ABSTRACT STEINER POINTS FOR CONVEX POLYTOPES

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Let $\mathscr{P}^d$ denote the set of all convex polytopes, degenerate or not, in d-dimensional Euclidean space E^d. An abstract Steiner point for convex polytopes in E^d is a mapping $S : \mathscr{P}^d \to E^d$ satisfying

$$S(P+Q) = S(P)+S(Q) \text{ for all } P, Q \in \mathscr{P}^d, \tag{1}$$

addition on the left being Minkowski addition of convex sets, and

$$S(\sigma(P)) = \sigma(S(P)) \tag{2}$$

for all $P \in \mathscr{P}^d$ and all similarity transformations σ in E^d. (A similarity means any composition of dilatations and isometries.)

In the present paper we shall characterize the abstract Steiner points for convex polytopes in two and three dimensions. As a simple consequence of this characterization we find that an abstract Steiner point S for convex polytopes in E^d, $d = 2, 3$, satisfying $S(P) \in P$ for all $P \in \mathscr{P}^d$, is the usual Steiner point. For the definition and properties of this point see [**3**].

For any vertex A of $P \in \mathscr{P}^d$, let $V(A)$ denote the closed convex spherical polytope in S^{d-1}, consisting of all outer normal directions for supporting hyperplanes of P through A. The *external angle* u at the vertex A is now defined as $u = \omega(V(A))$, where ω denotes the normalized surface measure of S^{d-1}.

Let μ be any mapping from the set $\mathscr{V}^{d-1}$ of closed convex spherical polytopes in S^{d-1} to the real numbers satisfying

$$\mu(V \cup W) = \mu(V)+\mu(W), \tag{3}$$

whenever V, W, $V \cup W \in \mathscr{V}^{d-1}$, and the polytopes V and W are without common interior points,

$$\mu(\delta(V)) = \mu(V), \tag{4}$$

for all isometries δ of S^{d-1}, and

$$\mu(S^{d-1}) = 1. \tag{5}$$

The set of all these mappings μ will be denoted by M.

For any $\mu \in$ M we define an abstract Steiner point S_μ by the formula

$$S_\mu(P) = \sum_{i=1}^{n} \mu(V(A_i))\, p_i, \tag{6}$$

where p_i denotes the position vector of the vertex A_i, $(i = 1, \ldots, n)$ of $P \in \mathscr{P}^d$. For the proof of (1) and (2) for S_μ see [**3**; p.1296].

Examples of such mappings μ are obtained as follows. Let Φ denote the set of functions $\phi : [0, 1] \to \mathbf{R}$ satisfying

$$\phi(u+v) = \phi(u)+\phi(v) \tag{7}$$

Received 26 February, 1970; revised 30 October, 1970.

whenever $u, v, u+v \in [0, 1]$, and

$$\phi(1) = 1. \tag{8}$$

For any $\phi \in \Phi$ we obtain a mapping $\mu_\phi \in \mathrm{M}$ by putting

$$\mu_\phi(V) = \phi(\omega(V)),\ V \in \mathscr{V}^{d-1},$$

and consequently an abstract Steiner point

$$S_\phi(P) = \sum_{i=1}^{n} \phi(u_i)\, p_i, \tag{9}$$

where as before the u_i ($i = 1, \ldots, n$) denote the external angles. If ϕ is the identity, S_ϕ is the usual Steiner point.

THEOREM 1. *Let S be an abstract Steiner point for convex polytopes in E^d, $d = 2, 3$. Then there exists a unique $\phi \in \Phi$ such that $S = S_\phi$.*

Proof. The uniqueness follows if one considers a right triangle. The proof of the existence is divided into two parts.

Case 1, $d = 2$

Notice that the value of an abstract Steiner point at a segment is the midpoint. We can now conclude that two abstract Steiner points S_1 and S_2 are identical if they coincide on the set of right triangles. Indeed, an interior altitude of a triangle divides the triangle into two right triangles, the sum of which equals the sum of the triangle and the altitude. This implies that S_1 and S_2 take the same values for all triangles, and hence for all convex polygons, because every convex polygon is expressible as a sum of segments and triangles.

By $\mathscr{T}_u$ we denote the set of right triangles with external angles $u, \frac{1}{4}, \frac{3}{4}-u$, $u \in]\frac{1}{4}, \frac{1}{2}[$. As any two triangles from $\mathscr{T}_u$ can be mapped onto one another by similarity transformations, there exists precisely one triple $(\phi_1(u), \phi_2(u), \phi_3(u)) \in \mathbf{R}^3$ such that

$$\phi_1(u)+\phi_2(u)+\phi_3(u) = 1, \tag{10}$$

and

$$S(T) = \phi_1(u)\,\mathbf{p}_1+\phi_2(u)\,\mathbf{p}_2+\phi_3(u)\,\mathbf{p}_3 \tag{11}$$

for all $T \in \mathscr{T}_u$, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ being position vectors for the vertices of T corresponding to the exterior angles $u, \frac{1}{4}, \frac{3}{4}-u$.

Now choose a $T \in \mathscr{T}_u$ such that the foot of the altitude H to the hypotenuse is situated at the origin. The altitude H divides T into two triangles T_1, $T_2 \in \mathscr{T}_u$, and we know that

$$T+H = T_1+T_2.$$

If $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ denote position vectors for the vertices of T corresponding to the external angles $u, \frac{1}{4}, \frac{3}{4}-u$, this relation gives

$$\begin{aligned}&\phi_1(u)\,\mathbf{p}_1+\phi_2(u)\,\mathbf{p}_2+\phi_3(u)\,\mathbf{p}_3+\tfrac{1}{2}\,\mathbf{p}_2\\ &= \phi_1(u)\,\mathbf{p}_1+\phi_3(u)\,\mathbf{p}_2+\phi_1(u)\,\mathbf{p}_2+\phi_3(u)\,\mathbf{p}_3,\end{aligned}$$

which implies that

$$\phi_2(u)+\tfrac{1}{2} = \phi_1(u)+\phi_3(u).$$

From this and (10) we obtain

$$\phi_2(u) = \tfrac{1}{4},\ \phi_1(u)+\phi_3(u) = \tfrac{3}{4}.$$

Moreover, for reasons of symmetry, we have

$$\phi_1(\tfrac{3}{4}-u) = \phi_3(u).$$

We have now proved that

There exists a mapping $\phi_1 :]\tfrac{1}{4},\tfrac{1}{2}[\to \mathbf{R}$ *such that, for all* $T \in \mathcal{T}_u$,

$$S(T) = \phi_1(u)\,\mathbf{p}_1+\tfrac{1}{4}\mathbf{p}_2+\phi_1(\tfrac{3}{4}-u)\,\mathbf{p}_3, \tag{12}$$

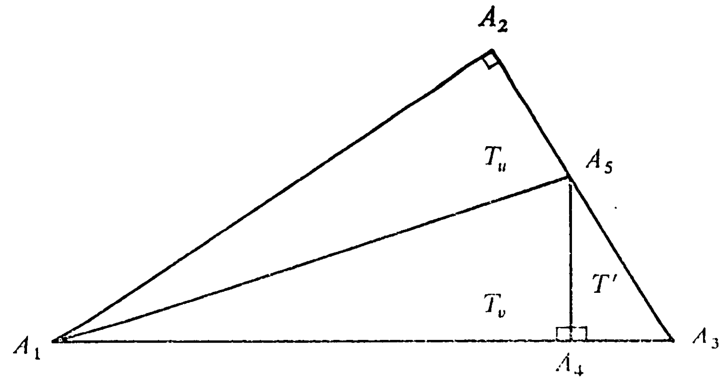
$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ *being position vectors for the vertices of* T *corresponding to the external angles* $u, \tfrac{1}{4}, \tfrac{3}{4}-u$. *Moreover,* ϕ_1 *satisfies* $\phi_1(u)+\phi_1(\tfrac{3}{4}-u) = \tfrac{3}{4}$; *in particular* $\phi_1(\tfrac{3}{8}) = \tfrac{3}{8}$.

We next prove that ϕ_1 satisfies the functional equation

$$\phi_1(u+v-\tfrac{1}{2})+\tfrac{1}{2} = \phi_1(u)+\phi_1(v), \tag{13}$$

whenever $u, v, u+v-\tfrac{1}{2} \in]\tfrac{1}{4},\tfrac{1}{2}[$.

Suppose that $u, v, u+v-\tfrac{1}{2} \in]\tfrac{1}{4},\tfrac{1}{2}[$.



The figure shows a right triangle $T = A_1A_2A_3 \in \mathcal{T}_{u+v-\frac{1}{2}}$, which is divided into three right triangles $T_u \in \mathcal{T}_u$, $T_v \in \mathcal{T}_v$, $T' \in \mathcal{T}_{u+v-\frac{1}{2}}$, such that the external angles at A_1 for the triangles T, T_u, T_v are respectively $u+v-\tfrac{1}{2}$, u, v. The origin of the plane is chosen to be A_5, and if we let $\mathbf{p}_i$ denote position vectors for the points A_i, $i = 1, \ldots, 4$, we have

$$T+[0,\mathbf{p}_1]+[0,\mathbf{p}_4] = T_u+T_v+T'.$$

Substituting (12) in the analogous expression for S, we obtain (13).

We now claim that ϕ_1 *is the restriction to* $]\tfrac{1}{4},\tfrac{1}{2}[$ *of a function* $\phi \in \Phi$ *such that* $S = S_\phi$.

To see this, define $\phi' :]0,\tfrac{1}{4}[\to \mathbf{R}$ by

$$\phi'(u) = \tfrac{1}{2}-\phi_1(\tfrac{1}{2}-u) \text{ for } u \in]0,\tfrac{1}{4}[.$$

From (13) it follows that ϕ' satisfies the functional equation

$$\phi'(u+v) = \phi'(u)+\phi'(v), \quad \text{whenever} \quad u, v, u+v \in \,]0, \tfrac{1}{4}[.$$

Furthermore $\phi'(\frac{1}{8}) = \frac{1}{8}$. This function ϕ' can be extended in a unique way to a function $\phi \in \Phi$, and we shall check that $\phi = \phi_1$ in $]\frac{1}{4}, \frac{1}{2}[$. For $u \in \,]\frac{1}{4}, \frac{1}{2}[$ we have

$$\phi(u) = 2\phi(\tfrac{1}{8})+\phi(u-\tfrac{1}{4}) = 2\phi'(\tfrac{1}{8})+\phi'(u-\tfrac{1}{4}) = \tfrac{3}{4}-\phi_1(\tfrac{3}{4}-u) = \phi_1(u).$$

To this $\phi \in \Phi$ we construct by (9) the abstract Steiner point S_ϕ, and (12) tells us that S and S_ϕ take on the same value for right triangles. This is sufficient to ensure that $S = S_\phi$.

Case 2, $d = 3$

We first assert that two abstract Steiner points S_1 and S_2 are identical, if they coincide on all convex polygons in E^3. To see this, let P be a three-dimensional convex polytope with j-faces $P_i^j, i = 1, \ldots, f_j$, where f_j is the number of j-faces $j = 0, 1, 2$. The reflection of P in the origin is denoted by $\tilde{P}$. The following formula, which is due to G. C. Shephard [**5**; p.600],

$$\sum_{i=1}^{f_0} P_i^0 + \sum_{i=1}^{f_2} P_i^2 + \tilde{P} = \sum_{i=1}^{f_1} P_i^1 + P,$$

shows that $S_1(P) = S_2(P)$.

Let π denote an arbitrary plane containing the origin. If P is a convex polygon in π, we have $S(P) \in \pi$ because P is invariant under the reflection in π. Therefore the restriction S^* of S to π is an abstract Steiner point in π, and the previous discussion shows that $S^* = S_\phi$ for a $\phi \in \Phi$. However, the external angle at a vertex is independent of the dimension of the space in which the polytope is considered, so that S and S_ϕ coincide on convex polygons in π. This, however, extends to all convex polygons in E^3, and hence by the first remark we have $S = S_\phi$.

Remark. Theorem 1 implies that any $\mu \in \mathrm{M}$ can be written in the form $\mu = \phi \circ \omega$ with a unique $\phi \in \Phi$ if d equals 2 or 3. This result is well known, and it is known too that it does not extend to higher dimensions. These problems are treated in the simpler case of Euclidean space in the monograph of H. Hadwiger [**1**; chapter 2]. Thus our theorem cannot be extended to dimensions higher than three. We do not know whether (6) gives the most general abstract Steiner point in this case.

It is well known that $\phi \in \Phi$ is continuous, and is then the identity, if ϕ is bounded in some subinterval of $[0, 1]$. Under this assumption S_ϕ is the usual Steiner point. However, if S is an abstract Steiner point which is bounded on the set of convex polytopes in the unit ball, it is not difficult to see that the corresponding $\phi \in \Phi$ is bounded on a suitable subinterval of $[0, 1]$. This proves the following theorem.

THEOREM 2. *Let S be an abstract Steiner point for convex polytopes in E^d, $d = 2, 3$, and suppose that S is bounded on the set of polytopes in the unit ball. Then S is the usual Steiner point.*

Remark. The boundedness condition of the theorem is fulfilled if S satisfies the geometrically natural condition $S(P) \in P$ for all $P \in \mathscr{P}^d$.

Let $\mathscr{C}^d$ denote the set of convex bodies in E^d topologised by the Hausdorff metric.

THEOREM 3. *Let $S : \mathscr{C}^d \to E^d$, $d = 2, 3$, be any mapping which satisfies*

(*a*) $S(K+L) = S(K)+S(L)$ for all $K, L \in \mathscr{C}^d$,

(*b*) $S(\sigma(K)) = \sigma(S(K))$ *for all* $K \in \mathscr{C}^d$ *and all similarity transformations* σ *in* E^d,

(*c*) S *is continuous.*

Then S is the Steiner point.

Proof. The Steiner point itself is a continuous mapping from $\mathscr{C}^d$ to E^d (*cf.* [3]), so it suffices to show that it coincides with S on the dense subset $\mathscr{P}^d$ of $\mathscr{C}^d$. This, however, follows from theorem 2 because S is bounded on the compact set of convex bodies in the unit ball.

Theorem 3 contains the uniqueness theorem of Shephard in [4] and its generalization to three dimensions. K. A. Schmitt's paper [2] which generalizes the uniqueness theorem of Shephard to all dimensions is insufficient, because the proof of (11) on p. 390 is incomplete.

References

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