A transformation from Hausdorff to Stieltjes moment sequences

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Abstract

We introduce a non-linear injective transformation $\mathcal{T}$ from the set of non-vanishing normalized Hausdorff moment sequences to the set of normalized Stieltjes moment sequences by the formula $\mathcal{T}[(a_n)]_n = 1/(a_1 \cdot \ldots \cdot a_n)$. Special cases of this transformation have appeared in various papers on exponential functionals of Lévy processes, partly motivated by mathematical finance. We give several examples of moment sequences arising from the transformation and provide the corresponding measures, some of which are related to q-series.

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1 Introduction and main results

In his fundamental memoir [23] Stieltjes characterized sequences of the form

$$s_n = \int_0^\infty x^n \, d\mu(x), \; n = 0, 1, 2, \ldots, \tag{1}$$

where μ is a non-negative measure on $[0, \infty[$, by certain quadratic forms being non-negative. These sequences are now called Stieltjes moment sequences. They are called normalized if $s_0 = 1$. A Stieltjes moment sequence is called *determinate*, if there is only one measure μ on $[0, \infty[$ such that (1) holds; otherwise it is called *indeterminate*. It is to be noticed that in the indeterminate case there are also solutions μ to (1), which are not supported by $[0, \infty[$, i.e. solutions to the

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corresponding Hamburger moment problem. However unless explicitly stated we only consider measures supported by $[0, \infty[$.

Later Hausdorff, cf. [19], characterized the Stieltjes moment sequences for which the measure is concentrated on the unit interval $[0, 1]$ by complete monotonicity. Both results can be found in [25] or in [4]. A Hausdorff moment sequence

$$a_n = \int_0^1 x^n \, d\mu(x), \; n = 0, 1, 2, \ldots, \tag{2}$$

is either non-vanishing (i.e. $a_n \neq 0$ for all n) or of the form $a_n = c\delta_{0n}$ with $c \geq 0$, where (δ_{0n}) is the sequence $(1, 0, 0, \ldots)$. The latter corresponds to the Dirac measure δ_0 with mass 1 concentrated at 0.

Our main result is the following construction of Stieltjes moment sequences from Hausdorff moment sequences.

Theorem 1.1 *Let (a_n) be a non-vanishing Hausdorff moment sequence. Then (s_n) defined by $s_0 = 1$ and $s_n = 1/(a_1 \cdot \ldots \cdot a_n)$ for $n \geq 1$ is a normalized Stieltjes moment sequence.*

The proof of Theorem 1.1, which will be given in Section 2, is rather constructive: We find explicitly a Stieltjes measure for those sequences (s_n), which are defined from the Hausdorff moment sequence of a finite linear combination of Dirac deltas. Finally we use that the set of finite linear combinations of Dirac deltas is dense in the set of positive measures supported in $[0, 1]$. To find the Stieltjes measure associated to a finite linear combination of Dirac deltas we use a technique whose philosophy goes back to Euler: Development of q-infinite products of several complex variables in power series—see for instance Chapter XVI or even Chapter X of his masterpiece *Introductio in Analysin Infinitorum*, in English version [17].

One can say that the proof could in principle have been found by Hausdorff or Stieltjes, if they had been motivated to search for such a non-linear result. We shall explain below that our motivation comes from recent work by Bertoin, Carmona, Petit and Yor on exponential functionals of Lévy processes, partly inspired by questions from mathematical finance.

Remark 1.2 If we replace the Hausdorff moment sequence (a_n) by $((1/c)a_n)$ with $c > 0$, then Theorem 1.1 gives the apparently more general result that $s_0 = 1, s_n = c^n/(a_1 \cdot \ldots \cdot a_n)$ for $n \geq 1$ is a Stieltjes moment sequence. Since however (c^n) is a Stieltjes moment sequence for any $c > 0$, and the product of two Stieltjes moment sequences is again a Stieltjes moment sequence (see below), we do not stress this more general version. In Section 3 we shall discuss the above transformation from non-vanishing normalized Hausdorff moment sequences to normalized Stieltjes moment sequences.

We recall that a function $\varphi :]0,\infty[\mapsto [0,\infty[$ is called *completely monotonic*, if it is C^∞ and $(-1)^k\varphi^{(k)}(s) \geq 0$ for $s > 0, k = 0, 1, \ldots$. By the Theorem of Bernstein we have

$$\varphi(s) = \int_0^\infty e^{-sx}\, d\alpha(x), \tag{3}$$

where α is a non-negative measure on $[0,\infty[$. Clearly $\varphi(0+) = \alpha([0,\infty[)$. If α is a non-zero finite measure, then $(a_n) = (\varphi(n))$ is a Hausdorff moment sequence such that $a_n \neq 0$ for all n, and the representing measure is the image measure of α under $x \mapsto \exp(-x)$. Conversely, any Hausdorff moment sequence (a_n) with $a_n \neq 0$ for all n is of the form

$$a_n = c\delta_{0n} + \int_0^1 x^n\, d\mu(x)$$

with $c \geq 0$ and $\mu(\{0\}) = 0, \mu \neq 0$, hence $a_n = \varphi(n), n \geq 1$ and $a_0 = c + \varphi(0+)$, where φ is given by (3), and α is the image measure of μ under $x \mapsto -\log x$.

Therefore Theorem 1.1 is essentially equivalent to the following result:

Theorem 1.3 *Let φ be a non-zero completely monotonic function. Then (s_n) defined by $s_0 = 1$ and $s_n = 1/(\varphi(1) \cdot \ldots \cdot \varphi(n))$ for $n \geq 1$ is a normalized Stieltjes moment sequence.*

Remark 1.4 If $\varphi(0+) < \infty$ the Theorems 1.1, 1.3 are equivalent, but it should be noticed that $\varphi(0+) = \infty$ is not excluded. The proof of Theorem 1.3 is given in Section 2.

The evaluation of φ at the integers can be replaced by the evaluation at the sequence $p+nq, n = 1, 2, \ldots$, where $p \geq 0, q > 0$ are real numbers. The conclusion is that $s_0 = 1,\ s_n = 1/(\varphi(p+q) \cdot \ldots \cdot \varphi(p+nq)), n \geq 1$ is a normalized Stieltjes moment sequence.

A Hausdorff moment sequence (2) is decreasing with $a_\infty := \lim_{n\to\infty} a_n = \mu(\{1\})$, and a completely monotonic function (3) is decreasing with $\varphi(\infty) := \lim_{s\to\infty} \varphi(s) = \alpha(\{0\})$. We shall now see how these quantities are related to the support of the representing measure(s) of the Stieltjes moment sequences of Theorem 1.1 and Theorem 1.3. The proof will be postponed to Section 2.

Theorem 1.5 *Let (a_n) (resp. φ) and (s_n) be as in* Theorem 1.1 (*resp.* Theorem 1.3).

If $a_\infty = 0$ (resp. $\varphi(\infty) = 0$) then any representing measure for (s_n) has unbounded support.

If $a_\infty = c > 0$ (resp. $\varphi(\infty) = c > 0$) then (s_n) is determinate and the support S of the uniquely determined representing measure satisfies $1/c \in S \subseteq [0, 1/c]$.

The sequence (s_n) is a Hausdorff moment sequence if and only if $a_\infty \geq 1$ (resp. $\varphi(\infty) \geq 1$).

Let $(\eta_t)_{t>0}$ be a convolution semigroup of sub-probabilities on $[0,\infty[$ with *Laplace exponent* or *Bernstein function* f given by

$$\int_0^\infty e^{-sx}\, d\eta_t(x) = e^{-tf(s)}, \quad s>0,$$

cf. [6],[9]. We recall that f has the integral representation

$$f(s) = a + bs + \int_0^\infty (1-e^{-sx})\, d\nu(x), \tag{4}$$

where $a,b \geq 0$ and the Lévy measure ν on $]0,\infty[$ satisfies the integrability condition $\int x/(1+x)\, d\nu(x) < \infty$. Note that $\eta_t([0,\infty[) = \exp(-at)$, so that $(\eta_t)_{t>0}$ consists of probabilities if and only if $a=0$.

In the following we shall exclude the Bernstein function identically equal to zero, which corresponds to the convolution semigroup $\eta_t = \delta_0, t>0$.

It is well-known and easy to see that $f(s)/s$ and $1/f(s)$ are completely monotonic functions, when f is a non-zero Bernstein function, viz. the Laplace transforms of the following measures

$$\lambda = b\delta_0 + (a + \nu(]x,\infty[))\, dY(x), \quad \kappa = \int_0^\infty \eta_t\, dt, \tag{5}$$

where Y denotes Lebesgue measure on $[0,\infty[$.

These two completely monotonic functions lead to the following known results as special cases of Theorem 1.3:

Corollary 1.6 ([13],[14],[24]). *Let f be a non-zero Bernstein function. Then $s_0 = 1, s_n = n!/(f(1)\cdot\ldots\cdot f(n))$ for $n\geq 1$ is a Stieltjes moment sequence.*

Corollary 1.7 ([11]). *Let f be a non-zero Bernstein function. Then $s_0 = 1, s_n = f(1)\cdot\ldots\cdot f(n)$ for $n \geq 1$ is a Stieltjes moment sequence.*

Theorems 1.1,1.3 were in fact found by searching for a result containing both Corollaries. In [11],[13], [14] the authors only consider Bernstein functions f with $a = f(0) = 0$.

It is stressed that our Theorems are more general than the results of the two Corollaries. As we shall see below in Example 2.4, the Hausdorff moment sequence (q^n) for $0<q<1$ leads to an indeterminate Stieltjes moment sequence, while the Stieltjes moment sequences of the Corollaries are always determinate as shown by the following remark.

Remark 1.8 The Stieltjes moment sequences of Corollary 1.6 and Corollary 1.7 are determinate as pointed out in [14] and [11].

First of all $s_n = n!$ is a determinate Stieltjes moment sequence of the exponential distribution $\exp(-x)\,dY(x)$. The determinacy follows from Carleman's criterion which states that the divergence of the series

$$\sum_{n=0}^{\infty} 1/\sqrt[2n]{s_n}$$

implies that the moment sequence is determinate (in the sense of Stieltjes), cf. [22]. By Stirling's formula the series in question is divergent. Since $s_n = n!/(f(1)\cdot\ldots\cdot f(n)) \le n!/(f(1))^n$, also this moment sequence is determinate. Since $f(s)/s \to b$ for $s \to \infty$, where b is the drift term in the representation (4), we see by Theorem 1.5 that the support of the representing measure for (s_n) is contained in $[0, 1/b]$. We also get that (s_n) is a Hausdorff moment sequence if and only if $b \ge 1$.

Since a Bernstein function f satisfies $f(s) \le f(1)s, s \ge 1$, we have $s_n := f(1)\cdot\ldots\cdot f(n) \le f(1)^n n!$, and the determinacy of (s_n) follows again by the criterion of Carleman. By Theorem 1.5 the support of the representing measure is contained in $[0, f(\infty)]$, and (s_n) is a Hausdorff moment sequence if and only if $f(\infty) \le 1$.

The proofs of the results in [13],[14],[11] use techniques from stochastic processes. To be more specific one considers a Lévy process $\xi = (\xi_t, t \ge 0)$ determined by the convolution semigroup $(\eta_t)_{t>0}$ corresponding to the non-zero Bernstein function f (with $f(0) = 0$), and one defines the exponential functional

$$I = \int_0^\infty \exp(-\xi_t)\,dt.$$

This random variable plays an important role in mathematical finance as well as in the study of the self-similar Markov processes obtained from ξ by a classical transformation of Lamperti, see [21]. In [13], [14], [24] it is proved that the stochastic variable I has the moments

$$\mathbb{E}(I^n) = \frac{n!}{f(1)\cdot\ldots\cdot f(n)}, \tag{6}$$

which is the Stieltjes moment sequence corresponding to the completely monotonic function $f(s)/s$.

To prove the result of [11] the authors introduce the strong Markov process $X = (X_t, t \ge 0)$ by

$$X_t = \exp(\xi_{\tau(t)}),\quad t \ge 0,$$

where the time-change $\tau(t)$ is defined by the identity

$$t = \int_0^{\tau(t)} \exp(\xi_s)\,ds.$$

They prove that the expectation of the variable $1/X_t$ is a completely monotonic function of t and thus the Laplace transform of a probability ρ. The moments of ρ are proved to be given by

$$\int_0^\infty x^n \, d\rho(x) = f(1) \cdot \ldots \cdot f(n), \tag{7}$$

which is the Stieltjes moment sequence corresponding to the completely monotonic function $1/f(s)$.

One should note that a non-zero Bernstein function f leads to the factorizations

$$1/s = [f(s)/s][1/f(s)], \quad Y = \lambda * \kappa \tag{8}$$

of respectively completely monotonic functions and measures, where we use the notation from (5). The paper [11] contains further information about the measures ρ given by (7). For further results about moments and exponential functionals see [12] and references therein.

In [20] Jacobsen and Yor consider an n-dimensional subordinator $(\xi_t)_{t>0}$ with non-vanishing Laplace exponent

$$\Phi(s) = \langle b, s\rangle + \int_{\mathbb{R}_+^n \setminus \{0\}} (1 - \exp(-\langle x, s\rangle)) \, d\nu(x), \quad s = (s_1, \ldots, s_n) \in \mathbb{R}_+^n, \tag{9}$$

where $b \in \mathbb{R}_+^n$ and ν is the Lévy measure. They prove that for any $s, t \in \mathbb{R}_+^n$, where $t \neq 0$, then

$$s_n = \prod_{k=1}^n \frac{k}{\Phi(s+kt)} \quad \text{and} \quad s_n = \prod_{k=1}^n \Phi(s+kt) \quad (s_0 = 1)$$

are Stieltjes moment sequences. These results are special cases of Theorem 1.3, because $\lambda \longrightarrow \Phi(s + \lambda t)$ is a non-vanishing Bernstein function.

In [11] Bertoin and Yor remarked that the determinacy in Remark 1.8 leads to a factorization of moments and distributions which is analogous to (8)

$$n! = [n!/(f(1) \cdot \ldots \cdot f(n))][f(1) \cdot \ldots \cdot f(n)], \quad \exp(-x) \, dY(x) = \hat{I} \diamond \rho. \tag{10}$$

Here $\hat{I}$ is the distribution of the stochastic variable I in (6), ρ is given by (7) and $\diamond$ denotes product convolution of measures on $[0, \infty[$. The product convolution $\mu \diamond \nu$ of two measures μ and ν on $[0, \infty[$ is defined as the image measure of $\mu \otimes \nu$ under the product mapping $s, t \mapsto st$. The second equation follows from the first since the n'th moment of the product convolution is the product of the n'th moments of the factors. Therefore the product convolution has the same moments as the exponential distribution, which is determinate.

Note that the second equation in (10) implies that neither $\hat{I}$ nor ρ has mass at zero.

The following result is an extension of Corollary 1.7.

Corollary 1.9 *Let f be a non-zero Bernstein function and let $c > 0$ be arbitrary. Then $s_0 = 1, s_n = (f(1) \cdot \ldots \cdot f(n))^c$ for $n \geq 1$ is a Stieltjes moment sequence, which is determinate for $c \leq 2$.*

Proof: It suffices to show that $1/f^c$ is completely monotonic, which follows since more generally $\varphi(f(s))$ is completely monotonic when φ is so, cf. [6]. Here we use the completely monotonic function $\varphi(s) = s^{-c}$. (One can also see that $1/f^c$ is the Laplace transform of the measure

$$\frac{1}{\Gamma(c)} \int_0^\infty t^{c-1} \eta_t \, dt,$$

which is the c'th convolution power of the potential kernel κ of the semigroup $(\eta_t)_{t>0}$ defined in (5).)

The criterion of Carleman used above shows the determinacy for $c \leq 2$. $\square$

Remark 1.10 There exist Bernstein functions f for which $s_n = (f(1) \cdot \ldots \cdot f(n))^c$ is indeterminate for $c > 2$. This is discussed in [5], and it proves that the assertion in Corollary 1.9 about determinacy is best possible.

2 Proofs

The set $\mathcal{S}$ of Stieltjes moment sequences (s_n) will be considered as a subset of $[0, \infty[^{\mathbb{N}_0}$ with the product topology. We need the following well-known fact about $\mathcal{S}$:

Lemma 2.1 *The set $\mathcal{S}$ is a closed set stable under pointwise sums, products and multiplication by non-negative scalars.*

Proof: We first recall that a sequence of real numbers $(s_n)_{n \geq 0}$ is called *positive definite* if all the symmetric matrices $(s_{i+j})_{0 \leq i,j \leq n}$ are non-negative, i.e.

$$\sum_{i=0}^{n} \sum_{j=0}^{n} s_{i+j} c_i c_j \geq 0 \text{ for all } (c_0, c_1, \ldots, c_n) \in \mathbb{R}^{n+1},$$

cf. [4]. The Theorem of Stieltjes tells that $(s_n) \in \mathcal{S}$ if and only if (s_n) and (s_{n+1}) are positive definite. This shows that $\mathcal{S}$ is a closed set. It is clearly stable under pointwise sums and multiplication by non-negative scalars, but it is also stable under pointwise products by the Theorem of Schur, cf. [4, p. 69]. The latter property is also a consequence of the following remark, which will be needed later: let (s_n) and (t_n) be two Stieltjes moment sequences of the measures μ and ν respectively. Then $(s_n t_n)$ is the moment sequence of the product convolution measure $\mu \diamond \nu$. $\square$

Lemma 2.2 *Let μ, ν be two measures on $[0, \infty[$ with moments of all orders and assume that μ is indeterminate, $\nu(\{0\}) = 0$ and $\nu \neq 0$. Then $\mu \diamond \nu$ is indeterminate.*

Proof: For a positive measure μ on the real line with moments of any order and corresponding sequence of orthonormal polynomials (p_n) we recall the following formula, where $z_0 \in \mathbb{C}$ is arbitrary

$$\inf \left\{ \int |p(x)|^2 \, d\mu(x) \mid p \in \mathbb{C}[x], p(z_0) = 1 \right\} = \left(\sum_{n=0}^{\infty} |p_n(z_0)|^2 \right)^{-1}, \tag{11}$$

cf. [1, p. 60].

A necessary and sufficient condition for μ to be indeterminate for the Hamburger moment problem is that the quantity (11) is strictly positive at $z_0 = i$, and if this is the case, then the function

$$\rho(z) = \left(\sum_{n=0}^{\infty} |p_n(z)|^2 \right)^{-1}$$

is strictly positive and continuous for $z \in \mathbb{C}$.

Let now μ be the measure of the lemma which by assumption is indeterminate for the Stieltjes moment problem and a fortiori for the Hamburger moment problem. Let μ' be an arbitrary of the measures on $[0, \infty[$ with the same moments as μ. Since the measures $\mu \diamond \nu$ and $\mu' \diamond \nu$ have the same moments (but we do not know if they are different), it is enough to prove that $\mu' \diamond \nu$ is indeterminate for a conveniently chosen μ'. We shall choose μ' such that $\mu'(\{0\}) = 0$, which is always possible for an indeterminate Stieltjes problem, cf. e.g. [8, Remark 2.2.2]. Without loss of generality we will therefore assume that $\mu(\{0\}) = 0$.

By assumption about ν there exists $x_0 > 0$ belonging to the support of ν. For $0 < \varepsilon < x_0$ we then have $\nu(]x_0 - \varepsilon, x_0 + \varepsilon[) > 0$.

For $p \in \mathbb{C}[x]$ satisfying $p(i) = 1$ and $y \in]x_0 - \varepsilon, x_0 + \varepsilon[$ we consider the polynomial $q_y(x) := p(xy)$ which satisfies $q_y(i/y) = 1$. By formula (11) we have

$$\int |q_y(x)|^2 \, d\mu(x) \geq \rho(i/y),$$

hence

$$\int |p(t)|^2 \, d\mu \diamond \nu(t) \geq \int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(\int |p(xy)|^2 \mu(x) \right) d\nu(y) \geq \int_{x_0-\varepsilon}^{x_0+\varepsilon} \rho(i/y) \, d\nu(y).$$

Since the last term is strictly positive and independent of the polynomial p, it follows that $\mu \diamond \nu$ is indeterminate for the corresponding Hamburger moment problem. Then it is also indeterminate as a Stieltjes problem, unless it is the N-extremal solution with mass at zero, cf. [15]. However $\mu \diamond \nu(\{0\}) = \mu([0, \infty[)\nu(\{0\}) + \mu(\{0\})\nu(]0, \infty[) = 0$, so this possibility is excluded. $\square$

Remark 2.3 It can be proved that Lemma 2.2 holds under the weaker assumption that $\nu \neq c\delta_0, c \geq 0$. In fact, if $\nu(\{0\}) > 0$ then $\nu = \nu(\{0\})\delta_0 + \nu'$ with ν' satisfying the assumptions of the Lemma. Therefore $\mu \diamond \nu'$ is indeterminate. Since $\mu \diamond \nu = \mu([0,\infty[)\nu(\{0\})\delta_0 + \mu \diamond \nu'$, it follows that also $\mu \diamond \nu$ is indeterminate.

We now give some examples of Theorem 1.1, and we shall use these as building blocks in the proof.

Example 2.4 For $0 < q \leq 1$ let $a_n = q^n$ be the Hausdorff moment sequence corresponding to the Dirac measure δ_q concentrated at q. The claim of Theorem 1.1 for this sequence is that $s_n = q^{-\binom{n+1}{2}}$ is a Stieltjes moment sequence. This is clear for $q = 1$ but in fact true also for $q < 1$, since it is the moments of the density

$$v(x) = \frac{q^{1/8}}{\sqrt{2\pi \log(1/q)}} \frac{1}{\sqrt{x}} \exp\left[-\frac{(\log x)^2}{2\log(1/q)}\right], \; x > 0$$

which is closely related to a log-normal density. There are many probabilities on $[0,\infty[$ with the same moments as v, cf. [16] for a recent paper on this indeterminate Stieltjes moment problem.

The next example involves basic hypergeometric functions, for which we refer the reader to the monograph by Gasper and Rahman [18]. We recall the q-shifted factorials

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k), z \in \mathbb{C}, 0 < q < 1, n = 1, 2, \ldots, \infty$$

and $(z;q)_0 = 1$. Note that $(z;q)_\infty$ is an entire function of z.

Example 2.5 For $c > 0$ and $0 < q < 1$ the (non-normalized) Hausdorff moment sequence $a_n = 1 + cq^{n-1}, n \geq 0$ of the measure $\delta_1 + (c/q)\delta_q$ leads by Theorem 1.1 to the sequence $s_n = 1/(-c;q)_n$. This is a Stieltjes moment sequence of the following discrete probability

$$\mu = \frac{1}{(-c;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} c^k \delta_{q^k}.$$

In fact, by the q-binomial Theorem, cf. [18], we have

$$\int x^n \, d\mu(x) = \frac{1}{(-c;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} (cq^n)^k = \frac{(-cq^n;q)_\infty}{(-c;q)_\infty} = \frac{1}{(-c;q)_n}.$$

Since the measure μ has compact support, the Stieltjes moment sequence is determinate.

The next example is an extension of Example 2.5 but more involved, and it is therefore presented as a lemma. It is the main ingredient in the proof of Theorem 1.1.

Lemma 2.6 *Let* $p \geq 1, c_j > 0, 0 < q_j < 1, j = 1, \ldots, p$ *be given. Then* $s_0 = 1$,

$$s_n = \prod_{k=0}^{n-1} (1 + c_1 {q_1}^k + \cdots + c_p {q_p}^k)^{-1}, \quad n \geq 1$$

is a Stieltjes moment sequence.

Proof: Consider the entire function of p complex variables

$$f(z_1, \ldots, z_p) = \prod_{k=0}^{\infty} (1 + z_1 {q_1}^k + \cdots + z_p {q_p}^k).$$

The power series expansion of f can be written

$$f(z) = f(z_1, \ldots, z_p) = \sum_{\alpha} b_\alpha z^\alpha,$$

where we use the multi-index notation

$$z = (z_1, \ldots, z_p), \ \alpha = (\alpha_1, \ldots, \alpha_p), \ z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_p^{\alpha_p},$$

and the sum is over all integers $\alpha_1 \geq 0, \ldots, \alpha_p \geq 0$. The coefficients $b_\alpha = b_\alpha(q)$ of the power series are positive as sums of products of powers of $q_1, \ldots, q_p$.

Let

$$\mu = \frac{1}{f(c_1, \ldots, c_p)} \sum_{\alpha} b_\alpha c^\alpha \delta_{q^\alpha}.$$

Then μ is a probability measure with compact support.

The n'th moment of μ is

$$s_n = \frac{1}{f(c)} \sum_{\alpha} b_\alpha c^\alpha (q^\alpha)^n = \frac{f(c_1 {q_1}^n, \ldots, c_p {q_p}^n)}{f(c_1, \ldots, c_p)} = \prod_{k=0}^{n-1} (1 + c_1 {q_1}^k + \cdots + c_p {q_p}^k)^{-1}.$$

□

Proof of Theorem 1.1:

Any non-negative measure μ on $[0, 1]$ is weak limit of a sequence of discrete measures of the form $a_1 \delta_{x_1} + \cdots + a_p \delta_{x_p}$, where $a_j > 0, j = 1, \ldots, p$ and $0 < x_1 < x_2 < \cdots < x_p < 1$. By the closedness of $\mathcal{S}$ stated in Lemma 2.1, it is enough to prove Theorem 1.1 for discrete measures of this type, i.e. to prove that

$$s_n = \prod_{k=1}^{n} (a_1 {x_1}^k + \cdots + a_p {x_p}^k)^{-1}, \tag{12}$$

(with $s_0 = 1$) belongs to $\mathcal{S}$.

We have

$$s_n = (1/a_p)^n (x_p)^{-\binom{n+1}{2}} \prod_{k=1}^{n} \left(1 + \frac{a_1}{a_p}(\frac{x_1}{x_p})^k + \cdots + \frac{a_{p-1}}{a_p}(\frac{x_{p-1}}{x_p})^k\right)^{-1},$$

which is the pointwise product of 3 Stieltjes moment sequences, namely $(1/a_p)^n$, and moment sequences of the type discussed in Example 2.4 and Lemma 2.6. A representing measure is the product convolution of 3 corresponding representing measures. $\square$

Remark 2.7 The moment sequence (12) is indeterminate since the factor

$$(x_p)^{-\binom{n+1}{2}}$$

is an indeterminate moment sequence, cf. Lemma 2.2.

Remark 2.8 For a Stieltjes moment sequence (s_n) all the Hankel determinants

$$H_n = \det(s_{i+j})_{0 \le i,j \le n}, \quad H'_n = \det(s_{i+j+1})_{0 \le i,j \le n}$$

are non-negative. Conversely, if for a real sequence (s_n) we have $H_n > 0, H'_n > 0$ for all $n \ge 0$, then (s_n) is a Stieltjes moment sequence. Using the special form $s_n = 1/(a_1 \cdot \ldots \cdot a_n)$ we obtain two sequences of inequalities for a non-vanishing Hausdorff moment sequence (a_n).

We have not found a proof of Theorem 1.1 by verification of the positivity of the Hankel determinants.

Proof of Theorem 1.3:

We only have to prove the result for completely monotonic functions φ with $\varphi(0+) = \infty$, since it follows from Theorem 1.1 if $\varphi(0+) < \infty$. For $\varepsilon > 0$ the function $\varphi_\varepsilon(s) = \varphi(s + \varepsilon)$ is completely monotonic with $\varphi_\varepsilon(0+) = \varphi(\varepsilon) < \infty$, so

$$s_n(\varepsilon) = \frac{1}{\varphi(1+\varepsilon) \cdot \ldots \cdot \varphi(n+\varepsilon)}$$

is a Stieltjes moment sequence. The result now follows from the closedness of $\mathcal{S}$ letting ε tend to zero. $\square$

For the proof of Theorem 1.5 we need the following elementary result.

Lemma 2.9 *Let*

$$s_n = \int_0^\infty x^n \, d\mu(x)$$

be a Stieltjes moment sequence. If $a > 0$ belongs to the support of μ, then for $0 < \varepsilon < a$ there exists $A > 0$ such that

$$s_n \geq A(a-\varepsilon)^n, \quad n \geq 0.$$

The support of μ is contained in $[0, c]$ for some $c > 0$ if and only if there exists $K > 0$ such that

$$s_n \leq Kc^n, \quad n \geq 0. \tag{13}$$

Proof: If a belongs to the support of μ and $0 < \varepsilon < a$, then $A := \mu(]a-\varepsilon, a+\varepsilon[) > 0$ and

$$s_n \geq \int_{a-\varepsilon}^{a+\varepsilon} x^n \, d\mu(x) \geq A(a-\varepsilon)^n.$$

If the support of μ is contained in $[0, c]$, then clearly $s_n \leq Kc^n$ with $K = \mu([0, \infty[)$.

Conversely, if (13) holds there cannot be a point a in the support of μ with $a > c$ by the first part of the Lemma. $\square$

Proof of Theorem 1.5:

We shall only prove the results about Hausdorff moment sequences since the other results follow in the same way.

Suppose first that $a_\infty = 0$. For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $a_n \leq \varepsilon$ for $n \geq N$ and hence for such n

$$s_n \geq \frac{\varepsilon^N}{a_1 \cdot \ldots \cdot a_N} \left(\frac{1}{\varepsilon}\right)^n.$$

Since $\varepsilon > 0$ was arbitrary, it follows by Lemma 2.9 that the support of μ is unbounded.

Suppose next that $a_\infty = c > 0$. Then clearly $s_n \leq (1/c)^n$, which shows that the support S of μ is contained in $[0, 1/c]$, and then μ is determinate.

On the other hand, since $a_n \to c$ there exists to any $\varepsilon > 0$ an $N \in \mathbb{N}$ such that

$$s_n \geq \frac{(c+\varepsilon)^N}{a_1 \cdot \ldots \cdot a_N} \left(\frac{1}{c+\varepsilon}\right)^n, \quad n \geq N.$$

This shows by Lemma 2.9 that $1/c \in S$.

If finally $a_\infty \geq 1$, then S is a subset of the unit interval, so (s_n) is a Hausdorff moment sequence. Conversely, if (s_n) is a Hausdorff moment sequence and in particular decreasing, we get from $s_n \leq s_{n-1}$ that $a_n \geq 1$ and hence $a_\infty \geq 1$. $\square$

As an application of Theorem 1.5 and Theorem 1.1 we get:

Corollary 2.10 *For an arbitrary Hausdorff moment sequence (a_n) the sequence (s_n) defined by $s_0 = 1$ and $s_n = 1/((1+a_1) \cdot \ldots \cdot (1+a_n))$ for $n \geq 1$ is a Hausdorff moment sequence.*

For a non-negative measure μ on $[0,\infty[$ with moment sequence (s_n) the moment generating function is given as

$$\int_0^\infty \exp(tx)\, d\mu(x) = \sum_{n=0}^\infty \frac{t^n}{n!} s_n. \tag{14}$$

If the radius of convergence of the power series in (14) is > 0, then it is well-known that μ is determinate.

For the moment sequences under consideration we get the following simple result.

Theorem 2.11 *Let* (a_n) *(resp.* φ*) and* (s_n) *be as in* Theorem 1.1 (*resp.* Theorem 1.3).

If $\lim_{n\to\infty} na_n = R$ *(resp.* $\lim_{n\to\infty} n\varphi(n) = R$*) then* $R \in [0,\infty]$ *is the radius of convergence of the power series in* (14).

The proof is straightforward by considering the quotient of two consecutive terms of the power series.

Applying Theorem 2.11 the determinacy discussed in Remark 1.8 can also be obtained as a consequence of the finiteness of the moment generating function (14). This was also pointed out in [14] and [11]. We give the following precise statement.

Theorem 2.12 *Let* f *be a non-zero Bernstein function with the representation* (4). *The radius of convergence* R *of the power series in* (14) *is given by*

(i) $R = f(\infty)$ *if* $s_n = n!/(f(1)\cdot \ldots \cdot f(n))$.

(ii) $R = 1/b$ *if* $s_n = f(1)\cdot \ldots \cdot f(n)$.

3 Complements and examples

Given a non-vanishing Hausdorff moment sequence (a_n) with representing measure μ, then $(c\delta_{0n} + a_n)$ is again a non-vanishing Hausdorff moment sequence for any $c \geq -\mu(\{0\})$, and they all give rise to the same normalized Stieltjes moment sequence by the construction of Theorem 1.1.

We denote by $\mathcal{T}$ the transformation from the set $\mathcal{H}_*$ of non-vanishing normalized Hausdorff moment sequences $a = (a_n)$ to the set $\mathcal{S}_*$ of normalized Stieltjes moment sequences $s = (s_n)$ given by Theorem 1.1, viz.

$$s_n = \mathcal{T}[(a_n)]_n = 1/(a_1 \cdot \ldots \cdot a_n), \;\; n \geq 1. \tag{15}$$

Note that $\mathcal{T}$ is multiplicative, i.e.

$$\mathcal{T}[(a_n b_n)] = \mathcal{T}[(a_n)]\mathcal{T}[(b_n)]. \tag{16}$$

The image of $\mathcal{H}_*$ under $\mathcal{T}$ is the set of normalized Stieltjes moment sequences (s_n) for which $a_n = s_{n-1}/s_n, n \geq 1$, is a Hausdorff moment sequence (with $a_0 = 1$). It is clear that $\mathcal{T}$ is a bijection of $\mathcal{H}_*$ onto this set.

The image is different from $\mathcal{S}_*$. In fact $s_n = n!$ is a Stieltjes moment sequence which does not belong to $\mathcal{T}(\mathcal{H}_*)$. If this sequence would belong to the image of $\mathcal{T}$, then $a_n = 1/n, n \geq 1, a_0 = 1$ should be a Hausdorff moment sequence of a measure μ, hence

$$\int_0^1 (1-x)\, d\mu(x) = 0,$$

but this is only possible if $\mu = \delta_1$ which does not have the right moments. (One can also easily see that $a_n = 1/n, n \geq 1, a_0 > 1$ can never be a Hausdorff moment sequence.)

The example just given also shows that the transformation $\mathcal{T}$ cannot be extended to a transformation of $\mathcal{S}_*$ into itself by the formula (15), because $\mathcal{T}[(n!)]_n = (1! \cdot \ldots \cdot n!)^{-1}$ is not a Stieltjes moment sequence. The reason is that the second Hankel determinant is negative.

Let $(a_n) \in \mathcal{H}_*$ with representing measure μ, and suppose that $s = \mathcal{T}[(a_n)]$ is determinate with representing measure ν, which is then uniquely determined. The equation $a_{n+1}s_{n+1} = s_n, n \geq 0$ means that the measures $(x\, d\mu(x)) \diamond (x\, d\nu(x))$ and ν have the same moments, and since ν is assumed determinate we get

$$(x\, d\mu(x)) \diamond (x\, d\nu(x)) = \nu. \tag{17}$$

By Lemma 2.2 it follows that also the measure $x\, d\nu(x)$ is determinate. The process can now be iterated, and we find that all the measures $x^n\, d\nu(x), n \geq 0$ are determinate. Using a terminology from [7] one can say that the index of determinacy of ν is infinite. See [2] for a discussion of cases, where ν is determinate but $x\, d\nu(x)$ is indeterminate.

We calculate some further values of the transformation $\mathcal{T}$.

Example 3.1 For $a > 0$ we have the following normalized Hausdorff moment sequence

$$a_n = \frac{a}{a+n} = a\int_0^1 x^{a+n-1}\, dx.$$

The corresponding Stieltjes moment sequence is

$$s_n = \frac{(a+1)\cdot \ldots \cdot (a+n)}{a^n},$$

and therefore

$$s_n(a) := (a+1)\cdot \ldots \cdot (a+n),\ s_0(a) := 1$$

is likewise a Stieltjes moment sequence, which can be written $s_n(a) = (a+1)_n$ using the Pochhammer symbol.

The sequence $(s_n(a))$ gives the moments of the Gamma distribution with density $(1/\Gamma(a+1))x^a \exp(-x)$ for $x > 0$, so $(s_n(a))$ is in fact a Stieltjes moment sequence for any $a > -1$. Note that $(s_n(a)) \notin \mathcal{T}(\mathcal{H}_*)$ for $-1 < a \leq 0$.

Example 3.2 The Stieltjes moment sequence (s_n) from Lemma 2.6 is again a normalized Hausdorff moment sequence, because the representing measure is supported by $[0,1]$, see also Corollary 2.10. Therefore we can apply $\mathcal{T}$ to this sequence and get

$$\mathcal{T}[(s_n)]_n = \prod_{k=0}^{n-1}(1 + c_1{q_1}^k + \cdots + c_p{q_p}^k)^{n-k}, \quad n \geq 1. \tag{18}$$

In particular, for $c > 0, 0 < q < 1$ we have that

$$s_n = \prod_{k=0}^{n-1}(1 + cq^k)^{n-k}, \quad n \geq 1 \tag{19}$$

is a Stieltjes moment sequence.

We shall give the representing measure for the Stieltjes moment sequence (18). To do this we consider the entire function of p complex variables

$$g(z_1, \ldots, z_p) = \prod_{k=0}^{\infty}(1 + z_1{q_1}^k + \cdots + z_p{q_p}^k)^k.$$

The power series expansion of g can be written

$$g(z) = g(z_1, \ldots, z_p) = \sum_\alpha d_\alpha z^\alpha,$$

where we use the multi-index notation as in the proof of Lemma 2.6. The coefficients $d_\alpha = d_\alpha(q)$ of the power series are positive as sums of products of powers of $q_1, \ldots, q_p$.

For $\gamma > 0$ and $c_1, \ldots, c_p > 0$ we consider the probability measure

$$\mu_{\gamma,c} = \frac{1}{g(c_1, \ldots, c_p)} \sum_\alpha d_\alpha c^\alpha \delta_{\gamma q^\alpha},$$

which is concentrated on the interval $[0, \gamma]$.

The n'th moment of $\mu_{\gamma,c}$ is

$$s_n(\mu_{\gamma,c}) = \frac{1}{g(c)} \sum_\alpha d_\alpha c^\alpha (\gamma q^\alpha)^n = \gamma^n \frac{g(c_1{q_1}^n, \ldots, c_p{q_p}^n)}{g(c_1, \ldots, c_p)},$$

which can be written

$$s_n(\mu_{\gamma,c}) = \frac{\gamma^n}{\prod_{k=0}^{n-1}(1+c_1{q_1}^k+\cdots+c_p{q_p}^k)^k}\left(\prod_{k=0}^{\infty}(1+c_1{q_1}^{n+k}+\cdots+c_p{q_p}^{n+k})\right)^{-n}.$$

With

$$\gamma = \prod_{k=0}^{\infty}(1+c_1{q_1}^k+\cdots+c_p{q_p}^k)$$

we get that $(s_n(\mu_{\gamma,c}))$ is the moment sequence (18).

In particular for $p=1, c>0$ we get that

$$\mu_{(-c;q)_\infty,c} = \frac{1}{g(c)}\sum_{\alpha=0}^{\infty} d_\alpha c^\alpha \delta_{(-c;q)_\infty q^\alpha}, \tag{20}$$

has the moments (19), where

$$g(z) = \prod_{k=0}^{\infty}(1+zq^k)^k = \sum_{\alpha=0}^{\infty} d_\alpha z^\alpha.$$

Notice that the moment sequence (19) converges to the sequence

$$(1+c)^{\binom{n+1}{2}}$$

when $q \to 1$, which is the log-normal moment sequence for the base $q = 1/(1+c)$, cf. Example 2.4. Weak accumulation points of the measures $\mu_{(-c;q)_\infty,c}$, cf. (20), for $q \to 1$ will therefore be solutions to this log-normal moment sequence.

Example 3.3 Let $0 < q < 1$ and let (a_n) be the Hausdorff moment sequence

$$a_n = \frac{1}{\log(1/q)}\frac{1-q^n}{n} = \frac{1}{\log(1/q)}\int_q^1 x^n\,\frac{dx}{x},\ n \geq 1.$$

(Notice that the right-hand side is 1 for $n = 0$.) The Stieltjes moment sequence $(\log(1/q))^{-n}\mathcal{T}[(a_n)]_n$ is $s_n = n!/(q;q)_n$. This is a determinate moment sequence, and it corresponds via Corollary 1.6 to the Bernstein function $f(s) = 1 - q^s$. The corresponding measure was found in [10] and has the density

$$i(x) = \sum_{k=0}^{\infty}\exp(-xq^{-k})\frac{(-1)^k q^{\binom{n}{2}}}{(q;q)_\infty (q;q)_k}.$$

See [10] for references to work on DNA-duplication and on Transmission Control Protocols, where this density also appears, and [3] for an analytical study.

Example 3.4 Let (a_n) be a non-vanishing Hausdorff moment sequence of a measure μ, and let $p(x) = \sum_{j=0}^m c_j x^j$ be a polynomial with positive coefficients or more generally a polynomial which is non-negative on the interval $[0,1]$.

Then

$$\tilde{a}_n = \int_0^1 x^n p(x)\, d\mu(x) = \sum_{j=0}^m c_j a_{n+j}$$

is a new Hausdorff moment sequence and this leads to the following Stieltjes moment sequence

$$s_n = \prod_{k=1}^n \left(\sum_{j=0}^m c_j a_{k+j} \right)^{-1}.$$

Taking e.g. $p(x) = 1 \pm x$ we get that

$$s_n = \prod_{k=1}^n \frac{1}{a_k + a_{k+1}}, \quad s_n = \prod_{k=1}^n \frac{1}{a_k - a_{k+1}}$$

are Stieltjes moment sequences.

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