ORTHOGONAL POLYNOMIALS, L^2-SPACES AND ENTIRE FUNCTIONS.

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Abstract. We show that for determinate measures μ having moments of every order and finite index of determinacy, (i.e., a polynomial p exists for which the measure $|p|^2\mu$ is indeterminate) the space $L^2(\mu)$ consists of entire functions of minimal exponential type in the Cartwright class.

1. Introduction

Let $\mathcal{M}^*$ denote the set of positive Borel measures on the real line having moments of every order and infinite support. We are interested in finding conditions on $\mu \in \mathcal{M}^*$ such that $L^2(\mu)$ consists of entire functions in the following sense: (i) There exists a continuous linear injection $E : L^2(\mu) \to \mathcal{H}(\mathbb{C})$, where $\mathcal{H}(\mathbb{C})$ denotes the set of entire functions with the topology of compact convergence. (ii) For all $f \in L^2(\mu)$ we have $E(f) = f$ μ-a.e.. We say that E is a *realization* of $L^2(\mu)$ as entire functions. In the discussion of this problem we need for $\mu \in \mathcal{M}^*$ the corresponding sequence of orthonormal polynomials (p_n). It is uniquely determined by the orthonormality condition and the requirement that p_n is a polynomial of degree n with positive leading coefficient. The sequence of orthonormal polynomials depends only on the moments of μ, so if μ is *indeterminate*, i.e. there are other measures having the same moments as μ, all these measures lead to the same sequence (p_n).

When the measure μ is indeterminate, the Fourier expansion of $f \in L^2(\mu)$

$$\sum_{n=0}^{\infty} \left(\int f(t) p_n(t) d\mu(t) \right) p_n(z) \tag{1.1}$$

converges in $L^2(\mu)$ and uniformly on compact subsets of $\mathbb{C}$ to an entire function $F(f)(z)$, which is the orthogonal projection of f onto the closure in $L^2(\mu)$ of the set $\mathbb{C}[t]$ of polynomials. We recall that $z \mapsto (p_n(z))_n$ is an entire function with values in the Hilbert space ℓ^2, so in particular $(p_n^{(m)}(z))_n \in \ell^2$ for all $z \in \mathbb{C}$, $m \in \mathbb{N}$, cf. [4]. By a theorem of M. Riesz ([8], [1]) $F(f)$ is of minimal exponential

1991 Mathematics Subject Classification. 42C05, 44A60

Key words and phrases. Orthogonal polynomials. Index of determinacy. Entire functions.

The work of the second author was supported by DGICYT ref. PB 93-0926.

type. If the indeterminate measure μ is *Nevanlinna extremal* (*N-extremal* in short), which means that $\mathbb{C}[t]$ is dense in $L^2(\mu)$, then μ is discrete and $F(f)(x) = f(x)$ for $x \in \operatorname{supp}(\mu)$. This means that $F(f)$ is an extension of f to an entire function of minimal exponential type.

Furthermore $f \mapsto F(f)$ is a continuous injection of $L^2(\mu)$ into $\mathcal{H}(\mathbb{C})$. In fact, for any compact set $K \subseteq \mathbb{C}$ we find by (1.1) and Parsevals formula

$$\sup_{z\in K} |F(f)(z)| \leq \|f\|_2 \sup_{z\in K} \rho(z) ,$$

where

$$\rho(z) = \left(\sum_{k=0}^{\infty} |p_k(z)|^2\right)^{\frac{1}{2}}$$

is continuous. Riesz ([8]) also showed that

$$\int_{-\infty}^{\infty} \frac{\log \rho(t)}{1+t^2}\, dt < \infty ,$$

and it follows that

$$\int_{-\infty}^{\infty} \frac{\log^+ F(f)(t)}{1+t^2}\, dt < \infty .$$

For a survey of the interplay between entire functions and indeterminate moment problems see [2].

In the following we denote by $\mathcal{C}_0$ the class of entire functions f of minimal exponential type satisfying

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)| dt}{1+t^2} < \infty .$$

It is the functions in the Cartwright class which are of minimal exponential type.

In the case of an N-extremal measure μ we have thus seen that $L^2(\mu)$ consists of entire function of class $\mathcal{C}_0$. The function $F(f)$ given by (1.1) will be called the *canonical extension* of f.

The purpose of the present paper is to establish that also for certain determinate measures $\mu \in \mathcal{M}^*$ the space $L^2(\mu)$ consists of entire functions. A determinate measure μ with this property must necessarily be discrete, as we shall see below. It turns out that $L^2(\mu)$ consists of entire functions of class $\mathcal{C}_0$, if μ is a determinate measure of finite index, meaning that there exists a polynomial p such that the measure $|p|^2\mu$ is indeterminate. If k is the smallest possible degree of a polynomial p such that $|p|^2\mu$ is indeterminate, then $k-1$ is the index of μ. This concept was studied in previous papers of the authors, cf. [4], [5].

In the case of an N-extremal measure μ the canonical extension $F(f)$ of $f \in L^2(\mu)$ has the additional property that $F(p)(z) = p(z)$ for all $z \in \mathbb{C}$, when p is a polynomial. We shall see that this property cannot subsist in the determinate case. It will be replaced by a condition which involves discrete differential operators of the form

$$T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)},\ a_{l,j} \in \mathbb{C} \tag{1.2}$$

associated to a system (z_i, k_i), $i = 1, \cdots, N$ of mutually different points $z_i \in \mathbb{C}$ and multiplicities $k_i \in \mathbb{N}$. These operators act on entire functions F via the formula

$$T(F) = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} F^{(j)}(z_l) \ .$$

It is well-known that any T of the form (1.2) has a unique continuous extension from $\mathbb{C}[t]$ to $L^2(\mu)$ if μ is N-extremal. This extension $\widetilde{T}$ satisfies

$$\widetilde{T}(f) = T(F(f)) \, , \ f \in L^2(\mu) \ , \tag{1.3}$$

where $F(f)$ is the canonical extension of $f \in L^2(\mu)$. In fact, if $(q_n) \in \mathbb{C}[t]$ converges in $L^2(\mu)$ to $f \in L^2(\mu)$ then $q_n = F(q_n)$ converges in $\mathcal{H}(\mathbb{C})$ to $F(f)$ and hence $\lim_{n\to\infty} T(q_n) = T(F(f))$. We notice that $(T(p_n)) \in \ell^2$, and if $f \in L^2(\mu)$ has the Fourier expansion $\sum c_n p_n$ then

$$\widetilde{T}(f) = \sum_{n=0}^{\infty} c_n T(p_n) \ . \tag{1.4}$$

If μ is determinate then T given by (1.2) has a (unique) continuous extension from $\mathbb{C}[t]$ to $L^2(\mu)$ if and only if $(T(p_n)) \in \ell^2$. Although $(p_n(z)) \notin \ell^2$ for $z \notin \operatorname{supp}(\mu)$, it is possible to characterize the differential operators T for which $(T(p_n)) \in \ell^2$. This was done in [5]. For determinate measures μ of finite index there are "many" of these operators, see below, and we shall prove the following:

Theorem 1.1. *Let μ be a determinate measure of finite index. Then $L^2(\mu)$ consists of entire functions of class $\mathcal{C}_0$ via a continuous linear injection $E : L^2(\mu) \to \mathcal{H}(\mathbb{C})$ with the additional property that*

$$\widetilde{T}(f) = T(E(f)) \tag{1.5}$$

for all $f \in L^2(\mu)$ and all operators T of the form (1.2) for which $(T(p_n)) \in \ell^2$.

A realization $f \mapsto E(f)$ satisfying (1.5) is not uniquely determined. We give several different realizations, and to complete the paper, we characterize for given $f \in L^2(\mu)$ the set of entire functions F satisfying

$$\widetilde{T}(f) = T(F)$$

for all operators T such that $(T(p_n)) \in \ell^2$. All these functions F turn out to be of class $\mathcal{C}_0$.

2. Preliminary results

As claimed in the introduction it imposes severe restrictions on a determinate measure μ, if $L^2(\mu)$ consists of entire functions.

Proposition 2.1. *Let $\mu \in \mathcal{M}^*$ be determinate and assume that $E : L^2(\mu) \to \mathcal{H}(\mathbb{C})$ is a realization of $L^2(\mu)$ as entire functions. Then μ is a discrete measure, and for each $z \in \mathbb{C} \setminus \mathrm{supp}(\mu)$ there exists $p \in \mathbb{C}[t]$ such that $p(z) \neq E(p)(z)$.*

Proof. If the support S of μ is non-discrete we can choose $x_0 \in S$ and a compact subset $F \subseteq S \setminus \{x_0\}$ having accumulation points. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support vanishing on F and such that $f(x_0) = 1$. The extension $E(f)$ of f to an entire function must necessarily vanish identically, but this is a contradiction.

For a discrete determinate measure μ it is known that $\sum |p_n(z)|^2 = \infty$ for all $z \notin \mathrm{supp}(\mu)$. Fix $z \notin \mathrm{supp}(\mu)$ and let us assume that the realization E has the property $E(p)(z) = p(z)$ for all $p \in \mathbb{C}[t]$. We define a sequence S_n of continuous linear functionals on ℓ^2 by

$$S_n(c) = \sum_{k=0}^{n} c_k p_k(z), \; c = (c_n) \in \ell^2 \; .$$

For any $c \in \ell^2$ there exists $f \in L^2(\mu)$ such that

$$\sum_{k=0}^{n} c_k p_k \to f \;\text{ in }\; L^2(\mu) \; ,$$

and hence

$$S_n(c) = E\left(\sum_{k=0}^{n} c_k p_k\right)(z) \to E(f)(z) \; .$$

By the Banach-Steinhaus Theorem this implies that

$$\sup_n \|S_n\| = \left(\sum_{0}^{\infty} |p_k(z)|^2\right)^{\frac{1}{2}} < \infty \; ,$$

which is a contradiction. □

The determinate measures of finite index are discrete, and we shall realize $L^2(\mu)$ as entire functions for this class of measures.

The index of determinacy of a determinate measure μ was introduced and studied by the authors in [4]. This index checks the determinacy under multiplication by even powers of $|t - z|$ for z a complex number, and it is defined as

$$\mathrm{ind}_z(\mu) = \sup\{k \in \mathbb{N} \mid |t - z|^{2k}\mu \quad \text{is determinate}\}. \tag{2.1}$$

Using the index of determinacy, determinate measures can be classified as follows:

If μ is constructed from an N-extremal measure by removing the mass at $k + 1$ points in the support, then μ is determinate with

$$\mathrm{ind}_z(\mu) = \begin{cases} k, & \text{for } z \notin \mathrm{supp}(\mu) \\ k+1, & \text{for } z \in \mathrm{supp}(\mu). \end{cases} \tag{2.2}$$

For an arbitrary determinate measure μ the index of determinacy is either infinite for every z, or finite for every z. In the latter case the index has the form (2.2), and μ is derived from an N-extremal measure by removing the mass at $k+1$ points. Such an N-extremal measure is highly non-unique by a perturbation result of Berg and Christensen, cf. [3, Theorem 8].

Using that the index of determinacy is constant at complex numbers outside of the support of μ, we define the index of determinacy of μ by

$$\operatorname{ind}(\mu) := \operatorname{ind}_z(\mu), \quad z \notin \operatorname{supp}(\mu). \tag{2.3}$$

We stress that a measure μ of finite index is discrete and $\operatorname{ind}(\mu)+1$ is the smallest degree of a polynomial p such that $|p|^2\mu$ is indeterminate.

To each measure μ which is either N-extremal or determinate of finite index we associate an entire function F_μ with simple zeros at the points of $\operatorname{supp}(\mu)$. We recall from [4] that

$$F_\mu(w) = \exp\left(-w\sum_{n=0}^{\infty}\frac{1}{x_n}\right)\prod_{n=0}^{\infty}\left(1-\frac{w}{x_n}\right)\exp(\frac{w}{x_n}), \tag{2.4}$$

where $\{x_n : n \in \mathbb{N}\}$ is the support of μ. This function F_μ is the uniquely determined entire function of minimal exponential type having $\operatorname{supp}(\mu)$ as its set of zeros and satisfying $F_\mu(0) = 1$. In the above formulation we tacitly assume $0 \notin \operatorname{supp}(\mu)$. If however $0 \in \operatorname{supp}(\mu)$, the above expression for F_μ shall be multiplied with w and $\{x_n : n \in \mathbb{N}\} = \operatorname{supp}(\mu) \setminus \{0\}$.

That F_μ is of minimal exponential type follows by a theorem of M. Riesz [8], according to which the entire functions in the Nevanlinna matrix for an indeterminate moment problem are of minimal exponential type. The function F_μ is also in the Cartwright class.

Theorem 2.2. *Let μ be N-extremal. For each $f \in L^2(\mu)$ we have*

$$F(f)(z) = \sum_{x\in\operatorname{supp}(\mu)} \frac{F_\mu(z)}{F'_\mu(x)(z-x)} f(x)\ , \quad z \in \mathbb{C}\ ,$$

where the series converges uniformly on compact subsets of $\mathbb{C}$.

Proof. Without loss of generality we may assume that $0 \in \operatorname{supp}(\mu)$, so F_μ is proportional to the function D from the Nevanlinna matrix, cf. [1], and it is well known that

$$\sum_{n=0}^{\infty} p_n(z)p_n(x) = \frac{B(z)D(x) - B(x)D(z)}{z-x}\ ,$$

cf. [4], [7], where

$$B(z) = -1 + z\sum_{n=0}^{\infty} q_n(0)p_n(z)\ .$$

Here (q_n) denotes the sequence of polynomials of the second kind given by

$$q_n(z) = \int \frac{p_n(z) - p_n(x)}{z-x} d\mu(x)\ .$$

Since D vanishes on $\mathrm{supp}(\mu)$ we get

$$F(f)(z) = \int \left(\sum_{n=0}^{\infty} p_n(z)p_n(x) \right) f(x)d\mu(x) = -D(z) \int \frac{B(x)f(x)}{z-x} d\mu(x) \, ,$$

and

$$\frac{B(x)f(x)}{z-x} = -\frac{f(x)}{z-x} + \frac{xf(x)}{z-x} \sum_{n=0}^{\infty} q_n(0)p_n(x)$$

belongs to $L^1(\mu)$ because $\sum q_n(0)p_n(x) \in L^2(\mu)$.

The mass at $x \in \mathrm{supp}(\mu)$ is given by ([1, p. 114])

$$\mu(\{x\}) = \frac{-1}{B(x)D'(x)}$$

showing that

$$F(f)(z) = \sum_{x \in \mathrm{supp}(\mu)} \frac{D(z)}{D'(x)(z-x)} f(x)$$

and the series converges uniformly on compact subsets of $\mathbb{C}$. Since D and F_μ are proportional the result follows. □

From Theorem 2.2 it is easy to verify that the realization $F(L^2(\mu))$ is a Hilbert space of entire functions in the sense of de Branges, see [6, p. 57]. For details see Corollary 3.3 below.

In [5] we obtained the following result:

Theorem 2.3. *Let $\mu \in \mathcal{M}^*$ be determinate and let (p_n) be the sequence of orthonormal polynomials corresponding to μ. Let $(z_1, k_1), \dots, (z_N, k_N)$ be given, where the z's are different complex numbers and the k's are nonnegative integers. Putting $M = \sum_{l=1}^{N}(k_l + 1)$ and*

$$\mathcal{T} = \{T = \sum_{l=1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)} \mid a_{l,j} \in \mathbb{C}\}$$

we have:

(i) *If*

$$ind(\mu) \geq \left(\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l + 1) \right) - 1,$$

then the sequence $(T(p_n))$ belongs to ℓ^2 only in the trivial cases, i.e., if and only if T is a linear combination of Dirac deltas evaluated at points z_l which are mass points of the measure μ.

(ii) *If*

$$0 \leq ind(\mu) \leq \left(\sum_{l:\mu(\{z_l\})>0} k_l + \sum_{l:\mu(\{z_l\})=0} (k_l + 1) \right) - 2,$$

then,

$$dim\left\{T \in \mathcal{T} \mid (T(p_n)) \in \ell^2\right\} = M - ind(\mu) - 1 \geq 1.$$

Furthermore, $(T(p_n)) \in \ell^2$ *if and only if* $T(z^k F_\mu(z)) = 0$ *for* $k = 0, 1, \ldots, \mathrm{ind}(\mu)$.

Corollary 2.4. *Let* $\mu \in \mathcal{M}^*$ *be a determinate measure of finite index. For an operator* $T \in \mathcal{T}$ *we have* $(T(p_n)) \in \ell^2$ *if and only if* $T(z^k F_\mu(z)) = 0$ *for* $k = 0, 1, \cdots, \mathrm{ind}(\mu)$.

Proof. It is enough to consider the case (i), and to prove that the equations $T(z^k F_\mu(z)) = 0$ for $k \leq \mathrm{ind}(\mu)$ imply that T is a linear combination of Dirac deltas at mass points of μ. To simplify the notation we assume that the system is ordered such that there exist positive integers $0 \leq N_1 \leq N_2 \leq N$ for which

$$\begin{cases} \mu(\{z_l\}) > 0 \ \text{ and } \ k_l = 0 \ \text{ for } \ l = 1, \cdots, N_1 \\ \mu(\{z_l\}) > 0 \ \text{ and } \ k_l > 0 \ \text{ for } \ l = N_1 + 1, \cdots, N_2 \\ \mu(\{z_l\}) = 0 \ \text{ for } \ l = N_2 + 1, \cdots, N \ . \end{cases}$$

Using $F_\mu(z_l) = 0$ for $l = 1, \cdots, N_2$, the equations $T(z^k F_\mu(z)) = 0$ can be written

$$\sum_{l=N_1+1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)}(z^k F_\mu(z)) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)}(z^k F_\mu(z)) = 0 \ .$$

This system has

$$p := \sum_{l=N_1+1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l + 1)$$

variables $a_{l,j}$ and $\mathrm{ind}(\mu) + 1$ equations, and $p \leq \mathrm{ind}(\mu) + 1$ since we consider the case (i). We claim that the system of equations with $k \leq p - 1$ ($\leq \mathrm{ind}(\mu)$) has a non-singular matrix, and therefore the variables involved are 0, i.e.

$$T = \sum_{l=1}^{N_2} a_{l,0} \delta_{z_l} \ .$$

The columns of the matrix can be put together in blocks

$$\left\{\delta_{z_l}^{(j)}(z^k F_\mu(z))\right\}_{\substack{k=0,\cdots,p-1 \\ j=1,\cdots,k_l}} \ , \ l = N_1 + 1, \cdots, N_2$$

and

$$\left\{\delta_{z_l}^{(j)}(z^k F_\mu(z))\right\}_{\substack{k=0,\cdots,p-1 \\ j=0,\cdots,k_l}} \ , \ l = N_2 + 1, \cdots, N \ .$$

Since $F_\mu(z_l) = 0$, $F'_\mu(z_l) \neq 0$ for $l = N_1 + 1, \cdots, N_2$ and $F_\mu(z_l) \neq 0$ for $l = N_2 + 1, \cdots, N$, column operations show that these blocks are equivalent to the blocks

$$\left\{\delta_{z_l}^{(j)}(z^k)\right\}_{\substack{k=0,\cdots,p-1 \\ j=0,\cdots,k_l-1}} \ , \ \left\{\delta_{z_l}^{(j)}(z^k)\right\}_{\substack{k=0,\cdots,p-1 \\ j=0,\cdots,k_l}} \ .$$

The determinant of the matrix formed by these blocks is a variant of Vandermondes determinant and is non-zero. □

3. The determinate case

For a given measure $\mu \in \mathcal{M}^*$ of finite index of determinacy we denote by $\mathcal{D}(\mu)$ the set of operators of the form (1.2) for which $(T(p_n)) \in \ell^2$, allowing the system (z_i, k_i) and N to vary. It is an infinite dimensional vector space. Any $T \in \mathcal{D}(\mu)$ can be extended from $\mathbb{C}[t]$ to a continuous linear operator $\tilde{T}$ in the space $L^2(\mu)$ via Fourier expansions:

$$\tilde{T}(f) = \sum_n \left(\int_{\mathbb{R}} f(t) p_n(t) d\mu(t) \right) T(p_n), \quad \text{for } f \in L^2(\mu).$$

We choose different real numbers $x_0, \cdots, x_{\mathrm{ind}(\mu)}$ outside of the support of μ and consider the measure

$$\sigma = \mu + \sum_{i=0}^{\mathrm{ind}(\mu)} \delta_{x_i} \, . \tag{3.1}$$

From the above, cf. Theorem 3.9 (1) in [4], it follows that the measure σ is N-extremal.

Given a function $f \in L^2(\mu)$, we extend it to a function $\tilde{f}$ in the space $L^2(\sigma)$ in the following way

$$\tilde{f}(t) = \begin{cases} f(t), & \text{for } t \in \mathrm{supp}(\mu) \\ 0, & \text{for } t = x_i,\ i = 0, \cdots, \mathrm{ind}(\mu). \end{cases} \tag{3.2}$$

Clearly, $f \mapsto \tilde{f}$ is a linear isometry of $L^2(\mu)$ into $L^2(\sigma)$.

Since σ is N-extremal, $\tilde{f}$ has a canonical extension to an entire function of class $\mathcal{C}_0$ given by

$$F(\tilde{f})(z) = \sum_n \left(\int_{\mathbb{R}} \tilde{f}(t) q_n(t) d\sigma(t) \right) q_n(z), \tag{3.3}$$

where (q_n) is the sequence of orthonormal polynomials with respect to σ. We can now formulate:

Theorem 3.1. *Let μ be a determinate measure with finite index of determinacy $\mathrm{ind}(\mu)$. The mapping $E(f) := F(\tilde{f})$ given by (3.3) is a realization of $L^2(\mu)$ as entire functions of class $\mathcal{C}_0$ such that for any operator $T \in \mathcal{D}(\mu)$*

$$\widetilde{T}(f) = T(E(f)) \, , \quad f \in L^2(\mu) \, . \tag{3.4}$$

Proof. It is clear that $E(f) = F(\tilde{f})$ is a realization of $L^2(\mu)$ as entire functions of class $\mathcal{C}_0$.

The set of functions $f \in L^2(\mu)$ for which (3.4) holds is a closed subspace, and therefore it suffices to prove (3.4) for $f = \chi_{\{x\}}$, $x \in \mathrm{supp}(\mu)$, where χ_A denotes the indicator function of the set A. This is a consequence of the following result:

Proposition 3.2. *For $x \in \operatorname{supp}(\mu)$ we have*

$$E(\chi_{\{x\}})(z) = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)}\ ,\ z \in \mathbb{C}\ ,$$

where p is the unique monic polynomial of degree $\operatorname{ind}(\mu)+1$ which vanishes at $x_0, \cdots, x_{\operatorname{ind}(\mu)}$.

The function

$$\frac{F_\mu(z)}{F'_\mu(x)(z-x)}$$

is an entire function of class $\mathcal{C}_0$ equal to $\chi_{\{x\}}$ on $\operatorname{supp}(\mu)$ and we have

$$\widetilde{T}(\chi_{\{x\}}) = T(E(\chi_{\{x\}})) = T\left(\frac{F_\mu(z)}{F'_\mu(x)(z-x)}\right) \quad \textit{for } T \in \mathcal{D}(\mu)\ .$$

Proof. For $f = \chi_{\{x\}}$ we find

$$\begin{aligned} \tilde{f}(t) &= \begin{cases} f(t), & \text{if } t \in \operatorname{supp}(\mu) \\ 0, & \text{for } t = x_i,\ i = 0, \cdots, \operatorname{ind}(\mu). \end{cases} \\ &= \begin{cases} 1, & \text{for } t = x, \\ 0, & \text{otherwise.} \end{cases} \\ &= \chi_{\{x\}}(t). \end{aligned}$$

For $T \in \mathcal{D}(\mu)$ we denote by $\widetilde{T}$ and $\widetilde{T}_\sigma$ the continuous extensions of T from $\mathbb{C}[t]$ to $L^2(\mu)$ and $L^2(\sigma)$ respectively. We then have $\widetilde{T}(f) = \widetilde{T}_\sigma(\tilde{f})$ for $f \in L^2(\mu)$ because $\|f-p\|_{L^2(\mu)} \le \|\tilde{f}-p\|_{L^2(\sigma)}$ when $p \in \mathbb{C}[t]$, and in particular $\widetilde{T}(\chi_{\{x\}}) = \widetilde{T}_\sigma(\chi_{\{x\}})$ when $x \in \operatorname{supp}(\mu)$.

By Theorem 2.2 we have

$$F(\tilde{f})(z) = \frac{F_\sigma(z)}{F'_\sigma(x)(z-x)} = \frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)}\ ,$$

because $F_\sigma(z) = \beta p(z)F_\mu(z)$ for a certain constant β, and hence $F'_\sigma(x) = \beta p'(x)F_\mu(x) + \beta p(x)F'_\mu(x) = \beta p(x)F'_\mu(x)$. This gives by (1.3)

$$\widetilde{T}(\chi_{\{x\}}) = T\left(\frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)}\right)\ ,$$

but since

$$\frac{F_\mu(z)p(z)}{F'_\mu(x)p(x)(z-x)} = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + q(z)F_\mu(z)\ ,$$

where

$$q(z) = \frac{p(z)-p(x)}{F'_\mu(x)(z-x)p(x)}$$

is a polynomial of degree $\operatorname{ind}(\mu)$, we have $T(qF_\mu) = 0$ by Corollary 2.4, and the second assertion follows. $\square$

Corollary 3.3. *With the notation above we have*

$$E(f)(z) = \sum_{x\in\mathrm{supp}(\mu)} \frac{F_\mu(z)p(z)}{F_\mu'(x)p(x)(z-x)} f(x) \quad \text{for } f\in L^2(\mu)\ , \tag{3.5}$$

where the series converges uniformly on compact subsets of $\mathbb{C}$.

The realization $E(L^2(\mu)) \subseteq \mathcal{H}(\mathbb{C})$ *is a Hilbert space of entire functions in the sense of de Branges.*

Proof. Formula (3.5) follows immediately from Theorem 2.2 and Proposition 3.2. To see that $E(L^2(\mu))$ is a Hilbert space of entire functions in the sense of de Branges we shall verify the properties (H1)–(H3) from [6, p. 57]. We shall only comment on (H1): If $w\in\mathbb{C}\setminus\mathbb{R}$ is a zero of $E(f)$ we have

$$\sum_{x\in\mathrm{supp}(\mu)} \frac{f(x)}{F_\mu'(x)p(x)(w-x)} = 0\ ,$$

and hence for $z\neq w$

$$\begin{aligned} E\left(f(x)\frac{x-\overline{w}}{x-w}\right)(z) &= F_\mu(z)p(z) \sum_{x\in\mathrm{supp}(\mu)} \frac{f(x)}{F_\mu'(x)p(x)(z-x)}\left(1+\frac{w-\overline{w}}{x-w}\right) \\ &= E(f)(z) + F_\mu(z)p(z)(w-\overline{w})S(z), \end{aligned}$$

where

$$S(z) = \sum_{x\in\mathrm{supp}(\mu)} \frac{f(x)}{F_\mu'(x)p(x)}\left(\frac{1}{(z-x)(x-w)} + \frac{1}{(z-w)(w-x)}\right).$$

Therefore we get

$$E\left(f(x)\frac{x-\overline{w}}{x-w}\right)(z) = E(f)(z)\frac{z-\overline{w}}{z-w},$$

which shows (H1). □

In Theorem 3.1, to get an extension of $f\in L^2(\mu)$ to an entire function, we add mass points to the measure μ until we reach an N-extremal measure σ. We next extend f by zero to a function in $L^2(\sigma)$, and use its canonical extension to an entire function. However, there is a different way to obtain N-extremal measures from a determinate measure μ having finite index of determinacy. We prove that this approach can also be used to find entire extensions of functions in $L^2(\mu)$, such that (3.4) holds.

For a determinate measure μ with finite index of determinacy $\mathrm{ind}(\mu)$, we take a polynomial

$$R(t) = \prod_{l=1}^{N}(t-z_l)^{k_l+1}\ ,\quad \text{with} \quad \sum_{l=1}^{N}(k_l+1) = \mathrm{ind}(\mu)+1\ ,$$

where $z_l \notin \text{supp}(\mu)$, $l = 1, \cdots, N$.

It follows from Lemma 2.1 in [5] that $\sigma = |R|^2\mu$ is an indeterminate measure, but the measure $|R(t)/(t-z_1)|^2 \mu$ is determinate. According to Lemma A in Section 3 of [4], we conclude that the measure $\sigma = |R|^2\mu$ is N-extremal.

Given a function $f \in L^2(\mu)$, we define $f^\natural \in L^2(\sigma)$ by $f^\natural = f/R$. Since σ is N-extremal, $f^\natural$ has a canonical extension $F(f^\natural)$ and we define

$$E(f)(z) := R(z)F(f^\natural)(z). \tag{3.6}$$

Theorem 3.4. *Let μ be a determinate measure of finite index and let R be as above. Then $L^2(\mu)$ is realized as entire functions of class $\mathcal{C}_0$ via* (3.6), *and it has the property*

$$\widetilde{T}(f) = T(E(f)) \,, \quad f \in L^2(\mu) \tag{3.7}$$

for any discrete differential operator $T \in \mathcal{D}(\mu)$.

Proof. The set of functions $f \in L^2(\mu)$ for which (3.7) holds is a closed subspace, and therefore it suffices to prove (3.7) for $f = \chi_{\{x\}}$, $x \in \text{supp}(\mu)$.

In this case $f^\natural(t) = (1/R(x))\chi_{\{x\}}(t)$, and since $F_\mu = F_\sigma$ we get

$$F(f^\natural)(z) = \frac{F_\mu(z)}{R(x)F'_\mu(x)(z-x)} \,,$$

hence

$$R(z)F(f^\natural)(z) = \frac{F_\mu(z)}{F'_\mu(x)(z-x)} + r(z)F_\mu(z),$$

where

$$r(z) = \frac{1}{R(x)F'_\mu(x)} \frac{R(z)-R(x)}{z-x}$$

is a polynomial of degree $\text{ind}(\mu)$. Now formula (3.7) follows from Corollary 2.4 and Proposition 3.2. □

Like in Corollary 3.3 we have

$$E(f)(z) = \sum_{x \in \text{supp}(\mu)} \frac{F_\mu(z)R(z)}{F'_\mu(x)R(x)(z-x)} f(x) \quad \text{for} \quad f \in L^2(\mu) \,.$$

The realization $E(L^2(\mu))$ is a Hilbert space in the sense of de Branges if R is a real polynomial.

For given $f \in L^2(\mu)$ we shall now describe the set of all entire functions F satisfying

$$\widetilde{T}(f) = T(F) \quad \text{for all} \quad T \in \mathcal{D}(\mu) \,. \tag{3.8}$$

Theorem 3.5. *Let μ be a determinate measure of finite index and let $f \in L^2(\mu)$.*

(i) *Given $(z_1, k_1), \cdots, (z_N, k_N)$, where $z_1, \cdots, z_N$ are different points of $\mathbb{C}$, $k_1, \cdots, k_N \in \mathbb{N}$, and assume that $0 \le N_2 \le N$ exists such that $z_l \in \mathrm{supp}(\mu)$ and $k_l > 0$ for $l = 1, \cdots, N_2$ and $z_l \notin \mathrm{supp}(\mu)$ for $l = N_2 + 1, \cdots, N$ and that*

$$\sum_{l=1}^{N_2} k_l + \sum_{l=N_2+1}^{N} (k_l + 1) = \mathrm{ind}(\mu) + 1 \ , \tag{3.9}$$

then there exists a unique entire function F satisfying (3.8) and the interpolation conditions

$$F^{(j)}(z_l) = \alpha_{l,j} \quad \begin{cases} j = 1, \cdots, k_l \ , \ l = 1, \cdots, N_2 \\ j = 0, \cdots, k_l \ , \ l = N_2 + 1, \cdots, N \end{cases} \tag{3.10}$$

where $\alpha_{l,j}$ are arbitrarily given. This entire function F is of class $\mathcal{C}_0$.

(ii) *If F is an entire function satisfying* (3.8)*, then $F + pF_\mu$, where p is any polynomial of degree not bigger than* $\mathrm{ind}(\mu)$*, are the only entire functions satisfying* (3.8)*. All of them are of class $\mathcal{C}_0$.*

Proof. (i) We first prove the existence. Assume that F is an entire function satisfying (3.8). From the hypothesis on the z_l's and since F_μ has simple zeros, we deduce that $F'_\mu(z_l) \neq 0$ for $l = 1, \cdots, N_2$ and $F_\mu(z_l) \neq 0$ for $l = N_2 + 1, \cdots, N$. Hence, if p denotes a polynomial, the equations

$$\delta_{z_l}^{(j)}(p(z)F_\mu)(z)) = F^{(j)}(z_l) - \alpha_{l,j} \ , \begin{cases} j = 1, \cdots, k_l, l = 1, \cdots, N_2 \\ j = 0, \cdots, k_l \ , \ l = N_2 + 1, \cdots, N \end{cases}$$

determine the quantities $p^{(j)}(z_l)$ uniquely for $j = 0, \cdots, k_l - 1, l = 1, \cdots, N_2$ and for $j = 0, \cdots, k_l, l = N_2 + 1, \cdots, N$. The hypothesis (3.9) guarantees that p is uniquely determined as a polynomial of degree $\le \mathrm{ind}(\mu)$. This means that $F - pF_\mu$ satisfies the interpolation conditions (3.10), and $F - pF_\mu$ still satisfies (3.8) by Corollary 2.4.

To prove uniqueness, assume that F and G are entire functions satisfying (3.8) and (3.10). We shall prove that $F(x) = G(x)$ for all $x \in \mathbb{C} \setminus (\mathrm{supp}(\mu) \cup \{z_{N_2+1}, \cdots, z_N\})$. This clearly implies $F \equiv G$. For x as above we consider the linear system

$$\sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)} \left(z^k F_\mu(z)\right) + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)} \left(z^k F_\mu(z)\right) = x^k F_\mu(x)$$

where $0 \le k \le \mathrm{ind}(\mu)$. The system is quadratic by (3.9), and it has a unique solution $(a_{l,j})$, cf. the proof of Corollary 2.4. This means that the operator

$$T := \sum_{l=1}^{N_2} \sum_{j=1}^{k_l} a_{l,j} \delta_{z_l}^{(j)} + \sum_{l=N_2+1}^{N} \sum_{j=0}^{k_l} a_{l,j} \delta_{z_l}^{(j)} - \delta_x$$

belongs to $\mathcal{D}(\mu)$, so $T(F) = T(G) = \widetilde{T}(f)$ by (3.8), but since F and G both satisfy (3.10) we conclude that $F(x) = G(x)$.

Since (3.8) has a solution F which is of class $\mathcal{C}_0$, the solution $F - pF_\mu$ from the existence part is again of class $\mathcal{C}_0$.

(ii) Let F, G be entire functions satisfying (3.8). The method in (i) shows that it is possible to find a polynomial p of degree $\leq \text{ind}(\mu)$ such that $G - pF_\mu$ satisfies the interpolation conditions

$$\delta_{z_l}^{(j)}(G - pF_\mu) = F^{(j)}(z_l)$$

with l, j as in (3.10). By the uniqueness assertion $G - pF_\mu = F$. □

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