal{G}$-algebra $C(X)$ has a $\mathcal{G}$-equivariant injective envelope $I_{\mathcal{G}}(C(X))$, such that for any other $\mathcal{G}$-equivariant injective envelope Z there is a unique $\mathcal{G}$-isomorphism $\psi\colon I_{\mathcal{G}}(C(X)) \to Z$ for which*

$$\begin{tikzcd}
I_{\mathcal{G}}(C(X)) \arrow[r, "\psi"] & Z \\
C(X) \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd} \tag{4.9}$$

commutes. We call the spectrum of $I_{\mathcal{G}}(C(X))$ the Furstenberg boundary *of $\mathcal{G}$.*

This works as for Hamana in [29, Lemma 3.8]:

Proof. Let $I_{\mathcal{G}}(C(X))$ be the $\mathcal{G}$-injective $\mathcal{G}$-rigid extension above and let Z be another $\mathcal{G}$-injective $\mathcal{G}$-essential ectension. We may extend the inclusions of $C(X)$ into one of the envelopes to the other envelope as in Diagram (4.10) by $\mathcal{G}$-injectivity:

$$\begin{tikzcd}
I_{\mathcal{G}}(C(X)) \arrow[r, dashed, "\psi"] & Z & & I_{\mathcal{G}}(C(X)) & Z \arrow[l, dashed, "\hat{\psi}"'] \\
C(X) \arrow[u, hook] \arrow[ur, hook] & & & & C(X) \arrow[u, hook] \arrow[ul, hook]
\end{tikzcd} \tag{4.10}$$

As $\hat{\psi} \circ \psi$ restricts to the identity on $C(X)$, it is the identity on $I_{\mathcal{G}}(C(X))$ by $\mathcal{G}$-rigidity, and in particulat $\hat{\psi}$ is surjective. Furthermore, $\hat{\psi}$ is injective by $\mathcal{G}$-essentiality of Z and hence a $\mathcal{G}$-isomorphism. □

4.2.3 Some Properties of Groupoid Boundaries

Consider a locally compact Hausdorff étale groupoid $\mathcal{G}$ with compact unit space X as above. Denote by $\tilde{X}$ its Furstenberg boundary, the spectrum of the $\mathcal{G}$-equivariant injective envelope $I_{\mathcal{G}}(C(X))$ of $C(X)$. Passing to the crossed product groupoid $\tilde{\mathcal{G}} := \tilde{X} \rtimes \mathcal{G}$ associated with the action of $\mathcal{G}$ on $C(\tilde{X})$, we treat the boundary as a second, larger groupoid $\tilde{\mathcal{G}}$ that contains $\mathcal{G}$ as a quotient. We call $\tilde{\mathcal{G}}$ the *boundary groupoid* of $\mathcal{G}$.

In this section we explore some properties that make $\tilde{\mathcal{G}}$ more tractable. The first is a generalisation of Lemma 2.3.4.

Proposition 4.2.16: All stabiliser groups of the boundary groupoid $\tilde{\mathcal{G}}$ of a Hausdorff étale groupoid $\mathcal{G}$ are amenable.

Proof. Consider the $\mathcal{G}$-algebra $L = \{f \in \ell^\infty(\mathcal{G}) \mid x \mapsto \|f|_{\mathcal{G}^x}\|_\infty \text{ is upper-semicontinuous}\}$ with left action of $\mathcal{G}$ on the argument. Identifying $C(X)$ as a unital sub-$\mathcal{G}$-algebra, we find a ucp $\mathcal{G}$-map $\varphi\colon L \to C(\tilde{X})$ by $\mathcal{G}$-injectivity. Passing to the fibre at any $x \in X$, we obtain a $\mathcal{G}_x^x$-equivariant ucp map $\varphi_x\colon \ell^\infty(\mathcal{G}^x) \to C(q^{-1}(x))$. For any $\tilde{x} \in q^{-1}(x)$, we may identify the stabiliser group $\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}$ of $\tilde{\mathcal{G}}$ at $\tilde{x}$ with a subgroup of $\mathcal{G}_x^x$ and obtain a $\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}$-equivariant unital *-homomorphism $\ell^\infty(\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}) \to \ell^\infty(\mathcal{G}^x)$ since each orbit of ${}_{\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}}\backslash^{\mathcal{G}^x}$ is in bijection with $\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}$. Then composition with the evaluation $\mathrm{eval}_{\tilde{x}}$ at $\tilde{x}$,

$$\ell^\infty(\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}) \to \ell^\infty(\mathcal{G}^x) \xrightarrow{\varphi_x} C(q^{-1}(x)) \xrightarrow{\mathrm{eval}_{\tilde{x}}} \mathbb{C}$$

gives a state on $\ell^\infty(\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}})$ that is invariant under the left action of $\tilde{\mathcal{G}}_{\tilde{x}}^{\tilde{x}}$. □

Let $q\colon \tilde{X} \to X$ be the quotient map obtained from the embedding $C(X) \hookrightarrow C(\tilde{X})$. The following is a generalisation of Lemma 3.3.5.

Proposition 4.2.17: Let $\tilde{X}$ be the Furstenberg boundary of a groupoid $\mathcal{G}$ with compact unit space X and $q\colon \tilde{X} \to X$ the continuous surjection obtained from the inclusion $C(X) \hookrightarrow C(\tilde{X})$. Let $Y \subseteq \tilde{X}$ be a closed $\mathcal{G}$-invariant subset such that $q(Y) = X$. Then $Y = \tilde{X}$.

Proof. As q restricted to Y is still surjective, we can embed $C(X)$ into $C(Y)$ and, by $\mathcal{G}$-injectivity of $C(\tilde{X})$, extend the embedding $C(X) \hookrightarrow C(\tilde{X})$ to a ucp map $\varphi\colon C(Y) \to C(\tilde{X})$. Then the composition with restriction to Y

$$C(\tilde{X}) \xrightarrow{\mathrm{res}_Y} C(Y) \xrightarrow{\varphi} C(\tilde{X})$$

is a $\mathcal{G}$-equivariant ucp map $C(\tilde{X}) \to C(\tilde{X})$ that restricts to the identity on the subalgebra $C(X)$. By $\mathcal{G}$-rigidity of the $\mathcal{G}$-equivariant injective envelope, it is therefore the identity on all of $C(\tilde{X})$. Hence, res_Y is injective and Y is dense. As it is also closed, we get $Y = \tilde{X}$. □

The boundary groupoid being ample is the étale groupoid analogon of the boundary of a group being Stonean by Proposition 2.2.13 and the remark before Proposition 2.2.20.

Proposition 4.2.18: Any boundary groupoid $\tilde{\mathcal{G}}$ is ample.

Proof. As $C(\tilde{X})$ is an injective C^*-algebra, $\tilde{X}$ is Stonean and in particular zero-dimensional. Any étale groupoid with totally disconnected unit space is ample. □

Proposition 4.2.19: If every orbit of a boundary groupoid $\tilde{\mathcal{G}}$ has at least two points, then the isotropy bundle $\mathrm{Iso}(\tilde{\mathcal{G}})$ of $\tilde{\mathcal{G}}$ is clopen.

If $\mathcal{G}$ already has orbits consisting of at least two points, then so does $\tilde{\mathcal{G}}$. We are of course tacitly avoiding the case where $\mathcal{G}$ is a group and $\mathcal{G}^{(0)}$ is a single point. Note that the proposition implies that $\tilde{G}$ is principal if and only if it is topologically principal in analogy to $\partial_F G$ being free if and only if it is topologically free as proven in Theorem 2.3.6.

Proof. For every groupoid the isotropy bundle is closed, so we have to show that it is open. Consider $\gamma \in \tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}}$. As $\tilde{\mathcal{G}}$ is ample, there exists a *full* open bisection B containing γ, that is, an open bisection with $s(B) = \tilde{X} = r(B)$ by Brix–Scarparo [10]. Then B defines a homeomorphism $\tilde{X} \to \tilde{X}$ by $r \circ (s|_{B'})^{-1}$ whose fixed point set F contains $s(\gamma) = \tilde{x} = r(\gamma)$. By Frolík's theorem [24] this fixed point set is open and $B \cap s^{-1}(F)$ is an open neighbourhood of γ contained in the isotropy. □

Remark. Proposition 4.2.19 of course works in general for every groupoid with Stonean unit space and orbits consisting of at least two points. So far we have been unable to drop this second condition, although it can be dropped for the case of crossed products by groups, where the existence of enough full bisections for the topological full group to be covering is trivial (see [10] for definitions). Note that it is however not necessary for every $\gamma \in \mathcal{G}$ to be contained in a full bisection in order to apply Frolík's theorem as above. It suffices that γ is contained in an open bisection B whose source $s(B)$ is again a Stonean space that contains the range $r(B)$.

Furthermore, we show that the notion of groupoid Furstenberg boundary generalises the boundaries of groups and even groups acting on spaces.

Proposition 4.2.20: Let $\mathcal{G} = X \rtimes G$ be the transformation groupoid of a discrete group G acting on a compact Hausdorff space X. Then the groupoid-equivariant injective envelope $I_{\mathcal{G}}(C(X))$ coincides with the group-equivariant injective envelope $I_G(C(X))$.

Hence, the groupoid Furstenberg boundary of a discrete group is its Furstenberg boundary.

Proof. First observe that $I_{\mathcal{G}} := I_{\mathcal{G}}(C(X))$ carries a G-action where $g.f(\tilde{x}) = f(s(\tilde{x}, g))$, with $g \in G$, $f \in I_{\mathcal{G}}$, and $\tilde{x}$ in the spectrum of $I_{\mathcal{G}}$. The action is continuous by continuity of the $\mathcal{G}$-action. Likewise, $I_G := I_G(C(X))$ carries a $\mathcal{G}$-action: As $C(X)$ embeds into the commutative C^*-algebra I_G, it is a $C(X)$-algebra and as before we may define a continuous action by $(x, g)[f]_x := [g.f]_{g.x}$ with $g \in G$, $f \in I_G$, and $x \in X$.

Using their injectivity in the respective category, we may therefore extend the embeddings of $C(X)$ into I_G and $I_{\mathcal{G}}$ to a ucp G-map $\varphi\colon I_{\mathcal{G}} \to I_G$ and a ucp $\mathcal{G}$-map $\psi\colon I_G \to I_{\mathcal{G}}$. We then calculate that for all $x \in X$

$$[\psi(g.f)]_{g.x} = \psi_{g.x}((x, g).[f]_x) = (x, g).\psi_x([f]_x) = [g.\psi(f)]_{g.x},$$

so ψ is also a ucp G-map. On the other hand, φ contains $C(X)$ in its multiplicative domain and is therefore a $C(X)$-map. Hence for all $x \in X$ we find that

$$\varphi_{g.x}((x, g).[f]_x) = \varphi_{g.x}\Big([g.f]_{g.x}\Big) = [\varphi(g.f)]_{g.x} = [g.\varphi(f)]_{g.x} = (x, g).\varphi_x([f]_x),$$

so φ is $\mathcal{G}$-equivariant.

Now $\psi \circ \varphi$ is a ucp $\mathcal{G}$-map $I_{\mathcal{G}} \to I_{\mathcal{G}}$ that fixes $C(X)$ and is therefore the identity by $\mathcal{G}$-rigidity of $I_{\mathcal{G}}$. Likewise, $\varphi \circ \psi$ is a ucp G-map $I_G \to I_G$ that fixes $C(X)$ and is the identity by G-rigidity of I_G and the two envelopes are isomorphic C^*-algebras. □

4.2.4 Simplicity of Groupoid C^*-algebras

Finally, we apply the theory of boundary groupoids to obtain a sufficient criterion for a groupoid $\mathcal{G}$ to have the intersection property in Theorem 4.2.25. In the case of a minimal groupoid, this provides a weaker sufficient criterion for C^*-simplicity than the widely used notion of *topological principality*.

We first relate the C^*-algebras of a groupoid and its boundary groupoid:

Lemma 4.2.21: *For a locally compact Hausdorff étale groupoid $\mathcal{G}$ with compact unit space there is a canonical embedding of $C^*_r(\mathcal{G})$ into the reduced C^*-algebra $C^*_r(\tilde{\mathcal{G}})$ of its boundary groupoid $\tilde{\mathcal{G}}$.*

Proof. Denote again the quotient map $\tilde{X} \to X$ by q. As a subset of $\mathcal{G} \times \tilde{X}$, $\tilde{\mathcal{G}}$ is given by $\left\{(\gamma, \tilde{x}) \;\middle|\; \gamma \in \mathcal{G}, \tilde{x} \in \tilde{X}, r(\gamma) = q(\tilde{x})\right\}$ and the range and source maps are given by $r((\gamma, \tilde{x})) = \tilde{x}$ and $s((\gamma, \tilde{x})) = \gamma^{-1}.\tilde{x}$. Furthermore denote by $Q\colon \tilde{\mathcal{G}} \to \mathcal{G}$ the surjective groupoid homomorphism given by $(\gamma, \tilde{x}) \mapsto \gamma$, which extends $q\colon \tilde{X} \to X$. Note that Q is proper so that $C_c(\mathcal{G})$ embeds into $C_c(\tilde{\mathcal{G}})$. Indeed, if $K \subset \mathcal{G}$ is a compact set, then the preimage $Q^{-1}(K) = \{(\gamma, \tilde{x}) \mid \gamma \in K\}$ is compact, since $\tilde{X}$ is.

Since Q is a groupoid homomorphism, precomposition with Q as a map $Q^*\colon C_c(\mathcal{G}) \hookrightarrow C_c(\tilde{\mathcal{G}})$ is compatible with the convolution and involution on both algebras. It is furthermore isometric with respect to the reduced norms, $\|f\|_{C^*_r(\mathcal{G})} = \|f \circ Q\|_{C^*_r(\tilde{\mathcal{G}})}$ for $f \in C_c(\mathcal{G})$, so that Q^* extends from $C_c(\mathcal{G})$ to an embedding of the associated reduced algebras: First note that Q gives a bijection $\tilde{\mathcal{G}}_{\tilde{x}} \to \mathcal{G}_{q(\tilde{x})}$ between the source fibres by $(\gamma, \gamma\tilde{x}) \mapsto \gamma$, which makes $U\colon \ell^2(\tilde{\mathcal{G}}_{\tilde{x}}) \to \ell^2(\mathcal{G}_{q(\tilde{x})})$ by $\delta_{(\gamma,\gamma\tilde{x})} \mapsto \delta_\gamma$, an isomorphism that intertwines the associated source fibre representations $\pi_{\tilde{x}}$ and π_x:

$$\begin{aligned} U\pi_{\tilde{x}}(f \circ Q)U^*\,\delta_\gamma &= U\pi_{\tilde{x}}(f \circ Q)\,\delta_{(\gamma,\gamma\tilde{x})} \\ &= U \sum_{\alpha \in \mathcal{G}_x} f \circ Q((\alpha, \alpha\tilde{x}))\,\delta_{(\alpha\gamma,\alpha\gamma\tilde{x})} \\ &= U \sum_{\alpha \in \mathcal{G}_x} f(\alpha)\,\delta_{(\alpha\gamma,\alpha\gamma\tilde{x})} \\ &= \sum_{\alpha \in \mathcal{G}_x} f(\alpha)\,\delta_{\alpha\gamma} \\ &= \pi_x(f)\,\delta_\gamma. \end{aligned}$$

Hence the norms of f and $f \circ Q$ coincide and we obtain an embedding of $C^*_r(\mathcal{G})$ into $C^*_r(\tilde{\mathcal{G}})$. □

To relate the intersection properties of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, we furthermore need the following technical lemma stating that the above embedding is compatible with the $\mathcal{G}$-action under ucp extensions:

Lemma 4.2.22: *Let $\mathcal{G}$ be a groupoid as above and $\tilde{\mathcal{G}}$ its boundary groupoid. Let π be a $*$-representation of $C^*_r(\mathcal{G})$ on $\mathcal{B}(\mathcal{H})$ whose restriction to $C(X)$ is injective, and let $\tilde{\pi}\colon C^*_r(\tilde{\mathcal{G}}) \to \mathcal{B}(\mathcal{H})$ be a ucp extension of π. Then the C^*-algebra E generated by $\tilde{\pi}(C(\tilde{X}))$ inside $\mathcal{B}(\mathcal{H})$ carries the structure of a $\mathcal{G}$-algebra, such that the restriction $\tilde{\pi}|_{C(\tilde{X})} \to E$ is $\mathcal{G}$-equivariant.*

Proof. Let $E := C^*(\tilde{\pi}(C(\tilde{X}))) \subseteq \mathcal{B}(\mathcal{H})$ be the C^*-algebra generated in $\mathcal{B}(\mathcal{H})$ by the image of $\tilde{\pi}$ on $C(\tilde{X})$. As $\tilde{\pi}$ restricts to π on $C(X)$ and is injective there, we may identify $C(X) \subseteq E$ as a subalgebra and since π is a *-homomorphism, $C(X)$ lies in the multiplicative domain of $\tilde{\pi}$, so that its action is central. Therefore E is a unital $C(X)$-algebra. For $\tilde{\pi}$ to be $\mathcal{G}$-equivariant, the

$\mathcal{G}$-action on $\tilde\pi(C(\tilde X)) \subseteq E$ is determined by

$$\gamma.[\tilde\pi(f)]_x = \tilde\pi_{\gamma.x}(\gamma.[f]_x)$$

for $f \in C(\tilde X)$ with $x = s(\gamma)$ and $\gamma.x = r(\gamma)$. Note that if $g_\gamma \in C_c(\mathcal{G})$ is supported on a bisection of $\mathcal{G}$ and $g_\gamma(\gamma) = 1$, then $g_\gamma * f * g_\gamma^*$ is in $C(\tilde X)$ and a quick calculation shows that

$$\big(g_\gamma * f * g_\gamma^*\big)(\tilde x) = \Big|g_\gamma\big((r|_B)^{-1}(\tilde x)\big)\Big|^2 \cdot f\big(s \circ (r|_B)^{-1}(\tilde x)\big), \tag{4.11}$$

where $B \subseteq \tilde{\mathcal{G}}$ is the bisection on which $g_\gamma \circ q$ is supported. Hence $\gamma.[f]_x = \big[g_\gamma * f * g_\gamma^*\big]_{\gamma.x}$ and since g_γ is in the multiplicative domain of $\tilde\pi$ we can define

$$\gamma.[\tilde\pi(f)]_x \coloneqq \tilde\pi_{\gamma.x}(\gamma.[f]_x) = \big[\tilde\pi(g_\gamma)\tilde\pi(f)\tilde\pi(g_\gamma)^*\big]_{\gamma.x}.$$

This extends to general $a \in E$ by

$$\gamma.[a]_x \coloneqq \big[\tilde\pi(g_\gamma)a\tilde\pi(g_\gamma)^*\big]_{\gamma.x} \tag{4.12}$$

and we proceed to show that this is a well-defined $\mathcal{G}$-action. First we show that $\gamma.[a]_x$ does not depend on the choice of a to represent $[a]_x$, that is, $\gamma.[a]_x = 0$ if $[a]_x = 0$ for $a \in \overline{C_0(X \setminus x)E}$. We may approximate a by finite sums of elements of the form $h\cdot\tilde\pi(f_1)\cdots\tilde\pi(f_n)$ with $h \in C_0(X \setminus x)$ and $f_i \in C(\tilde X)$ while n varies and as the action in Equation (4.12) depends continuously on a, it suffices to show that $\gamma.[h \cdot \tilde\pi(f_1)\cdots\tilde\pi(f_n)]_x = 0$. As $g_\gamma^* * g_\gamma$ and $g_\gamma * g_\gamma^*$ are supported on X and $\big[g_\gamma * g_\gamma^*\big]_{\gamma.x} = \mathbb{1}$, we may now calculate that

$$\begin{aligned}
\Big[\pi\big(g_\gamma\big)h\tilde\pi(f_1)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x &= \Big[\pi\big(g_\gamma * g_\gamma^*\big)\Big]_{\gamma.x}\Big[\pi\big(g_\gamma\big)h\tilde\pi(f_1)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x \\
&= \Big[\pi\big(g_\gamma * g_\gamma^*\big)\pi\big(g_\gamma\big)h\tilde\pi(f_1)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x \\
&= \Big[\pi\big(g_\gamma\big)\pi\big(g_\gamma^* * g_\gamma\big)h\tilde\pi(f_1)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x \\
&= \Big[\pi\big(g_\gamma\big)h\tilde\pi(f_1)\pi\big(g_\gamma\big)^*\pi\big(g_\gamma\big)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x \\
&= \Big[\pi\big(g_\gamma\big)h\tilde\pi(f_1)\pi\big(g_\gamma\big)^*\Big]_{\gamma.x}\Big[\pi\big(g_\gamma\big)\tilde\pi(f_2)\pi\big(g_\gamma\big)^*\Big]_x \\
&= \tilde\pi_{\gamma.x}(\gamma.[hf_1]_x)\tilde\pi_{\gamma.x}(\gamma.[f_2]_x) = 0
\end{aligned}$$

for $n = 2$ and analogously for arbitrary n. Incidentally, this also shows how the action gives homomorphisms between the appropriate fibers, that is,

$$\gamma.[ab]_x = \gamma.[a]_x\gamma.[b]_x.$$

Next we show that $\gamma.[a]_x$ is independent of the choice of g_γ as above. Suppose g_γ and g'_γ are as described supported on open bisections B and B' and evaluate to one at γ. As $B \cap B'$ is an open neighbourhood of γ, the partial homeomorphisms $s \circ (r|_B)^{-1}$ and $s \circ (r|_{B'})^{-1}$ coincide on the neighbourhood $r(B \cap B')$ of $r(\gamma)$ and by Equation (4.11) we find that

$$\big(g_\gamma * f * g_\gamma^*\big) - \Big(g'_\gamma * f * \big(g'_\gamma\big)^*\Big)(\tilde x) = \underbrace{\Big(\Big|g_\gamma\big((r|_B)^{-1}(x)\big)\Big|^2 - \Big|g'_\gamma\big((r|_B)^{-1}(x)\big)\Big|^2\Big)}_{\in C_0(X\setminus r(\gamma))}\cdot f\big(s \circ (r|_B)^{-1}(\tilde x)\big).$$

and therefore

$$\left[g_\gamma * f * g_\gamma^*\right]_{r(\gamma)} = \left[g'_\gamma * f * \left(g'_\gamma\right)^*\right]_{r(\gamma)}.$$

By approximating arbitrary $a \in E$ by finite sums of elements of the form $h \cdot \tilde{\pi}(f_1) \cdots \tilde{\pi}(f_n)$ as above, we conclude that the same holds for arbitrary a.

As $\left[g_\gamma^* * g_\gamma\right]_x = \mathbb{1}$ and g_γ^* is a valid choice for $g_{\gamma^{-1}}$, the action of γ is invertible by acting with γ^{-1} and is therefore by $*$-isomorphisms. Furthermore, as $g_\eta * g_\gamma$ for composable η and $\gamma \in \mathcal{G}$ is supported on a bisection and evaluates to one at $\eta\gamma$, it is a valid choice for $g_{\eta\gamma}$ so that the action is compatible with composition in $\mathcal{G}$.

Finally, we show that the action is continuous. Let $\gamma_\lambda \to \gamma$ and $[a_\lambda]_{x_\lambda} \to [a]_x$ with $x_\lambda = s(\gamma_\lambda)$ and $x = s(\gamma)$. We have to prove that $\gamma_\lambda.[a_\lambda]_{x_\lambda} \to \gamma.[a]_x$. As the elements of E are exactly the continuous sections in the associated bundle, the convergence $[a_\lambda]_{x_\lambda} \to [a]_x$ is equivalent to $\left\|[a_\lambda - a]_{x_\lambda}\right\|_{x_\lambda} \to 0$ and it therefore suffices to show that $\gamma_\lambda.[a]_{x_\lambda} \to \gamma.[a]_x$. We do this first for $a = \tilde{\pi}(f)$ and then generalise to arbitrary a as before. As $\tilde{\pi}$ is $\mathcal{G}$-equivariant we find that

$$\gamma_\lambda.[\tilde{\pi}(f)]_{x_\lambda} = \tilde{\pi}_{\gamma_\lambda.x_\lambda}(\gamma_\lambda.[f]_{x_\lambda}) \to \tilde{\pi}_{\gamma.x}(\gamma.[f]_x) = \gamma.[\tilde{\pi}(f)]_x.$$

With the action by γ being via *-homomorphism, the same holds with $\tilde{\pi}(f)$ replaced by a linear combination of finite products of this form. Since these are dense, convergence $\gamma_\lambda.[a_\lambda]_{x_\lambda} \to \gamma.[a]_x$ may be tested against such sections: The net converges to $\gamma.[a]_x$, if and only if for all $\epsilon > 0$, $n, k \in \mathbb{N}$, and $f_{1,1}, \ldots, f_{n,k} \in C(\tilde{X})$ with $\|\gamma.[a]_x - [\sum_{i=1}^k \tilde{\pi}(f_{1,i}) \cdots \tilde{\pi}(f_{n,i})]_{\gamma.x}\|_{\gamma.x} < \epsilon$ we eventually have $\|\gamma_\lambda.[a_\lambda]_{x_\lambda} - [\sum_{i=1}^k \tilde{\pi}(f_{1,i}) \cdots \tilde{\pi}(f_{n,i})]_{\gamma_\lambda.x_\lambda}\|_{\gamma_\lambda.x_\lambda} < \epsilon$.

Let $b = \sum_{i=1}^k \tilde{\pi}(f_{1,i}) \cdots \tilde{\pi}(f_{n,i})$ such that $[b]_{\gamma.x}$ approximates $\gamma.[a]_x$ up to ϵ. Then

$$\begin{aligned}\left\|\gamma_\lambda.[a]_{x_\lambda} - [b]_{\gamma_\lambda.x_\lambda}\right\|_{\gamma_\lambda.x_\lambda} &= \left\|[a]_{x_\lambda} - \gamma_\lambda^{-1}.[b]_{\gamma_\lambda.x_\lambda}\right\|_{x_\lambda} \\ &\leq \left\|[a]_{x_\lambda} - \left[\tilde{\pi}(g_\gamma)^* b \tilde{\pi}(g_\gamma)\right]_{x_\lambda}\right\|_{x_\lambda} + \left\|\left[\tilde{\pi}(g_\gamma)^* b \tilde{\pi}(g_\gamma)\right]_{x_\lambda} - \gamma_\lambda^{-1}.[b]_{\gamma_\lambda.x_\lambda}\right\|_{x_\lambda}.\end{aligned}$$

By the arguments above the right-hand term vanishes as $\lambda \to \infty$ while the left-hand term is the norm of a section and hence depends upper semi-continuously on x_λ. Hence

$$\limsup_{\lambda \to \infty} \left\|\gamma_\lambda.[a]_{x_\lambda} - [b]_{\gamma_\lambda.x_\lambda}\right\|_{\gamma_\lambda.x_\lambda} \leq \left\|[a]_x - \left[\tilde{\pi}(g_\gamma)^* b \tilde{\pi}(g_\gamma)\right]_x\right\|_x = \left\|[a]_x - \gamma^{-1}.[b]_{\gamma.x}\right\|_x < \epsilon,$$

so the action is continuous. □

We are now able to show that a groupoid $\mathcal{G}$ inherits the intersection property from its boundary groupoid $\tilde{\mathcal{G}}$, generalising one implication of Theorem 3.3.2.

Lemma 4.2.23: *Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with compact unit space and $\tilde{\mathcal{G}}$ its boundary groupoid. If $\tilde{\mathcal{G}}$ has the intersection property, then so does $\mathcal{G}$.*

Proof. Let $\pi\colon C_\mathrm{r}^*(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a representation of $C_\mathrm{r}^*(\mathcal{G})$ with $\ker(\pi) \cap C(X) = \{0\}$. By Arveson extension we may find a ucp extension $\tilde{\pi}\colon C_\mathrm{r}^*(\tilde{\mathcal{G}}) \to \mathcal{B}(H)$ and we denote by $D = C^*(\tilde{\pi}(C_\mathrm{r}^*(\tilde{\mathcal{G}})))$ the sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ generated by $\tilde{\pi}(C_\mathrm{r}^*(\mathcal{G}))$. We consider $E := C^*(\tilde{\pi}(C(\tilde{X}))) \subseteq D$ as in Lemma 4.2.22 and endow it with the $\mathcal{G}$-algebra structure defined there.

As E contains $C(X)$ as a sub-$\mathcal{G}$-algebra, we may find a ucp $\mathcal{G}$-map $\varphi\colon E \to C(\tilde{X})$ by extending the inclusion of $C(X)$. Then $\varphi \circ \tilde{\pi}|_{C(\tilde{X})}$ is a $\mathcal{G}$-map fixing $C(X)$, hence by rigidity of $C(\tilde{X})$ it is the identity. Therefore, both $\varphi|_{\tilde{\pi}(C(\tilde{X}))}$ and $\tilde{\pi}|_{C(\tilde{X})}$ are isometries and since the multiplicative domain of φ coincides with $\overline{\operatorname{span}}\{u \mid \|u\| = 1,\ \varphi(u) \text{ unitary}\}$ and contains $\tilde{\pi}(u)$ for $u \in C(\tilde{X})$ unitary which generate E as a C^*-algebra, φ is a *-homomorphism.

However, $\tilde{\pi}|_{C(\tilde{X})}$ might fail to be a *-homomorphism, if φ has non-trivial kernel. This is alleviated as follows: We consider $F = \overline{\ker(\varphi) \cdot \tilde{\pi}(C_c(\tilde{\mathcal{G}}))}$ and show that it is an ideal in D, or equivalently $\tilde{\pi}(C_c(\tilde{\mathcal{G}}))\ker(\varphi) \subseteq \overline{\ker(\varphi)\tilde{\pi}(C_c(\tilde{\mathcal{G}}))}$. Fix $a \in \ker(\varphi)$, as well as $g \in C_c(\mathcal{G})$ real-valued and vanishing outside of an open bisection B. We want to show that $\pi(g)a$ is contained in $\ker(\varphi)\pi(C_c(\mathcal{G}))$. Let $h = \sqrt[3]{g}$. Then $h^* * h$ is supported on the unit space X, since g is supported on a bisection and given by

$$(h^* * h)(x) = \begin{cases} |h(\gamma_x)|^2 & x \in s(B) \\ 0 \end{cases}$$

where γ_x is the unique arrow in B such that $Bx = \{\gamma_x\}$. Hence $h * h^* * h = g$ and $\pi(g)a = \pi(h * h^* * h)a$. Since $h^* * h \in C(X)$ acts centrally on E, we find that $\pi(g)a = \pi(h * h^* * h)a = (\pi(h) \cdot a \cdot \pi(h)^*)\pi(h)$.

We proceed to argue that $\varphi(\pi(h)a\pi(h)^*)$ vanishes, or equivalently $[\pi(h)a\pi(h)^*]_x \in \ker(\varphi_x)$ for all $x \in X$. First assume $x \notin r(B)$, or $x \in r(B)$ with $xB = \gamma$ and $h(\gamma) = 0$. Then let $k = \sqrt[3]{h}$ and note that $k^* * k \in C_0(X \setminus x)$, so

$$[\pi(h)a\pi(h)^*]_x = [\pi(kk^*)\pi(k)a\pi(h)^*]_x = [\pi(kk^*)]_x[\pi(k)a\pi(h)^*]_x = 0.$$

On the other hand, if $x \in r(B)$ with $xB = \gamma$ but $h(\gamma) \neq 0$, we may rescale h to $h' := h/h(\gamma)$. Then by definition of the $\mathcal{G}$-action

$$h(\gamma)^2 \cdot \gamma.[a]_{s(\gamma)} = h(\gamma)^2[\pi(h')a\pi(h')^*]_x = [\pi(h)a\pi(h)^*]_x.$$

Now, since φ is a $\mathcal{G}$-map, $\gamma.[a]_{s(\gamma)}$ is in the kernel of φ_x exactly if $[a]_{s(\gamma)}$ is in the kernel of $\varphi_{s(\gamma)}$, which holds since $a \in \ker(\varphi)$. So, for $g \in C_c(\mathcal{G})$ real-valued and supported on a bisection, we find that

$$\pi(g)a = (\pi(h) \cdot a \cdot \pi(h)^*)\pi(h) \in \ker(\varphi) \cdot \pi(C_c(\mathcal{G})),$$

and as such g span $C_c(\mathcal{G})$ densely we may conclude that $\pi(C_c(\mathcal{G}))\ker(\varphi) \subseteq \overline{\ker(\varphi)\pi(C_c(\mathcal{G}))}$. Note that $C_c(\mathcal{G}) \cdot C(\tilde{X})$ spans $C_c(\tilde{\mathcal{G}})$ densely and that $C(\tilde{X})\ker(\varphi) \subseteq \ker(\varphi)$, so that the collection of all ga for g and a as above spans a dense subset of $\tilde{\pi}(C_c(\tilde{\mathcal{G}}))\ker(\varphi)$. Therefore F is an ideal in D whose elements are exactly these fixed by left multiplication with an approximate unit of $\ker(\varphi) \subseteq E$. Hence $F \cap E = \ker(\varphi)$.

Denoting by Φ the quotient map $D \to D/F$, we consider the ucp map $\Phi \circ \tilde{\pi}$. As φ by design factors through the restriction of Φ to E to some $\bar{\varphi}$, and $\bar{\varphi} \circ (\Phi \circ \tilde{\pi})|_{C(\tilde{X})} = \mathrm{id}_{C(\tilde{X})}$ we may now conclude that $(\Phi \circ \tilde{\pi})|_{C(\tilde{X})}$ is a *-homomorphism since $\bar{\varphi}$ is an injective left inverse on $E/F \subseteq D/F$. Additionally, $(\Phi \circ \tilde{\pi})|_{C^*_{\mathrm{r}}(\mathcal{G})} = \Phi \circ \pi$ is a *-homomorphism , so both $C(\tilde{X})$ and $C^*_{\mathrm{r}}(\mathcal{G})$ belong to the multiplicative domain of $\Phi \circ \tilde{\pi}$. As their product is dense in $C^*_{\mathrm{r}}(\tilde{\mathcal{G}})$ it follows that $\Phi \circ \tilde{\pi}$ itself is a *-homomorphism. However, $\Phi \circ \tilde{\pi}$ is faithful on $C(\tilde{X})$ since $\bar{\varphi} \circ (\Phi \circ \tilde{\pi})|_{C(\tilde{X})} = \mathrm{id}_{C(\tilde{X})}$, and by the intersection property of $\tilde{\mathcal{G}}$, it is itself faithful. As $\tilde{\pi}$ extends π, we may conclude that π is faithful on $C^*_{\mathrm{r}}(\mathcal{G})$. We conclude that $\mathcal{G}$ has the intersection property. □

Recall that Kawabe's characterisation [38] of C^*-simplicity of discrete groups acting on compact spaces from Theorems 3.3.4 and 3.3.6 generalises Kennedy's results [40] from Theorem 2.3.16 by identifying it with the absence of *recurrent* amenable sections of subgroups in the stabiliser subgroups of the action. Given a discrete group G, both notions rely on endowing the set $\mathrm{Sub}(G)$ of subgroups of G equipped with the Chabauty topology, that is, the topology of pointwise convergence of indicator functions, as well as the action of G by conjugating subgroups. Recall from Definition 2.3.12 that a *recurrent* subgroup of a group G is a subgroup H, such that the closure of its orbit under the G-action does not contain the trivial subgroup $\{e\}$. The analogous notion in Kawabe's work, where G acts on a compact space X, considers amenable subgroups of the point stabilisers $\mathrm{Stab}(x)$ of the action of G on X, equipped with an action of G by conjugation. To keep track of the basepoint, denote a subgroup $H \leq \mathrm{Stab}(x)$ as (x, H) in $X \times \mathrm{Sub}(G)$ with the action of $g \in G$ by $g.(x, H) = (g.x, gHg^{-1})$ and the product topology. We called such a subgroup *dynamically recurrent*, if the closure of its orbit does not contain the trivial subgroup $(x, \{e\})$ at the same basepoint. For G acting minimally on X, Kawabe shows that the absence of dynamically recurrent amenable subgroups in the stabilisers is again equivalent to simplicity of the reduced crossed product $C(X) \rtimes G$, while in general it is equivalent to the intersection property of every closed, G-invariant subset of X.

In the following we provide a generalisation of one of Kawabe's results from Theorems 3.3.4 and 3.3.6 to étale groupoids, establishing a new sufficient criterion for the intersection property and consequently for C^*-simplicity. Let $\mathrm{Sub}(\mathcal{G})$ denote the space of all subgroups of the isotropy groups of $\mathcal{G}$, that is, the disjoint union of the $\mathrm{Sub}(\mathcal{G}_x^x)$ ranging over all $x \in \mathcal{G}^{(0)}$. Recall that the Chabauty topology is a topology on the power set of $\mathcal{G}$ with a subbasis $\mathcal{B}$ given by

$$\mathcal{B} = \left\{ O_U, O'_K \;\middle|\; U \subset \mathcal{G} \text{ open}, K \subset \mathcal{G} \text{ compact} \right\}$$

where

$$O_U = \left\{ Y \subset \mathcal{G} \;\middle|\; Y \cap U \neq \emptyset \right\} \qquad \text{and} \qquad O'_K = \left\{ Y \subseteq \mathcal{G} \;\middle|\; Y \cap K = \emptyset \right\}.$$

As $\mathcal{G}$ is not discrete, this no longer coincides with the topology of pointwise convergence of indicator functions. We endow $\mathrm{Sub}(\mathcal{G})$ with the subspace topology as a subset of the power set of $\mathcal{G}$ with the Chabauty topology. As before we may equip $\mathrm{Sub}(\mathcal{G})$ with an action of $\mathcal{G}$ by conjugation, where an arrow $\gamma \in \mathcal{G}$ acts on a subgroup $(s(\gamma), H_{s(\gamma)})$ of $\mathcal{G}_{s(\gamma)}^{s(\gamma)}$ by conjugating it to the subgroup $(r(\gamma), \gamma H_{s(\gamma)} \gamma^{-1})$ of $\mathcal{G}_{r(\gamma)}^{r(\gamma)}$. For ease of notation we will usually drop the fiber as it is implicit and simply write $\gamma.H_{s(\gamma)} = \gamma H_{s(\gamma)} \gamma^{-1}$.

Definition 4.2.24: Let $\mathcal{G}$ be a locally compact étale Hausdorff groupoid. Choosing a subgroup $H_x \leq \mathcal{G}_x^x$ of the isotropy group $\mathcal{G}_x^x$ at every unit x, we call $\Lambda = \{H_x \mid x \in \mathcal{G}^{(0)}\}$ a *section of isotropy subgroups*. We furthermore call the section *recurrent* if the closure of its $\mathcal{G}$-orbits under conjugation does not contain all trivial subgroups of the isotropy groups. That is, if there is $x_0 \in \mathcal{G}^{(0)}$ such that $\{x_0\}$ cannot be approximated by subsets of the form $\gamma_\lambda H_{x_\lambda} \gamma_\lambda^{-1}$ for $\gamma_\lambda \in \mathcal{G}_{x_\lambda}$ and $H_{x_\lambda} \in \Lambda$. We call a section of isotropy subgroups Λ *amenable* if all contained subgroups $H_x \in \Lambda$ for all $x \in \mathcal{G}^{(0)}$ are amenable.

Similar to the setting of crossed products in Theorem 3.3.4, the absence of amenable recurrent sections of isotropy subgroups forces the boundary groupoid to be principal and provides therefore a sufficient criterion for the intersection property of $\mathcal{G}$.

Theorem 4.2.25: *Let $\mathcal{G}$ be a Hausdorff étale groupoid with compact unit space that does not have recurrent amenable sections of isotropy subgroups, and in which the orbit of any unit in the groupoid contains at least two points. Then $\mathcal{G}$ has the intersection property.*

Proof. As before let $\tilde{\mathcal{G}}$ be the boundary groupoid of $\mathcal{G}$, let $Q\colon \tilde{\mathcal{G}} \to \mathcal{G}$ be the continuous groupoid homomorphism given by $(\gamma, \tilde{x}) \mapsto \gamma$, and denote the unit spaces of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ by X and $\tilde{X}$, respectively. Assume that $\mathcal{G}$ has no amenable recurrent sections of subgroups in the isotropy. Note that all isotropy groups $\tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}}$ of the boundary groupoid are amenable by Proposition 4.2.16, hence so are the corresponding subgroups $Q(\tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}})$ of $\mathcal{G}$. Let $\Lambda = \left\{Q(\tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}}) \mid \tilde{x} \in \tilde{X}\right\}$ and note that its orbit closure $\overline{\mathcal{G}.\Lambda}$ contains the trivial subgroups $\{\{x\} \mid x \in X\}$ at every unit of $\mathcal{G}$, since it contains an amenable section of subgroups in the isotropy which is by assumption not recurrent. However, Λ is already invariant since $\gamma.\tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}} = \tilde{\mathcal{G}}^{\gamma.\tilde{x}}_{\gamma.\tilde{x}}$ and is furthermore closed: Let $\Phi\colon \tilde{X} \to \mathrm{Sub}(\mathcal{G})$ be the isotropy map $\tilde{x} \mapsto Q(\tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}})$. Then $\Lambda = \Phi(\tilde{X})$ is the range of Φ, and since $\tilde{X}$ is compact and $\mathrm{Sub}(\mathcal{G})$ Hausdorff, Λ is closed provided that Φ is continuous. To verify this, we calculate

$$\begin{aligned} \Phi^{-1}(O_U) &= \left\{\tilde{x} \,\middle|\, \tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}} \cap Q^{-1}(U) \neq \emptyset\right\} & \text{and} \qquad \Phi^{-1}(O'_K) &= \left\{\tilde{x} \,\middle|\, \tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}} \cap Q^{-1}(K) = \emptyset\right\} \\ &= s\Big(\mathrm{Iso}\big(\tilde{\mathcal{G}}\big) \cap Q^{-1}(U)\Big) & &= \tilde{X} \setminus s\Big(\mathrm{Iso}\big(\tilde{\mathcal{G}}\big) \cap Q^{-1}(K)\Big). \end{aligned}$$

As every orbit in $\mathcal{G}$ has at least two points, $\mathrm{Iso}(\tilde{\mathcal{G}})$ is clopen by Proposition 4.2.19, and furthermore s is an open surjection while $Q^{-1}(U)$ and $Q^{-1}(K)$ are open and closed, respectively. Hence, Φ is continuous and $\Lambda = \Phi(\tilde{X})$ is closed. Now, Λ contains $\{\{x\} \mid x \in X\}$, so for every $x \in X$ there is some $\tilde{x} \in \tilde{X}$ with $q(\tilde{x}) = x$ and $\tilde{G}^{\tilde{x}}_{\tilde{x}} = \{\tilde{x}\}$. That is, there are enough trivial isotropies in $\tilde{\mathcal{G}}$ to cover X along q. Hence $Z := \{\tilde{x} \in \tilde{X} \mid \tilde{\mathcal{G}}^{\tilde{x}}_{\tilde{x}} = \{\tilde{x}\}\}$ is a $\mathcal{G}$-invariant subset of $\tilde{X}$ with $q(Z) = X$. Furthermore, Z is closed, since its complement $\tilde{X} \setminus Z = s(\mathrm{Iso}(\tilde{\mathcal{G}}) \setminus \tilde{X})$ is the image of an open set under the open, surjective map s. Thus $Z = \tilde{X}$ by Proposition 4.2.17, so $\tilde{\mathcal{G}}$ is principal and hence has the intersection property. By Lemma 4.2.23 we conclude that $\mathcal{G}$ inherits the intersection property from $\tilde{\mathcal{G}}$. □

Although slightly weaker, the above theorem looks more familiar when phrased in terms of dynamically recurrent subgroups instead of sections of subgroups:

Definition 4.2.26: Let $\mathcal{G}$ be a locally compact étale Hausdorff groupoid and $x \in \mathcal{G}^{(0)}$ a unit. A subgroup $H \leq \mathcal{G}^x_x$ of the isotropy $\mathcal{G}^x_x$ at x is called a *dynamically recurrent* subgroup of the isotropy group, if the closure of the $\mathcal{G}$-orbit of H under the conjugation action does not contain the trivial subgroup $\{x\}$ of $\mathcal{G}^x_x$.

To shorten notation, we call such $H \leq \mathcal{G}^x_x$ a dynamically recurrent subgroup of $\mathcal{G}$, rather than a dynamically recurrent subgroup of an isotropy group of $\mathcal{G}$.

Corollary 4.2.27: *Let $\mathcal{G}$ be a Hausdorff étale groupoid with compact unit space that does not have dynamically recurrent amenable subgroups, and in which the orbit of any unit in the groupoid contains at least two points. Then the restriction of $\mathcal{G}$ to any closed invariant subset of $\mathcal{G}^{(0)}$ has the intersection property.*

Proof. The absence of dynamically recurrent (amenable) subgroups implies absence of recurrent (amenable) sections of subgroups in the isotropy: If $\Lambda = \{H_y \mid y \in \mathcal{G}^{(0)}\}$ was a recurrent section of subgroups, then its orbit closure would not contain $\{x\}$ for some unit $x \in \mathcal{G}^{(0)}$. Hence the orbit closure of H_x, a subset of the orbit closure of Λ, would not contain $\{x\}$ either and therefore H_x would be dynamically recurrent.

Noting that this implication holds for every restricion of $\mathcal{G}$ to a closed invariant subset of $\mathcal{G}^{(0)}$, we may apply Theorem 4.2.25 to any such restriction and obtain the desired conclusion. □

Remark. A groupoid $\mathcal{G}$ for which every restriction of $\mathcal{G}$ to a closed invariant subset of $\mathcal{G}^{(0)}$ has the intersection property is said to have the *residual intersection property*, see for example [6, Def 3.8].

We finally apply Theorem 4.2.25 to describe C^*-simplicity of $\mathcal{G}$. Recall, for example from [6, Thm A], that a groupoid is C^*-simple exactly if it has the intersection property and is minimal, that is, if every orbit in $\mathcal{G}^{(0)}$ is dense. While minimality is a straightforward property of the groupoid, the intersection property is hard to describe without passing to the associated C^*-algebras, and is therefore in applications often replaced by the stronger assumption of topological principality. See further the works by Archbold–Spielberg [3], Kawamura–Tomiyama [39], and Sierakowski [55] for crossed products and Brown–Clark–Farthing–Sims [11], as well as Bönicke–Li [6] for groupoids. Therefore, we may freely restrict to minimal groupoids when concerned with C^*-simplicity.

Corollary 4.2.28: *Let $\mathcal{G}$ be a* minimal *Hausdorff étale groupoid with compact unit space that does not have recurrent amenable sections of isotropy subgroups. Then $\mathcal{G}$ is C^*-simple.*

Proof. If $\mathcal{G}^{(0)}$ consists of a single point then $\mathcal{G}$ is a discrete group and C^*-simple by Kennedy's characterisation of Theorem 2.3.16. If $\mathcal{G}^{(0)}$ has more than one point, then every orbit has at least two points since it is dense. Therefore, $\mathcal{G}$ has the intersection property by Corollary 4.2.27 and is C^*-simple since it is also minimal. □

Question 1: Does the converse of Theorem 4.2.25 hold? More precisely, let $\mathcal{G}$ be a Hausdorff étale groupoid with compact unit space. Is the absence of recurrent amenable sections of isotropy subgroups in $\mathcal{G}$ *equivalent* to the intersection property of $\mathcal{G}$? Is the absence of dynamically recurrent amenable subgroups of $\mathcal{G}$ furthermore equivalent to the *residual* intersection property of $\mathcal{G}$?

Corollary 4.2.29: *Let $\mathcal{G}$ be a minimal Hausdorff étale groupoid with compact unit space. If any isotropy group of $\mathcal{G}$ is C^*-simple, then $C^*_r(\mathcal{G})$ is also simple.*

Proof. If $\mathcal{G}^x_x$ is C^*-simple for some unit $x \in \mathcal{G}^{(0)}$, then $\mathcal{G}^x_x$ does not contain recurrent amenable subgroups by Theorem 2.3.16. Therefore, any amenable subgroup H_x of $\mathcal{G}^x_x$ contains $\{x\}$ in its orbit closure. As any amenable section of subgroups in the isotropy will contain an amenable subgroup of $\mathcal{G}^x_x$, its orbit closure will contain $\{x\}$ and therefore all trivial subgroups in the isotropy by minimality. Hence $\mathcal{G}$ does not have recurrent amenable sections of subgroups in the isotropy and is therefore C^*-simple. □

Note that for a second-countable groupoid topological principality, where a dense set of units in $\mathcal{G}^{(0)}$ has trivial isotropy, implies not having recurrent amenable sections of subgroups in the isotropy, since the choices of H_x must be the trivial subgroup for a dense set of units, approximating any other trivial subgroup of the isotropy bundle. For minimal $\mathcal{G}$, topological principality furthermore implies the absence of dynamically recurrent amenable subgroups:

Lemma 4.2.30: *Let $\mathcal{G}$ be a minimal Hausdorff étale groupoid with compact unit space that has at least one trivial isotropy group. Then $\mathcal{G}$ has no dynamically recurrent amenable subgroups.*

Proof. Let $x \in \mathcal{G}^{(0)}$ be a unit. We show that conjugates of any given isotropy group $\mathcal{G}_x^x$ already approximate trivial subgroups, so that any subgroup of G_x^x will likewise have to approximate a trivial subgroup. Assume that $\mathcal{G}_{x_0}^{x_0} = \{x_0\}$ is the trivial isotropy at x_0. Since $\mathcal{G}$ is minimal, there are $\gamma_\lambda \in \mathcal{G}_x$ such that $x_\lambda := r(\gamma_\lambda) \xrightarrow{\lambda\to\infty} x_0$ and we show that $\mathcal{G}_{x_\lambda}^{x_\lambda} \to \mathcal{G}_{x_0}^{x_0} = \{x_0\}$ in the Chabauty topology. Note that $y_\lambda \to y$ does not in general imply $G_{y_\lambda}^{y_\lambda} \to \mathcal{G}_y^y$. Any open set $U \subseteq \mathcal{G}$ containing x_0 contains a neighbourhood of x_0 in $\mathcal{G}^{(0)}$, as the latter is open, so it eventually contains x_λ and therefore intersects $G_{x_\lambda}^{x_\lambda}$. On the other hand, every compact set $K \subseteq \mathcal{G}$ that does not contain x_0 intersects the closed isotropy in a compact set $K \cap \operatorname{Iso}(\mathcal{G})$ not containing x_0. As the range map is continuous, $r(K \cap \operatorname{Iso}(\mathcal{G}))$ is a compact subset of $\mathcal{G}^{(0)}$ that does not contain x_0 and therefore $\mathcal{G}_{x_\lambda}^{x_\lambda}$ will eventually not intersect K when $x_\lambda \notin r(K \cap \operatorname{Iso}(\mathcal{G}))$. Together this shows that $\gamma_\lambda \mathcal{G}_x^x \gamma_\lambda^{-1} \to \{x_0\}$ and therefore the same holds for every (amenable) subgroup H_x of $\mathcal{G}_x^x$. As $\mathcal{G}$ is minimal, x is contained in the orbit closure of x_0 and therefore $\{x\}$ is contained in the orbit closure of $\{x_0\}$, which is again contained in the orbit closure of H_x. □

Remark. In the case of minimality, having at least one trivial isotropy subgroup is of course equivalent to being topologically principal for a second-countable groupoid.

For *amenable* minimal groupoids, it is known that C^*-simplicity is equivalent to topological principality, see e.g. [6, Corollary 3.12]. The following Lemma shows that in the above equivalence one can replace topological principality with the absence of dynamically recurrent amenable subgroups.

Lemma 4.2.31: *Let $\mathcal{G}$ be a (not necessarily minimal) locally compact Hausdorff étale groupoid with compact unit space. Assume that every isotropy group of $\mathcal{G}$ is amenable. Then $\mathcal{G}$ is topologically principal if it has no dynamically recurrent amenable subgroups. If $\mathcal{G}$ is furthermore minimal, the two notions are equivalent.*

Proof. Assuming that $\mathcal{G}$ is not topologically principal, we show that it contains a dynamically recurrent amenable subgroup in the isotropy. Since the interior of the isotropy $\operatorname{Iso}(\mathcal{G})$ of $\mathcal{G}$ does not coincide with the clopen unit space $\mathcal{G}^{(0)}$, we may choose a nonempty open set U contained in $\operatorname{Iso}(\mathcal{G})^\circ \setminus \mathcal{G}^{(0)}$. Take K' any compact subset of $\mathcal{G}$ that contains U, then K given by $(K' \cap \operatorname{Iso}(\mathcal{G})) \setminus \mathcal{G}^{(0)}$ is a compact subset containing U that is disjoint from $\mathcal{G}^{(0)}$. Let $x \in s(U)$. We show that $\mathcal{G}_x^x$ is dynamically recurrent. If it was not, conjugates of $\mathcal{G}_x^x$ would approximate the trivial subgroup $\{x\}$, that is, there would be x_λ in the orbit of x such that $\mathcal{G}_{x_\lambda}^{x_\lambda} \to \{x\}$. However, as x is not contained in the compact set K but is contained in the open set $s(U)$, both $\mathcal{O}'_K$ and $\mathcal{O}_{s(U)}$ are neighbourhoods of $\{x\}$, but if $\mathcal{G}_{x_\lambda}^{x_\lambda} \in \mathcal{O}_{s(U)}$, then $x_\lambda \in s(U) \subseteq s(K)$, so $\mathcal{G}_{x_\lambda}^{x_\lambda}$ intersects K and therefore $\mathcal{G}_{x_\lambda}^{x_\lambda} \notin \mathcal{O}'_K$. In other words, the neighbourhood $\mathcal{O}_{s(U)} \cap \mathcal{O}'_K$

of $\{x\}$ does not contain any isotropy groups, so $\{x\}$ cannot be approximated by conjugates of $\mathcal{G}_x^x$ which necessarily are isotropy groups. Therefore, $\mathcal{G}_x^x$ is dynamically recurrent and by assumption amenable.

If $\mathcal{G}$ is minimal, the converse implication is given in Lemma 4.2.30. □

Remark. Note for a minimal groupoid $\mathcal{G}$ *in which all isotropy groups are amenable* the existence of a dynamically recurrent amenable subgroup of $\mathcal{G}$ is equivalent to the existence of a recurrent amenable section of subgroups in the isotropy, since the orbit closure of any amenable subgroup in the isotropy will be an amenable section of subgroups in the isotropy by minimality and the latter is recurrent exactly if the former is. In general, it is unclear whether amenability is a closed property in the Chabauty topology on $\mathrm{Sub}(\mathcal{G})$.

Question 2: Does the equivalence of C^*-simplicity and topological principality extend from minimal, amenable groupoids to minimal groupoids for which all isotropy groups are amenable?

If Question 1 were to be answered in the positive, so would Question 2.

4.3 Elek Algebras

This section is based on [7], to appear in Mathematica Scandinavica, in which we reformulate a construction by Gábor Elek, which associates C^*-algebras with uniformly recurrent subgroups, in the language of groupoid C^*-algebras. Apart from simplifying several proofs from Elek's original papers and adding the converse direction to his characterisation of nuclearity, this allows us to apply the results from Section 4.2.4 to find new examples of simple Elek algebras.

4.3.1 Introduction

Defined by Glasner and Weiss [26], *uniformly recurrent subgroups*, or URSs for short, have recently drawn a lot of attention in the world of C^*-algebras since Kennedy [40] characterised C^*-simplicity of a discrete group G as the absence of non-trivial amenable uniformly recurrent subgroups. Another relation between uniformly recurrent subgroups and C^*-algebras was given by a construction of Elek [20], who defined a C^*-algebra closely tied to the dynamics of a finitely-generated discrete group G acting on one of its uniformly recurrent subgroups Z. His construction, the completion of the algebra of "local kernels" on the Schreier graph of a subgroup in the URS Z in a regular representation, takes more of the combinatorial nature of Z into account than the crossed product of G acting on $C(Z)$ does. This construction is thereby very well-suited for finding C^*-algebras with desired properties by rephrasing such properties at the combinatorial level of URSs. For example, Elek obtains a C^*-algebra with both a uniformly amenable and a nonuniformly amenable trace, by providing a Schreier graph with the corresponding properties for URSs.

We recast Elek's construction from the viewpoint of groupoid C^*-algebras, which allows us to simplify and extend the ties between properties of the URS and its associated algebra. Using this new angle, we are able to prove that Elek's sufficient criterion for nuclearity of his C^*-algebras, the so-called *local property A*, is in fact also necessary, providing an equivalent characterisation of nuclearity of $C_{\mathrm{r}}^*(Z)$.

After recalling the necessary terminology in Section 4.3.2, we construct in Section 4.3.3 an étale groupoid for a given URS Z whose reduced C^*-algebra is canonically isomorphic to Elek's C^*-algebra $C^*_\Gamma(Z)$. In Section 4.3.4 we relate this groupoid to the dynamics of the action of G on Z by conjugation. Finally, in Section 4.3.5, we use the new framework to give simpler proofs for some of Elek's results on simplicity and nuclearity of the associated C^*-algebras and provide the now equivalent characterisation of nuclearity.

4.3.2 Preliminaries

We recall the definition of Elek's C^*-algebras associated with uniformly recurrent subgroups. Let G be a finitely-generated discrete group. Let $\mathrm{Sub}(G)$ be the space of its subgroups, equipped as in Section 2.3.3 with the topology of pointwise convergence of the characteristic functions associated with the subsets and left G-action $g.H = gHg^{-1}$ by conjugation, where $H \in \mathrm{Sub}(G)$ and $g \in G$. Recall that for a discrete group G, this topology is also known as the Chabauty topology or Fell topology on $\mathrm{Sub}(G)$. As convergence in the topology is a pointwise condition, and conjugation with a fixed element g is simply a relabelling of G, the action of G on $\mathrm{Sub}(G)$ is continuous. Recall from Definition 2.3.9 that a *uniformly recurrent subgroup* (URS) of G is a closed, G-invariant subspace of $\mathrm{Sub}(G)$ on which the action is *minimal*, that is, on which every orbit is dense. By Elek's terminology the URS is called *generic*, if the stabiliser of any subgroup $H \in Z$ is as small as possible, namely H itself.

Note that every normal subgroup of G trivially defines a URS that consists of a single point, and furthermore every closed, invariant subset of $\mathrm{Sub}(G)$ contains a URS by Lemma 2.3.14. The number of distinct URSs in a given countable discrete group G can vary wildly between just the trivial normal subgroups $\{e\}$ and G (for a "Tarski monster" group) and uncountably many non-isomorphic URSs (for $\mathbb{F}_2$), as shown by Glasner and Weiss [26, Example 1.9, Theorem 5.1].

Fixing a finite, symmetric system of generators Q of G, to each $H \in \mathrm{Sub}(G)$ we may assign a rooted, labeled graph $S^Q_\Gamma(H)$ called its *Schreier graph*, which has vertex set G/H, root H, and for every $gH \in G/H$ and $q \in Q$ an edge from gH to qgH labeled by q. The group G acts on the vertex set G/H of $S^Q_\Gamma(H)$ by left multiplication, or, equivalently, by following along the edges that spell out g in terms of the generators Q. We denote the shortest-path metric on a graph S by d, or d_S if there is ambiguity, and likewise the balls of radius R around a vertex $x \in S$ by $B_R(x)$ or $B_R(S, x)$. On the space S^Q_Γ of Schreier graphs associated with subgroups of G, we introduce a metric by

$$d_{S^Q_\Gamma}(S_1, S_2) := 2^{-r} \tag{4.13}$$

for $S_1 = S^Q_\Gamma(H_1)$ and $S_2 = S^Q_\Gamma(H_2)$ two graphs in S^Q_Γ and r the largest integer such that $B_r(S_1, H_1)$ and $B_r(S_2, H_2)$ are root-label isomorphic. This space carries a left G-action, where $g.S^Q_\Gamma(H) = S^Q_\Gamma(gHg^{-1})$, that is, g acts by changing the root. Elek identifies the graphs $S^Q_\Gamma(H)$ for which the orbit closure of H forms a URS as those where any root-label isomorphism class of balls is repeated with at most bounded distance from any point in the graph (see [20, Proposition 2.1] for the full description), and the graphs for which it forms a generic URS as those where vertices with large isomorphic balls are sufficiently far apart (see [20, Proposition 2.3]).

To associate a C^*-algebra with a given URS Z, Elek considers its *local kernel algebra* $\mathbb{C}Z$ formed by the *local* kernels $K\colon G/H \times G/H \to \mathbb{C}$ of finite width on $S^Q_\Gamma(H)$ for some subgroup

$H \in Z$. A kernel K on $S^Q_\Gamma(H)$ is of width R, if $K(p,q) = 0$ for any two vertices $p, q \in S^Q_\Gamma(H)$ with $d(p,q) > R$. It is furthermore *local* with width R, if $K(p, g.p) = K(q, g.q)$ for any p and q with root-label isomorphic R-balls and $g \in G$ of length at most R. Equipped with pointwise addition, convolution $KL(p,q) = \sum_z K(p,z)L(z,q)$, and involution $K^*(p,q) = \overline{K(q,p)}$ for local kernels K and L and p, q, and z in G/H, the local kernel algebra forms a *-algebra. Up to isomorphism, this algebra does not depend on the choice of root $H \in Z$. A "regular" representation of $\mathbb{C}Z$ on $\ell^2(G/H)$ is given by $(Kf)(p) = \sum K(p,q)f(q)$ for $f \in \ell^2(G/H)$. The reduced C^*-algebra $C^*_r(Z)$ of Z is the completion of $\mathbb{C}Z$ in the norm induced by this representation.

Several C^*-algebraic properties of $C^*_r(Z)$ can be read off of the URS Z and its Schreier graph. Genericity of Z implies simplicity of $C^*_r(Z)$, as discussed in Section 4.3.5. Furthermore, a *local* version of Yu's property A for the Schreier graph $S^Q_\Gamma(H)$, for any subgroup H in the URS Z, is equivalent to nuclearity of $C^*_r(Z)$. Recall that a Schreier graph $S^Q_\Gamma(H)$ for $H \in Z \subseteq G$ has Elek's *local property A* if there is a sequence of *local* functions

$$\rho^n \colon G/H \to \ell^2(G/H), \quad p \mapsto \rho^n_p,$$

such that $\|\rho^n_p\|_2 = 1$, while $d(p,q) \leq n$ implies $\|\rho^n_p - \rho^n_q\| \leq 1/n$. As with kernels, *locality* of a function $\rho \colon G/H \to \ell^2(G/H)$ means that there is some $R > 0$ such that ρ_p is supported in the R-ball $B_R(p)$ centred at p, and whenever θ is a root-label isomorphism $B_R(p) \to B_R(q)$, we have $\rho_q \circ \theta = \rho_p$.

4.3.3 URS Algebras as Groupoid Algebras

Construction of the Groupoid

For a given uniformly recurrent subgroup, we construct a groupoid whose regular representation models Elek's construction on the local kernel algebra.

Let Z be a uniformly recurrent subgroup of a discrete group G, fix $H \in Z$, and let $S = S^Q_\Gamma(H)$ be its Schreier graph. For any $n \in \mathbb{N}$ we define an equivalence relation on the vertices $V(S)$ of S by

$$p \sim_n q \Leftrightarrow B_n(S,p) \cong_{r,l} B_n(S,q);$$

that is, if the n-balls around p and q are isomorphic under an isomorphism preserving the roots and labels. Such a root-label isomorphism is necessarily unique, since the roots are unique and every vertex is uniquely described by the labels on any path from the root to said vertex. Likewise, if a root-label isomorphism of n-balls on a Schreier graph has a fixed point, the two balls coincide and the root-label isomorphism is the identity, since we may describe every vertex in the balls by paths relative to the fixed point. The equivalence relation on the vertices $p, q \in V(S)$ can alternatively be formulated in terms of the group elements g_p and g_q of G describing paths from the root of S to p and q, respectively: With the above we have $p \sim_n q$ if and only if $g_p.S$ and $g_q.S$ are 2^{-n}-close in the metric $d_{S^Q_\Gamma}$ on S^Q_Γ introduced in Equation (4.13). Let $E_n = V(S)/\sim_n$ denote the finite set of equivalence classes of $\sim_n$ equipped with the discrete topology and the obvious connecting maps $e_{n+1} \colon E_{n+1} \to E_n$. Note that these sets do not depend on the choice of $H \in Z$:

Proposition 4.3.1: Let H and H' be two groups contained in a URS Z of a discrete, finitely-generated group G. Then H and H' have the same sets of equivalence classes of rooted, labeled n-balls in their associated Schreier graphs.

Proof. Let Q be a generating set of G, and let E_n and E'_n be the sets of root-label equivalence classes of n-balls in $S = S_G^Q(H)$ and $S' = S_G^Q(H')$. We show that for every $p \in V(S)$ there is a $p' \in V(S')$ such that $B_n(S, p) \cong_{r,l} B_n(S', p')$. If $p = gH$, then since Z is uniformly recurrent, the subgroup gHg^{-1} is in the orbit closure of H', and therefore $S_G^Q(gHg^{-1})$ is in the orbit closure of S'. Hence there is some $g' \in G$ such that

$$B_n(S, p) \cong_{r,l} B_n(S_G^Q(gHg^{-1}), gHg^{-1}) \cong_{r,l} B_n(S', g'H')$$

and $p' = g'H'$ gives $E_n \subseteq E'_n$. The claimed equality follows by symmetry. □

We endow the projective limit $\varprojlim E_n$ with the subspace topology of the product topology on $\prod_{n\in\mathbb{N}} E_n$. Then, as in [20, Lemma 6.1], it is easy to check that $\varprojlim E_n$ is homeomorphic to Z as a subspace of $\mathrm{Sub}(G)$, which in turn is either a Cantor space or a finite discrete set. It is noteworthy that under this identification, the orbit of H in Z is exactly described by those elements in $\varprojlim E_n$ that can be represented by the equivalence classes $[p]_n$ of a *fixed* vertex $p \in S$. This is depicted in Figure 4.1. The other elements, as in Figure 4.2, describe subgroups in the orbit closure, but not the orbit, of H.

We employ this description of the space Z to construct an ample Hausdorff étale groupoid $\mathcal{G}_Z$ with unit space $\mathcal{G}_Z^0$ homeomorphic to Z, whose reduced groupoid C^*-algebra is $C_r^*(Z)$. As a set, the groupoid $\mathcal{G}_Z$ is identified with a subset of the projective limit $\varprojlim F_n$ of the finite, discrete sets

$$F_n = \bigsqcup_{[x_n]\in E_n} B_n(S, x_n) \sqcup \{\infty_n\}. \tag{4.14}$$

Here $B_n(S, x_n)$ denotes the n-ball in S that is determined uniquely by $[x_n]$, even if there is a choice in the representing vertex x_n. To avoid this choice of representing elements, a pair $([x_n], y)$ with $y \in B_n(S, x_n)$ as in Equation (4.14) can be more readily expressed as a pair $([x_n], g)$ as in Equation (4.15) below, where g is any chosen path from x_n to y inside $B_n(S, x_n)$, up to $g \approx_{x_n} g'$ if both paths lead to the same vertex in S; that is, if $g.x_n = g'.x_n$:

$$F_n \cong \bigsqcup_{[x_n]\in E_n} \{g \in G \mid l(g) \leq n\}/{\approx_{x_n}} \sqcup \{\infty_n\}. \tag{4.15}$$

Implicit in this description is our later identification of $\mathcal{G}_Z$ with a quotient of the transformation groupoid $Z \rtimes G$. In this picture, ∞_n fills in for choices of vertices that are not contained in $B_n(S, x_n)$, or, respectively, for those g whose length as a word in the generators exceeds n. The connecting maps $F_{n+1} \to F_n$ are then given by

$$([x_{n+1}], g) \mapsto \begin{cases} (e_{n+1}([x_{n+1}]), g) & \text{if } d(x_{n+1}, g.x_{n+1}) \leq n \\ \infty_n & \text{else} \end{cases}$$
$$\infty_{n+1} \mapsto \infty_n.$$

Now let $\mathcal{G}_Z := \varprojlim F_n \setminus \{\infty\}$ with $\infty = (\infty_1, \infty_2, \ldots)$, equipped with the subspace topology of the projective limit, which itself is equipped with the subspace topology of the product

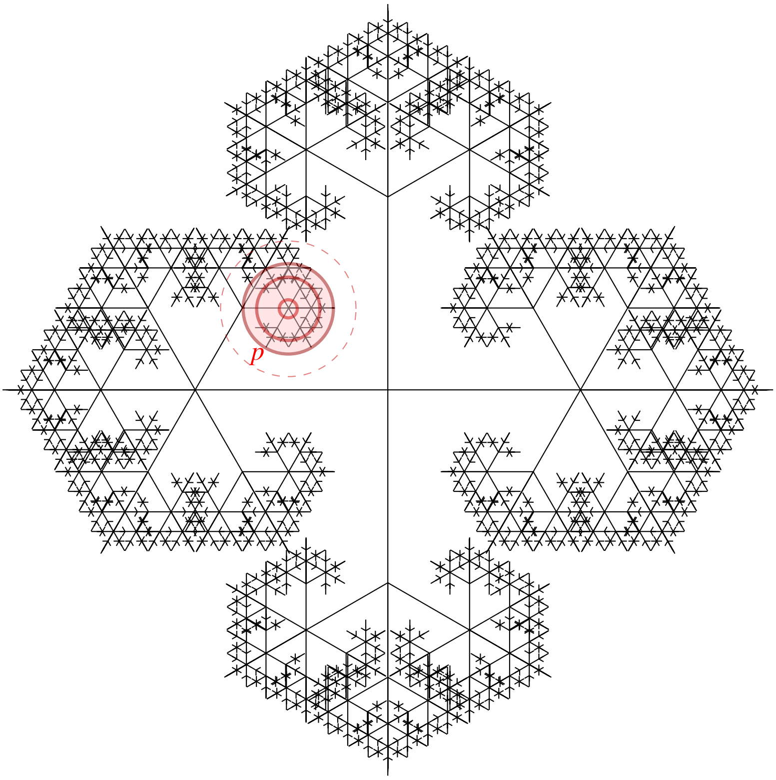

Figure 4.1: Elements of $\varprojlim E_n$ are described by a sequence of isomorphism classes of n-balls in some fixed Schreier graph $S_\Gamma^Q(H)$ for a group $H \in Z$. Writing $([p_0]_0, [p_1]_1, [p_2]_2, \ldots) \in \varprojlim E_n$, we denote by $[p_n]_n$ the equivalence class in E_n represented by the n-ball around a vertex $p_n \in S_\Gamma^Q(H)$. To form a valid element, the sequence of vertices $(p_n)_n$ has to be compatible in the sense that $[p_i]_i = [p_j]_i$ for all $i \leq j$. A particularly easy way to achieve this is by choosing a constant vertex p such that $p_n = p$ for all n. If a path from the root of $S_\Gamma^Q(H)$ to p is described by the group element g, then the element $([p]_0, [p]_1, [p]_2, \ldots)$ is sent to the group $g.H$ under the homeomorphism between $\varprojlim E_n$ and Z. This is reflected in the fact that changing the root of $S_\Gamma^Q(H)$ to p will yield the Schreier graph $S_\Gamma^Q(g.H)$. The Schreier graphs depicted are chosen for ease of drawing with no particular group in mind. Labels and loops are omitted for readability.

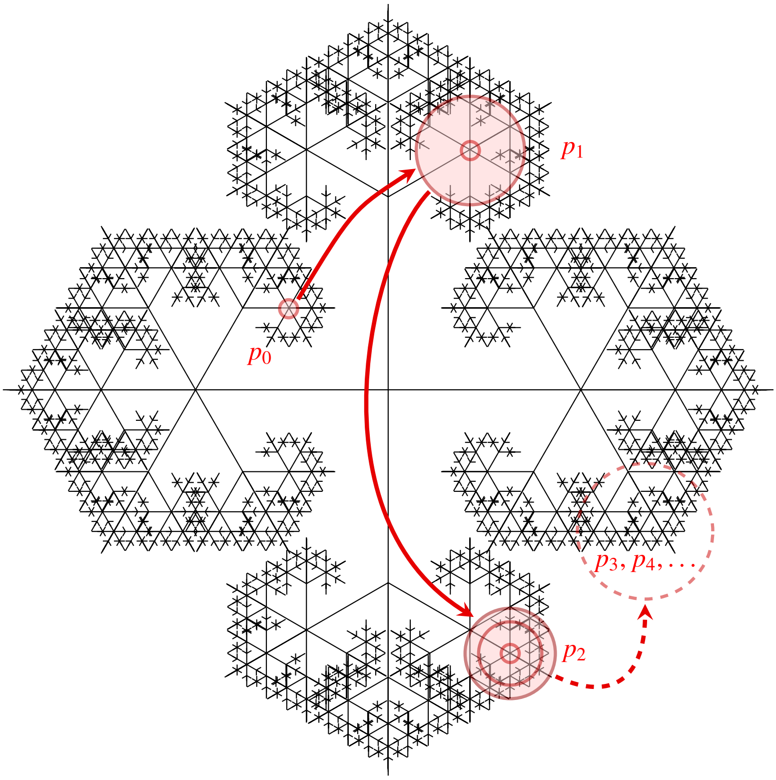

Figure 4.2: There might be elements $([p_0]_0, [p_1]_1, [p_2]_2, \ldots) \in \varprojlim E_n$ that are not described by a single, constant vertex $p \in V(S^Q_\Gamma(H))$, but by a choice of vertices p_n such that the i-ball around any p_j for $j \geq i$ is root-label-isomorphic to the i-ball around the vertex p_i, without eventually being constant. These isomorphism classes of balls assemble into a new Schreier graph, whose n-ball around the root is root-label-isomorphic to the n-ball around p_n. If $g_n \in G$ describes a path from the root of $S^Q_\Gamma(H)$ to p_n, then then $g_n.H$ will converge to another group $K \in Z$ and the Schreier graph assembled from the balls around the vertices p_n is $S^Q_\Gamma(K)$. Consequently, $([p_0]_0, [p_1]_1, [p_2]_2, \ldots)$ is sent to K under the homeomorphism between $\varprojlim E_n$ and Z.

topology on $\prod_{n\in\mathbb{N}} F_n$. Identifying E_n with a subset of F_n by sending $[x_n]$ to $([x_n], e)$, we may identify $Z \cong \varprojlim E_n$ with a subset of $\mathcal{G}_Z$. To simplify notation, we write $(x, g) \in \varprojlim E_n \times G$ for the element of $\varprojlim F_n$ that it represents when making a choice of representing vertex $[x_n] \in E_n$ for every n while identifying $([x_n], g)$ with ∞_n whenever $l(g) > n$. Note that (x, g) and (x, g') with $l(g') \geq l(g)$, represent the same equivalence class in $\varprojlim F_n$ if $g.x_{l(g')} = g'.x_{l(g')}$. The topology on $\mathcal{G}_Z$ is then equivalently given by the metric $d_{\mathcal{G}_Z}$, where for x and y in Z and $g, g' \in G$,

$$d_{\mathcal{G}_Z}((x, g), (y, g')) = 2^{-N}$$

for N maximal such that $([x_N]_N, g.x_N)$ and $([y_N]_N, g'.y_N)$ coincide in F_N.

We define the range and source maps as $r(x, g) = (x, e)$ and $s(x, g) = (g.x, e)$ respectively, with $g.x := ([g.x_{l(g)}]_0, [g.x_{l(g)+1}]_1, \ldots)$. This is merely acting entry-wise with g, but for the action to be well-defined we first need to left-shift the representing elements by $l(g)$ steps because for two vertices p and q, $p \sim_n q$ does not imply that $g.p \sim_n g.q$, but $p \sim_{n+l(g)} q$ does, as depicted in Figure 4.3. Clearly, such a left-shift of representing elements does not change the described element of $\varprojlim E_n$. If we define an action of G on $\varprojlim E_n$ by $g.x = s(x, g)$ as above, the mentioned homeomorphism between $\varprojlim E_n$ and Z is equivariant.

The range and source maps map $\varprojlim F_n$ onto the subspace where the group element is simply e, that is, onto the subspace $\varprojlim E_n$ in our earlier identification. The unit space $\mathcal{G}_Z^0 = r(\mathcal{G}_Z) = s(\mathcal{G}_Z)$ is therefore homeomorphic to Z. Elements of the range fibre $\mathcal{G}_Z^x$ have the form (x, g) for $g \in G$, and the identification of (x, g) and (x, g') with $l(g') \geq l(g)$ if $g.x_{l(g')} = g'.x_{l(g')}$ is exactly the definition of $\cong_{x_{l(g')}}$. Note that, depending on the choice of g, it might be that $d(x_{l(g)}, g.x_{l(g)})$ is strictly less than $l(g)$, in which case there is another $g' \in G$ of length $d(x_{l(g)}, g.x_{l(g)})$, such that (x, g) and (x, g') denote the same arrow. If there is no such g', we call g an element *of minimal length*.

If (x, g) and (y, g') are composable, that is, if $x = g'.y$, then we define the composition $(y, g')(x, g)$ to be (y, gg'). Note that $s(y, gg') = s(x, g)$, since $([x_0]_0, [x_1]_1, \ldots)$ coincides with the N-fold left-shift of representing elements $([x_N]_0, [x_{N+1}]_1, \ldots)$ in $\mathcal{G}_Z^0$ for any $N \in \mathbb{N}$. We see that any $x \in \mathcal{G}_Z^0$ is a unit in $\mathcal{G}_Z$ when written as (x, e) for $e \in G$ the neutral element, and consequently the equivalence class represented by (x, g) has as its inverse the class represented by $(g.x, g^{-1})$.

Intuitively, an arrow with range x such that x is represented by the equivalence classes of a single, constant vertex in the Schreier graph is thought of as a path from x to $g.x$, but $g.x_n$ is only well-defined for $n \geq l(g)$, in which case $g.x_n$ determines a unique isomorphism class of $(n - d)$-balls and thereby a class in E_{n-d}, where $d = d(x_n, g.x_n)$ as in Figure 4.3. An element of $\mathcal{G}_Z^x$ is therefore represented by a pair (x, g) with g of minimal length, and we can describe (x, g) as consistent choices of vertices $g.x_n$ in the balls described by the classes $[x_n]_n$.

This structure indeed gives rise to a groupoid:

Proposition 4.3.2: Let Z be a uniformly recurrent subgroup of a discrete, finitely-generated group G. Then the associated construction of $\mathcal{G}_Z$ gives rise to an ample minimal Hausdorff étale groupoid with unit space homeomorphic to Z.

Proof. It is easy to see that the range and source maps,

$$r(x, g) = x, \qquad s(x, g) = g.x,$$

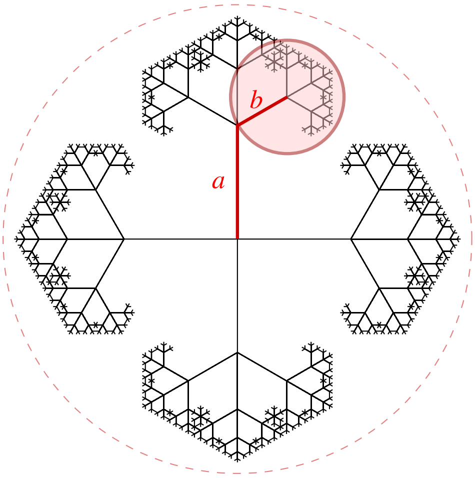

Figure 4.3: If x_n is a class ◌ in E_n, then γx_n determines a class ● in $E_{n-l(\gamma)}$. Here, $n = 5$ is depicted with $\gamma = ba$ for a, b two generators.

together with the composition and inversion

$$(y, g')(x, g) = (y, gg'), \qquad (x, g)^{-1} = (g.x, g^{-1}),$$

indeed turn $\mathcal{G}_Z$ into a groupoid. Equipping $\mathcal{G}_Z$ with the locally compact Hausdorff subspace topology of $\varprojlim F_n$, we have to check that the defined operations are continuous. Consider the basic open sets

$$U_{e_N,g} = \{(x, g) \in \mathcal{G}_Z \mid [x_N]_N = e_N\}$$

that fix $g \in G$ and $e_N \in E_N$ for some N. The range map is obviously continuous, as any basic open set of $\varprojlim E_n$ can be turned into a union of basic open sets of $\varprojlim F_n$ by letting g vary. Inversion is continuous since the action of G on Z is, which in turn makes the source map continuous. To see that the composition is continuous, we fix $e_N \in E_N$ and $g \in G$ and find the preimage of $U_{e_N,g}$ under the composition map. For $(x, h)(h^{-1}.x, h')$ to be contained in $U_{e_N,g}$, we first need that $x \in U_{e_N,h}$, and secondly that $h' = h^{-1}g$. Hence the desired preimage is described by the intersection of

$$\bigcup_{h \in G} U_{e_N,h} \times U_{e_0,h^{-1}g}$$

with the subspace $\mathcal{G}_Z^{(2)}$ of composable pairs in $\mathcal{G}_Z \times \mathcal{G}_Z$ and therefore open in $\mathcal{G}_Z^{(2)}$. Here we use e_0 to denote the unique equivalence class in E_0.

To see that the range map is a local homeomorphism, note that it restricts to a homeomorphism onto its image on every basic open set $U_{e_N,g}$, and $\mathcal{G}_Z$ is therefore étale. Finally, $\mathcal{G}_Z$ is ample as it is an étale groupoid with totally disconnected unit space.

As the orbits of Z and $\mathcal{G}_Z^{(0)}$ coincide, $\mathcal{G}_Z$ is minimal, concluding the proof. ☐

By Proposition 4.3.2, the groupoid $\mathcal{G}_Z$ does not depend on a choice of subgroup $H \in Z$ to construct the underlying Schreier graph.

Identification of the Associated Algebras

In the following, we show that Elek's algebra associated with a URS Z of a discrete, finitely-generated group G is isomorphic to the reduced C^*-algebra of our associated groupoid $\mathcal{G}_Z$ via a canonical isomorphism. We first identify the local kernel algebra $\mathbb{C}Z$ of Z with a dense subset of $C_c(\mathcal{G}_Z)$ and then show that Elek's representation of $\mathbb{C}Z$ arises as the restriction of the regular representations of $\mathcal{G}_Z$ along this embedding.

Let $K \in \mathbb{C}Z$ be a local kernel, seen as a function on $V(S) \times V(S)$, where $S = S^Q_\Gamma(H)$ is the Schreier graph of a group $H \in Z$. Recall that there is a minimal $N \in \mathbb{N}$ called the *width* of K such that K vanishes on $(p, q) \in G/H \times G/H$ if the distance between p and q is more than N and such that $B_N(S, p) \cong_{r,l} B_N(S, q)$ implies that $K(p, gp) = K(q, gq)$ if $l(g) \leq N$; that is, K only depends on the root-label isomorphism class of N-balls.

Lemma 4.3.3: *Let Z be a URS of a discrete, finitely-generated group G, and let $\mathcal{G}_Z$ be the associated groupoid. The local kernel algebra $\mathbb{C}Z$ of Z is in bijection with the dense subalgebra of locally constant functions in $C_c(\mathcal{G}_Z)$. Equipped with convolution and involution, this is an isomorphism of *-algebras.*

Proof. Since $\mathcal{G}_Z$ is equipped with the subspace topology of a projective limit of discrete sets, we may describe an open set of $\mathcal{G}_Z$ by fixing a group element g and the first n coordinates $[x_n] \in E_n$ while letting $[x_m] \in E_m$ for $m > n$ vary. Together, these form a basis of the topology of $\mathcal{G}_Z$. If a kernel K on $G/H \times G/H$ for a subgroup $H \in Z$ is local of width N, this means that K only depends on $(gH, g'H)$ up to the equivalence class of gH in E_N. Therefore, even though the range and source of an element $([x_n], g) \in G$ might not be described by a single, constant choice of vertices in $S^Q_\Gamma(H)$ like in Figure 4.1, we can obtain a function f_K on $\mathcal{G}_Z$ from a kernel K by picking a representing vertex in G/H corresponding to the component in E_N as in the next paragraph and this choice is made uniformly on a neighbourhood as described above. This gives a way to translate back and forth between kernels in $\mathbb{C}Z$ and locally constant functions in $C_c(\mathcal{G}_Z)$, and indeed we built $\mathcal{G}_Z$ such that the algebraic operations are compatible.

Let K be a local kernel of width N, and define a function $f_K \in C_c(\mathcal{G}_Z)$ given by $f_K((x, g)) = K(x_M, gx_M)$, where $M = \max\{N, l(g)\}$. If g is chosen of minimal length, we may pick $M = N$. Equivalently, f_K evaluates (x, g) at its component $([x_N]_N, g)$ in F_N and assigns $K(x_N, gx_N)$ to it, where $K(\infty_N) = 0$. This is well-defined, as the vertex x_N is given up to root-label isomorphism of N-balls. The function f_K is continuous, because it is uniformly locally constant: It is constant on any 2^{-N}-ball in $\mathcal{G}_Z$. To see that f_K is compactly supported, note that the embedding $\mathcal{G}_Z \hookrightarrow \varprojlim F_n$ is the one-point compactification of $\mathcal{G}_Z$. Therefore, a set $U \subseteq \mathcal{G}_Z$ is relatively compact exactly if there is some $M \in \mathbb{N}$ such that no element of U has F_M-component ∞_M. Equivalently, U as a subset of $\varprojlim F_n$ does not intersect the 2^{-M}-ball centred at ∞. As f_K is supported outside of the 2^{-N}-ball centred at ∞ for N the width of K, it is compactly supported.

Conversely, any locally constant function $f \in C_c(\mathcal{G}_Z)$ defines a kernel K_f in $\mathbb{C}Z$. As f is locally constant, for each $g \in \mathcal{G}_Z$, we can pick a ball centred at g on which f is constant. Then finitely many of such balls cover the support of f, and we may pick $N \in \mathbb{N}$ such that these have radius at least 2^{-N}. Since two 2^{-N}-balls in $\mathcal{G}_Z$ or $\varprojlim F_n$ are either disjoint or equal, f is constant on any 2^{-N}-ball and supported outside of the 2^{-N}-ball of ∞. Given such f, we may now define a local kernel K_f with width at most N as follows: For a fixed vertex

$p \in V(S)$, consider the unit $[[p]] := ([p]_0, [p]_1, \dots) \in \mathcal{G}_Z^0$ and define $K_f(p, gp) := f([[p]], g)$. If $gp = g'p$, then $([[p]], g) = ([[p]], g')$, so K_f is a well-defined kernel. As the F_N-component of $([[p]], g)$ is ∞_N if $d(p, gp) > N$, we have $K_f(p, gp) = 0$ in that case. Furthermore, if the N-balls $B_N(S, p)$ and $B_N(S, q)$ are root-label-isomorphic for $p, q \in V(S)$ and $l(g) \leq N$, then $([[p]], g)$ and $([[q]], g)$ are 2^{-N}-close, since their first N components coincide. As f is constant on 2^{-N}-balls, we have $K_f(p, gp) = K_f(q, gq)$, so K_f is local of width N.

It is easy to check that $f_{K_f} = f$ and $K_{f_K} = K$, so the local kernel algebra $\mathbb{C}Z$ of Z is in bijection with the subset of locally constant functions in $C_c(\mathcal{G}_Z)$, which is dense in $C_c(\mathcal{G}_Z)$, as $\mathcal{G}_Z$ is totally disconnected.

Furthermore, the *-algebra structure of $\mathbb{C}Z$ is preserved under the described inclusion: For $p \in V(S)$ and $g \in G$, we have $s([[p]], g) = [[g.p]]$ and the orbit of $[[p]]$ in $\mathcal{G}_Z$ is $\{[[q]] \mid q \in V(S)\}$. We calculate

$$\begin{aligned} f_{L*K}([[p]], g) = L * K(p, gp) &= \sum_{q \in V(S)} L(p, q) K(q, gp) \\ &= \sum_{([[p]], g') \in \mathcal{G}_Z^{[[p]]}} f_L([[p]], g') f_K([[g'p]], g(g')^{-1}) = f_L * f_K([[p]], g) \end{aligned}$$

and

$$(f_K)^*([[p]], g) = \overline{f_K([[gp]], g^{-1})} = \overline{K(gp, p)} = K^*(p, gp) = f_{K^*}([[p]], g).$$

We conclude that $f_{L*K} = f_L * f_K$ and $(f_K)^* = f_{K^*}$ on the subset

$$\{([[p]], g) \mid p \in V(S), g \in G\} \subseteq \mathcal{G}_Z$$

of arrows in $\mathcal{G}_Z$, whose range is given by the equivalence classes of a single, constant vertex p in S. This subset is dense, as the orbit of S is dense in $S_G^Q(Z)$, and as the functions in question are continuous, they coincide on all of $\mathcal{G}_Z$. Hence $\mathbb{C}Z$ embeds into $C_c(\mathcal{G}_Z)$ by a *-homomorphism onto the sub-*-algebra of locally constant functions as claimed. □

Recall that Elek's algebra $C^*_{\mathrm{r}}(Z)$ is the completion of $\mathbb{C}Z$ in the norm arising from the faithful representation π on $\mathcal{B}(\ell^2(G/H))$ given by

$$(\pi(K)h)(gH) = \sum_{g'H \in G/H} K(gH, g'H) h(g'H)$$

for $h \in \ell^2(G/H)$. We show that this norm arises as the restriction of the reduced norm on $C_c(\mathcal{G}_Z)$ along the canonical embedding, and furthermore that $\mathbb{C}Z$ is dense in $C_c(\mathcal{G}_Z)$ with respect to the reduced norm. Consequently, $C^*_{\mathrm{r}}(Z)$ is canonically isomorphic to the reduced groupoid C^*-algebra $C^*_{\mathrm{r}}(\mathcal{G}_Z)$.

Theorem 4.3.4: *Let Z be a uniformly recurrent subgroup of a discrete, finitely-generated group G and $\mathcal{G}_Z$ the associated groupoid. The C^*-algebras $C^*_r(Z)$ and $C^*_r(\mathcal{G}_Z)$ are isomorphic via the isomorphism extending the canonical embedding of the local kernel algebra $\mathbb{C}Z$ as the subalgebra of locally constant functions in $C_c(\mathcal{G}_Z)$.*

Recall that for an étale groupoid $\mathcal{G}_Z$, the reduced norm on $C_c(\mathcal{G}_Z)$ is given by $\|f\| = \sup_x \|\pi_x(f)\|$ with the representations $\pi_x\colon C_c(\mathcal{G}_Z) \to \mathcal{B}(\ell^2((\mathcal{G}_Z)_x))$ defined by

$$(\pi_x(f)\xi)(g) = \sum_{g' \in (\mathcal{G}_Z)_{s(g)}} f(g(g')^{-1})\xi(g')$$

for $x \in \mathcal{G}_Z^0$, $g \in (\mathcal{G}_Z)_x$, and $\xi \in \ell^2((\mathcal{G}_Z)_x)$.

Proof. We show that the norm on $\mathbb{C}Z$ defined by Elek coincides with the reduced norm on $C_c(\mathcal{G}_Z)$ restricted to the subset of locally constant functions, which we identified with $\mathbb{C}Z$ in Lemma 4.3.3. In fact, we even show that Elek's representation π of $\mathbb{C}Z$ and the groupoid representation π_x of any $x \in \mathcal{G}_Z^0$ are equivalent. Let us first consider the easiest case of $x = [[H]]$, the unit represented by the root in $S = S_\Gamma^Q(H)$, the Schreier graph of $H \leq G$. We obtain a map $V(S) \to (\mathcal{G}_Z)_{[[H]]}$ by $gH \mapsto ([[gH]], g^{-1})$, mapping a vertex gH in S to the arrow in $\mathcal{G}_Z$ that is described by any path from H to gH. This is obviously surjective and is well-defined, as the arrows $([[gH]], g^{-1})$ and $([[g'H]], (g')^{-1})$ are identified if $gH = g'H$. It is furthermore injective: If $([[gH]], g^{-1}) = ([[g'H]], (g')^{-1})$, then the N-balls centred at gH and $g'H$ are root-label isomorphic via an isomorphism mapping H to H if $N > l(g) + l(g')$. But a root-label-isomorphism of N-balls in Schreier graphs is the identity if it has a fixed point; hence, $gH = g'H$. We have thus established a bijection between $(\mathcal{G}_Z)_{[[H]]}$ and $G/H = V(S)$. This yields a unitary $T\colon \ell^2(V(S)) \to \ell^2((\mathcal{G}_Z)_{[[H]]})$ that intertwines the representations π and $\pi_{[[H]]}$ on $\mathbb{C}Z$ as follows. Let $h \in \ell^2(V(S))$. Then

$$\pi(K)h(gH) = \sum_{g'H \in V(S)} K(gH, g'H)h(g'H),$$

and so

$$\begin{aligned} T(\pi(K)h)([[gH]], g^{-1}) &= \sum_{g'H \in V(S)} K(gH, g'H)h(g'H) \\ &= \sum_{g'H \in V(S)} f_K([[gH]], g'g^{-1})h(g'H) \\ &= \sum_{g'H \in V(S)} f_K([[gH]], g'g^{-1})(Th)([[g'H]], (g')^{-1}) \\ &= \sum_{g \in (\mathcal{G}_Z)_{[[H]]}} f_K(([[gH]], g^{-1})g^{-1})(Th)(g) \\ &= (\pi_{[[H]]}(f_K)Th)([[gH]], g^{-1}), \end{aligned}$$

and therefore the representations π and $\pi_{[[H]]}$ define identical reduced norms on the local kernel algebra $\mathbb{C}Z$ as a subalgebra of $C_c(\mathcal{G}_Z)$.

Morally, this already implies that all source-fibre representations π_x for $x \in \mathcal{G}_Z^0$ are unitarily equivalent to π, since the groupoid $\mathcal{G}_Z$ does not depend on the choice of $H \in Z$ in its construction, and for every $x \in \mathcal{G}_Z^0$ there is a unique subgroup in Z which is mapped to x under the homeomorphism $Z \cong \mathcal{G}_Z^0$. For completeness we nevertheless show that all reduced representations of $\mathcal{G}_Z$ are equivalent.

As in any groupoid, the representations π_x are unitarily equivalent for any two units x that share an orbit. Therefore we only need to consider π_x for $x \in \mathcal{G}_Z^0$ that corresponds to a subgroup H' in $Z\backslash G.H$. For fixed $K \in \mathbb{C}Z$ of width R and $\epsilon > 0$, we may pick $h \in \ell^2((\mathcal{G}_Z)_{[[H]]})$ of norm one such that

$$\|\pi_{[[H]]}(f_K)h\|_2 > \|\pi_{[[H]]}(f_K)\| - \epsilon$$

and h is supported on arrows $([[gH]], g^{-1})$ with $l(g) \leq N$ for some $N \in \mathbb{N}$. As the orbit of H' in Z is dense, there is a unit y in the orbit of x in $\mathcal{G}_Z^0$ such that y and $[[H]]$ are $2^{-(N+R)}$-close; that is, their E_{N+R}-components coincide. Hence, for every arrow $([[gH]], g^{-1}) \in (\mathcal{G}_Z)_{[[H]]}$ with $l(g) \leq N+R$, there is a *unique* arrow $(g.y, g^{-1}) \in (\mathcal{G}_Z)_y$ that is described by the same g^{-1}, since two paths of length less than $N+R$ starting at H end in the same vertex exactly if the analogous paths in the isomorphic $(N+R)$-ball of y do. This yields a bijection between the subspaces of $(\mathcal{G}_Z)_{[[H]]}$ and $(\mathcal{G}_Z)_y$ described by elements of G with length at most $N+R$. Extending by zero, we transport $h \in \ell^2((\mathcal{G}_Z)_{[[H]]})$ along this bijection to a function $h' \in \ell^2((\mathcal{G}_Z)_y)$ with $1 = \|h\|_2 = \|h'\|_2$. Noting that $\pi_{[[H]]}(f_K)h \in \ell^2((\mathcal{G}_Z)_{[[H]]})$ is supported on $([[gH]], g^{-1})$ with $l(g) \leq N+R$, we may likewise transport this to a function in $\ell^2((\mathcal{G}_Z)_y)$ of the same norm. It is now easy to see that this function will just be $\pi_y(f_K)h'$, whence

$$\|\pi_y(f_K)\| \geq \|\pi_y(f_K)h'\|_2 > \|\pi_{[[H]]}(f_K)\| - \epsilon,$$

and by unitary equivalence

$$\|\pi_x(f_K)\| > \|\pi_{[[H]]}(f_K)\| - \epsilon.$$

By symmetry we obtain the converse direction, implying that all norms on $C_c(\mathcal{G}_Z)$ induced by reduced representations are identical and coincide on $\mathbb{C}Z \subseteq C_c(\mathcal{G}_Z)$ with the reduced norm of $\mathbb{C}Z$ as given by Elek's representation π.

To conclude that the completion of $\mathbb{C}Z$ in this norm is the same as the completion of $C_c(\mathcal{G}_Z)$ in the reduced norm, and hence $C_r^*(\mathcal{G}_Z)$, we finally show that $\mathbb{C}Z$ is dense in $C_c(\mathcal{G}_Z)$ in the reduced norm. Let $f \in C_c(\mathcal{G}_Z)$. As f is compactly supported, it vanishes on the 2^{-R}-ball around $\infty \in \varprojlim F_n$ for some R, so that $f([[g'H]], g^{-1}) = 0$ if g is chosen of minimal length and yet $l(g) > R$. We therefore find for $h \in \ell^2((\mathcal{G}_Z)_{[[H]]})$ that

$$\begin{aligned}
\|\pi_{[[H]]}(f)h\|_2^2 &= \sum_{gH\in G/H} \left|(\pi_{[[H]]}(f)h)([[gH]], g^{-1})\right|^2 \\
&= \sum_{gH\in G/H} \Bigg| \sum_{g'H\in B_R(S,gH)} f([[gH]], g'g^{-1})h([[g'H]], (g')^{-1})\Bigg|^2 \\
&\leq \|f\|_\infty^2 \sum_{gH\in G/H} \Bigg| \sum_{g'H\in B_R(S,gH)} h([[g'H]], (g')^{-1})\Bigg|^2 \\
&\leq \|f\|_\infty^2 \sum_{gH\in G/H} \Bigg| \sum_{g'H\in B_R(S,gH)} |h([[g'H]], (g')^{-1})|\Bigg|^2 \\
&\leq \|f\|_\infty^2 \sum_{gH\in G/H} \Bigg| \sum_{g'\in G, l(g')\leq R} h^{(g')}([[gH]], g^{-1})\Bigg|^2 \\
&\leq (|Q|+1)^{2R}\|f\|_\infty^2\|h\|_2^2,
\end{aligned}$$

with $h^{(g')}([[gH]], g^{-1}) = |h([[g'gH]], (g'g)^{-1})|$ denoting copies of $|h|$ shifted by g'. Note that passing to the absolute value is necessary, since while $g'gH$ for $\{g' \in G \mid l(g') \leq R\}$ cover

$B_r(S, gG)$, they might not do so uniquely. The last inequality stems from the fact that there are fewer than $(|Q| + 1)^R$ different g' of length at most R in G, and $\|h^{(g')}\|_2 = \|h\|_2$. Hence $\|f\|_r \leq (|Q| + 1)^{2R}\|f\|_\infty$ for all $f \in C_c(\mathcal{G}_Z)$ supported outside of the 2^{-R}-ball around ∞ with $\|f\|_r = \|\pi_{[[H]]}(f)\|$ the unique reduced norm. For any $\epsilon > 0$, by partitioning $\mathcal{G}_Z$ into open 2^{-N}-balls for large N, we may approximate f in $C_c(\mathcal{G}_Z)$ up to ϵ by a function f_ϵ that is constant on every 2^{-N}-ball. As two balls are either disjoint or one is contained in the other, we may choose $f_\epsilon(g)$ supported outside of the 2^{-R}-ball of ∞ for large N, such that $\|f - f_\epsilon\|_r \leq (|Q| + 1)^{2R}\|f - f_\epsilon\|_\infty = \epsilon(|Q| + 1)^{2R}$.

As $\mathbb{C}Z$ is dense in $C_c(\mathcal{G}_Z)$ in the reduced norm, which restricts to Elek's norm, the embedding extends to a *-isomorphism as claimed. □

In summary, for any uniformly recurrent subgroup Z we have constructed an ample minimal étale Hausdorff groupoid with unit space homeomorphic to Z, such that the reduced C^*-algebras $C^*_r(Z)$ of Z and $C^*_r(\mathcal{G}_Z)$ of $\mathcal{G}_Z$ coincide. This enables us to examine $C^*_r(Z)$ using tools for groupoid C^*-algebras.

4.3.4 The Transformation Groupoid

In this section we establish a relationship between our newly-defined groupoid $\mathcal{G}_Z$ associated with a uniformly recurrent subgroup Z of a finitely-generated discrete group G and the transformation groupoid $Z \rtimes G$ associated with the action of G on Z by conjugation.

Recall that as a space, the transformation groupoid $Z \rtimes G$ is simply the cartesian product $Z \times G$ equipped with the product topology. The unit space is given by the subspace $Z \times \{e\}$ and identified with Z, the range and source of an arrow (H, g) are respectively given by H and $g^{-1}.H$, while the product of two composable arrows is $(H, g)(g^{-1}.H, h) = (H, gh)$. This turns $Z \rtimes G$ into a Hausdorff étale groupoid, which for a URS Z is furthermore ample and minimal.

To distinguish our groupoid $\mathcal{G}_Z$ from $Z \rtimes G$, we describe the range fibres $\mathcal{G}_Z^H$ further. Recall that the homeomorphism between Z and $\varprojlim E_n$ describes every group $H \in Z$ as a sequence of isomorphism classes of balls in $S^Q_\Gamma(H')$ for an arbitrary $H' \in Z$. In particular, the isomorphism class in E_n associated with H is given by the n-ball around the root in $S^Q_\Gamma(H)$. Two arrows $([[H]], g)$ and $([[H]], h)$ in $\mathcal{G}_Z$ will then coincide if and only if the paths in $S^Q_\Gamma(H)$ that start at the root and are described by g and h end at the same vertex. That is, they coincide if $gH = hH$, or equivalently $h^{-1}g \in H$, and hence the range fibre $\mathcal{G}_Z^H$ at H is in bijection with G/H via $([[H]], g) \mapsto gH$. To simplify this notation and remove the ambiguity in the choice of g, we denote $([[H]], g)$ as (H, gH) for the remainder of this section. From this identification we obtain a surjective map $q\colon Z \rtimes G \to \mathcal{G}_Z$ via $(H, g) \mapsto (H, g^{-1}H)$. Note that the range fibres $(Z \rtimes G)^H$ of $Z \rtimes G$ are simply given by G. Then the map q is a quotient map of topological groupoids:

Proposition 4.3.5: Let Z be a URS of a finitely-generated discrete group G, and let $\mathcal{G}_Z$ be the associated groupoid. Let $q\colon Z \rtimes G \to \mathcal{G}_Z$ be the map given by $(H, g) \mapsto (H, g^{-1}H)$ for a subgroup $H \in Z$ and a group element $g \in G$. Then q is a continuous, open, and surjective groupoid homomorphism. In particular, $\mathcal{G}_Z$ is a quotient of the transformation groupoid $Z \rtimes G$.

Proof. We first check that q is a groupoid homomorphism. Indeed, q preserves sources and ranges, since

$$\begin{aligned} & q(s(H,g)) = q(g^{-1}.H,e) = (g^{-1}.H,g^{-1}.H) = s(H,g^{-1}H) = s(q(H,g)), \\ \text{and} \qquad & q(r(H,g)) = q(H,e) = (H,H) = r(H,g^{-1}H) = r(q(H,g)). \end{aligned}$$

Hence the images of two composable arrows (H,g) and $(g^{-1}.H,h)$ in $Z \rtimes G$ are composable in $\mathcal{G}_Z$ and we calculate

$$\begin{aligned} q(H,g)q(g^{-1}.H,h) &= (H,g^{-1}H)(g^{-1}.H,h^{-1}(g^{-1}.H)) \\ &= (H,h^{-1}g^{-1}H) \\ &= q(H,gh) \\ &= q((H,g),(g^{-1}.H,h)). \end{aligned}$$

Furthermore, q is surjective, since any arrow (H,gH) in $\mathcal{G}_Z$ clearly has preimage (H,g^{-1}) in $Z \rtimes G$.

We continue by showing that q is continuous and open. Consider the basis of the topology of $\mathcal{G}_Z$ given by the open sets

$$U_{H,N,g} = \left\{(K,gK) \mid d_{S^Q_\Gamma}(S^Q_\Gamma(H),S^Q_\Gamma(K)) \leq 2^{-N}\right\}$$

indexed by a subgroup $H \in Z$, an element $g \in G$ and $N \in \mathbb{N}$ with $l(g) \leq N$, and consisting of all arrows (K,gK) such that the N-balls around the root in $S^Q_\Gamma(H)$ and $S^Q_\Gamma(K)$ coincide. Equivalently, a group element $h \in G$ with $l(h) \leq 2N$ is contained in K if and only if it is contained in H. Likewise, we fix a basis

$$V_{H,N,g} = \left\{(K,g) \mid d_{S^Q_\Gamma}(S^Q_\Gamma(H),S^Q_\Gamma(K)) \leq 2^{-N}\right\}$$

of the topology of $Z \rtimes G$.

It is now easy to see that q is open: Let $(K,g^{-1}K) \in q(V_{H,N,g})$. Then $U_{K,M,g^{-1}}$ for $M = \max\{N,l(g)\}$ is a neighbourhood of $(K,g^{-1}K)$ contained in $q(V_{H,N,g})$.

Conversely,

$$q^{-1}(U_{H,N,g^{-1}}) = \left\{(K,gk) \mid d_{S^Q_\Gamma}(S^Q_\Gamma(H),S^Q_\Gamma(K)) \leq 2^{-N},\ k \in K\right\},$$

and we claim that this is open in $Z \rtimes G$. Indeed, for every $(K,gk) \in q^{-1}(U_{H,N,g^{-1}})$ the neighbourhood $V_{K,M,gk}$ is contained in $q^{-1}(U_{H,N,g^{-1}})$, where $M = \max\{N,l(k)\}$, so that any subgroup in the 2^{-M}-ball around K is guaranteed to contain the element k. We conclude that q is the desired quotient map. □

Remark. Up to the inversion of g, the map q can be thought of as dividing out the subgroup H from its range fibre $(Z \rtimes G)^H$ at every $H \in Z$. The inversion is necessary, since we chose to define our groupoid $\mathcal{G}_Z$ with the opposite conventions of those commonly used for transformation groupoids, as this matches Elek's definition more closely. If we had defined the groupoid $\mathcal{G}_Z$ as the opposite of the current definition, the groupoid homomorphism q in the proof of Proposition 4.3.5 would not need the inversion of g, but to identify $\mathbb{C}Z$ with a subalgebra of $C_c(\mathcal{G}_Z)$ in Lemma 4.3.3 we would first need to pass to the opposite to cancel the effects. In particular, the isomorphism $K \mapsto f_K$ of Lemma 4.3.3 would satisfy $f_{L*K} = f_K * f_L$ rather than $f_{L*K} = f_L * f_K$. An alternative way of defining q in Proposition 4.3.5 would therefore be to first pass to the *opposite groupoid* $(Z \rtimes G)^{\mathrm{op}}$ via the canonical groupoid isomorphism and then divide out the base group from every source fibre without inversion. We thank the anonymous reviewer for pointing this out.

Note that the quotient map q can only be injective if Z is the trivial uniformly recurrent subgroup containing only the trivial subgroup, and in that case $C^*_{\mathrm{r}}(Z)$ coincides with the reduced group algebra of G. In general, if Z is a singleton consisting of a normal subgroup N, the C^*-algebras $C^*_{\mathrm{r}}(Z)$ and $C^*_{\mathrm{r}}(\mathcal{G}_Z)$ will be isomorphic to the reduced group algebra of the quotient group G/N, since $\mathcal{G}_Z$ will be the transformation groupoid of G/N acting on a singleton. In every nontrivial case, when Z is not just the trivial subgroup, the associated groupoid $\mathcal{G}_Z$ is different from the transformation groupoid.

4.3.5 Simplicity and Nuclearity

Characterisations of Simplicity and Nuclearity

As before let Z be a URS of a finitely-generated discrete group G, and let $\mathcal{G}_Z$ be the associated groupoid. We employ our description of the Elek algebras as groupoid algebras to give simplified proofs of Elek's characterisations, explaining why they arise in the language of groupoids.

Proposition 4.3.6: Let Z be a uniformly recurrent subgroup of a finitely-generated, discrete group. If Z is generic, then the associated groupoid $\mathcal{G}_Z$ is principal.

Proof. Suppose $\mathcal{G}_Z$ is not principal. Then there is a unit $x = ([x_0]_0, [x_1]_1, \ldots)$ and an arrow in the isotropy $(\mathcal{G}_Z)^x_x$ that is not a unit. Therefore there is a group element $g \in G$ such that $x = g.x$ but $(x, e) \neq (x, g)$. In particular, $g.x_N \neq x_N$ for large N, while $B_{N-l(g)}(S, x_N) \cong_{r,l} B_{N-l(g)}(S, g.x_N)$ and $d(x_N, g.x_N) \leq l(g)$. By [20, Proposition 2.3], Z is not generic, concluding the proof. □

As an immediate corollary, we reproduce [20, Theorem 7].

Corollary 4.3.7: *Let Z be a uniformly recurrent subgroup of a finitely-generated, discrete group. If Z is generic, then its reduced C^*-algebra $C^*_{\mathrm{r}}(Z)$ is simple.*

Proof. By Proposition 4.3.6, $\mathcal{G}_Z$ is a minimal principal étale groupoid, and every such groupoid has a simple reduced C^*-algebra by Proposition 4.1.17. □

Regarding nuclearity, we are able to add the converse direction to Elek's characterisation [20, Theorem 8]. We first describe when our groupoids are amenable:

Theorem 4.3.8: *Let Z be a uniformly recurrent subgroup of the finitely-generated discrete group G, and let $\mathcal{G}_Z$ be the groupoid associated with Z. The Schreier graph $S^Q_\Gamma(H)$ of any group $H \in Z$ has local property A if and only if $\mathcal{G}_Z$ is (topologically) amenable.*

Proof. We first show that local property A of the Schreier graph $S^Q_\Gamma(H)$ implies topological amenability of $\mathcal{G}_Z$. Let $\mathcal{G}_Z$ be constructed from $H \in Z$ and let $\rho^n \colon G/H \to \ell^2(G/H)$ implement local property A of $S^Q_G(H)$. Let R_n describe the locality of ρ^n as in Section 4.3.2. Let x be a unit of $\mathcal{G}_Z$ and let x_{R_n} denote any element of the equivalence class of E_{R_n} forming the R_n-component of x. Define $f_n \in C_c(\mathcal{G}_Z)$ by $f_n(x, g) = \rho^n_{x_{R_n}}(gx_{R_n})$. This is independent of the choice of x_{R_n}, as the ρ^n are locally defined and of width at most R_n. In addition, ρ^n is continuous as it is constant on 2^{-R_n}-balls and compactly supported as it vanishes on the 2^{-R_n}-ball of ∞. Then, using the fact that $\rho^n_{x_{R_n}}$ is supported in the R_n-ball centred at x_{R_n},

$$\sum_{(x,g)\in\mathcal{G}_Z^x} |f_n(x, g)|^2 = \sum_{y\in B_{R_n}(x_{R_n})} |\rho^n_{x_{R_n}}(y)|^2 = \|\rho^n_{x_{R_n}}\|_2^2 = 1.$$

Furthermore, we calculate

$$f_n * f_n^*(x, g) = \sum_{(g.x,g')\in\mathcal{G}_Z^{g.x}} f_n((x, g)(g.x, g'))\overline{f_n}(g.x, g'),$$

where we may restrict to $l(g') \leq R_n$, as $f_n(g.x, g')$ vanishes otherwise:

$$\begin{aligned} f_n * f_n^*(x, g) &= \sum_{(g.x,g')\in\mathcal{G}_Z^{g.x}, l(g')\leq R_n} \rho^n_{x_{R_n+l(g)}}(g'gx_{R_n+l(g)})\overline{\rho^n_{gx_{R_n+l(g)}}}(g'gx_{R_n+l(g)}) \\ &= \langle \rho^n_{x_{R_n+l(g)}}, \rho^n_{gx_{R_n+l(g)}} \rangle, \end{aligned}$$

using again the assumptions on the support of $\rho^n_{gx_{R_n+l(g)}}$. However, for fixed $gH, g'H \in G/H$ we have

$$\left|1 - \langle \rho^n_{gH}, \rho^n_{gH} \rangle\right| = \left|\langle \rho^n_{gH} - \rho^n_{gH}, \rho^n_{gH} \rangle\right| \leq \|\rho^n_{gH} - \rho^n_{gH}\|_2 \cdot \|\rho^n_{gH}\|_2 \overset{(\dagger)}{\leq} 1/n \cdot 1 \xrightarrow{n\to\infty} 0,$$

where the estimate (†) holds for large n such that $d(u, v) \leq n$. But as $l(g)$ is bounded on compact sets, the same estimate may be used uniformly on any compact set for sufficiently large n, so that $f_n * f_n^*$ converges to 1 uniformly on compact subsets. These functions witness the (topological) amenability of $\mathcal{G}_Z$ as in the original definition by Renault, condition 3 in Definition 4.1.15.

Conversely, suppose that $\mathcal{G}_Z$ is topologically amenable. By the equivalent characterisation 2 of amenability in Definition 4.1.15, we may assume that there is a sequence $f_n \in C_c(\mathcal{G}_Z)$, such that

$$\sum_{(x,g)\in\mathcal{G}_Z^x} |f_n(x, g)|^2 \xrightarrow{n\to\infty} 1 \tag{4.16}$$

uniformly on compact subsets of $\mathcal{G}_Z^0$ as a function of $x \in \mathcal{G}_Z^0$, and

$$\sum_{h\in\mathcal{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2 \xrightarrow{n\to\infty} 0 \tag{4.17}$$

uniformly on compact subsets of $\mathcal{G}_Z$ as a function of g. To obtain local maps ρ^n_x as in the definition of local property A, we first, for any $\epsilon > 0$, approximate f_n by (uniformly) locally constant functions $f_{n,\epsilon} \in C_c(\mathcal{G}_Z)$ such that $\|f_n - f_{n,\epsilon}\|_\infty < \epsilon$, choosing $f_{n,\epsilon}$ to be zero on the largest possible ball centred at ∞.

We next construct a sequence $\epsilon_n \searrow 0$ such that f_{n,ϵ_n} satisfies the convergence properties above: As for fixed n there is $T_n \in \mathbb{N}$ such that both f_n and $f_{n,\epsilon}$ for every $\epsilon > 0$ vanish on the 2^{-T_n}-ball of ∞, we may restrict summations like in Equation (4.16) and Equation (4.17) to arrows described by group elements g with length $l(g)$ at most T_n, such that the respective sums become finite with fewer than $(|Q| + 1)^{T_n}$ terms. First, observe that

$$\begin{aligned}\left| \sum_{(x,g)\in\mathcal{G}_Z^x} |f_n(x,g)|^2 - \sum_{(x,g)\in\mathcal{G}_Z^x} |f_{n,\epsilon}(x,g)|^2 \right| &\leq \sum_{\substack{(x,g)\in\mathcal{G}_Z^x,\\ l(g)\leq T_n}} (|f_n(x,g)| + \epsilon)^2 - |f_n(x,g)|^2 \\ &\leq (|Q| + 1)^{T_n}(2\|f_n\|_\infty + \epsilon)\epsilon \xrightarrow{\epsilon\to 0} 0,\end{aligned}$$

with the convergence uniform in $x \in \mathcal{G}_Z^0$. Then, to verify the condition of Equation (4.17), we calculate

$$\begin{aligned}\sum_{h\in\mathcal{G}_Z^{r(g)}} |f_{n,\epsilon}(g^{-1}h) - f_{n,\epsilon}(h)|^2 &\leq \sum_{h\in\mathcal{G}_Z^{r(g)},l(g)\leq T_n} (|f_n(g^{-1}h) - f_n(h)| + 2\epsilon)^2 \\ &\leq (4\epsilon^2 + 8\epsilon\|f_n\|_\infty)(|Q| + 1)^{T_n} + \sum_{h\in\mathcal{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2 \\ &\xrightarrow{\epsilon\to 0} \sum_{h\in\mathcal{G}_Z^{r(g)}} |f_n(g^{-1}h) - f_n(h)|^2,\end{aligned}$$

with the convergence uniform in $g \in \mathcal{G}_Z$. Picking $\epsilon_n \to 0$ such that

$$\begin{aligned} & 1/n \geq (|Q| + 1)^{T_n}(2\|f_n\|_\infty + \epsilon_n)\epsilon_n \\ \text{and} \quad & 1/n \geq (4\epsilon_n^2 + 8\epsilon_n\|f_n\|_\infty)(|Q| + 1)^{T_n}\end{aligned}$$

does the trick. To such f_{n,ϵ_n} we may assign $\rho^n\colon G/H \to \ell^2(G/H)$ given by $\rho^n_{gH}(g'gH) = f_{n,\epsilon_n}([[gH]], g')$.

Picking R_n such that f_{n,ϵ_n} is constant on any 2^{-R_n}-ball, we see that ρ^n_{gH} is supported on $B_{R_n}(gH)$, since f_{n,ϵ_n} vanishes on the 2^{-R_n}-neighbourhood of ∞. Similarly, $\rho^n_{gH}(g'gH) = \rho^n_{hH}(g'hH)$ for $l(g') \leq R_n$ if the R_n-balls around gH and hH are isomorphic, as (gH, g') and (hH, g') are 2^{-R_n}-close in that case, and we conclude that ρ_n is local of width R_n.

Furthermore, we compute that

$$\|\rho^n_{gH}\|_2^2 = \sum_{g'H\in G/H} |\rho^n_{gH}(g'H)|^2 = \sum_{([[gH]],g')\in\mathcal{G}_Z^{[[gH]]}} |f_{n,\epsilon_n}([[gH]], g')|^2 \xrightarrow{n\to\infty} 1$$

uniformly on $\mathcal{G}_Z^0$. Likewise,

$$\begin{aligned}
&\|\rho^n_{gH} - \rho^n_{g'H}\|_2^2 \\
&= \sum_{hH \in G/H} \left|\rho^n_{gH}(hH) - \rho^n_{g'H}(hH)\right|^2 \\
&= \sum_{hH \in G/H} \left|f_{n,\epsilon_n}([[gH]], hg^{-1}) - f_{n,\epsilon_n}([[g'H]], h(g')^{-1})\right|^2 \\
&= \sum_{hH \in G/H} \left|f_{n,\epsilon_n}\Big(([[gH]], g'g^{-1})([[g'H]], h(g')^{-1})\Big) - f_{n,\epsilon_n}([[g'H]], h(g')^{-1})\right|^2 \\
&= \sum_{h \in \mathcal{G}_Z^{[[g'H]]}} \left|f_{n,\epsilon_n}\Big(([[gH]], g'g^{-1})h\Big) - f_{n,\epsilon_n}(h)\right|^2 \xrightarrow{n\to\infty} 0
\end{aligned}$$

uniformly for $([[g'H]], g(g')^{-1})$ in compact subsets of $\mathcal{G}_Z$. In particular, the convergence is uniform when the pair (g, g') is taken from a subset with bounded difference; that is, if there is some $N > 0$ such that $l(g(g')^{-1}) < N$. More importantly, this means that for any $N \in \mathbb{N}$, we may choose $n_0 \in \mathbb{N}$ such that $\|\rho^n_{gH} - \rho^n_{g'H}\|_2^2 \leq 1/N$, whenever $d_{S^Q_\Gamma(H)}(gH, g'H) \leq N$ and $n \geq n_0$. The analogous statement holds after replacing ρ^n_x with the normed function $\hat{\rho}^n_x = \rho^n_x / \|\rho^n_x\|_2$, since

$$\|\hat{\rho}^n_{gH} - \hat{\rho}^n_{g'H}\|_2 \leq \frac{1}{\|\rho^n_{gH}\|_2}\Big(\|\rho^n_{gH} - \rho^n_{g'H}\|_2 + \Big|\|\rho^n_{gH}\|_2 - \|\rho^n_{g'H}\|_2\Big|\Big),$$

while $\|\rho^n_{gH}\|_2$ and $\Big|\|\rho^n_{gH}\|_2 - \|\rho^n_{g'H}\|_2\Big|$ converge uniformly to 1 and 0, respectively. Then, after relabelling, $\hat{\rho}^n$ witnesses local property A of $S^Q_\Gamma(H)$, concluding the proof. □

For étale groupoids there is a clear relation between amenability and nuclearity of their reduced C^*-algebras, which directly translates to our case. See for example [57, Section 2] for a brief, but broader overview of the different notions of groupoid amenability. This implies the converse direction of [20, Theorem 8]:

Corollary 4.3.9: *Let Z be a uniformly recurrent subgroup and $H \in Z$. The graph $S^Q_\Gamma(H)$ has local property A if and only if the C^*-algebra $C^*_r(Z)$ is nuclear.*

Proof. As the groupoid $\mathcal{G}_Z$ associated with Z is étale, $C^*_r(\mathcal{G}_Z)$ is nuclear if and only if $\mathcal{G}_Z$ is (topologically) amenable by [2, Corollary 6.2.14], as stated after Proposition 4.1.16. By Theorem 4.3.8 this is the case if and only if the Schreier graph $S^Q_\Gamma(H)$ has local property A. □

4.3.6 Applications of Groupoid Simplicity

As non-trivial uniformly recurrent subgroups are difficult to construct, Elek's description of which Schreier graphs come from uniformly recurrent subgroups (see [20, Proposition 2.1]) gives an interesting new way to construct examples combinatorially. Elek gives such examples (see [20, Sections 5.2, 10.1]), all of which give rise to simple C^*-algebras, since they are generic.

Below we provide examples of non-generic URSs, whose associated C^*-algebras can nonetheless be proven to be simple via the groupoid picture. To this end, we employ Corollary 4.2.29: If a minimal groupoid with compact unit space has at least one C^*-simple isotropy group, the reduced groupoid C^*-algebra is simple.

For ease of notation, we describe our examples as quotients of the transformation groupoids as described in Section 4.3.4, but skip passing to the opposite, since a groupoid and its opposite are isomorphic and hence so are their reduced C^*-algebras: For a discrete group G with a subgroup $H \leq G$ such that its orbit closure $Z := \overline{G.H}$ forms a uniformly recurrent subgroup, we consider the transformation groupoid $Z \rtimes G$ associated with the action of G on Z and divide out by the equivalence relation given by

$$(g, K) \sim (h, K) \Leftrightarrow gK = hK$$

for two group elements $g, h \in G$ and a subgroup $K \in Z$. We write $\mathcal{G}'_Z := (Z \rtimes G)/\sim$ so that $C^*_{\mathrm{r}}(\mathcal{G}_Z) \cong C^*_{\mathrm{r}}(\mathcal{G}'_Z)$.

Note that the isotropy group of $Z \rtimes G$ at unit (e, K) for $K \in Z$ is given by the normaliser $N(K)$ of K, that is, exactly these group elements that fix K under conjugation. After passing to the quotient $\mathcal{G}'_Z$, the isotropy group at (e, K) is $N(K)/K$, which is sometimes called the *Weyl group* of K. Any uniformly recurrent subgroup Z for which a contained subgroup $H \in Z$ has C^*-simple Weyl group will therefore give rise to a C^*-simple Elek algebra by Corollary 4.2.29.

Consider the dihedral group $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$, which arises as the semidirect product of the action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$, where the non-trivial element $[1]_2$ acts by inversion. The following arguments also work slightly more generally with $\mathbb{Z}_3$ replaced by $\mathbb{Z}_{2k+1}$ for any $k \in \mathbb{N}$. Let F be any non-trivial C^*-simple discrete group, for example, the non-abelian free group in two generators. Then let $G = (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$ and $H = \mathbb{Z}_2 \subseteq G$ be the canonical copy of $\mathbb{Z}_2$ inside $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$, which sits inside G. In other words, $H = \{(([0]_3, x), e_F) \in G \mid x \in \mathbb{Z}_2\}$ with $[0]_3$ the neutral element of $\mathbb{Z}_3$ and e_F the neutral element of F.

Calculating the action of an arbitrary element of G on H as

$$((n, x), f)H((n, x), f)^{-1} = \{e_G, ((2n, [1]_2), e_F)\}$$

for $n \in \mathbb{Z}_3$, $x \in \mathbb{Z}_2$, and $f \in F$, we see that the normaliser of H in G is $\mathbb{Z}_2 \times F$ seen as a subset of G. The orbit of H in $\mathrm{Sub}(G)$ contains exactly three distinct subgroups, described by the three choices of $n \in \mathbb{Z}_3$. As it is finite, the orbit is closed in $\mathrm{Sub}(G)$, whence the orbit of H is a URS of G. The Weyl group of $H \leq G$ is simply $(\mathbb{Z}_2 \times F)/\mathbb{Z}_2 \cong F$, which is simple by assumption on F. We conclude that $H \leq G$ provides an example of a simple Elek algebra that neither arises from a *generic* URS, nor as the group C^*-algebra of a C^*-simple group, as would be the case if H were normal in G.

In the above example we avoided the subtleties of finding a uniformly recurrent subgroup by providing a group with *finite* orbit under conjugation. However, the example can be adapted to have infinite orbit without too much work, but will no longer be finitely-generated and therefore not be covered by Elek's framework as discussed in [20]. Nonetheless, it stands to reason that the appropriate quotient $\mathcal{G}'_Z$ of the transformation groupoid gives rise to a generalised Elek algebra associated with a URS Z of a not-necessarily finitely-generated discrete group G.

Let G be the infinite direct sum

$$G := \bigoplus_{\infty} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$$

with subgroup $H := \bigoplus_{\infty} \mathbb{Z}_2$, again for a non-trivial C^*-simple group F. Since the conjugation is component-wise, the normaliser of H is readily identified as $\bigoplus_{\infty} \mathbb{Z}_2 \times F$, and its Weyl group as $\bigoplus_{\infty} F$, which is C^*-simple. We merely have to verify that H is uniformly recurrent in G. First note that the convergence in the Chabauty topology of $\mathrm{Sub}(G)$ is component-wise in $\mathrm{Sub}((\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F)$, and $\mathbb{Z}_2$ has finite, discrete orbit in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$. Further, note that the orbit of H in G is given by a component-wise choice of conjugate of $\mathbb{Z}_2$ in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$. That is, the orbit contains exactly the subgroups of the form $\bigoplus_{i=1}^{\infty} K_i$ with each K_i one of the three conjugates of $\mathbb{Z}_2$ in $(\mathbb{Z}_3 \rtimes \mathbb{Z}_2) \times F$ while $K_i = \mathbb{Z}_2$ for all but finitely many i. As the set $\mathcal{K}$ of all subgroups of G that are of the form $\bigoplus_{i=1}^{\infty} K_i$ for K_i arbitrary conjugates of $\mathbb{Z}_2$ is compact in $\mathrm{Sub}(G)$, it contains the orbit closure of H, and as the orbit of H is clearly dense, they coincide. Since any group of the form $\bigoplus_{i=1}^{\infty} K_i$ is conjugate to $\bigoplus_{i=1}^{\infty} K_i'$ if and only if $K_i = K_i'$ for all but finitely many i, any subgroup contained in $\mathcal{K}$ again has dense orbit in $\mathcal{K}$, so the orbit closure of H is minimal and therefore a uniformly recurrent subgroup. In conclusion, the orbit closure of $H \leq G$ is an example of a non-generic, infinite URS giving rise to a simple (generalised) Elek algebra.

Further Research

5.1 A Converse for the Main Result

Foremost is of course Question 1, asking whether the sufficient condition of Theorem 4.2.25 is in fact equivalent. To this end, consider first the proof of Theorem 3.3.2, in particular the arguments for the intersection property of the G-action on X implying the intersection property of the G-action on the spectrum $\tilde{X}$ of the equivariant injective envelope. We would like to supply a converse to Lemma 4.2.23 by showing that if a groupoid $\mathcal{G}$ has the intersection property, so does its boundary groupoid $\tilde{\mathcal{G}}$. Assuming that $\mathcal{G}$ has the intersection property and $I \lhd C^*_{\mathrm{r}}(\tilde{\mathcal{G}})$ is an ideal which intersects $C(\tilde{X})$ trivially, the proof of Theorem 3.3.2 suggests constructing a ucp map $\varphi\colon C^*_{\mathrm{r}}(\tilde{G})/I \to C(\tilde{X})$ such that $\varphi \circ \pi$ is the faithful conditional expectation of $C^*_{\mathrm{r}}(\tilde{\mathcal{G}})$ onto $C(\tilde{X})$ for π the projection onto $C^*_{\mathrm{r}}(\tilde{G})/I$. For crossed products this worked as follows: Since $I \lhd C(\tilde{X}) \rtimes_r G$ intersects $C(\tilde{X})$ only trivially, it also intersects $C(X)$ only trivially, so $C(X) \rtimes_r G \cap I = \{0\}$ by the intersection property of $C(X) \rtimes_r G$. Hence $C(X) \rtimes_r G$ embeds into $(C(\tilde{X}) \rtimes_r G)/I$ and we may extend the canonical expectation $E_X\colon C(X) \rtimes_r G \to C(X) \subseteq C(\tilde{X})$ along this embedding by injectivity of $C(\tilde{X})$. More precisely, equipping $C(X) \rtimes_r G$ and $C(\tilde{X}) \rtimes_r G$ with a G-action by adjoining with the unitaries λ_g, the embedding is G-equivariant and we may extend E_X to a G-equivariant ucp map $\varphi\colon (C(\tilde{X}) \rtimes_r G)/I \to C(\tilde{X})$. As $(\varphi \circ \pi)|_{C(\tilde{X})}$ is a G-equivariant ucp self-map, it is the identity by G-rigidity of $C(\tilde{X})$ and on the other hand, it restricts to the canonical conditional expectation onto $C^*_{\mathrm{r}}(G) \subseteq C(X) \rtimes_r G \subseteq C(\tilde{X}) \rtimes_r G$. Then, as $C(\tilde{X})$ is contained in the multiplicative domain of $\varphi \circ \pi$, and $C(\tilde{X})$ and $C^*_{\mathrm{r}}(G)$ span $C(\tilde{X}) \rtimes_r G$ densely, we may conclude that $\varphi \circ \pi$ is the canonical conditional expectation $E_{\tilde{X}}$ of $C(\tilde{X}) \rtimes_r G$ onto $C(\tilde{X})$, which is faithful.

In the groupoid setting, constructing a ucp map $\varphi\colon C^*_{\mathrm{r}}(\tilde{\mathcal{G}})/I \to C(\tilde{X})$ that restricts to the identity on $C(\tilde{X}) \subseteq C^*_{\mathrm{r}}(\tilde{\mathcal{G}})/I$ and to the canonical conditional expectation on $C^*_{\mathrm{r}}(\mathcal{G}) \subseteq C^*_{\mathrm{r}}(\tilde{\mathcal{G}})/I$ would yield the analogous result. However, we cannot obtain it in the same way, as there is no reasonable $\mathcal{G}$-action on $C^*_{\mathrm{r}}(\mathcal{G})$ or $C^*_{\mathrm{r}}(\tilde{\mathcal{G}})$! In fact, recall an action of $\mathcal{G}$ is necessarily on a $C(X)$-algebra, and although $C(X)$ embeds into $C^*_{\mathrm{r}}(\mathcal{G})$ and $C^*_{\mathrm{r}}(\tilde{\mathcal{G}})$, it is usually not central. Even worse, a $\mathcal{G}$-action on a C^*-algebra A requires an abundance of ideals I_x, one for each $x \in X$ given as $\overline{C_0(X \setminus \{x\})A}$, to define the fibres $A_x = A/I_x$. But $C^*_{\mathrm{r}}(\mathcal{G})$ might even be simple, so it might not have any proper, closed, two-sided ideals at all, so every fibre would either be all of $C^*_{\mathrm{r}}(\mathcal{G})$ or zero. By a partition of unity argument, at most one ideal I_x can be trivial, or else the action of $C(X)$ is degenerate. Hence a simple C^*-algebra can only be fibred if every fibre but one is zero, and a groupoid can only act on such an algebra if it has a unit which is fixed by all its arrows. In particular, any such $\mathcal{G}$-action on $C^*_{\mathrm{r}}(\mathcal{G})$ would not restrict to the canonical $\mathcal{G}$-action on $C(X)$, unless $\mathcal{G}$ was a group to begin with.

Consequently, we cannot simply adapt the proof of Theorem 3.3.2 in a straightforward way, as there is no sense in talking about a $\mathcal{G}$-equivariant ucp map $\varphi\colon C^*_{\mathrm{r}}(\tilde{\mathcal{G}})/I \to C(\tilde{X})$. Note, however, that $\mathcal{G}$-equivariance is only required when talking about the restriction of φ to $\pi(C(\tilde{X}))$, which indeed carries a $\mathcal{G}$-action, where it is used to conclude that $(\varphi \circ \pi)|_{C(\tilde{X})}$ is the identity by $\mathcal{G}$-rigidity. A construction of $\varphi\colon C^*_{\mathrm{r}}(\tilde{\mathcal{G}})/I \to C(\tilde{X})$ which extends the canonical conditional expectation of $C^*_{\mathrm{r}}(\mathcal{G})$ and is $\mathcal{G}$-equivariant when restricted to $\pi(C(\tilde{X}))$ by other means would consequently show that the boundary groupoid has the intersection property if and only if the original groupoid does.

Recall that we presented a different proof strategy for the analogous statement for group C^*-algebras in Theorem 2.3.1. As for a group C^*-algebra the action is on a single point and hence minimal, simplicity of the associated C^*-algebra coincides with the intersection property. It crucially relies on Lemma 2.3.3, which applies to the groupoid context, if the analogue of Lemma 2.3.2 holds. We formulate this as Question 3:

Question 3: Let $\mathcal{G}$ be a Hausdorff étale groupoid with compact unit space and $\tilde{\mathcal{G}}$ its boundary groupoid. Is there a canonical embedding

$$C^*_{\mathrm{r}}(\mathcal{G}) \subseteq C^*_{\mathrm{r}}(\tilde{\mathcal{G}}) \subseteq I(C^*_{\mathrm{r}}(\mathcal{G}))$$

respecting the embedding of $C^*_{\mathrm{r}}(\mathcal{G})$ into $I(C^*_{\mathrm{r}}(\mathcal{G}))$?

Indeed, if Question 3 was answered in the positive, it immediately follows that the boundary groupoid has the intersection property if and only if the original groupoid does, using Lemma 2.3.3 as in the proof of Theorem 2.3.1. Hamana's approach suggests that a variant of the monotone complete crossed product of Definition 2.2.21 for groupoids would supply a proof, but we have not succeeded in providing such a construction.

5.2 Ozawa's Conjecture

The work of Kalantar–Kennedy [36] resolved Ozawa's conjecture in the case of reduced group C^*-algebras. The conjecture [47] states that every exact C^*-algebra A embeds *tightly* into a nuclear C^*-algebra $N(A)$ in the sense that $N(A)$ sits between A and its injective envelope $I(A)$:

$$A \subseteq N(A) \subseteq I(A).$$

Ozawa proved this for the reduced C^*-algebras of free groups [47, Corollary 2], using the notion of hyperbolic boundary, and Kalantar–Kennedy generalised his result to any reduced group C^*-algebra by replacing the boundary with the more general Furstenberg boundary:

Proposition 5.2.1 ([36, Theorem 1.3]): Let G be an *exact* discrete group. Then $C(\partial_F G) \rtimes_r G$ is nuclear and

$$C^*_{\mathrm{r}}(G) \subseteq C(\partial_F G) \rtimes_r G \subseteq I(C^*_{\mathrm{r}}(G)).$$

Proof. The inclusion the C^*-algebras in Proposition 5.2.1 are an immediate consequence of Lemma 2.3.2 since $C^*_{\mathrm{r}}(G) = \mathbb{C} \rtimes_r G$ while $C(\partial_F G) = I_G(\mathbb{C})$. On the other hand, Kalantar and Kennedy prove that if G is exact, then the action of G on $\partial_F G$ is amenable, whence $C(\partial_F G) \rtimes_r G$ is nuclear. Their proof [36, Theorem 4.5] passes through the action of G on the Stone-Cech compactification βG of G, which is amenable if G is exact. In fact, an amenable action on any compact space is sufficient and having such an action is equivalent to exactness. The idea is as follows: If X is a compact space with G-action, then by G-injectivity of $C(\partial_F G)$ there is a ucp G-map $\psi\colon C(X) \to C(\partial_F G)$. Consider $\partial_F G$ as the subset of point measures in the space $\mathcal{M}(\partial_F G)$ of Radon probability measures on $\partial_F G$. Then the adjoint of ψ yields a G-equivariant map $\psi^*\colon \partial_F G \subseteq \mathcal{M}(\partial_F G) \to \mathcal{M}(X) \subseteq \mathcal{P}(X)$ for $\mathcal{P}(X)$ the space of probability measures on X. By a result of Caprace and Monod [16, Proposition 9], the action of G on $\mathcal{P}(X)$ is amenable, and pulling an invariant mean back to $\partial_F G$ with the G-equivariant map ψ^* yields amenability of the action of G on $\partial_F G$. □

As even étale groupoids offer a powerful model of C^*-algebras, a positive resolution of Question 3 would be an important step towards the resolution of Ozawa's conjecture. Exactness of groupoid C^*-algebras is treated by Anantharaman-Delaroche [1]. In particular, a locally compact groupoid $\mathcal{G}$ is called *amenable at infinity*, if there is an $\mathcal{G}$-space Y with proper base point map such that $Y \rtimes_r \mathcal{G}$ is amenable, see Definitions 3.4 and 2.5 of [1]. This is a sufficient, criterion for $C^*_{\mathrm{r}}(\mathcal{G})$ to be exact [1, Corollary 6.4], but not necessarily equivalent.

5.3 Groupoids with Trivial Boundary

For a discrete group G, the injective envelope $I_G(\mathbb{C})$ is trivial, that is, $\mathbb{C}$, if and only if the group G is amenable, see [36, Proposition 3.2]. The analogous statement already fails for crossed products, as for X a G-space $I_G(C(X)) = C(X)$ implies that $C(X)$ is injective, which is in no way necessary for the action of G on X to be amenable. In some sense, the smallest possible candidate for $I_G(C(X))$ is the non-equivariant injective envelope $I(C(X))$ and alternatively triviality of the equivariant injective envelope could be formulated as $I_G(C(X)) = I(C(X))$. It remains interesting to see which properties of a groupoid force any of these to be the case:

Question 4: For which étale groupoids $\mathcal{G}$ with compact unit space $\mathcal{G}^{(0)}$ do $I_{\mathcal{G}}(C(\mathcal{G}^{(0)}))$ and $C(\mathcal{G}^{(0)})$ coincide so that $\tilde{\mathcal{G}} = \mathcal{G}$? For which $\mathcal{G}$ do $I_{\mathcal{G}}(C(\mathcal{G}^{(0)}))$ and $I(C(\mathcal{G}^{(0)}))$ coincide?

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