ON A CONJECTURE OF CLARK AND ISMAIL

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Abstract. Let $\Phi_m(x) = -x^m\psi^{(m)}(x)$, where ψ denotes the logarithmic derivative of Euler's gamma function. Clark and Ismail prove in a recently published article that if $m \in \{1, 2, ..., 16\}$, then $\Phi_m^{(m)}$ is completely monotonic on $(0, \infty)$, and they conjecture that this is true for all natural numbers m. We disprove this conjecture by showing that there exists an integer m_0 such that for all $m \geq m_0$ the function $\Phi_m^{(m)}$ is not completely monotonic on $(0, \infty)$.

Keywords. Completely and absolutely monotonic functions, polygamma functions, infinite series, inequalities.

1. Introduction

Let $f : (a, b) \to \mathbf{R}$ be a function, which has derivatives of all orders. Then, f is called absolutely monotonic, if

$$f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in (a, b) \quad \text{and} \quad n = 0, 1, 2, ... \tag{1.1}$$

And, f is said to be completely monotonic, if

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x \in (a, b) \quad \text{and} \quad n = 0, 1, 2, ... \tag{1.2}$$

The connection between these classes of functions is obvious: f is completely monotonic on (a, b) if and only if $x \mapsto f(-x)$ is absolutely monotonic on $(-b, -a)$.

Absolutely and completely monotonic functions have remarkable applications. In view of their importance in probability theory, numerical analysis, potential theory, and other fields these functions have been studied by many authors. We refer to [14, Chapter IV], where the basic properties of absolutely and completely monotonic functions are collected. Interesting historical facts on these and related classes of functions can be found in [4] and [13, Section 82]. A detailed list of references on completely monotonic functions is given in in [2] and [3].

In this paper, we are concerned with functions, which are defined on $(0, \infty)$. Therefore, throughout, 'f is absolutely (resp. completely) monotonic' means that (1.1) (resp. (1.2)) holds with $a = 0$ and $b = \infty$. An important characterization of completely monotonic functions was given by Bernstein, who proved that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. See [14, p. 161].

The logarithmic derivative of Euler's gamma function, $\psi = \Gamma'/\Gamma$, is known in the literature as digamma or psi function. The derivatives ψ', ψ'', ... are called polygamma functions. The following integral and series representations are valid for $x > 0$ and $n \in \mathbf{N}$:

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}} dt = (-1)^{n+1} n! \sum_{k=0}^\infty \frac{1}{(x + k)^{n+1}}. \tag{1.3}$$

See [1, p. 260]. We note that (1.3) implies that for all $n \in \mathbf{N}$ the function $|\psi^{(n)}|$ is completely monotonic.

In a recently published paper, Clark and Ismail [5] introduce the functions

$$G_m(x) = x^m \psi(x) \quad \text{and} \quad \Phi_m(x) = -x^m \psi^{(m)}(x).$$

They prove that $G_m^{(m+1)}$ is completely monotonic for $m = 1, 2, ...$ and that $\Phi_m^{(m)}$ is completely monotonic for $m = 1, 2, ..., 16$. Clark and Ismail conjecture that $\Phi_m^{(m)}$ is completely monotonic for all natural numbers m. It is the aim of this paper to disprove this conjecture. Indeed, in Section 3 we establish:

Theorem 1.1. *There exists an integer m_0 such that for all $m \geq m_0$ the function $\Phi_m^{(m)}$ is not completely monotonic.*

It might be surprising that in the proof of Theorem 1.1 a key role is played by the function $H(x) = \sum_{k=1}^{\infty}(1/k)\sin(x/k)$, which was studied by Hardy and Littlewood in 1936. In the next section we investigate the behaviour of $H(x)$ for large x.

2. A function of Hardy and Littlewood

Some problems on Lambert summability led Hardy and Littlewood [8] to the function

$$H(x) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(\frac{x}{k}), \quad x \in \mathbf{C}. \tag{2.1}$$

It is an odd entire function, and inserting the power series for the sine function we get the representation

$$H(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\zeta(2k)}{(2k-1)!} x^{2k-1}, \tag{2.2}$$

showing that H is of exponential type:

$$|H(x)| \leq \frac{\pi^2}{6} \sinh(|x|) \leq \frac{\pi^2}{12} \exp(|x|), \quad x \in \mathbf{C}. \tag{2.3}$$

Using that $|\sin(x)| \leq |x|$ for $x \in \mathbf{R}$ we obtain the elementary bound

$$|H(x)| \leq \frac{\pi^2}{6}|x|, \quad x \in \mathbf{R},$$

where the constant factor $\pi^2/6$ is best possible. Since the power series in (2.2) is related to the sine series one might expect that H is bounded on the real axis. This is not so as it was shown by Hardy and Littlewood using a number theoretic approach. They constructed a sequence $y_k \to \infty$ such that $H(y_k) > C\big(\log\log y_k\big)^{1/2}$. Flett [6] continued the study of H and established that for every $\epsilon > 0$:

$$H(x) = O((\log x)^{3/4}(\log\log x)^{\epsilon+1/2}), \quad x \to \infty.$$

A simple calculation gives $(-1)^k H'(k\pi) > 0$ for $k = 0, 1, ...$ This implies that H attains a local maximum in $[2n\pi, (2n+1)\pi]$ and a local minimum in $[(2n+1)\pi, (2n+2)\pi]$, $n = 0, 1, ...$

Our proof of Theorem 1.1 depends on the fact that H can attain arbitrary large negative values. Such a result is not mentioned in [8], [6], but is important for the main result of this paper, since we prove that the conjecture of Clark and Ismail is equivalent to the inequality $H(x) \geq -\pi/2$ for $x > 0$. A computer plot reveals that $H(x) > -0.5$ for $0 < x < 1000$; see [7]. We also remark that $H(x_0) = 0$, where $x_0 = 48.2...$ and that $H(x) > 0$ for $0 < x < x_0$.

We show below that the proof in [8] can be modified to show that there exists a sequence $x_k \to \infty$ such that $H(x_k) < -C\big(\log\log x_k\big)^{1/2}$. The result of Hardy and Littlewood and the corresponding result for large negative values can be expressed by the statement

$$H(x) = \Omega_{\pm}((\log\log x)^{1/2}),\ x \to \infty.$$

That such an extension is possible was observed by Pétermann [10, p. 73]. In this and subsequent work he proved $\Omega_{\pm}$-estimates for classes of functions including the Hardy-Littlewood function; see [11] and [12].

For $x > 0$ we define

$$H_*(x) = \sum_{k=1}^{[x]} \frac{1}{k}\sin(\frac{x}{k}) \quad \text{and} \quad H^*(x) = \sum_{k=[x]+1}^{\infty} \frac{1}{k}\sin(\frac{x}{k}),$$

where $[x]$ denotes the greatest integer not greater than x. Let $x \geq 2$. Since $y \mapsto (1/y)\sin(x/y)$ is decreasing on $[[x], \infty)$, we get

$$\int_{[x]+1}^{\infty} \frac{1}{y}\sin(\frac{x}{y})dy \leq H^*(x) \leq \int_{[x]}^{\infty} \frac{1}{y}\sin(\frac{x}{y})dy. \tag{2.4}$$

The sine integral is defined by $\mathrm{Si}(x) = \int_0^x \sin(t)/t dt$. Substituting $t = x/y$ we conclude from (2.4) that

$$0.65... = \mathrm{Si}(2/3) \leq \mathrm{Si}(x/([x]+1)) \leq H^*(x) \leq \mathrm{Si}(x/[x]) \leq \mathrm{Si}(3/2) = 1.32....$$

In particular, we obtain $\lim_{x\to\infty} H^*(x) = \mathrm{Si}(1) = 0.94....$

Let M be the set of natural numbers q such that all prime factors of q are of the form $4n+1$, that is,

$$M = \{1, 5, 13, 17, 25, 29, 37, 41, 45, ...\},$$

and let $N(n) = \#\{q \in M | q \leq n\}$, where $n \geq 0$. In [9] Landau proved that

$$B(x) \sim bx/\sqrt{\log x}, \quad x \to \infty,$$

where $B(x)$ denotes the number of integers $n \leq x$, which can be written as the sum of two squares, and $b > 0$ is an explicit constant. From this we can deduce that there exists a positive constant C^* such that

$$N(n) \geq C^* n/\sqrt{\log n}, \quad n > 1, \tag{2.5}$$

as stated in [8].

For $k \in \mathbf{N}$ we define

$$K = K(k) = \prod_{\substack{q=1\\ q\in M}}^{4k+1} q \quad \text{and} \quad x_j = (4j+3)K\frac{\pi}{2}, \quad \text{where} \quad j = 1, ..., K. \tag{2.6}$$

Then we have:

Theorem 2.1. *For every $k \in \mathbf{N}$ there exists an integer $j_k \in \{1, ..., K(k)\}$ such that*

$$H_*(x_{j_k}) \leq a - b\sqrt{\log k},$$

where $a, b > 0$ are real constants independent of k.

Proof. Let $k \in \mathbf{N}$ and $j \in \{1, ..., K\}$. We write $H_*(x_j) = A_j + B_j$ with

$$A_j = \sum_{n=1}^{K} \frac{1}{n} \sin(\frac{x_j}{n}) \quad \text{and} \quad B_j = \sum_{n=K+1}^{[x_j]} \frac{1}{n} \sin(\frac{x_j}{n})$$

and claim that

$$|B_j| \leq C_1, \tag{2.7}$$

where C_1 is independent of k. Applying the mean value theorem we get

$$B_j - \int_{K+1}^{[x_j]+1} \frac{1}{y} \sin(\frac{x_j}{y}) dy = \sum_{n=K+1}^{[x_j]} \int_n^{n+1} \Big(\frac{1}{n} \sin(\frac{x_j}{n}) - \frac{1}{y} \sin(\frac{x_j}{y}) \Big) dy$$

$$= \sum_{n=K+1}^{[x_j]} \int_n^{n+1} (n-y) \Big(-\frac{1}{\xi^2} \sin(\frac{x_j}{\xi}) - \frac{x_j}{\xi^3} \cos(\frac{x_j}{\xi}) \Big) dy,$$

where $\xi = \xi_n(y) \in (n, n+1)$. Using $0 = \mathrm{Si}(0) \leq \mathrm{Si}(x) \leq \mathrm{Si}(\pi) = 1.85...$ for $x \geq 0$, we obtain

$$|B_j| \leq \Big| \int_{K+1}^{[x_j]+1} \frac{1}{y} \sin(\frac{x_j}{y}) dy \Big| + \sum_{n=K+1}^{[x_j]} \Big(\frac{1}{n^2} + \frac{x_j}{n^3} \Big)$$

$$\leq \Big| \mathrm{Si}\Big(\frac{x_j}{K+1}\Big) - \mathrm{Si}\Big(\frac{x_j}{[x_j]+1}\Big) \Big| + \frac{\pi^2}{6} + x_j \sum_{n=K+1}^{\infty} \frac{1}{n^3}$$

$$\leq \mathrm{Si}(\pi) + \frac{\pi^2}{6} + \frac{x_j}{2K^2} \leq \mathrm{Si}(\pi) + \frac{\pi^2}{6} + \frac{(4K+3)\pi}{4K} \leq C_1.$$

Next, we set $A_j = a_j + a_j^*$ with

$$a_j = \sum_{\substack{n=1 \\ n|K}}^{K} \frac{1}{n} \sin(\frac{x_j}{n}) \quad \text{and} \quad a_j^* = \sum_{\substack{n=1 \\ n\nmid K}}^{K} \frac{1}{n} \sin(\frac{x_j}{n}).$$

If $n|K$, then $(4j+3)K/n$ is of the form $4p+3$ ($p \in \mathbf{Z}$), so that $\sin(x_j/n) = -1$. This implies

$$a_1 = ... = a_K = -\sum_{\substack{n=1 \\ n|K}}^{K} \frac{1}{n}.$$

Further,

$$-a_1 \geq \sum_{\substack{q=1 \\ q\in M}}^{4k+1} \frac{1}{q} = \sum_{n=1}^{4k+1} \frac{N(n) - N(n-1)}{n} \geq \sum_{n=1}^{4k} \frac{N(n)}{n(n+1)}. \tag{2.8}$$

Applying (2.5) we conclude from (2.8):

$$-a_1 \geq \frac{1}{2} + C^* \sum_{n=2}^{4k} \frac{1}{(n+1)\sqrt{\log n}}$$

$$\geq \frac{1}{2} + C^* \int_3^{4k+2} \frac{dx}{x\sqrt{\log x}} = \frac{1}{2} + 2C^* \big(\sqrt{\log(4k+2)} - \sqrt{\log 3} \big).$$

This shows that there exists a constant $C_2 > 0$ independent of k such that

$$a_1 \leq -C_2 \sqrt{\log k}. \tag{2.9}$$

Setting $x = K\pi/(2n)$ we have

$$2\sin(2x)\sin((4j+3)x) = \cos((4j+1)x) - \cos((4j+5)x).$$

Summing yields

$$2\sin(2x)\sum_{j=1}^{K}\sin(\frac{x_j}{n}) = \cos(5x) - \cos((4K+5)x).$$

Let $1 < n \leq K$ and $n \nmid K$. Then

$$\Big|\sum_{j=1}^{K}\sin(\frac{x_j}{n})\Big| \leq \frac{1}{|\sin(2x)|} = \frac{1}{|\sin(K\pi/n)|}. \tag{2.10}$$

We can write $K = nd + r$ with $1 \leq r < n$. This leads to

$$|\sin(K\pi/n)| = \sin(r\pi/n) \geq \sin(\pi/n) \geq 2/n. \tag{2.11}$$

Applying (2.10) and (2.11) we get

$$\sum_{j=1}^{K} a_j^* = \sum_{\substack{n=1\\ n\nmid K}}^{K}\frac{1}{n}\sum_{j=1}^{K}\sin(\frac{x_j}{n}) \leq \sum_{\substack{n=1\\ n\nmid K}}^{K}\frac{1}{n}\cdot\frac{n}{2} \leq \frac{K}{2}. \tag{2.12}$$

From (2.9) and (2.12) we obtain

$$\frac{1}{K}\sum_{j=1}^{K} A_j = a_1 + \frac{1}{K}\sum_{j=1}^{K} a_j^* \leq a_1 + \frac{1}{2} \leq \frac{1}{2} - C_2\sqrt{\log k}.$$

This reveals that for at least one $j_k \in \{1, ..., K\}$ we have $A_{j_k} \leq (1/2) - C_2\sqrt{\log k}$. This result combined with (2.7) gives that there exists a constant $C_3 > 0$ independent of k such that $H_*(x_{j_k}) = A_{j_k} + B_{j_k} \leq C_3 - C_2\sqrt{\log k}$. The proof of Theorem 2.1 is complete. □

Let (j_k) be the sequence given in Theorem 2.1 and let $x_{j_k} = (4j_k + 3)K(k)\pi/2$. From (2.6) we get

$$x_{j_k} \leq (4K+3)K\frac{\pi}{2} \quad \text{and} \quad K < (4k+1)^{4k+1},$$

which implies that there exists a number $k_0 \in \mathbf{N}$ such that for all $k \geq k_0$:

$$\log\log x_{j_k} \leq 2\log k.$$

Since H^* is bounded on $[0, \infty)$, we get for $k \geq k_0$:

$$H(x_{j_k}) = H_*(x_{j_k}) + H^*(x_{j_k}) \leq -C\sqrt{\log\log x_{j_k}}, \tag{2.13}$$

where $C > 0$ is a constant independent of k and $\lim_{k\to\infty} x_{j_k} = \infty$.

3. Proof of Theorem 1.1

Let $\Phi_m(x) = -x^m\psi^{(m)}(x)$. In order to establish Theorem 1.1 we make use of the following integral formula, which is given in [5]:

$$\Phi_m^{(m)}(x) = \int_0^\infty e^{-xt}t^m f_m(t)dt,$$

where

$$f_m(x) = \frac{d^m}{dx^m}\frac{x^m}{1-e^{-x}}. \tag{3.1}$$

We show that there exists an integer m_0 such that for all $m \geq m_0$ the function f_m attains negative values on $(0, \infty)$, although $f_m(x) \geq 0$ for $x \geq 2\log 2$, $m = 1, 2, \ldots$. See the Appendix.

The generating function for the Bernoulli numbers yields for $|x| < 2\pi$:

$$\frac{x}{1-e^{-x}} = \frac{x}{e^x - 1} + x = 1 + \frac{x}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} x^k.$$

Using the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$ we obtain

$$f_m(x) = \frac{m!}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!} (k)_m x^{k-1},$$

and since $B_{2k+1} = 0$ for $k \geq 1$, we get

$$f_m(x) = \frac{m!}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (2k)_m x^{2k-1}. \tag{3.2}$$

Thus, for $|x| < 2m\pi$:

$$\frac{1}{m!} f_m(\frac{x}{m}) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{(2k)_m}{m^{2k-1} m!} x^{2k-1}. \tag{3.3}$$

Let $k \geq 1$ be a fixed integer. Since

$$\lim_{m\to\infty} \frac{(2k)_m}{m^{2k-1} m!} = \frac{1}{(2k-1)!}$$

we conclude that formally the expression on the right-hand side of (3.3) converges to

$$\frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{x^{2k-1}}{(2k-1)!} = s(x), \quad \text{say.}$$

To give a rigorous proof of

$$\lim_{m\to\infty} \frac{1}{m!} f_m(\frac{x}{m}) = s(x), \tag{3.4}$$

we fix $A > 0$ and prove the uniform convergence for complex numbers x with $|x| \leq A$. Let $N \in \mathbf{N}$ such that $A < 2N\pi$ and let $|x| \leq A$. Then we get for $m \geq N$ that $|x/m| \leq A/N < 2\pi$. We have for $k \in \mathbf{N}$:

$$\frac{(m+1)_{2k-1}}{m^{2k-1}} = \prod_{j=1}^{2k-1} \Big(1 + \frac{j}{m}\Big) \leq \prod_{j=1}^{2k-1} \Big(1 + \frac{j}{N}\Big) = \frac{(N+2k-1)!}{N! N^{2k-1}}.$$

This leads to

$$\frac{|B_{2k}|}{(2k)!} \frac{(m+1)_{2k-1}}{m^{2k-1}} \frac{|x|^{2k-1}}{(2k-1)!} \leq \frac{|B_{2k}|}{(2k)!} \frac{(N+2k-1)!}{N!(2k-1)!} \Big(\frac{A}{N}\Big)^{2k-1} = c_k, \quad \text{say.}$$

Using

$$\lim_{k\to\infty} \sqrt[2k]{\frac{|B_{2k}|}{(2k)!}} = \frac{1}{2\pi} \quad \text{and} \quad \lim_{k\to\infty} \sqrt[2k]{\frac{(2k+N-1)!}{(2k-1)!}} = 1$$

we obtain

$$\lim_{k\to\infty} \sqrt[2k]{c_k} = \frac{A}{2N\pi} < 1,$$

which implies that $\sum_{k=1}^{\infty} c_k$ is convergent. Let $\epsilon > 0$. We choose $k_0 \in \mathbf{N}$ such that

$$\sum_{k=k_0}^{\infty} c_k < \epsilon \quad \text{and} \quad \sum_{k=k_0}^{\infty} \frac{|B_{2k}|}{(2k)!} \frac{A^{2k-1}}{(2k-1)!} < \epsilon. \tag{3.5}$$

Then we get

$$\Big| \frac{1}{m!} f(\frac{x}{m}) - s(x) \Big| \le \Big(\sum_{k=1}^{k_0-1} + \sum_{k=k_0}^{\infty} \Big) \frac{|B_{2k}|}{(2k)!} \Big| \frac{(m+1)_{2k-1}}{m^{2k-1}} - 1 \Big| \frac{|x|^{2k-1}}{(2k-1)!}. \tag{3.6}$$

The second sum in (3.6) can be majorized by the two series given in (3.5) with sum less than 2ϵ. The first sum converges uniformly to zero, so it is less than ϵ for sufficiently large m.

Let H be the function defined in (2.1). Inserting Euler's formula $\zeta(2k) = (-1)^{k-1} B_{2k} 2^{2k-1} \pi^{2k}/(2k)!$ ($k \in \mathbf{N}$) in the expression (2.2), we obtain the identity

$$\frac{1}{2} + \frac{1}{\pi} H(\frac{x}{2\pi}) = s(x), \tag{3.7}$$

which shows that $H(x)$ is bounded below by $-\pi/2$ for positive x if and only if $s(x) \ge 0$ for $x > 0$. It follows from (2.13) and (3.7) that s attains negative values on $(0, \infty)$. We further claim that the conjecture of Clark and Ismail is equivalent to $s(x) \ge 0$ for $x > 0$. In fact, if $\Phi_m^{(m)}$ is completely monotonic for all m, then f_m is non-negative on $(0, \infty)$ for all m, so that (3.4) yields $s(x) \ge 0$ for $x > 0$. Conversely, if $s(x) \ge 0$ for $x > 0$, then the formulas

$$f_m(x) = \int_0^{\infty} e^{-t} t^m s(xt) dt, \quad |x| < 2\pi, \tag{3.8}$$

and

$$f_m(x) \ge 0, \quad x \ge 2 \log 2, \tag{3.9}$$

which are proved in the Appendix, imply that f_m is non-negative on $(0, \infty)$, so that $\Phi_m^{(m)}$ is completely monotonic.

4. Remarks and open problems

(1) Computer experiments suggest that f_m is non-negative on the interval $(0, 2 \log 2)$ for $m = 17, ..., 40$. We conjecture that the smallest positive integer m^* such that f_{m^*} attains negative values is 'rather large'. In particular, it remains an open problem to determine all positive integers m such that $\Phi_m^{(m)}$ is completely monotonic.

(2) We define for $\alpha \in \mathbf{R}$ and $m \in \mathbf{N}$:

$$\Delta_{\alpha,m}(x) = x^{\alpha} |\psi^{(m)}(x)|, \quad x > 0. \tag{4.1}$$

Since the product of completely monotonic functions is also completely monotonic, we obtain: if $\alpha \le 0$ and $m \in \mathbf{N}$, then $\Delta_{\alpha,m}$ is completely monotonic. Next, let $\alpha > 0$ and let $\Delta_{\alpha,m}$ be completely monotonic. Then we get for $x > 0$:

$$x^{m+1-\alpha} \Delta'_{\alpha,m}(x) = \alpha x^m |\psi^{(m)}(x)| - x^{m+1} |\psi^{(m+1)}(x)| \le 0.$$

We have

$$\lim_{x \to \infty} x^k |\psi^{(k)}(x)| = (k-1)!, \quad k \in \mathbf{N}, \tag{4.2}$$

(see [1, p. 260]), so that we obtain $(m-1)!(\alpha-m)\le 0$ or $\alpha\le m$. However, Theorem 1.1 and the identity

$$(-1)^n(\Phi_m^{(m)})^{(n)}=(-1)^{n+m}\Delta_{m,m}^{(m+n)},\quad n\ge 0,\ m\ge 1,$$

reveal that for large m the inequality $\alpha\le m$ is not a sufficient condition for the complete monotonicity of $\Delta_{\alpha,m}$. It is an open problem to determine all $(\alpha,m)\in\mathbf{R}^+\times\mathbf{N}$ such that $\Delta_{\alpha,m}$ is completely monotonic.

(3) For all real numbers α and positive integers m we have: the function $\Delta_{\alpha,m}$ (as defined in (4.1)) is *not* absolutely monotonic.

To prove this we assume (for a contradiction) that there exist a real number α and a positive integer m such that $\Delta_{\alpha,m}$ is absolutely monotonic. We set $\Delta=\Delta_{\alpha,m}$. Using

$$\lim_{x\to 0}x^{k+1}|\psi^{(k)}(x)|=k!,\quad k\in\mathbf{N},$$

(see [1, p. 260]), we obtain

$$\lim_{x\to 0}x^{m+2-\alpha}\Delta'(x)=m!(\alpha-m-1)\ge 0. \tag{4.3}$$

Let $p=[\alpha]+1-m$. From (4.3) we conclude that $p>1$. Applying the Leibniz rule we get

$$\Delta^{(p)}(x)=\sum_{j=0}^{p}\binom{p}{j}(-1)^{p-j}(\alpha-j+1)_j x^{\alpha-j}|\psi^{(p+m-j)}(x)|.$$

Let $0\le j\le p$. Then $0\le\alpha-j<p+m-j$, so that (4.2) gives

$$\lim_{x\to\infty}x^{\alpha-j}|\psi^{(p+m-j)}(x)|=0.$$

This implies that $\lim_{x\to\infty}\Delta^{(p)}(x)=0$. By assumption, $\Delta^{(p+1)}(x)\ge 0$ for $x>0$. Hence, $\Delta^{(p)}$ is increasing on $(0,\infty)$, which leads to $\Delta^{(p)}(x)\le 0$ for $x>0$. Thus, $\Delta^{(p)}\equiv 0$ on $(0,\infty)$, so that we obtain $x^\alpha|\psi^{(m)}(x)|=Q(x)$, where Q is a polynomial of degree $r\le p-1$. Differentiation gives

$$\alpha-x\frac{|\psi^{(m+1)}(x)|}{|\psi^{(m)}(x)|}=x\frac{Q'(x)}{Q(x)}. \tag{4.4}$$

Applying (4.2) we conclude from (4.4) that $\alpha=m+r$. Hence,

$$x^m|\psi^{(m)}(x)|=\sum_{j=0}^{r}a_jx^{-j},\quad\text{say.}$$

But this contradicts the (uniquely determined) asymptotic expansion

$$x^m|\psi^{(m)}(x)|\sim(m-1)!+\frac{m!}{2x}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+m-1)!}{(2k)!}x^{-2k},\quad x\to\infty.$$

See [1, p. 260].

(4) Let $G_m(x)=x^m\psi(x)$. In Section 1 we mentioned that $G_m^{(m+1)}$ is completely monotonic for all $m\in\mathbf{N}$. This result can be generalized. Let $m,n\in\mathbf{N}$. The function $G_m^{(n)}$ is completely monotonic if and only if $n>m$ and $m+n$ is odd.

Using the Leibniz rule and the recurrence formula $\psi^{(k)}(x+1)=\psi^{(k)}(x)+(-1)^kk!x^{-k-1}$ we obtain for $1\le n\le m$ and $x>0$:

$$x^{n-m}G_m^{(n)}(x)=(m-n+1)_n\psi(x+1)-(m-n)_n\frac{1}{x}+\sum_{j=0}^{n-1}\binom{n}{j}(m-j+1)_jx^{n-j}\psi^{(n-j)}(x+1).$$

This leads to

$$\lim_{x\to 0} x^{n-m+1} G_m^{(n)}(x) = -(m-n)_n, \quad 1 \le n < m, \tag{4.5}$$

and

$$\lim_{x\to 0} G_m^{(m)}(x) = -\gamma m!, \tag{4.6}$$

where $\gamma = 0.57221...$ denotes Euler's constant. And, if $n > m$, then we get for $N \ge 0$ and $x > 0$:

$$(-1)^N \big(G_m^{(n)}(x)\big)^{(N)} = (-1)^{m+n+1}(N+n)! \sum_{k=1}^{\infty} \frac{k^m}{(x+k)^{N+n+1}}. \tag{4.7}$$

See [5]. From (4.5) and (4.6) we conclude that if $1 \le n \le m$, then $G_m^{(n)}$ is not completely monotonic. And (4.7) implies that if $n > m$, then $G_m^{(n)}$ is completely monotonic if and only if $m + n$ is odd.

5. Appendix

The Laguerre polynomial L_m of degree m is defined by

$$L_m(x) = \frac{e^x}{m!} \frac{d^m}{dx^m}(e^{-x}x^m) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{k!} x^k.$$

We get

$$e^{-kx} L_m(kx) = \frac{1}{m!} \frac{d^m}{dx^m}(e^{-kx}x^m).$$

This yields the following connection to f_m, as given in (3.1):

$$f_m(x) = m! \sum_{k=0}^{\infty} e^{-kx} L_m(kx), \quad x > 0.$$

By an inequality due to Szegö (see [14, p. 168]) we have $|L_m(x)| \le e^{x/2}$ for $x \ge 0$, so that we obtain for $x > 0$:

$$\Big|\sum_{k=1}^{\infty} e^{-kx} L_m(kx)\Big| \le \sum_{k=1}^{\infty} (e^{-x/2})^k = \frac{e^{-x/2}}{1 - e^{-x/2}}.$$

The right-hand side is ≤ 1 if and only if $x \ge 2\log 2$, so that $L_m(0) = 1$ implies that $f_m(x) \ge 0$ for $x \ge 2\log 2$, as claimed in (3.9).

For $|x| < 2\pi$ we have the power series expansion (3.2) for $f_m(x)$, and inserting Euler's formula for $\zeta(2k)$ we obtain

$$f_m(x) = \frac{\Gamma(m+1)}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\zeta(2k)}{(2k-1)!} \Gamma(m+2k) \Big(\frac{x}{2\pi}\Big)^{2k-1}.$$

Using Euler's integral for the gamma function, interchanging sum and integral, and applying (3.7) we get

$$f_m(x) = \int_0^\infty e^{-t} t^m \Big(\frac{1}{2} + \frac{1}{\pi} H\big(\frac{xt}{2\pi}\big)\Big) dt = \int_0^\infty e^{-t} t^m s(xt) dt,$$

which proves (3.8). We remark that the interchanging is allowed by dominated convergence because $|x| < 2\pi$ and inequality (2.3) holds.

An examination of the function f_m shows that $f_m(x)$ starts as an increasing function at $x = 0$ with $f_m(0) = m!/2$, $f_m'(0) = (m+1)!/12$, and it oscillates crossing the line $y = m!$ a number of times. It approaches $m!$ from above or below depending on the parity of m for $x \to \infty$. Computer

experiments suggest that it crosses $m-1$ times. The oscillation close to $x=0$ becomes very wild as m becomes very large. In fact, the oscillation of $H(x)$ and hence of $s(x)$ for large x is reflected in the oscillation of $f_m(x)$ for x close to zero because of (3.4).
Using the power series expansion of s we obtain easily

$$\int_0^\infty e^{-xt}s(t)dt = \frac{1}{x(1-e^{-1/x})} - 1, \quad x > 0.$$

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References

[1] M. Abramowitz and I.A. Stegun (eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1965.
[2] H. Alzer and C. Berg, Some classes of completely monotonic functions, Ann. Acad. Scient. Fennicae **27** (2002), 445-460.
[3] H. Alzer and C. Berg, Some classes of completely monotonic functions, II, submitted to The Ramanujan Journal.
[4] R. P. Boas, Signs of derivatives and analytic behavior, Amer. Math. Monthly **78** (1971), 1085-1093.
[5] W.E. Clark and M.E.H. Ismail, Inequalities involving gamma and psi functions, Anal. Appl. **1** (2003), 129-140.
[6] T.M. Flett, On the function $\sum_{n=1}^{\infty}(1/n)\sin(t/n)$, J. London Math. Soc. **25** (1950), 5-19.
[7] W. Gautschi, The Hardy-Littlewood function: an exercise in slowly convergent series, in preparation.
[8] G.H. Hardy and J.E. Littlewood, Notes on the theory of series (XX): On Lambert series, Proc. London Math. Soc. (2) **41** (1936), 257-270.
[9] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, Arch. Math. (3) **13** (1908), 305-312.
[10] Y.-F.S. Pétermann, About a theorem of Paolo Codecà's and Ω-estimates for arithmetical convolutions, J. Number Theory **30** (1988), 71-85.
[11] Y.-F.S. Pétermann, About a theorem of Paolo Codecà's and Ω-estimates for arithmetical convolutions: addendum, J. Number Theory **36** (1990), 322-327.
[12] Y.-F.S. Pétermann, About a theorem of Paolo Codecà's and Omega estimates for arithmetical convolutions, Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat. (4) **17** (1990), 343-353.
[13] A.W. Roberts and D.E. Varberg, Convex Functions, Academic Press, New York, 1973.
[14] D.V. Widder, The Laplace Transform, Princeton Univ. Press, 1946.