DST HOMEWORK ASSIGNMENT 2:  
MYCIELSKI’S THEOREM; THE SPACE OF SUBSETS; BOREL CODES

**Homework**: This is the second of three mandatory homework assignments. You must hand it in at the beginning of lecture on Friday, December 21, 2012.

**Exercise 1**

**Theorem.** Let $E$ be an equivalence relation$^1$ on a Polish space $X$. Suppose that $E$ is meagre as a subset of $X \times X$, when the latter is given the product topology. Then there is a continuous injection $f : 2^\omega \to X$ such that $(\forall x, y \in 2^\omega) x \neq y \implies f(x) \notin f(y)$, i.e., $f(2^\omega)$ meets each $E$ equivalence class in at most one point.

**Hint:** Fix a decreasing sequence $U_n \subseteq X^2$ of dense open sets such that $E \cap \bigcap_{n \in \omega} U_n = \emptyset$.

Recursively (on $\ell h(s)$, say) define a Cantor scheme $(V_s)_{s \in 2^{<\omega}}$ of open sets in $X$ such that

1. $V_s \supseteq V_i$ for all $s \in 2^{<\omega}$, $i \in \{0, 1\}$;
2. diam$(V_s) \leq 2^{-\ell h(s)}$;
3. For all $n \in \omega$ and $s, t \in 2^{<\omega}$ with $\ell h(s) = \ell h(t) = n$, if $t \neq s$ then $V_s \times V_t \subseteq U_n$.

**Exercise 2**

(a) Let $\mathcal{P}(A)$ denote the powerset of $A$ (i.e., the set of all subsets of $A$), and for $x, y \in \mathcal{P}(A)$, let $x \triangle y = x \setminus y \cup y \setminus x$ be the symmetric difference. Show that if $A = \{a_n : n \in \omega\}$ is a countable set, then

$$d(x, y) = 2^{-\min\{n \in \omega : a_n \in x \triangle y\}}$$

(where as per our usual convention $\min(\emptyset) = \infty$) defines a complete metric on $\mathcal{P}(A)$. Show that the set

$$\text{FIN}(A) = \{x \in \mathcal{P}(A) : x \text{ is finite}\}$$

is dense in $\mathcal{P}(A)$, and conclude that $(\mathcal{P}(A), d)$ is a Polish metric space.

(b) Show that $x_n \to x$ in $\mathcal{P}(A)$ if and only if for all $m \in \omega$ we have:

$$a_m \in x \implies (\exists N)(\forall n \geq N)a_m \in x_n$$

and

$$a_m \notin x \implies (\exists N)(\forall n \geq N)a_m \notin x_n.$$

(c) Consider now $\mathcal{P}(\omega^{<\omega})$. Show that Tree($\omega$) $\subseteq \mathcal{P}(\omega^{<\omega})$ is a closed subset of $\mathcal{P}(\omega^{<\omega})$. Is the set of finite branching trees closed? What about the set of pruned trees? Or the set of wellfounded trees?

$^1$i.e., a reflexive, symmetric and transitive binary relation
EXERCISE 3

This exercise requires some preparatory definitions.

**Definition.** A *Borel code* is a pair \((T, f)\), where \(T \in \text{Tree}(\omega)\) is a well-founded, \(T \supseteq \{\emptyset\}\), and where

\[
f : \{ t \in T : t \text{ is terminal} \} \to \omega
\]
is a function. (Thus a Borel code can be thought of as a well-founded tree where each terminal node has been labeled by some \(n\).) The *rank* of the Borel code \((T, f)\), denoted \(\text{rk}(T, f)\), is the rank of \(\emptyset \in T\), i.e., \(\text{rk}(T, f) = \rho_T(\emptyset)\). (See Kechris 2.E.)

Let \(X\) be a Polish space, and fix a basis \((U_n)_{n \in \omega}\) for the topology. Let \((T, f)\) be a Borel code. We define recursively for \(t \in T\) the sets \(B^t(T, f)\) as follows: If \(t \in T\) is terminal in \(T\), then \(B^t(T, f) = U_{f(t)}\). If \(t \in T\) is not terminal, we define

\[
B^t(T, f) = \bigcap\{ X \setminus B^{t \downarrow n}(T, f) : n \in \omega \land t \downarrow n \in T \}.
\]

Finally, we define \(B(T, f) = B^\emptyset(T, f)\).

(a) Prove that \(B(T, f) \in \Pi^0_{\text{rk}(T, f)}\).
(b) Prove that if \(A \subseteq X\) is \(\Pi^0_\alpha\) then \(A = B(T, f)\) for some Borel code \((T, f)\) of rank \(\alpha\).

Hint: In both cases, use induction on ordinals.