SOME CLASSES OF COMPLETELY MONOTONIC FUNCTIONS, II

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Abstract. A function $f : (0, \infty) \to \mathbf{R}$ is said to be completely monotonic if $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$ and $n = 0, 1, 2, ...$ In this paper we present several new classes of completely monotonic functions. Our functions have in common that they are defined in terms of the classical gamma, digamma, and polygamma functions. Moreover, we apply one of our monotonicity theorems to prove a new inequality for prime numbers. Some of the given results extend and complement theorems due to Bustoz & Ismail, Clark & Ismail, and other researchers.

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1. Introduction

There exists an extensive literature on inequalities for special functions. In particular, many authors published numerous interesting inequalities for Euler's gamma function:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (x > 0),$$

its logarithmic derivative, known as digamma or psi function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt \quad (x > 0;\ \gamma = \text{Euler's constant}), \tag{1.1}$$

and the polygamma functions:

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty e^{-xt} \frac{t^n}{1 - e^{-t}} dt \quad (x > 0;\ n = 1, 2, ...). \tag{1.2}$$

We refer to the survey paper [24] and the references given therein. Bustoz & Ismail [15], Ismail, Lorch & Muldoon [27], and other researchers proved that certain inequalities for these functions follow from monotonicity and limit properties of functions, which involve Γ, ψ, or $\psi^{(n)}$. And, remarkably, often such functions are not only decreasing or convex, but even completely monotonic.

We recall that a function $f : (0, \infty) \to \mathbf{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x > 0 \quad \text{and} \quad n = 0, 1, 2, ... \tag{1.3}$$

Dubourdieu [21] pointed out that if a non-constant function f is completely monotonic, then strict inequality holds in (1.3). It is known (Bernstein's Theorem) that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. See [44, p. 161].

Completely monotonic functions appear naturally in various fields, like, for example, probability theory and potential theory. The main properties of these functions are given in [44, Chapter IV]. We also refer to [5], where a detailed list of references on completely monotonic functions can be found.

It is the aim of this paper to provide several new classes of completely monotonic functions. The functions we study have in common that they are defined in terms of gamma, digamma, and polygamma functions. In the next section we collect some lemmas. Our monotonicity theorems are stated and proved in sections 3 and 4. And, finally, in section 5 we show that one of our monotonicity results can be applied to establish a new Bonse-type inequality for prime numbers.

2. Lemmas

The formulas given in Lemma 2.1 and Lemma 2.2 (as well as many more) can be found, for example, in [1, Chapter 6], [7, Chapter 1], and [22, Chapter I].

Lemma 2.1. *For all $x > 0$ we have*

$$\log \Gamma(x) = \int_0^\infty \Big[(x-1)e^{-t} - \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}\Big] \frac{dt}{t}, \tag{2.1}$$

$$\psi(x) - \log x = \int_0^\infty e^{-xt} \Big(\frac{1}{t} - \frac{1}{1 - e^{-t}}\Big) dt, \tag{2.2}$$

$$\psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}} \quad (n = 0, 1, ...), \tag{2.3}$$

$$(-1)^{n+1} \psi^{(n)}(x) = n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} \quad (n = 1, 2, ...). \tag{2.4}$$

Lemma 2.2. *We have for $x \to \infty$:*

$$\log \Gamma(x) \sim (x - 1/2) \log x - x + \frac{1}{2} \log(2\pi) + \frac{1}{12x} + ..., \tag{2.5}$$

$$\psi(x) \sim \log x - \frac{1}{2x} - \frac{1}{12x^2} + ..., \tag{2.6}$$

$$(-1)^{n+1} \psi^{(n)}(x) \sim \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + ... \quad (n = 1, 2, ...). \tag{2.7}$$

The following monotonicity theorem is proved in [4].

Lemma 2.3. *Let n be a natural number and c be a real number. The function $x \mapsto x^c|\psi^{(n)}(x)|$ is strictly decreasing on $(0,\infty)$ if and only if $c \le n$.*

The first part of the next lemma is well-known. The second part can be proved by applying Leibniz' rule and induction. And, a proof for the third part is given in [9, p. 66] and [12, p. 83].

Lemma 2.4. (i) *The sum, the product, and the pointwise limit of completely monotonic functions are also completely monotonic.*
(ii) *Let $h : (0,\infty) \to (0,\infty)$. If $(-\log h)'$ is completely monotonic, then h is also completely monotonic.*
(iii) *Let $f : (0,\infty) \to \mathbf{R}$ and $g : (0,\infty) \to (0,\infty)$. If f and g' are completely monotonic, then $x \mapsto f(g(x))$ is also completely monotonic.*

The following inequality for completely monotonic functions is due to Kimberling [30].

Lemma 2.5. *Let $f : [0,\infty) \to (0,1]$ be continuous. If f is completely monotonic, then*

$$f(x)f(y) \le f(x+y) \quad (x,y \ge 0). \tag{2.8}$$

Let $a,b \in \mathbf{R}$ and $0 < u,v \in \mathbf{R}$. The family of Gini means is defined by

$$G_{a,b}(u,v) = \Big(\frac{u^a+v^a}{u^b+v^b}\Big)^{1/(a-b)} \quad (a \ne b) \tag{2.9}$$

and

$$G_{a,a}(u,v) = \exp\Big(\frac{u^a\log u + v^a \log v}{u^a+v^a}\Big). \tag{2.10}$$

For $(a,b) = (1,0)$ we get the arithmetic, for $(a,b) = (0,0)$ the geometric, and for $(a,b) = (0,-1)$ the harmonic means of u and v. In [18] the authors offer a solution of the comparison problem for Gini means. For our purposes the following inequality is helpful.

Lemma 2.6. *Let a,b,r,s be real numbers with*

$$r+s \le a+b, \quad \min(r,s) \le \min(a,b), \quad \textit{and} \quad \min(a,b,r,s) \ge 0.$$

Then we have for all $x,y > 0$:

$$G_{r,s}(x,y) \le G_{a,b}(x,y).$$

Further, we need a Petrović-type inequality for the function $x \mapsto 1/[e^{1/x}-1]$.

Lemma 2.7. *Let $\sigma(x) = 1/[e^{1/x}-1]$. Then we have for all $b_k > 0$ $(k = 1,...,m)$:*

$$\sum_{k=1}^m \sigma(b_k) - \sigma\Big(\sum_{k=1}^m b_k\Big) + \frac{m-1}{2} \ge 0. \tag{2.11}$$

Proof. We denote the expression on the left-hand side of (2.11) by S_m and establish $S_m \geq 0$ by induction on m. We have $S_1 = 0$. The representation

$$x^4[e^{1/x}-1]^3e^{-1/x}\sigma''(x) = 1+2x+(1-2x)e^{1/x} = \sum_{k=2}^{\infty}\frac{k-1}{(k+1)!}x^{-k}$$

implies that σ' is strictly increasing on $(0,\infty)$. Let $y>0$ be fixed. We define for $x>0$:

$$\Lambda(x) = \sigma(x)+\sigma(y)-\sigma(x+y)+\frac{1}{2}.$$

Then

$$\Lambda'(x) = \sigma'(x)-\sigma'(x+y) < 0$$

and

$$\Lambda(x) > \sigma(y)+\frac{1}{2}+\lim_{s\to\infty}[\sigma(s)-\sigma(s+y)].$$

Since

$$\begin{aligned}&\lim_{s\to\infty}[\sigma(s)-\sigma(s+y)]\\ &= \lim_{t\to 0}\frac{1}{t}\Big[\frac{t}{e^t-1}-(1+ty)\frac{t/(1+ty)}{\exp(t/(1+ty))-1}\Big] = -y,\end{aligned}$$

we obtain

$$\Lambda(x) > \sigma(y)+\frac{1}{2}-y = \frac{1}{2(e^{1/y}-1)}\sum_{k=2}^{\infty}\frac{k-1}{(k+1)!}y^{-k} > 0.$$

This proves $S_2 \geq 0$. Now, let $S_m \geq 0$. Then we get

$$S_{m+1} = S_m + \sigma(b_{m+1}) + \sigma\Big(\sum_{k=1}^{m} b_k\Big) - \sigma\Big(b_{m+1}+\sum_{k=1}^{m} b_k\Big) + \frac{1}{2} \geq S_m \geq 0.$$

This completes the proof of Lemma 2.7. □

As usual, we denote by p_n the n-th prime number and by $\pi(n)$ the number of primes not exceeding n. Further, θ denotes Tchebyschef's theta function defined by $\theta(p_n) = \sum_{k=1}^{n}\log p_k$. The next lemma is given in [35].

Lemma 2.8. *For all integers $n \geq 59$ we have*

$$\log p_{n+1} < \log n + \log\log n + \frac{\log\log n - 0.4}{\log n}.$$

Proofs for the following inequalities can be found in [37] and [38].

Lemma 2.9. *For all integers $n \geq 20$ we have*

$$n(\log n + \log\log n - 3/2) < p_n < n(\log n + \log\log n - 1/2).$$

Lemma 2.10. *For all integers $n \geq 3$ we have*

$$n\Big(\log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n}\Big) \leq \theta(p_n),$$

and for all $n \geq 126$ we have

$$\theta(p_n) \leq n\Big(\log n + \log\log n - 1 + \frac{\log\log n - 1.9185}{\log n}\Big).$$

Lemma 2.11. *For all integers $n \geq 59$ we have*
$$\frac{n}{\log n}\Big(1+\frac{1}{2\log n}\Big) < \pi(n).$$

The book [32] contains many more inequalities for $\pi(n)$ and other arithmetical functions.

3. Gamma function

The generalized multinomial coefficient is defined for real numbers $r_1, \dots, r_m$ with $r_k > -1$ ($k = 1, \dots, m$) and $r_1 + \ldots + r_m > -1$ by
$$\binom{r_1+\ldots+r_m}{r_1,\ldots,r_m} = \frac{\Gamma(r_1+\ldots+r_m+1)}{\Gamma(r_1+1)\cdots\Gamma(r_m+1)}.$$
Further, we define for positive real numbers $a_1, \dots, a_m$ and x:
$$\Delta(a_1,\ldots,a_m;x) = \frac{s^{sx+1/2}}{(2\pi x)^{(m-1)/2}\prod_{k=1}^m a_k^{a_k x+1/2}} \quad \text{with} \quad s = \sum_{k=1}^m a_k.$$
In [16] it is proved that for all integers $k > 2$ the sequence $m \mapsto \Delta(k-1,1;m)/\binom{km}{m}$ is decreasing for $m = 1, 2, \ldots$. Recently, Clark & Ismail [17] presented a substantial extension. They established that the function

$$J(a_1,\ldots,a_m;x) = \Delta(a_1,\ldots,a_m;x)\Big/\binom{(a_1+\ldots+a_m)x}{a_1x,\ldots,a_mx} \quad (a_k>0;\ k=1,\ldots,m) \tag{3.1}$$

is decreasing on $(0, \infty)$ with respect to x and converges to 1 as $x \to \infty$. We show that this result can be generalized.

Theorem 3.1. *The function $x \mapsto J(a_1, \ldots, a_m; x)$, as defined in* (3.1), *is completely monotonic.*

Proof. From Lemma 2.4 (ii) we conclude that it suffices to show that
$$\begin{aligned}\phi(x) &= (-\log J(x))' \\ &= -\sum_{k=1}^m a_k\psi(a_kx+1) + s\psi(sx+1) + \frac{m-1}{2x} - s\log s + \sum_{k=1}^m a_k\log a_k\end{aligned}$$
is completely monotonic. In [17] it is proved that J is decreasing on $(0, \infty)$, so that we obtain $\phi(x) = -J'(x)/J(x) \geq 0$ for $x > 0$. Let $n \geq 1$ be an integer. Then we get for $x > 0$:
$$(-1)^n\phi^{(n)}(x) = \sum_{k=1}^m a_k^{n+1}|\psi^{(n)}(a_kx+1)| - s^{n+1}|\psi^{(n)}(sx+1)| + \frac{(m-1)n!}{2x^{n+1}}.$$
Using (1.2) and

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)}\int_0^\infty e^{-xt}t^{r-1}dt \quad (x>0;\ \ r>0), \tag{3.2}$$

we obtain
$$(-1)^n\phi^{(n)}(x) = \int_0^\infty e^{-xt}t^n\Big[\sum_{k=1}^m \sigma(a_k/t) - \sigma\Big(\sum_{k=1}^m a_k/t\Big) + \frac{m-1}{2}\Big]dt,$$
where $\sigma(x) = 1/[e^{1/x}-1]$. Applying Lemma 2.7 we conclude that $(-1)^n\phi^{(n)}(x) \geq 0$. $\square$

In 1985, Kershaw & Laforgia [29] investigated the monotonicity of the function

$$g_a(x) = x^a\big(\Gamma(1+1/x)\big)^x \quad (a \in \mathbf{R}) \tag{3.3}$$

for $a = 0$ and $a = 1$. They proved that g_0 and $1/g_1$ are decreasing on $(0,\infty)$. We ask for all real parameters a and b such that g_a and $1/g_b$ are completely monotonic. The final result depends on the following concepts.

An important subclass of completely monotonic functions consists of the Stieltjes transforms defined as the class of functions $f : (0,\infty) \to \mathbf{R}$ of the form

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t},$$

where $a \geq 0$ and μ is a nonnegative measure on $[0,\infty)$ with $\int d\mu(t)/(1+t) < \infty$; see [9]. These functions play a significant role in the theory of generalized gamma convolutions introduced by Thorin, cf. [41], [42], [13] as well as in recent work [5], [10].

A Stieltjes transform as above obviously has a holomorphic extension to the cut plane $\mathcal{A} = \mathbf{C} \setminus (-\infty, 0]$ by the same formula, and for $z = x + iy \in \mathcal{A}$ we find

$$\operatorname{Im} f(z) = -\int_0^\infty \frac{y\, d\mu(t)}{(x+t)^2 + y^2},$$

so $\operatorname{Im} f(z) \leq 0$ for $\operatorname{Im} z > 0$.

In the Addenda and Problems in [2, p. 127] it is stated that if a function F is holomorphic in the cut plane $\mathcal{A}$, and satisfies $\operatorname{Im} F(z) \leq 0$ for $\operatorname{Im} z > 0$ and $F(x) \geq 0$ for $x > 0$, then F is a Stieltjes transform. A proof is written out in [8]. This is closely related to the theory of Pick functions, see [20], which are holomorphic functions F in the upper half-plane with $\operatorname{Im} F(z) \geq 0$ for $\operatorname{Im} z > 0$.

Theorem 3.2. *Let g_a be the function defined in* (3.3). *Then, g_a is completely monotonic if and only if $a \leq 0$. And, $1/g_b$ is completely monotonic if and only if $b \geq 1$. Furthermore, g_0 and $1/g_1$ are Stieltjes transforms.*

Proof. We have for $x > 0$:

$$g_a'(x) = \frac{g_a(x)}{x}[a + u(x)], \tag{3.4}$$

where

$$u(x) = x \log \Gamma(1+1/x) - \psi(1+1/x).$$

If g_a is completely monotonic, then (3.4) and $\lim_{x\to\infty} u(x) = 0$ imply that $a \leq 0$.
Next, let $a \leq 0$. Since $x \mapsto x^a$ is completely monotonic, we conclude from Lemma 2.4 (i) and the identity $g_a(x) = x^a g_0(x)$ that it suffices to show that g_0 is completely monotonic. We prove below the stronger result that it is a Stieltjes transform, but we prove first the complete monotonicity directly. Differentiation gives

$$(-\log g_0(x))' = -\frac{u(x)}{x} = v(x), \quad \text{say}.$$

Using (2.3) and (2.4) we obtain

$$-v'(x) = \frac{1}{x} w(x), \tag{3.5}$$

where

$$w(x) = -1 + \frac{1}{x^2}\psi'(1/x) = \sum_{k=1}^\infty \frac{1}{(1+kx)^2}.$$

This implies that w is completely monotonic, so that Lemma 2.4 (i) and (3.5) yield that $-v'$ is also completely monotonic. Since $v(x) \geq \lim_{t\to\infty} v(t) = 0$ for $x > 0$, we conclude from Lemma 2.4 (ii) that g_0 is completely monotonic.
Next, let $1/g_b$ be completely monotonic. Then we get for $x > 0$:

$$\left(\frac{1}{g_b(x)}\right)' = -\frac{1}{xg_b(x)}[b + u(x)] \leq 0,$$

Applying (2.5) and (2.6) we obtain $\lim_{x\to 0} u(x) = -1$, which implies $b \geq 1$.
Conversely, if $b \geq 1$ the complete monotonicity of $1/g_b$ follows immediately from that of $1/g_1$, and we shall establish the stronger result that g_0 and $1/g_1$ are Stieltjes transforms.

It is a classical result about Stieltjes transforms, that if φ is a Stieltjes transform not identically zero, then $1/(x\varphi(x))$ is again a Stieltjes transform, see [8], [28], [36], so it is sufficient to prove that g_0 is a Stieltjes transform.

In [11] it is proved that

$$h(z) = \frac{\log\Gamma(z+1)}{z}, \quad z \in \mathbf{C} \setminus (-\infty, -1]$$

is a Pick function with the integral representation

$$c + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) M(t)\, dt,$$

where

$$c = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \arctan\frac{1}{k}\right) = -0.30164...$$

and $M : \mathbf{R} \to [0, \infty)$ is defined as $M(t) = 0$ for $t \geq -1$ and

$$M(t) = \frac{k-1}{-t} \quad \text{for} \quad t \in [-k, -k+1), \;\; k = 2, 3, \ldots.$$

Note that $1/2 \leq M(t) \leq 1$ for $t \leq -1$. We also point out that h has a removable singularity for $z = 0$ with value $\psi(1) = -\gamma$ and for the derivative of h we find

$$h'(x) = \int_1^{\infty} \frac{M(-t)}{(x+t)^2}\, dt,$$

which shows that h is increasing on $(-1, \infty)$ with a completely monotonic derivative (on $(0, \infty)$).

We find

$$\operatorname{Im}\frac{\log\Gamma(z+1)}{z} = \int_{-\infty}^{-1} \frac{y}{(t-x)^2+y^2} M(t)\, dt, \quad z = x+iy, \;\; y > 0,$$

and hence

$$0 < \operatorname{Im}\frac{\log\Gamma(z+1)}{z} < \int_{-\infty}^{-1} \frac{y}{(t-x)^2+y^2}\, dt = \int_{-\infty}^{-1-x} \frac{y}{u^2+y^2}\, du < \pi.$$

Therefore, also $z \mapsto (\Gamma(z+1))^{1/z}$ is a Pick function because the sign of its imaginary part is determined by

$$\sin\left(\operatorname{Im}\frac{\log\Gamma(z+1)}{z}\right),$$

which is non-negative. Since the transformation $z \mapsto 1/z$ maps the upper half-plane into the lower half-plane and vice versa, a Pick function composed with $1/z$ has negative imaginary part in the upper half-plane, and it follows that $(\Gamma(1+1/z))^z$ is a Stieltjes transform. $\square$

Remark. By a result going back to Hirsch [26], if φ is a Stieltjes transform not identically zero, then $1/\varphi(1/x)$ is again a Stieltjes transform. For a proof see also [8]. It follows by Theorem 3.2 that $(\Gamma(1+x))^{-1/x}$ and $(\Gamma(1+x))^{1/x}/x$ are Stieltjes transforms. Like above it is easy to prove that $x^a(\Gamma(1+x))^{-1/x}$ is completely monotonic if and only if $a \le 0$ and $(\Gamma(1+x))^{1/x}/x^a$ is completely monotonic if and only if $a \ge 1$.

Inspired by Stirling's formula, Muldoon [33] studied the monotonicity behaviour of the function

$$H_{a,b}(x) = [x^a(e/x)^x\Gamma(x)]^b \quad (a, b \in \mathbf{R};\ b \ne 0). \tag{3.6}$$

He proved in 1978: if $a \le 1/2$ and $b > 0$, then $H_{a,b}$ is completely monotonic. Moreover, he used this theorem to present an interesting characterization of the gamma function via the notation of complete monotonicity. In 1986, Ismail, Lorch & Muldoon [27] showed: if $a \ge 1$ and $b = -1$, then $H_{a,b}$ is completely monotonic. Our next theorem complements these results.

Theorem 3.3. *Let $H_{a,b}$ be the function defined in* (3.6). *Then, $H_{a,b}$ is completely monotonic if and only if either $a \le 1/2$ and $b > 0$ or $a \ge 1$ and $b < 0$.*

Proof. First, we assume that $H_{a,b}$ is completely monotonic. Then we get for $x > 0$:

$$x(\log H_{a,b}(x))' = b[a + x(\psi(x) - \log x)] \le 0. \tag{3.7}$$

From (2.3) and (2.6) we obtain

$$\lim_{x\to 0} x[\psi(x) - \log x] = -1 \quad \text{and} \quad \lim_{x\to\infty} x[\psi(x) - \log x] = -1/2,$$

so that (3.7) gives that either $a \le 1/2$ and $b > 0$ or $a \ge 1$ and $b < 0$.

To show that $H_{a,b}$ is completely monotonic we use Lemma 2.4 (ii). Differentiation yields for $x > 0$:

$$(-\log H_{a,b}(x))' = -b[a/x + \psi(x) - \log x] = \tau_{a,b}(x), \quad \text{say}.$$

Applying (2.2) and (3.2) we get for $n \ge 0$:

$$(-1)^n\tau_{a,b}^{(n)}(x) = -b\int_0^\infty e^{-xt}t^n\rho_a(t)dt,$$

where

$$\rho_a(t) = a + \frac{1}{t} - \frac{1}{1-e^{-t}}.$$

Let $t > 0$. For $a \ge 1$ we have

$$\rho_a(t) \ge \rho_1(t) = \frac{e^t - 1 - t}{t(e^t - 1)} > 0,$$

and for $a \le 1/2$ we obtain

$$-\rho_a(t) \ge -\rho_{1/2}(t) = \left(\sum_{k=2}^\infty \frac{k-1}{2(k+1)!}t^k\right)/(e^t - 1) > 0.$$

This implies that $\tau_{a,b}$ and $H_{a,b}$ are completely monotonic. □

Many authors extensively investigated various inequalities and monotonicity properties of the ratio $\Gamma(x+a)/\Gamma(x+b)$ and related expressions. We define

$$Q_{a,b}(x) = \frac{(\Gamma(x+a+1))^{1/(x+a)}}{(\Gamma(x+b+1))^{1/(x+b)}} \quad (0 \le a, b \in \mathbf{R}). \tag{3.8}$$

In 1989, Sándor [39] established that $Q_{1,0}$ is decreasing on $(1,\infty)$. We now provide an extension of this result.

Theorem 3.4. *Let $Q_{a,b}$ be defined in (3.8). If $c>0$ is a real number, then $x\mapsto (Q_{a,b}(x))^c$ is completely monotonic if and only if $a\geq b$.*

Proof. Let $a\geq b\geq 0$ and $c>0$. We define

$$\eta(x)=(-\log{(Q_{a,b}(x))^c})'.$$

Let $n\geq 0$ be an integer. Differentiation gives for $x>0$:

$$\frac{1}{c}(-1)^n\eta^{(n)}(x)=(-1)^n\delta^{(n)}(x+b)-(-1)^n\delta^{(n)}(x+a), \tag{3.9}$$

where

$$\delta(x)=\big(\frac{1}{x}\log\Gamma(x+1)\big)'=\frac{1}{x}\psi(x+1)-\frac{1}{x^2}\log\Gamma(x+1).$$

Using (2.1) we get

$$-\delta'(x)=2\int_0^\infty\frac{t^2}{e^t-1}\chi(xt)dt,$$

where

$$\chi(x)=\frac{1}{x^3}-e^{-x}\Big(\frac{1}{2x}+\frac{1}{x^2}+\frac{1}{x^3}\Big).$$

Applying (3.2) we obtain

$$\chi(x)=\frac{1}{2}\int_0^\infty e^{-xt}t^2dt-\frac{1}{2}\int_0^\infty e^{-x(t+1)}dt$$
$$-\int_0^\infty e^{-x(t+1)}tdt-\frac{1}{2}\int_0^\infty e^{-x(t+1)}t^2dt=\frac{1}{2}\int_0^1 e^{-xt}t^2dt.$$

Thus, we get for $x>0$:

$$(-1)^n\chi^{(n)}(x)=\frac{1}{2}\int_0^1 e^{-xt}t^{n+2}dt,$$

which leads to

$$(-1)^{n+1}\delta^{(n+1)}(x)=\int_0^\infty\frac{t^{n+2}}{e^t-1}\int_0^1 e^{-xts}s^{n+2}dsdt>0. \tag{3.10}$$

This implies that $x\mapsto(-1)^n\delta^{(n)}(x)$ is decreasing on $(0,\infty)$, so that (3.9) implies that η is completely monotonic.
Next, we assume that $(Q_{a,b})^c$ is completely monotonic. Then we get for $x>0$:

$$(\log{(Q_{a,b}(x))^c})'=c[\delta(x+a)-\delta(x+b)]\leq 0. \tag{3.11}$$

From (3.10) we conclude that δ is strictly decreasing on $(0,\infty)$, so that (3.11) gives $a\geq b$. □

Remark. The function $\log Q_{a,b}$ $(a\geq b\geq 0)$ is completely monotonic. In fact, using the integral representation of the Pick function $h(z)=(1/z)\log\Gamma(z+1)$, we find with the notation above that

$$\delta(x)=h'(x)=\int_1^\infty\frac{M(-t)}{(x+t)^2}\,dt$$

is completely monotonic, and therefore

$$\log Q_{a,b}(x)=h(x+a)-h(x+b)$$

$$= \int_{-\infty}^{-1} \left(\frac{1}{t-x-a} - \frac{1}{t-x-b} \right) M(t)\, dt = \int_1^{\infty} \frac{a-b}{(a+x+t)(b+x+t)} M(-t)\, dt,$$

showing that $\log Q_{a,b}$ is completely monotonic for $a \geq b$. From this formula we get another proof that $(Q_{a,b})^c$ is completely monotonic for every $c > 0$ because

$$(Q_{a,b}(x))^c = \exp(c \log Q_{a,b}(x)) = \sum_{k=0}^{\infty} \frac{c^k}{k!} (\log Q_{a,b}(x))^k$$

is completely monotonic by Lemma 2.4 (i).

Corollary. *Let a and b be real numbers with $a > b \geq 0$. Then we have for all $x, y \geq 0$:*

$$\lambda_{a,b} \leq \frac{Q_{a,b}(x+y)}{Q_{a,b}(x) Q_{a,b}(y)} < \nu_{a,b}, \tag{3.12}$$

with the best possible constants

$$\lambda_{a,b} = \frac{(\Gamma(b+1))^{1/b}}{(\Gamma(a+1))^{1/a}} \quad \text{and} \quad \nu_{a,b} = 1. \tag{3.13}$$

Proof. Applying Theorem 3.4 and the asymptotic formula (2.5) it follows that $Q_{a,b}$ is strictly decreasing on $[0, \infty)$ with $\lim_{x \to \infty} Q_{a,b}(x) = 1$. Therefore, we obtain for $x, y \geq 0$:

$$\frac{Q_{a,b}(x+y)}{Q_{a,b}(x)} \frac{1}{Q_{a,b}(y)} < 1.$$

This proves the right-hand side of (3.12) with $\nu_{a,b} = 1$. And, since $x \mapsto Q_{a,b}(x)/Q_{a,b}(0)$ satisfies the assumption of Lemma 2.5, we conclude from (2.8) that the left-hand side of (3.12) is valid with $\lambda_{a,b}$ as given in (3.13). If $x = 0$, then equality holds in the first inequality of (3.12). Further, we have

$$\lim_{y \to \infty} \lim_{x \to \infty} \frac{Q_{a,b}(x+y)}{Q_{a,b}(x) Q_{a,b}(y)} = 1.$$

Hence, the constant bounds in (3.13) are both best possible. □

Remark. Since $Q_{a,b}$ $(a \neq b)$ is strictly monotonic on $[0, \infty)$, we obtain: if a, b are integers with $a > b \geq 0$, then the diophantine equation

$$\begin{aligned} &((k+a)!)^{(k+b)(n+a)(n+b)} ((n+b)!)^{(n+a)(k+a)(k+b)} \\ &= ((n+a)!)^{(n+b)(k+a)(k+b)} ((k+b)!)^{(k+a)(n+a)(n+b)} \quad (0 \leq k, n \in \mathbf{Z}) \end{aligned} \tag{3.14}$$

has only the trivial solution $k = n$. This extends a result due to Sándor [40], who studied the equation (3.14) for the special case $a = 1$, $b = 0$.

In 1997, Merkle [31] proved that $x \mapsto (\Gamma(x))^2/\Gamma(2x)$ is log-convex on $(0, \infty)$. By the duplication formula of Legendre this statement is equivalent to $\Gamma(x)/\Gamma(x + 1/2)$ being log-convex for $x > 0$. However, the quotient $\Gamma(x)/\Gamma(x+a)$ is completely monotonic for $a \geq 0$ and in particular log-convex on $(0, \infty)$. By Lemma 2.4 (ii) it is enough to prove that $\psi(x+a) - \psi(x)$ is completely monotonic, which follows from (1.1). See also Theorem 4.3 below.

The result of Merkle is also a special case of the following theorem.

Theorem 3.5. *Let a, b, α, β be real numbers with $a > b > 0$. The function $x \mapsto (\Gamma(ax))^{\alpha}/(\Gamma(bx))^{\beta}$ is completely monotonic if and only if $\alpha \leq 0$ and $\alpha a = \beta b$.*

Proof. Let

$$y(x) = \frac{(\Gamma(ax))^\alpha}{(\Gamma(bx))^\beta} \quad \text{and} \quad z(x) = (\log y(x))' = \alpha a \psi(ax) - \beta b \psi(bx).$$

We suppose that y is completely monotonic. Then y is also log-convex; see [23]. Thus, z is non-positive and increasing on $(0,\infty)$. We have the representation

$$z(x) = \alpha a[\psi(ax) - \log(ax)] - \beta b[\psi(bx) - \log(bx)] + \alpha a \log a - \beta b \log b + (\alpha a - \beta b)\log x.$$

Applying (2.6) we get: if $\alpha a - \beta b > 0$, then $\lim_{x\to\infty} z(x) = \infty$. And, if $\alpha a - \beta b < 0$, then $\lim_{x\to\infty} z(x) = -\infty$. Hence, $\alpha a - \beta b = 0$. This leads to

$$z(x) = \alpha a[\psi(ax) - \log(ax) - \psi(bx) + \log(bx) + \log(a/b)]$$

and $\lim_{x\to\infty} z(x) = \alpha a \log(a/b) \le 0$. Thus, $\alpha \le 0$.
Conversely, let $\alpha \le 0$ and $\alpha a = \beta b$. It suffices to show that $-z$ is completely monotonic. Let

$$\omega(x) = \psi(ax) - \psi(bx) \quad \text{and} \quad \kappa(x) = x^n |\psi^{(n)}(x)| \quad (n \in \mathbf{N}).$$

Applying Lemma 2.3 we get for $x > 0$:

$$x^n(-1)^n \omega^{(n)}(x) = \kappa(bx) - \kappa(ax) > 0.$$

In particular, ω is decreasing, so that we obtain

$$\omega(x) \ge \lim_{t\to\infty} \omega(t) = \log(a/b) > 0 \quad (x > 0).$$

Thus, ω is completely monotonic. The identity $-z(x) = -\alpha a \omega(x)$ reveals that $-z$ is also completely monotonic. ☐

Remark. There do not exist real numbers α, β (with $\alpha \ne \beta$) such that $q_{\alpha,\beta}(x) = \Gamma(x^\alpha)/\Gamma(x^\beta)$ is completely monotonic. We assume (for a contradiction) that $q_{\alpha,\beta}$ is completely monotonic. Since $q'_{\alpha,\beta}(1) = (\beta - \alpha)\gamma$, where $\gamma = 0.5772...$ denotes Euler's constant, we get $\beta < \alpha$. If $\beta < 0 = \alpha$ or $\beta = 0 < \alpha$, then $q'_{\alpha,\beta}$ attains positive values on $(0,\infty)$. And, if $\beta < 0 < \alpha$, then $\lim_{x\to 0} q_{\alpha,\beta}(x) = 0$. Next, let $\beta > 0$. From $(\log q_{\alpha,\beta}(x))' \le 0$ we obtain

$$0 < \frac{\alpha}{\beta} \le x^{\beta-\alpha} \frac{\psi(x^\beta)}{\psi(x^\alpha)} \quad (x \text{ large}).$$

This contradicts $\lim_{x\to\infty} x^{\beta-\alpha}\psi(x^\beta)/\psi(x^\alpha) = 0$. And, if $\alpha < 0$, then

$$x^{\beta-\alpha} \frac{\psi(x^\beta)}{\psi(x^\alpha)} \le \frac{\alpha}{\beta} \quad (x \text{ small}),$$

which is false since $\lim_{x\to 0} x^{\beta-\alpha}\psi(x^\beta)/\psi(x^\alpha) = \infty$.

Let

$$P_{a,b}(u,v;x) = \frac{\Gamma(x+u)}{\Gamma(x+v)} \exp[(v-u)\psi(x + G_{a,b}(u,v))] \quad (a,b \in \mathbf{R};\ 0 < u,v \in \mathbf{R}), \tag{3.15}$$

where $G_{a,b}(u,v)$ denotes the family of Gini means defined in (2.9) and (2.10). Bustoz & Ismail [15] proved that $x \mapsto P_{1,0}(s,1;x)$ $(0 < s < 1)$ is completely monotonic. We determine all Gini means such that the function given in (3.15) is completely monotonic.

Theorem 3.6. *The function $x \mapsto P_{a,b}(u,v;x)$, as defined in* (3.15)*, is completely monotonic for all $u,v \in \mathbf{R}$ with $v > u > 0$ if and only if $a + b \ge 1$ and $\min(a,b) \ge 0$.*

Proof. Let $P(x) = P_{a,b}(u, v; x)$. We may suppose that $a \geq b$. First, we assume that P is completely monotonic for all $u, v \in \mathbf{R}$ with $v > u > 0$. Then we get for $x > 0$:

$$0 \geq \frac{P'(x)}{P(x)} = \psi(x+u) - \psi(x+v) + (v-u)\psi'(x + G_{a,b}(u,v)) = \Lambda_{a,b}(u, v; x), \quad \text{say.}$$

We have

$$\Lambda_{a,b}(u, v; x)\Big|_{u=v} = \frac{\partial}{\partial u}\Lambda_{a,b}(u, v; x)\Big|_{u=v} = \frac{\partial^2}{\partial u^2}\Lambda_{a,b}(u, v; x)\Big|_{u=v} = 0.$$

This leads to

$$0 \leq \frac{\partial^3}{\partial u^3}\Lambda_{a,b}(u, v; x)\Big|_{u=v} = \frac{3\psi''(x+v)}{4v}\Big[1 - a - b + \frac{v\psi'''(x+v)}{3\psi''(x+v)}\Big].$$

The formulas (2.4) and (2.7) yield $\psi'' < 0$ and $\lim_{x\to\infty} \psi'''(x)/\psi''(x) = 0$, so that we obtain $a+b \geq 1$. Next, let $b < 0$. Since P is decreasing on $(0, \infty)$ with $\lim_{x\to\infty} P(x) = 1$, we get

$$P(x) \geq 1 \quad \text{for} \quad x > 0 \quad \text{and} \quad v > u > 0. \tag{3.16}$$

We have

$$\lim_{u\to 0} G_{a,b}(u, v) = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{e^{v\psi(x)}}{x} = 0.$$

This leads to

$$\lim_{x\to 0}\lim_{u\to 0} P(x) = 0,$$

which contradicts (3.16). Hence, $b \geq 0$.
Conversely, let $a \geq b \geq 0$, $a + b \geq 1$, $v > u > 0$, and $x > 0$. From (1.1) and (1.2) we get

$$(-\log P(x))' = \int_0^\infty e^{-xt}\frac{1}{1-e^{-t}}\Theta_{a,b}(u, v; t)dt, \tag{3.17}$$

where

$$\Theta_{a,b}(u, v; t) = e^{-ut} - e^{-vt} - (v-u)t\exp[-tG_{a,b}(u, v)].$$

Applying Lemma 2.6 with $r = 1$ and $s = 0$ we obtain for $v > u > 0$:

$$G_{a,b}(u, v) \geq G_{1,0}(u, v) = \frac{u+v}{2}.$$

Thus,

$$\Theta_{a,b}(u, v; t) \geq \Theta_{1,0}(u, v; t) \tag{3.18}$$

$$= 2e^{-(u+v)t/2}\Big[\sinh\frac{(v-u)t}{2} - \frac{(v-u)t}{2}\Big] > 0 \quad (t > 0).$$

From (3.17) and (3.18) we conclude that P is completely monotonic. □

Remark. In [15] it is proved that $x \mapsto 1/P_{0,0}(s, 1; x)$ $(0 < s < 1)$ is strictly decreasing on $(0, \infty)$. It remains an open problem to determine all real parameters a and b such that $x \mapsto 1/P_{a,b}(u, v; x)$ is completely monotonic for all $u, v \in \mathbf{R}$ with $v > u > 0$.

4. Digamma and polygamma functions

The digamma function is strictly increasing from $-\infty$ to ∞ on $(0,\infty)$, in contrast to $|\psi^{(n)}| = (-1)^{n+1}\psi^{(n)}$ ($n \in \mathbf{N}$) which is completely monotonic. However, as we prove now, under certain conditions for the parameters a_k and b_k the linear combination $\sum_{k=1}^m a_k\psi(b_k x)$ is completely monotonic.

Theorem 4.1. *Let a_k and b_k ($k = 1, ..., m$) be real numbers such that $a_1 \geq ... \geq a_m$ and $b_1 \geq ... \geq b_m > 0$. The function $x \mapsto \sum_{k=1}^m a_k\psi(b_k x)$ is completely monotonic if and only if $\sum_{k=1}^m a_k = 0$ and $\sum_{k=1}^m a_k \log b_k \geq 0$.*

Proof. Let $\chi(x) = \sum_{k=1}^m a_k\psi(b_k x)$. First, we show: if $\sum_{k=1}^m a_k = 0$ and $\sum_{k=1}^m a_k \log b_k \geq 0$, then χ is completely monotonic. Let $n \geq 1$ be an integer and $x > 0$. Then (1.2) leads to

$$(-1)^n\chi^{(n)}(x) = \sum_{k=1}^m a_k b_k^n (-1)^n\psi^{(n)}(b_k x) = -\int_0^\infty e^{-xt}t^{n-1}\sum_{k=1}^m a_k\eta(t/b_k)dt, \tag{4.1}$$

where $\eta(x) = x/[1-e^{-x}]$. Since $\eta'(x) = e^{-x}(1-e^{-x})^{-2}[e^x - 1 - x] > 0$ for $x > 0$, we conclude that η is increasing on $(0,\infty)$. Hence, we have $a_1 \geq ... \geq a_m$ and $\eta(t/b_1) \leq ... \leq \eta(t/b_m)$ $(t > 0)$, so that Tchebyschef's inequality (see [25, p. 43]) yields

$$\sum_{k=1}^m a_k\eta(t/b_k) \leq \frac{1}{m}\sum_{k=1}^m a_k \sum_{k=1}^m a_k\eta(t/b_k) = 0 \quad (t > 0). \tag{4.2}$$

From (4.1) and (4.2) we obtain $(-1)^n\chi^{(n)}(x) \geq 0$ for $n \geq 1$ and $x > 0$. It remains to show that χ is non-negative. We have

$$\chi(x) = \sum_{k=1}^m a_k[\psi(b_k x) - \log(b_k x)] + \sum_{k=1}^m a_k \log b_k.$$

From (2.6) we get $\lim_{x\to\infty}\chi(x) = \sum_{k=1}^m a_k \log b_k \geq 0$. Since χ is decreasing it follows that $\chi(x) \geq 0$ for $x > 0$.
Conversely, let χ be completely monotonic. Then we obtain for $x > 0$:

$$0 \leq \chi(x) = \sum_{k=1}^m a_k[\psi(b_k x) - \log(b_k x)] + \sum_{k=1}^m a_k \log b_k + \sum_{k=1}^m a_k \log x \tag{4.3}$$

and

$$x\chi'(x) = \sum_{k=1}^m a_k b_k x\psi'(b_k x) \leq 0. \tag{4.4}$$

Using (2.6) we conclude from (4.3) that $\sum_{k=1}^m a_k \geq 0$; and since $\lim_{x\to\infty} x\psi'(x) = 1$, we get from (4.4) that $\sum_{k=1}^m a_k \leq 0$. Thus, $\sum_{k=1}^m a_k = 0$. Applying (2.6) again we obtain from (4.3) that $\sum_{k=1}^{m,} a_k \log b_k \geq 0$. This completes the proof of Theorem 4.1. □

Our next theorem has been motivated by the asymptotic formulas (2.6) and (2.7). Let

$$\lambda_{a,b}(x) = \psi(x+a) - \log(x+b) \quad (0 \leq a, b \in \mathbf{R}) \tag{4.5}$$

and

$$\mu_{a,b}(x) = |\psi^{(n)}(x+a)| - \frac{(n-1)!}{(x+b)^n} \quad (0 \leq a, b \in \mathbf{R};\ n \in \mathbf{N}). \tag{4.6}$$

We determine all parameters a and b such that these functions are completely monotonic.

Theorem 4.2. *Let $\lambda_{a,b}$ and $\mu_{a,b}$ be defined in* (4.5) and (4.6), *respectively.*
(i) *The function $\lambda_{a,b}$ is completely monotonic if and only if $a-b\geq 1/2$. And, $-\lambda_{a,b}$ is completely monotonic if and only if $a-b\leq 0$.*
(ii) *The function $\mu_{a,b}$ is completely monotonic if and only if $a-b\leq 0$. And, $-\mu_{a,b}$ is completely monotonic if and only if $a-b\geq 1/2$.*

Proof. Since the proofs of (i) and (ii) are very similar, we only establish the first part. Let $\lambda_{a,b}$ be completely monotonic. From $\lambda'_{a,b}(x)\leq 0$ we obtain for $x>0$:

$$b-a\leq \frac{1}{\psi'(x+a)}-(x+a).$$

Using (2.7) we get $\lim_{t\to\infty} 1/\psi'(t)-t=-1/2$. Thus, $b-a\leq -1/2$.
Next, let $a-b\geq 1/2$. Differentiation gives for $x>0$:

$$-\lambda'_{a,b}(x)=\int_0^\infty e^{-xt}\Delta_{a,b}(t)dt, \tag{4.7}$$

where

$$\Delta_{a,b}(t)=e^{-bt}-\frac{te^{-at}}{1-e^{-t}}.$$

Since

$$\Delta_{a,b}(t)\geq \Delta_{a,a-1/2}(t)=\frac{2e^{-at}}{1-e^{-t}}\Big[\sinh\frac{t}{2}-\frac{t}{2}\Big]>0\quad (t>0),$$

we conclude that $-\lambda'_{a,b}$ is completely monotonic. We have $\lambda_{a,b}(x)\geq \lim_{t\to\infty}\lambda_{a,b}(t)=0$ for $x>0$. This implies that $\lambda_{a,b}$ is also completely monotonic.
We assume that $-\lambda_{a,b}$ is completely monotonic. Then $\lambda'_{a,b}$ is also completely monotonic, so that (4.7) yields $\Delta_{a,b}(t)\leq 0$ for all $t>0$. This leads to

$$b-a\geq \frac{\log(1-e^{-t})-\log t}{t}=\delta(t),\quad \text{say.}$$

Since $\lim_{t\to\infty}\delta(t)=0$, we get $b-a\geq 0$.
Conversely, let $a-b\leq 0$. Then we have for $t>0$:

$$\Delta_{a,b}(t)\leq \Delta_{a,a}(t)=\frac{e^{-at}}{1-e^{-t}}[1-t-e^{-t}]<0.$$

so that (4.7) implies that $\lambda'_{a,b}$ is completely monotonic. Since $\lambda_{a,b}(x)\leq \lim_{t\to\infty}\lambda_{a,b}(t)=0$ for $x>0$, we conclude that $-\lambda_{a,b}$ is also completely monotonic. □

We study now the complete monotonicity of certain differences of digamma and polygamma functions.

Theorem 4.3. *Let $n\geq 0$ be an integer and let α, a, b be real numbers with $a,b>0$ and $a\neq b$. The function $x\mapsto(-1)^n x^\alpha[\psi^{(n)}(x+a)-\psi^{(n)}(x+b)]$ is completely monotonic if and only if $\alpha\leq 0$ and $a>b$.*

Proof. We define

$$W_\alpha(x)=(-1)^n x^\alpha[\psi^{(n)}(x+a)-\psi^{(n)}(x+b)]. \tag{4.8}$$

First, we suppose that W_α is completely monotonic. Since $-\psi$ and $|\psi^{(n)}|$ $(n \geq 1)$ are strictly decreasing on $(0,\infty)$, we conclude from $W_\alpha(x) \geq 0$ for $x > 0$ that $a > b$. Next, let $\alpha > 0$. Then, $W_\alpha(0) = 0$, so that $W'_\alpha(x) \leq 0 \leq W_\alpha(x)$ $(x > 0)$ implies $W_\alpha \equiv 0$. This leads to $a = b$. Thus, we have $\alpha \leq 0$.
Conversely, let $\alpha \leq 0$ and $a > b$. Using (1.1) and (1.2) we obtain

$$W_0(x) = \int_0^\infty e^{-xt} t^n \frac{e^{-bt} - e^{-at}}{1 - e^{-t}} dt. \tag{4.9}$$

Thus, W_0 is completely monotonic. Applying Lemma 2.4 (i) we conclude from $W_\alpha(x) = x^\alpha W_0(x)$ that W_α is also completely monotonic. □

Remarks. (1) A function $u : (0,\infty) \to \mathbf{R}$ is called star-shaped, if

$$u(\epsilon x) \leq \epsilon u(x) \quad \text{for all} \quad x > 0 \quad \text{and} \quad \epsilon \in (0,1). \tag{4.10}$$

And, if

$$u(x) + u(y) < u(x+y) \quad \text{for all} \quad x, y > 0, \tag{4.11}$$

then u is said to be strictly superadditive. If (4.11) holds with '$\leq$' instead of '$<$', then u is called superadditive. It is easy to show that a star-shaped function is also superadditive. Trimble, Wells & Wright [43] investigated a subclass of completely monotonic functions. They proved: let

$$v(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

where the integral converges for $x > 0$ and ϕ is a non-constant function, which is non-negative and increasing on $(0,\infty)$. Then $1/v$ is star-shaped and strictly superadditive.
Let W_α be the function defined in (4.8). If $n \geq 0$, $\alpha \leq -1$, $a > b > 0$, then (3.2), (4.9), and the convolution theorem for Laplace transforms yield

$$W_\alpha(x) = \int_0^\infty e^{-xt} \phi(t) dt,$$

where

$$\phi(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t s^n (t-s)^{-\alpha-1} \frac{e^{-bs} - e^{-as}}{1 - e^{-s}} ds$$

is non-constant, non-negative, and increasing on $(0,\infty)$. This implies that (4.10) and (4.11) hold with $u = 1/W_\alpha$.
(2) The following companion of Theorem 4.3 is valid. The function

$$F_n(x) = (-1)^n [\psi^{(n)}(1 + 1/(1+x)) - \psi^{(n)}(1 - 1/(1+x))] \quad (0 \leq n \in \mathbf{Z})$$

is completely monotonic. To prove this we apply (1.1) and (1.2) to get the formula

$$F_n(x) = 2 \int_0^\infty \frac{t^n}{e^t - 1} \sinh(\frac{t}{x+1})\, dt,$$

which shows the assertion. In fact, using the power series expansion of sinh it is clear that $\sinh(t/(x+1))$ is a completely monotonic function of x for each $t > 0$.

In 1999, Palumbo [34] published an elegant inequality for the polygamma functions. He proved that

$$\psi^{(n)}(x)\psi^{(n)}(x+y+z) - \psi^{(n)}(x+y)\psi^{(n)}(x+z) \geq 0$$

holds for all integers $n \geq 1$ and all positive real numbers x, y, z. The following theorem provides an extension of this result.

Theorem 4.4. *Let n be a natural number and let a_1, a_2, b_1, b_2 be non-negative real numbers such that* $\max(a_1, a_2) \geq \max(b_1, b_2)$. *The function*

$$x \mapsto \psi^{(n)}(x+a_1)\psi^{(n)}(x+a_2) - \psi^{(n)}(x+b_1)\psi^{(n)}(x+b_2) \tag{4.12}$$

is completely monotonic if and only if $a_1 + a_2 \leq b_1 + b_2$.

Proof. We assume that $a_1 \geq a_2 \geq 0$, $b_1 \geq b_2 \geq 0$, and $a_1 \geq b_1$. Let K be the function defined in (4.12). First, we prove: if $a_1 + a_2 \leq b_1 + b_2$, then K is completely monotonic. Using (1.2) and the convolution theorem for Laplace transforms we get

$$K(x) = \frac{1}{2}\int_0^\infty e^{-xt}\int_0^t \frac{s^n(t-s)^n}{(1-e^{-s})(1-e^{-(t-s)})} \times[\Lambda(a_1,a_2;s,t) - \Lambda(b_1,b_2;s,t)]dsdt, \tag{4.13}$$

where

$$\Lambda(u,v;s,t) = e^{-(t-s)u-sv} + e^{-su-(t-s)v}.$$

Let $0 \leq s \leq t$. Partial differentiation yields

$$\frac{\partial}{\partial b_2}\Lambda(b_1,b_2;s,t) = -se^{-(t-s)b_1-sb_2} - (t-s)e^{-sb_1-(t-s)b_2} \leq 0.$$

Since $b_2 \geq a_1 + a_2 - b_1$, we obtain

$$\Lambda(b_1,b_2;s,t) \leq \Lambda(b_1, a_1+a_2-b_1; s,t) = e^{(2s-t)b_1-s(a_1+a_2)} + e^{(t-2s)b_1-(t-s)(a_1+a_2)} = \Theta(a_1,a_2;b_1;s,t), \quad \text{say.} \tag{4.14}$$

We have $2b_1 \geq a_1 + a_2$, so that we get

$$\frac{\partial}{\partial b_1}\Theta(a_1,a_2;b_1;s,t) = (2s-t)e^{(t-2s)b_1+(s-t)(a_1+a_2)}[e^{(2s-t)[2b_1-(a_1+a_2)]} - 1] \geq 0.$$

This leads to

$$\Theta(a_1,a_2;b_1;s,t) \leq \Theta(a_1,a_2;a_1;s,t) = \Lambda(a_1,a_2;s,t). \tag{4.15}$$

From (4.14) and (4.15) we obtain

$$\Lambda(a_1,a_2;s,t) - \Lambda(b_1,b_2;s,t) \geq 0 \quad (0 \leq s \leq t),$$

so that (4.13) implies that K is completely monotonic.
Next, let K be completely monotonic. We define $\tau_a(x) = |\psi^{(n)}(x+a)| - (n-1)!/(x+a)^n$. Then we get

$$K(x) = \tau_{a_1}(x)\tau_{a_2}(x) - \tau_{b_1}(x)\tau_{b_2}(x) + (n-1)!\Big(\frac{\tau_{a_1}(x)}{(x+a_2)^n} + \frac{\tau_{a_2}(x)}{(x+a_1)^n} - \frac{\tau_{b_1}(x)}{(x+b_2)^n} - \frac{\tau_{b_2}(x)}{(x+b_1)^n}\Big) + ((n-1)!)^2\Big(\frac{1}{(x+a_1)^n(x+a_2)^n} - \frac{1}{(x+b_1)^n(x+b_2)^n}\Big).$$

Using the limit relations

$$\lim_{x\to\infty} x^{n+1}\tau_a(x) = \frac{1}{2}n! \quad (a > 0)$$

and

$$\lim_{x\to\infty} x^{2n+1}\Big(\frac{1}{(x+a_1)^n(x+a_2)^n} - \frac{1}{(x+b_1)^n(x+b_2)^n}\Big) = n(b_1+b_2-a_1-a_2),$$

we obtain

$$0 \leq \lim_{x \to \infty} x^{2n+1} K(x) = (n-1)!n!(b_1 + b_2 - a_1 - a_2).$$

This leads to $b_1 + b_2 \geq a_1 + a_2$. □

Remark. Applying Lemma 2.4 (i) and (iii) we get for all integers $n \geq 1$: if $a = 1$, $b \in \mathbf{N}$, or $0 \leq a \leq 1$, $b = 1$, then $x \mapsto |\psi^{(n)}(x^a)|^b$ is completely monotonic. We ask for all positive integers n and real numbers a, b such that this function is completely monotonic.

5. Prime numbers

If $H_{a,b}$ denotes the function defined in (3.6), then we conclude from Theorem 3.3:

$$(\log H_{1,-1}(x))' < 0 \quad \text{and} \quad (\log H_{1/2,1}(x))' < 0 \quad (x > 0),$$

or, equivalently,

$$\log x - \frac{1}{x} < \psi(x) < \log x - \frac{1}{2x} \quad (x > 0). \tag{5.1}$$

Proofs for (5.1) are also given in [3], [6], and [19]. In [19] the authors provide a statistical application of (5.1). We show that (5.1) plays a role in the proof of a new number theoretic inequality involving the digamma function and prime numbers.

In 1907, Bonse [14] presented two inequalities, which compare the product of the first n primes with powers of the $(n+1)$-th prime number:

$$p_{n+1}^2 < p_1 p_2 \cdots p_n \quad (n \geq 4) \quad \text{and} \quad p_{n+1}^3 < p_1 p_2 \cdots p_n \quad (n \geq 5). \tag{5.2}$$

Further inequalities of this type can be found in the monograph [32, p. 246]. It is natural to look for refinements of (5.2) by replacing the exponents 2 and 3 by expressions which depend on n. The following elegant sharpening of (5.2) was established by Panaitopol [35] in 2000:

$$p_{n+1}^{n-\pi(n)} < p_1 p_2 \cdots p_n \quad (n \geq 2). \tag{5.3}$$

It might be a bit surprising that the digamma function can be used to improve (5.3) for all $n \geq 11$.

Theorem 5.1. *Let α be a real number. The inequality*

$$p_{n+1}^{n(1-\alpha/\psi(n))} < p_1 p_2 \cdots p_n \tag{5.4}$$

holds for all integers $n \geq 2$ if and only if $\alpha \geq 1$.

Proof. Inequality (5.4) is equivalent to

$$0 < \frac{1}{n}\theta(p_n) - \Big(1 - \frac{\alpha}{\psi(n)}\Big) \log p_{n+1} = f_\alpha(n), \quad \text{say.} \tag{5.5}$$

Here, θ denotes the Tchebyschef function defined in section 2. Let $\alpha \geq 1$ and $n \geq 89$. Applying the second inequality of (5.1) and Lemmas 2.8 and 2.10 we obtain

$$\begin{aligned} f_\alpha(n) \geq f_1(n) &\geq \log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n} \\ &- \Big(1 - \frac{1}{\log n - 1/(2n)}\Big)\Big(\log n + \log\log n + \frac{\log\log n - 0.4}{\log n}\Big) \end{aligned} \tag{5.6}$$

$$= \frac{ng(\log n) + 5000 \log n + 8727}{5000(\log n)(2n \log n - 1)},$$

where

$$g(x) = x(10^4 \log x - 17454) + 10^3(10 \log x - 4).$$

The function g is strictly increasing on $(0, \infty)$. Since $g(\log n) \geq g(\log 89) = 69.90...$, we conclude from (5.6) that $f_1(n) > 0$ for $n \geq 89$. A direct calculation reveals that $f_1(n)$ is positive for $n = 2, 3, ..., 88$, too. This implies that inequality (5.5) is valid for all real numbers $\alpha \geq 1$ and all integers $n \geq 2$.

Next, we assume that (5.4) holds for all $n \geq 2$. Then we obtain

$$\begin{aligned}(5.7) \qquad \alpha \geq \psi(n)\Big(1 - \frac{\theta(p_n)}{n \log p_{n+1}}\Big) &\geq \psi(n)\Big(1 - \frac{\theta(p_n)}{n \log p_n}\Big) \\ &= a(n)b(n)[c(n) - d(n)],\end{aligned}$$

where

$$a(n) = \frac{\psi(n)}{\log n}, \quad b(n) = \frac{\log n}{\log p_n}, \quad c(n) = \log p_n - \log n - \log\log n, \quad d(n) = \frac{\theta(p_n)}{n} - \log n - \log\log n.$$

Using (5.1) and Lemmas 2.9 and 2.10 we get the limit relations

$$(5.8) \qquad \lim_{n\to\infty} a(n) = 1, \quad \lim_{n\to\infty} b(n) = 1, \quad \lim_{n\to\infty} c(n) = 0, \quad \lim_{n\to\infty} d(n) = -1.$$

From (5.7) and (5.8) we conclude that $\alpha \geq 1$. □

Remark. Let

$$h(n) = \pi(n) - \frac{n}{\psi(n)}.$$

Applying the left-hand side of (5.1) and Lemma 2.11 we obtain for $n \geq 59$:

$$\frac{h(n)}{n} > \frac{1}{\log n} + \frac{1}{2(\log n)^2} - \frac{1}{\log n - 1/n} = \frac{(n-2)\log n - 1}{2(\log n)^2(n \log n - 1)} > 0.$$

A simple calculation gives that $h(n) > 0$ is also true for $n = 7, 8$ and $n = 11, 12, ..., 58$. This implies that (5.4) with $\alpha = 1$ improves inequality (5.3) for all $n \geq 11$.

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