

Complete Monotonicity of a Difference Between the Exponential and Trigamma Functions and Properties Related to a Modified Bessel Function

Feng Qi* and Christian Berg

Abstract. In the paper, the authors find necessary and sufficient conditions for a difference between the exponential function $\alpha e^{\beta/t}$, $\alpha, \beta > 0$, and the trigamma function $\psi'(t)$ to be completely monotonic on $(0, \infty)$. While proving the complete monotonicity, the authors discover some properties related to the first order modified Bessel function of the first kind I_1 , including inequalities, monotonicity, unimodality, and convexity.

Mathematics Subject Classification (2010). Primary 26A48, 33C10; Secondary 26A51, 33B10, 33B15, 44A10.

Keywords. Complete monotonicity, difference, trigamma function, exponential function, necessary and sufficient condition, inequality, monotonicity, convexity, unimodality, modified Bessel function.

1. Introduction

Recall from [8, Chapter XIII], [14, Chapter 1] and [15, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1.1)$$

for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$.

The exponential function $e^{1/z}$ for $z \in \mathbb{C}$ with $z \neq 0$ can be expanded into the Laurent series

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k}, \quad z \neq 0. \quad (1.2)$$

*Corresponding author.

Submitted on 22 October 2012; Accepted on 9 December 2012.

In [16, Theorem 2.1], derivatives of the function $e^{1/t}$ were inductively and explicitly derived to be

$$(e^{1/t})^{(i)} = (-1)^i e^{1/t} \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k \quad (1.3)$$

for $i \in \mathbb{N}$ and $t \neq 0$, where

$$a_{i,k} = \binom{i}{k} \binom{i-1}{k} k! \quad (1.4)$$

for $0 \leq k \leq i-1$. The series (1.2) and (1.3) both show that the function $e^{1/t}$ is completely monotonic on $(0, \infty)$.

The classical Euler gamma function $\Gamma(z)$ and the digamma function $\psi(z)$ may be defined respectively by

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, du \quad (1.5)$$

and

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (1.6)$$

for $\Re z > 0$. It is well known that for $n \in \mathbb{N}$

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1-e^{-u}} e^{-zu} \, du \quad (1.7)$$

for $\Re z > 0$, see [1, p. 260, 6.4.1], and that

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}} \quad (1.8)$$

for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, see [1, p. 260, 6.4.10]. These formulas show that the trigamma function $\psi'(t)$ is completely monotonic on $(0, \infty)$.

In [11, Lemma 2], the function

$$h(t) = e^{1/t} - \psi'(t), \quad (1.9)$$

a difference between two completely monotonic functions, was stated to be completely monotonic on $(0, \infty)$. But since the complete monotonicity was not used in [4] and there was a mistake in the proof in [11], the statement on complete monotonicity of $h(t)$ was removed from the formally published version [4]. However, the validity of the inequality

$$\psi'(t) < e^{1/t} - 1 \quad (1.10)$$

on $(0, \infty)$ was obtained and applied in [4, Lemma 2].

Recently, the complete monotonicity of $h(t)$ was established in [13, Theorem 1.1].

The first aim of this paper is to provide a simpler proof for [13, Theorem 1.1] and to find a necessary and sufficient condition for the function

$$h_\alpha(t) = \alpha e^{1/t} - \psi'(t) \quad (1.11)$$

to be completely monotonic on $(0, \infty)$. The first main result is formulated as Theorem 3.1 below.

It is natural to consider the more general function

$$h_{\alpha, \beta}(t) = \alpha e^{\beta/t} - \psi'(t) \quad (1.12)$$

for $\alpha, \beta > 0$ on $(0, \infty)$. The second aim of this paper is to discover necessary and sufficient conditions for the function $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$. The second main result is formulated as Theorems 4.1 and 6.1 below.

While discovering necessary and sufficient conditions for the functions $h(t)$, $h_{\alpha}(t)$, and $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$, some properties such as monotonicity, unimodality, convexity, and inequalities related to the first order modified Bessel function of the first kind $I_1(t)$ are established, where

$$I_{\nu}(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left(\frac{z}{2}\right)^{2k+\nu} \quad (1.13)$$

for $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$.

In the final section of this paper, we list several remarks about inequalities related to $I_1(t)$.

2. Lemmas

We need the following lemmas.

Lemma 2.1 ([13, Theorem 1.2]). *For $k \in \{0\} \cup \mathbb{N}$ and $z \neq 0$, let*

$$H_k(z) = e^{1/z} - \sum_{m=0}^k \frac{1}{m!} \frac{1}{z^m}. \quad (2.1)$$

For $\Re z > 0$, the function $H_k(z)$ has the integral representation

$$H_k(z) = \frac{1}{k!(k+1)!} \int_0^{\infty} {}_1F_2(1; k+1, k+2; t) t^k e^{-zt} dt, \quad (2.2)$$

where the hypergeometric series is given by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} \quad (2.3)$$

for $b_i \notin \{0, -1, -2, \dots\}$ using the shifted factorial $(a)_0 = 1$ and

$$(a)_n = a(a+1) \cdots (a+n-1) \quad (2.4)$$

for $n > 0$ and any complex number a .

Lemma 2.2 ([15, p. 161, Theorem 12b]). *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that*

$$f(x) = \int_0^{\infty} e^{-xt} d\mu(t), \quad (2.5)$$

where μ is a positive measure on $[0, \infty)$ such that the integral converges for $0 < x < \infty$.

Remark 2.1. Lemma 2.2 means that a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of a positive measure μ .

Lemma 2.3 ([3]). *Let a_k and b_k for $k \in \{0\} \cup \mathbb{N}$ be real numbers and the power series*

$$A(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{and} \quad B(x) = \sum_{k=0}^{\infty} b_k x^k \quad (2.6)$$

be convergent on $(-R, R)$ for some $R > 0$. If $b_k > 0$ and the ratio $\frac{a_k}{b_k}$ is (strictly) increasing for $k \in \mathbb{N}$, then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing on $(0, R)$.

Remark 2.2. Lemma 2.3 have several different proofs and various applications. For more information, please refer to the first sentence after [2, p. 582, Lemma 2.1], the papers [9, 12], and closely related references therein. We emphasize that Lemma 2.3 has been generalized in [6, Lemma 2.2].

3. Complete monotonicity of $h_\alpha(t)$

In this section, we will supply a simpler proof for the complete monotonicity of $h(t)$ and find a necessary and sufficient condition for $h_\alpha(t)$ to be completely monotonic on $(0, \infty)$.

Theorem 3.1. *The function $h(t)$ defined by (1.9) is completely monotonic on $(0, \infty)$ and*

$$\lim_{t \rightarrow \infty} h(t) = 1. \quad (3.1)$$

The function $h_\alpha(t)$ defined by (1.11) is completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 1$.

Proof. From the integral representation (1.7) or the series expansion (1.8), it is obvious that $\lim_{t \rightarrow \infty} \psi'(t) = 0$. So the limit (3.1) follows immediately.

From the recurrence formula

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n \frac{n!}{z^{n+1}}, \quad n \geq 0, \quad (3.2)$$

see [1, p. 260, 6.4.7], we see that

$$\begin{aligned} h(t) - h(t+1) &= e^{1/t} - e^{1/(t+1)} + \psi'(t+1) - \psi'(t) \\ &= e^{1/t} - e^{1/(t+1)} - \frac{1}{t^2} \\ &= -\frac{1}{t^2} + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{1}{t^k} - \frac{1}{(t+1)^k} \right] \\ &= \frac{1}{6t^3(t+1)^3} + \sum_{k=4}^{\infty} \frac{1}{k!} \left[\frac{1}{t^k} - \frac{1}{(t+1)^k} \right] \end{aligned}$$

$$= \frac{1}{6t^3(t+1)^3} + \sum_{k=4}^{\infty} \frac{1}{(k-1)!} \int_0^1 \frac{dx}{(t+x)^{k+1}}.$$

Since $\frac{1}{t^3(t+1)^3}$ and $\frac{1}{(t+x)^k}$ are completely monotonic with respect to $t \in (0, \infty)$ for $k \geq 1$ and $x \geq 0$, so is the difference $h(t) - h(t+1)$. Accordingly,

$$h(t) - h(t+n+1) = \sum_{k=0}^n [h(t+k) - h(t+k+1)]$$

is completely monotonic on $(0, \infty)$, and so is the pointwise limit

$$\lim_{n \rightarrow \infty} [h(t) - h(t+n+1)] = h(t) - 1.$$

If the function $h_\alpha(t)$ is completely monotonic on $(0, \infty)$, then its first derivative should be non-positive, that is,

$$-\frac{\alpha}{t^2} e^{1/t} - \psi''(t) \leq 0,$$

which can be rearranged as

$$\alpha \geq -\frac{t^2 \psi''(t)}{e^{1/t}} \rightarrow -\lim_{t \rightarrow \infty} [t^2 \psi''(t)] = 1$$

as $t \rightarrow \infty$, where the limit

$$\lim_{x \rightarrow \infty} [(-1)^{k+1} x^k \psi^{(k)}(x)] = (k-1)!, \quad (3.3)$$

see [10, p. 81, (41)], is used. As a result, the condition $\alpha \geq 1$ is necessary.

Since

$$\alpha e^{1/t} - \psi'(t) = (\alpha - 1)e^{1/t} + h(t)$$

and both of the functions $e^{1/t}$ and $h(t)$ are completely monotonic on $(0, \infty)$, the condition $\alpha \geq 1$ is also sufficient. Theorem 3.1 is thus proved. \square

4. Complete monotonicity of $h_{\alpha, \beta}(t)$

In this section, with the help of Theorem 3.1, we will discover necessary and sufficient conditions for $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$.

Theorem 4.1. For $\alpha, \beta > 0$,

1. the function $h_{1, \beta}(t)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \geq 1$;
2. if $\beta \geq 1$ and $\alpha\beta \geq 1$, the function $h_{\alpha, \beta}(t)$ is completely monotonic on $(0, \infty)$;
3. a necessary condition for the function $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$ is $\alpha\beta \geq 1$;
4. if $0 < \beta < 1$, the condition

$$\alpha\beta \geq \max_{u \in (0, \infty)} F_\beta(u) > 1, \quad (4.1)$$

where

$$F_\beta(u) = \frac{u}{1 - e^{-u}} \frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})} \quad (4.2)$$

on $(0, \infty)$ with

$$\lim_{u \rightarrow 0^+} F_\beta(u) = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} F_\beta(u) = 0 \quad (4.3)$$

for all $\beta > 0$, is necessary and sufficient for $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0, \infty)$.

Proof. Combining the integral representations (1.7) for $n = 1$ and (2.2) for $k = 0$ leads to

$$\begin{aligned} h_{\alpha,\beta}(z) &= \alpha \left[1 + \int_0^\infty \frac{I_1(2\sqrt{u})}{\sqrt{u}} e^{-zu/\beta} du \right] - \int_0^\infty \frac{u}{1-e^{-u}} e^{-zu} du \\ &= \alpha + \int_0^\infty \left[\alpha \sqrt{\frac{\beta}{u}} I_1(2\sqrt{\beta u}) - \frac{u}{1-e^{-u}} \right] e^{-zu} du \\ &= \alpha + \int_0^\infty \left[\alpha \beta \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} - \frac{u}{1-e^{-u}} \right] e^{-zu} du \end{aligned} \quad (4.4)$$

for $\Re z > 0$.

Since, by Theorem 3.1, the function $h(t) = h_{1,1}(t)$ is known to be completely monotonic on $(0, \infty)$, it follows from Lemma 2.2 that the inequality

$$\frac{I_1(2\sqrt{u})}{\sqrt{u}} \geq \frac{u}{1-e^{-u}} \quad (4.5)$$

holds true for $u > 0$.

An easy calculation gives

$$\frac{I_1(2\sqrt{u})}{\sqrt{u}} = 1 + \frac{u}{2} + \frac{u^2}{12} + \frac{u^3}{144} + o(u^3)$$

and

$$\frac{u}{1-e^{-u}} = 1 + \frac{u}{2} + \frac{u^2}{12} - \frac{u^4}{720} + o(u^4)$$

for $u \rightarrow 0$. Consequently,

$$\alpha \beta \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} - \frac{u}{1-e^{-u}} = \alpha \beta - 1 + \frac{\alpha \beta^2 - 1}{2} u + \frac{\alpha \beta^3 - 1}{12} u^2 + o(u^2) \quad (4.6)$$

for $u \rightarrow 0$. By (4.5), (4.6), and the fact that the function $\frac{I_1(2s)}{s}$ is strictly increasing on $(0, \infty)$, it follows that the inequality

$$\frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} \geq \frac{u}{1-e^{-u}} \quad (4.7)$$

is valid on $(0, \infty)$ if and only if $\beta \geq 1$. As a result, by Lemma 2.2, the function $h_{1,\beta}(t)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \geq 1$.

If $\beta \geq 1$ and $\alpha \beta \geq 1$, then

$$\alpha \beta \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} \geq \frac{I_1(2\sqrt{u})}{\sqrt{u}} \geq \frac{u}{1-e^{-u}} \quad (4.8)$$

is valid on $(0, \infty)$, so, by Lemma 2.2 once again, the function $h_{\alpha,\beta}(t)$ is completely monotonic on $(0, \infty)$ for $\beta \geq 1$ and $\alpha \beta \geq 1$.

By (4.6) or by a straightforward computation, it is easy to obtain that

$$\lim_{u \rightarrow 0^+} \left[\alpha \beta \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} - \frac{u}{1 - e^{-u}} \right] = \alpha \beta - 1. \quad (4.9)$$

So $\alpha \beta \geq 1$ is a necessary condition for the function $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$.

When $\beta < 1$, the complete monotonicity of $h_{\alpha, \beta}(t)$ is equivalent to the inequality

$$\alpha \beta \geq \frac{u}{1 - e^{-u}} \frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})}$$

for all $u \in (0, \infty)$. It is not difficult to see that the function $\frac{u}{1 - e^{-u}}$ is increasing on $(0, \infty)$, with

$$\lim_{u \rightarrow 0^+} \frac{u}{1 - e^{-u}} = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{u}{1 - e^{-u}} = \infty,$$

and that the function $\frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})}$ is decreasing on $(0, \infty)$ for any given number $\beta > 0$, with

$$\lim_{u \rightarrow 0^+} \frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})} = 1 \quad \text{and} \quad \lim_{u \rightarrow \infty} \frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u})} = 0.$$

On the other hand, the function $F_\beta(u)$ may be rearranged as

$$F_\beta(u) = \frac{e^u}{e^u - 1} \frac{u\sqrt{\beta u}}{\sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} (\sqrt{\beta u})^{2k+1}} = \frac{e^u}{e^u - 1} \frac{u}{\sum_{k=0}^{\infty} \frac{\beta^k}{k!(k+1)!} u^k}$$

and it tends to 0 as $u \rightarrow \infty$. Accordingly, the condition (4.1) and the limits in (4.3) are obtained. The proof of Theorem 4.1 is complete. \square

5. Monotonicity and unimodality of $F_\beta(u)$

In this section, we will find monotonicity, unimodality, and convexity of the function $F_\beta(u)$. Some of these properties will be used in Theorem 6.1 below to recover necessary and sufficient conditions for $h_{\alpha, \beta}(t)$ to be completely monotonic on $(0, \infty)$.

Theorem 5.1. *When $\beta \geq 1$, the function $F_\beta(u)$ defined by (4.2) is decreasing on $(0, \infty)$; when $0 < \beta < 1$, it is unimodal and its reciprocal $\frac{1}{F_\beta(u)}$ is convex on $(0, \infty)$.*

Proof. For simplicity, we consider the reciprocal

$$\frac{1}{F_\beta(u)} = \frac{1 - e^{-u}}{u} \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} = \frac{1}{e^u} \frac{e^u - 1}{u} \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} \triangleq G_\beta(u). \quad (5.1)$$

By (1.13) and the power series expansion of e^u , we have

$$\frac{e^u - 1}{u} \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} = \sum_{n=0}^{\infty} \frac{u^n}{(n+1)!} \sum_{n=0}^{\infty} \frac{\beta^n}{n!(n+1)!} u^n = \sum_{n=0}^{\infty} a_n u^n,$$

where

$$a_n = \sum_{k=0}^n \frac{\beta^k}{k!(k+1)!(n+1-k)!} = \frac{1}{(n+1)!} \sum_{k=0}^n \binom{n+1}{k} \frac{\beta^k}{(k+1)!}.$$

Hence,

$$G_\beta(u) = \frac{\sum_{n=0}^{\infty} a_n u^n}{\sum_{n=0}^{\infty} u^n / n!}.$$

When $\beta \geq 1$, let $c_n = n!a_n$ for $n \in \{0\} \cup \mathbb{N}$. It is clear that $c_0 = 1$, $c_1 = \frac{1+\beta}{2}$, and $c_2 = \frac{1}{3} + \frac{\beta}{2} + \frac{\beta^2}{6}$ satisfy $c_0 \leq c_1 \leq c_2$. For $n \geq 2$,

$$\begin{aligned} c_{n+1} - c_n &= \frac{1}{n+2} \sum_{k=0}^{n+1} \binom{n+2}{k} \frac{\beta^k}{(k+1)!} - \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k} \frac{\beta^k}{(k+1)!} \\ &= \frac{\beta^{n+1}}{(n+2)!} + \sum_{k=2}^n \binom{n}{k-2} \frac{\beta^k}{k(k+1)!} - \frac{1}{(n+2)(n+1)} \\ &\geq \frac{\beta^2}{12} - \frac{1}{(n+2)(n+1)} \\ &\geq 0. \end{aligned}$$

In other words, when $\beta \geq 1$, the sequence $c_n = n!a_n$ is increasing. From this and Lemma 2.3, it follows that, when $\beta \geq 1$, the function $G_\beta(u)$ is increasing on $(0, \infty)$. Equivalently, when $\beta \geq 1$, the function $F_\beta(u)$ is decreasing on $(0, \infty)$.

A direct computation yields

$$G'_\beta(u) = \frac{1}{e^u} \left\{ \left[\frac{e^u - 1}{u} \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} \right]' - \frac{e^u - 1}{u} \frac{I_1(2\sqrt{\beta u})}{\sqrt{\beta u}} \right\} = \frac{\sum_{n=0}^{\infty} b_n u^n}{\sum_{n=0}^{\infty} u^n / n!},$$

where $b_n = (n+1)a_{n+1} - a_n$ satisfy

$$b_0 = a_1 - a_0 = \frac{\beta - 1}{2}, \quad b_1 = 2a_2 - a_1 = \frac{\beta^2 - 1}{6},$$

and for $n \geq 2$

$$\begin{aligned} b_n &= \frac{1}{n!} \left[\sum_{k=0}^{n+1} \frac{1}{n+2} \binom{n+2}{k} \frac{\beta^k}{(k+1)!} - \sum_{k=0}^n \frac{1}{n+1} \binom{n+1}{k} \frac{\beta^k}{(k+1)!} \right] \\ &= \frac{1}{n!} \left[\frac{\beta^{n+1}}{(n+2)!} - \frac{1}{(n+1)(n+2)} + \sum_{k=2}^n \binom{n}{k-2} \frac{\beta^k}{k(k+1)!} \right]. \end{aligned}$$

When $0 < \beta < 1$, let $C_n = n!b_n$. Then

$$C_0 = \frac{\beta - 1}{2} < C_1 = \frac{\beta^2 - 1}{6}$$

and for $n \geq 1$

$$C_{n+1} - C_n = \frac{\beta^{n+1}}{(n+2)!} \left(\frac{\beta}{n+3} - 1 + \frac{n}{2} \right) + \frac{2}{(n+1)(n+2)(n+3)} + \sum_{k=3}^n \binom{n}{k-3} \frac{\beta^k}{k(k+1)!}.$$

Therefore,

$$C_2 - C_1 = \frac{\beta^2(\beta-2)}{24} + \frac{1}{12} > 0$$

and for $n \geq 2$

$$C_{n+1} - C_n > \frac{\beta^{n+1}}{(n+2)!} \frac{\beta}{n+3} > 0.$$

Consequently, when $0 < \beta < 1$, the sequence $C_n = n!b_n$ is strictly increasing. By Lemma 2.3 again, it follows that $G'_\beta(u)$ is strictly increasing, and so $G_\beta(u) = \frac{1}{F_\beta(u)}$ is convex, on $(0, \infty)$ for $0 < \beta < 1$. Since

$$b_n \geq -\frac{1}{(n+2)!} + \frac{\beta^3}{72(n-1)!}$$

for $n \geq 3$, we get for such n

$$\begin{aligned} \sum_{n=0}^{\infty} b_n u^n &\geq b_0 + b_1 u + b_2 u^2 - \sum_{n=3}^{\infty} \frac{u^n}{(n+2)!} + \frac{\beta^3}{72} \sum_{n=3}^{\infty} \frac{u^n}{(n-1)!} \\ &> b_0 + b_1 u + b_2 u^2 - \frac{e^u}{u^2} + \frac{\beta^3 u}{72} (e^u - 1 - u), \end{aligned}$$

thus

$$G'_\beta(u) > \frac{b_0 + b_1 u + b_2 u^2}{e^u} - \frac{1}{u^2} + \frac{\beta^3 u}{72} \left(1 - \frac{1+u}{e^u} \right) \rightarrow \infty$$

as $u \rightarrow \infty$. On the other hand, when $0 < \beta < 1$,

$$G'_\beta(0) = b_0 = \frac{\beta-1}{2} < 0.$$

As a result, when $0 < \beta < 1$, by the above proved monotonicity of $G'_\beta(u)$, the derivative $G'_\beta(u)$ has a unique zero, and so the positive function $G_\beta(u)$ has a unique minimum, on $(0, \infty)$. In other words, when $0 < \beta < 1$, the positive function $F_\beta(u)$ has a unique maximum on $(0, \infty)$. Theorem 5.1 is proved. \square

6. Recovery of complete monotonicity of $h_{\alpha,\beta}(t)$

In this section, with the aid of Theorem 5.1, we will simply recover necessary and sufficient conditions in Theorem 4.1.

Theorem 6.1. *For $\alpha, \beta > 0$, the function $h_{\alpha,\beta}(t)$ defined by (1.12) is completely monotonic on $(0, \infty)$ if and only if the condition (4.1) holds true.*

Proof. By Lemma 2.2 and the integral representation (4.4), we easily see that the condition (4.1) is necessary and sufficient. Theorem 6.1 is thus proved. \square

Corollary 6.1. *When $\beta \geq 1$, the function $h_{\alpha,\beta}(t)$ defined by (1.12) is completely monotonic on $(0, \infty)$ if and only if $\alpha \geq 1$.*

Proof. This follows from monotonicity of $F_\beta(u)$ established in Theorem 5.1, the first limit in (4.3), and Theorem 6.1. \square

7. Inequalities for the modified Bessel function

By observing the proof of Theorem 4.1 or by Theorem 5.1, we conclude the following inequalities which give new lower bounds for I_1 .

Theorem 7.1. *The inequalities*

$$\alpha I_1(x) > \frac{(x/2)^3}{1 - e^{-(x/2)^2}} \quad (7.1)$$

and

$$I_1(x) \geq \frac{\frac{1}{\beta} \left(\frac{x}{2}\right)^3}{1 - \exp\left[-\frac{1}{\beta} \left(\frac{x}{2}\right)^2\right]} \quad (7.2)$$

are valid on $(0, \infty)$ if and only if $\alpha \geq 1$ and $\beta \geq 1$.

Proof. The inequality (7.2) follows from replacing u by $\frac{1}{\beta} \left(\frac{x}{2}\right)^2$ in (4.7) and further simplifying.

The inequality (7.1) follows from combining the second part of Theorem 3.1 with the formula (4.4) for $h_{\alpha,1} = h_\alpha$ and Lemma 2.2.

The inequality (7.2) can also be deduced from Theorem 5.1. The proof of Theorem 7.1 is complete. \square

8. Remarks

In this section, we would like to compare the inequalities (7.1) and (7.2) with other known ones.

Remark 8.1. When $\beta = 1$, the inequality (7.2) is quite good for $0 < x < 2$, but for large x the right-hand side of (7.2) increases like $\frac{x^3}{8}$, while $I_1(x)$ increases faster than any power of x .

Remark 8.2. We refer to the following known inequalities

$$I_1(x) > \frac{1 - x/2}{2 + x} x e^x, \quad (8.1)$$

cf. [7, (6.23)], and

$$I_1(x) > \frac{x}{2} \left(1 + \frac{x^2}{j_{1,1}^2}\right)^{j_{1,1}^2/8}, \quad (8.2)$$

cf. [5, (3.20)], for $x > 0$, where $j_{1,1} = 3.83\dots$ is the first zero of J_1 and

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k! \Gamma(\nu + k + 1)} \quad (8.3)$$

for $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ is the Bessel function of the first kind. Looking at graphs of the lower bounds in (7.2), (8.1), and (8.2) reveals that the lower bound in (7.2) for $\beta = 1$ is the largest of the three for $0 < x < 5$.

Remark 8.3. The inequality (7.1) and the second part of Theorem 3.1 are equivalent to each other, so are the inequality (7.2) and the necessary and sufficient condition $\beta \geq 1$ in Theorem 4.1 for $h_{1,\beta}(t)$ to be completely monotonic on $(0, \infty)$.

Acknowledgements

The authors would like to thank Árpád Baricz for providing the references [2, 5, 7] and commenting on the inequality (7.1) in Theorem 7.1.

References

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.
- [2] A. Baricz, *Bounds for modified Bessel functions of the first and second kinds*, Proc. Edinburgh Math. Soc. **53** (2010), 575–599; Available online at <http://dx.doi.org/10.1017/S0013091508001016>.
- [3] M. Biernacki and J. Krzyż, *On the monotony of certain functionals in the theory of analytic functions*, Annales Univ. Mariae Curie-Skłodowska A **9** (1955), 135–147.
- [4] B.-N. Guo and F. Qi, *Refinements of lower bounds for polygamma functions*, Proc. Amer. Math. Soc. (2012), in press; Available online at <http://dx.doi.org/10.1090/S0002-9939-2012-11387-5>.
- [5] E. K. Ifantis and P. D. Siafarikas, *Inequalities involving Bessel and modified Bessel functions*, J. Math. Anal. Appl. **147** (1990), no. 1, 214–227; Available online at [http://dx.doi.org/10.1016/0022-247X\(90\)90394-U](http://dx.doi.org/10.1016/0022-247X(90)90394-U).
- [6] S. Koumandos and H. L. Pedersen, *On the asymptotic expansion of the logarithm of Barnes triple Gamma function*, Math. Scand. **105** (2009), no. 2, 287–306.
- [7] Y. L. Luke, *Inequalities for generalized hypergeometric functions*, J. Approx. Theory **5** (1972), no. 1, 41–65; Available online at [http://dx.doi.org/10.1016/0021-9045\(72\)90028-7](http://dx.doi.org/10.1016/0021-9045(72)90028-7).
- [8] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
- [9] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika **44** (1997), 278–301.
- [10] F. Qi and B.-N. Guo, *Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic*, Adv. Appl. Math. **44** (2010), no. 1, 71–83; Available online at <http://dx.doi.org/10.1016/j.aam.2009.03.003>.
- [11] F. Qi and B.-N. Guo, *Refinements of lower bounds for polygamma functions*, available online at <http://arxiv.org/abs/0903.1966>.

- [12] F. Qi and A. Sofo, *An alternative and united proof of a double inequality for bounding the arithmetic-geometric mean*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **71** (2009), no. 3, 69–76.
- [13] F. Qi and S.-H. Wang, *Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions*, available online at <http://arxiv.org/abs/1210.2012>.
- [14] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, de Gruyter Studies in Mathematics **37**, De Gruyter, Berlin, Germany, 2010.
- [15] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
- [16] X.-J. Zhang, F. Qi, and W.-H. Li, *Properties of three functions relating to the exponential function and the existence of partitions of unity*, Int. J. Open Probl. Comput. Sci. Math. **5** (2012), no. 3, 122–127.

Feng Qi*

College of Mathematics

Inner Mongolia University for Nationalities

Tongliao City

Inner Mongolia Autonomous Region, 028043

China

e-mail: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: <http://qifeng618.wordpress.com>

Christian Berg

Department of Mathematics

University of Copenhagen

Universitetsparken 5, DK-2100

Denmark

e-mail: berg@math.ku.dk