WHAT'S MY PAPER IN THE PNAS ABOUT? AND WHY DID I COMPARE A MAD FAMILY TO A LOTTERY TICKET?

Newspapers around the world have recently been writing about a 50 years old mathematical "riddle" that my former postdoc David Schrittesser and I have solved, and published the solution in The Proceedings of the National Academy of Science of the USA (PNAS). On the next 3–4 pages, I will try to give a slightly more detailed discussion of what this research is about, and why it is interesting.

Initially, I aimed for these pages to be readable with a typical high school mathematics background. But as you get beyond the first page or so, the mathematics I describe goes well beyond high school level, and you probably have to arm yourself with considerable patience if you're not an expert.

A mathematical puzzle (or three)

Can you solve the following mathematical problems?

(1) Six people enter a room. Prove that either there is a group of 3 people in the room who either had never met before entering the room, or there is a group of 3 people in the room who all knew each other before going into the room.

(2) The Infinite Lottery Ticket: A ticket in this lottery game has infinitely many rows, one row for each number x in the interval $[0, 1]$. The x'th row consists of infinitely many natural numbers (positive integers),

$$x(0), x(1), x(2), \dots$$

and the rule is that two different rows on a valid ticket can't have infinitely many numbers in common. The lotto authority draws infinitely many positive integers. A ticket wins if there is a row on the ticket that has infinitely many numbers in common with the numbers that have been drawn. Is there a lottery ticket that wins this game every time?

(3) Do the previous two problems have anything to do with each other?

Cantor's Set Theory, Cohen's Forcing, and the Continuum Hypothesis

The result in the PNAS article belongs to an area of mathematics called *Set Theory*.

Set theory is a theory of the concept of infinity in mathematics, especially of infinite numbers ("infinite cardinal numbers") and infinite processes ("infinite ordinal numbers"). Set theory was introduced to the world by the German mathematician Georg Cantor in the 1870s, in response to an increasing need for a more sophisticated approach to infinity and infinite processes in mathematics.

An *example* of an infinite process is the calculation of the decimals in the number π. There are infinitely many decimal places in π, with no apparent pattern. We have algorithms that allow us to compute as many decimal places of π as we would like (if we have enough time or computing power), in what is called an *infinite approximation process*.

Set theory is a universal theory for all mathematics. All other kinds of mathematics, including geometry, algebra, number theory, etc., can be interpreted in set theory. Therefore, in the late 1800s, a majority of leading mathematicians agreed that set theory was the *foundation of mathematics*.

In Cantor's investigation of infinite numbers, he discovered that there are different degrees of infinity. It is clear that there are infinitely many positive integers

$$1, 2, 3, 4, 5, \ldots,$$

but it turns out that if we look at the set of real numbers (i.e., numbers that can be represented by an infinite decimal expansion), this set has a higher degree of infinity. Cantor, however, could not figure out what degree of infinity the set of real numbers has. This led him to formulate the so-called **Continuum Hypothesis**, which postulates that the infinity of the set of real numbers has the lowest degree of infinity that is *greater than the set of integer's infinity*. The continuum hypothesis became a driving force in much mathematical research in the 20th century.

The continuum hypothesis was "solved" in 1963 by the American mathematician Paul Cohen. Paul Cohen proved that the accepted principles of set theory, called the *axioms* of the set theory, are insufficient to prove or disprove the continuum hypothesis. Cohen did this by introducing a technique called *Forcing*, which makes it possible to construct various mathematical universes (called *models*). Cohen showed that if there is a mathematical universe in which the continuum hypothesis is true, then there is one where it is false, and vice versa.

The solution of the Continuum Hypothesis was such big news that it was featured prominently in the New York Times! See

https://timesmachine.nytimes.com/timesmachine/1963/11/14/87346441.html?pageNumber=35

Solovay's model: Area, volume, measurability, and regularity

After the continuum hypothesis, the next big unsolved questions on the wish list of set theorists were questions about *regularity and measurability*.

The axioms of the set theory are self-evident[1], except (possibly) the last of them, called *the Axiom of Choice*. The Axiom of Choice implies that not all subsets of the plane, or subsets of the 3-dimensional Euclidian space, can be assigned an area or volume in a meaningful way[2]. The question was whether this "measurability problem" was solely due to the Axiom of Choice, i.e., whether it was possible to avoid the (non-)measurability problem if the Axiom of Choice was dropped.

The American mathematician Robert Solovay solved the measurability problem for areas, volumes, etc., around 1969, showing that if we drop the Axiom of Choice, then there is a model (i.e. a mathematical universe) in which all subsets of the plane have a reasonable area, and all subsets of 3-dimensional Euclidean space have a reasonable volume (etc.).

[1] At least at first glance

[2] A famous consequence of Axiom of Choice, and the associated lack of a meaningful measure of volume for certain subsets of space, is the so-called Banach-Tarski Paradox, which in popular terms say that you can take an orange and cut it into finitely many pieces, and reassemble those finitely many pieces into *two* oranges that are the same size and volume as the original orange.

Back to the 3 puzzles: Ramsey Theory and Mad Families

Area and volume measurements are far from the only "measurability properties" used by mathematicians. Another important measurability property stems from combinatorics, specifically from the area of *Ramsey theory*.

Puzzle (1) above is actually a special case of a classic mathematical result called *Ramsey's Theorem*. More generally, Ramsey's theorem tells us that when we consider sufficiently large mathematical systems, order and structure ("regularity") begin (unexpectedly) to arise. In the puzzle, the structure that arises is that there must either be a group of 3 people who either all know each other already, or a group of 3 people none of whom knew each other already.

Mathematicians, of course, became interested in whether Ramsey's theorem is also true of infinite sets, in one form or another. It turns out that the Axiom of Choice again stands in the way of having Ramsey theoretic regularity extend to all subsets of the real line.

Shortly after Solovay had shown that there is a universe where the measurability problem can be solved, Adrian Mathias showed that the Ramsey theoretic regularity problem can also be solved. In fact, Solovay's model has total (Ramsey-theoretic) combinatorial regularity, Mathias showed.

Mathias solved the regularity problem by introducing a variant of Cohen's forcing, which today is called *Mathias forcing*. Mathias forcing is a method of building a new universe (model of set theory) from an old one, in such a way that the combinatorial regularity is higher in the new universe than in the one you start with.

And what does this have to do with puzzle (2) above? A lottery ticket that wins every time in the infinite lottery game described above is called a *mad family*. The word "mad" is short for "maximum almost disjoint".

The Axiom of Choice implies quite easily that such winning lottery tickets exist, that is, there are mad families. (Are there any mad families in Solovay's model? We didn't know that for a long, long time, but I solved that problem about 5 years ago: There aren't any.)

So why are these mad families, i.e. winning lottery tickets, interesting? They are interesting because they give us indirect information about what happens when you form a new universe (model of set theory) using Mathias forcing.

But if the universe already has total combinatorial (Ramsey theoretic) regularity (called "The Ramsey Property for all sets"), then there is nothing new to be gained by creating a new universe with Mathias forcing (it would seem). This led Mathias to ask the following: *In a universe where there is total combinatorial (Ramsey theoretic) regularity, is it the case that there are no mad families?*. In other word: Does the Ramsey property imply no mad families? It is *this* problem that David Schrittesser and I have now solved, some 50 years after it was first stated. The solution can be found here:

https://www.pnas.org/content/early/2019/08/28/1906183116/tab-article-info

What is it good for?

Today, it has become popular to ask what new research is good for. Do you get a better mobile phone from my research? Do planes use less fuel in the future because of my research? Does my research make sick people healthy again?

It cannot be ruled out that one day there will be applications of this kind, or applications that no one can yet imagine, but that is not the goal of my research. My research is pure *scientific* research, it is *not* technological research. Scientific research is driven by the desire to understand the world we live in better, which in my case means: To understand better the mathematics we use to describe the world. Technological research *depends* on the knowledge and understanding that scientific research brings. Roughly speaking: no science, no technological progress. Conversely, our opportunities to make scientific discoveries become better, the better our technological capabilities evolve, and if technological research stopped, scientific research would suffer badly. Therefore, technological and scientific progress go hand in hand, and therefore investment must be made in both.

Asger Törnquist, September 2019.