

Small eigenvalues of large Hankel matrices: The indeterminate case *

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Abstract

In this paper we characterize the indeterminate case by the eigenvalues of the Hankel matrices being bounded below by a strictly positive constant. An explicit lower bound is given in terms of the orthonormal polynomials and we find expressions for this lower bound in a number of indeterminate moment problems.

1 Introduction

Let α be a positive measure on \mathbf{R} with infinite support and finite moments of all orders

$$(1.1) \quad s_n = s_n(\alpha) = \int_{\mathbf{R}} x^n d\alpha(x).$$

With α we associate the infinite Hankel matrix $\mathcal{H}_\infty = \{H_{jk}\}$,

$$(1.2) \quad H_{jk} = s_{j+k}.$$

Let \mathcal{H}_N be the $(N+1) \times (N+1)$ matrix whose entries are H_{jk} , $0 \leq j, k \leq N$. Since \mathcal{H}_N is positive definite, then all its eigenvalues are positive. The large N asymptotics of the smallest eigenvalue, denoted as λ_N , of the Hankel matrix \mathcal{H}_N has been studied in papers by Szegő [11], Widom and Wilf [13], Chen and Lawrence [6]. See also the monograph by Wilf [14]. All the cases considered by these authors are determinate moment problems, and it was shown in each case that $\lambda_N \rightarrow 0$, and asymptotic results were obtained about how fast λ_N tends to zero.

The smallest eigenvalue can be obtained from the classical Rayleigh quotient:

$$(1.3) \quad \lambda_N = \min \left\{ \sum_{j=0}^N \sum_{k=0}^N s_{j+k} v_j v_k : \sum_{k=0}^N v_k^2 = 1, v_j \in \mathbf{R}, 0 \leq j \leq N \right\}.$$

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It follows that λ_N is a decreasing function of N .

The main result of this paper is Theorem 1.1, which we state next.

Theorem 1.1 *The moment problem associated with the moments (1.1) is determinate if and only if $\lim_{N \rightarrow \infty} \lambda_N = 0$.*

We shall compare this result with a theorem of Hamburger [8, Satz XXXI], cf. [1, p.83] or [10, p.70]

Let μ_N be the minimum of the Hankel form \mathcal{H}_N on the hyperplane $v_0 = 1$, i.e.

$$(1.4) \quad \mu_N = \min \left\{ \sum_{j=0}^N \sum_{k=0}^N s_{j+k} v_j v_k : v_0 = 1, v_j \in \mathbf{R}, 0 \leq j \leq N \right\},$$

and let μ'_N be the corresponding minimum for the moment sequence $s'_n = s_{n+2}$, $n \geq 0$, i.e.

$$\begin{aligned} \mu'_N &= \min \left\{ \sum_{j=0}^N \sum_{k=0}^N s_{j+k+2} v'_j v'_k : v'_0 = 1, v'_j \in \mathbf{R}, 0 \leq j \leq N \right\} \\ &= \min \left\{ \sum_{j=0}^{N+1} \sum_{k=0}^{N+1} s_{j+k} v_j v_k : v_0 = 0, v_1 = 1, v_j \in \mathbf{R}, 0 \leq j \leq N+1 \right\}. \end{aligned}$$

The theorem of Hamburger can be stated that the moment problem is determinate if and only if at least one of the limits $\lim_{N \rightarrow \infty} \mu_N$, $\lim_{N \rightarrow \infty} \mu'_N$ are zero.

It is clear from (1.3), (1.4) that $\mu_N \geq \lambda_N$ and similarly $\mu'_N \geq \lambda_{N+1}$. From these inequalities and Hamburger's theorem, we obtain the "only if" statement in Theorem 1.1. The "if" statement will be proved by finding a positive lower bound for the eigenvalues λ_N in the indeterminate case, cf. Theorem 1.2 below.

We think that Theorem 1.1 has the advantage over the theorem of Hamburger that it involves only the moment sequence (s_n) and not the shifted sequence (s_{n+2}) . In section 2 we give another proof of the "only if" statement to make the proof of Theorem 1.1 independent of Hamburger's theorem.

If

$$(1.5) \quad \pi_N(x) := \sum_{j=0}^N v_j x^j, \quad v_j \in \mathbf{R}$$

then a simple calculation shows that

$$(1.6) \quad \sum_{0 \leq j, k \leq N} s_{j+k} v_j v_k = \int_E \pi_N^2(x) d\alpha(x),$$

and

$$(1.7) \quad \sum_{k=0}^N v_k^2 = \int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi}.$$

We could also study the reciprocal of λ_N given by

$$(1.8) \quad \frac{1}{\lambda_N} = \max \left\{ \int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} : \pi_N, \int_E \pi_N^2(x) d\alpha(x) = 1 \right\}.$$

Let $\{p_k\}$ denote the orthonormal polynomials with respect to α , normalized so that p_k has positive leading coefficient.

We recall that the moment problem is indeterminate, cf. [1],[10], if and only if there exists a non-real number z_0 such that

$$(1.9) \quad \sum_{k=0}^{\infty} |p_k(z_0)|^2 < \infty.$$

In the indeterminate case the series in (1.9) actually converges for all z_0 in \mathbf{C} , uniformly on compact sets. In the determinate case the series in (1.9) diverges for all non-real z_0 and also for all real numbers except the at most countably many points, where α has a positive mass.

If we expand the polynomial (1.5) as a linear combination of the orthonormal system

$$\pi_N(x) = \sum_{j=0}^N c_j p_j(x),$$

then

$$\int_0^{2\pi} \left| \pi_N(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi} = \sum_{0 \leq j, k \leq N} c_j c_k \int_0^{2\pi} p_j(e^{i\theta}) p_k(e^{-i\theta}) \frac{d\theta}{2\pi} = \sum_{0 \leq j, k \leq N} \mathcal{K}_{jk} c_j c_k,$$

where we have defined

$$(1.10) \quad \mathcal{K}_{jk} = \int_0^{2\pi} p_j(e^{i\theta}) p_k(e^{-i\theta}) \frac{d\theta}{2\pi}.$$

Thus

$$(1.11) \quad \frac{1}{\lambda_N} = \max \left\{ \sum_{0 \leq j, k \leq N} \mathcal{K}_{jk} c_j c_k : c_j, \sum_{j=0}^N c_j^2 = 1 \right\}.$$

Since the eigenvalues of the matrix $(\mathcal{K}_{jk})_{0 \leq j, k \leq N}$ are positive, and their sum is its trace, then

$$(1.12) \quad \frac{1}{\lambda_N} \leq \sum_{k=0}^N \mathcal{K}_{kk} = \int_0^{2\pi} \sum_{k=0}^N \left| p_k(e^{i\theta}) \right|^2 \frac{d\theta}{2\pi}.$$

Thus in the case of indeterminacy,

$$(1.13) \quad \frac{1}{\lambda_N} \leq \int_0^{2\pi} \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty,$$

which shows that

$$(1.14) \quad \lim_{N \rightarrow \infty} \lambda_N \geq \left(\int_0^{2\pi} \frac{1}{\rho(e^{i\theta})} \frac{d\theta}{2\pi} \right)^{-1},$$

where

$$(1.15) \quad \rho(e^{i\theta}) = \left(\sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 \right)^{-1}.$$

We recall that for $z \in \mathbf{C} \setminus \mathbf{R}$ the number $\rho(z)/|z - \bar{z}|$ is the radius of the Weyl circle at z .

The above argument establishes the following result:

Theorem 1.2 *In the indeterminate case the smallest eigenvalue λ_N of the Hankel matrix \mathcal{H}_N is bounded below by the harmonic mean of the function ρ along the unit circle.*

We shall conclude this paper with examples, where we have calculated or estimated the quantity

$$(1.16) \quad \rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

This will be done for the moment problems associated with the Stieltjes-Wigert polynomials, cf. [4],[12], the Al-Salam-Carlitz polynomials [2], the symmetrized version of polynomials of Berg-Valent ([3]) leading to a Freud-like weight [5], and the q^{-1} -Hermite polynomials of Ismail and Masson [9].

If we introduce the coefficients of the orthonormal polynomials as

$$(1.17) \quad p_k(x) = \sum_{j=0}^k \beta_{k,j} x^j$$

then

$$\int_0^{2\pi} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{j=0}^k \beta_{k,j}^2,$$

and therefore

$$(1.18) \quad \rho_0 = \sum_{k=0}^{\infty} \sum_{j=0}^k \beta_{k,j}^2.$$

Another possibility for calculating ρ_0 is to use the entire functions B, D from the Nevanlinna matrix since it is well known that [1, p.123]

$$(1.19) \quad \sum_{k=0}^{\infty} |p_k(z)|^2 = \frac{B(z)D(\bar{z}) - D(z)B(\bar{z})}{z - \bar{z}}.$$

It follows that

$$(1.20) \quad \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 = \text{Im}\{B(e^{i\theta})D(e^{-i\theta})\} / \sin \theta.$$

2 Indeterminate Moment Problems

In this section we shall give a proof of Theorem 1.1 which is independent of Hamburgers result. We have already established that if $\lim_{N \rightarrow \infty} \lambda_N = 0$, then the problem is determinate. We shall next prove that if $\lambda_N \geq \gamma$ for all N , where $\gamma > 0$, then the problem is indeterminate. Since $1/\lambda_N \leq 1/\gamma$ for all N , and $1/\lambda_N$ is the biggest eigenvalue of the positive definite matrix $(\mathcal{K}_{jk})_{0 \leq j, k \leq N}$, we get

$$(2.1) \quad \sum_{0 \leq j, k \leq N} \mathcal{K}_{jk} c_j \bar{c}_k \leq \frac{1}{\gamma} \sum_{j=0}^N |c_j|^2,$$

for all vectors $(c_0, \dots, c_N) \in \mathbf{C}^{N+1}$. If we consider an arbitrary complex polynomial p of degree $\leq N$ written as $p(x) = \sum_{k=0}^N c_k p_k(x)$, the inequality (2.1) can be formulated

$$(2.2) \quad \int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \leq \frac{1}{\gamma} \int |p(x)|^2 d\alpha(x).$$

Let now z_0 be an arbitrary non-real number in the open unit disc. By the Cauchy integral formula

$$p(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(e^{i\theta})}{e^{i\theta} - z_0} e^{i\theta} d\theta,$$

and therefore

$$(2.3) \quad |p(z_0)|^2 \leq \int_0^{2\pi} |p(e^{i\theta})|^2 \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\theta} - z_0|^2} \frac{d\theta}{2\pi}.$$

Combined with (2.2) we see that there is a constant K such that for all complex polynomials p

$$(2.4) \quad |p(z_0)|^2 \leq K \int |p(x)|^2 d\alpha(x),$$

where $K = 1/(\gamma(1 - |z_0|^2))$.

This inequality implies indeterminacy in the following way. Applying it to the polynomial

$$p(x) = \sum_{k=0}^N p_k(\overline{z_0}) p_k(x),$$

we get

$$(2.5) \quad \sum_{k=0}^N |p_k(z_0)|^2 \leq K,$$

and since N is arbitrary, indeterminacy follows.

Remark. We see that the infinite positive definite matrix $\mathcal{K}_\infty = \{\mathcal{K}_{j,k}\}$ is bounded on ℓ^2 if and only if $\lambda_N \geq \gamma$ for all N for some $\gamma > 0$. Furthermore \mathcal{K}_∞ is of trace class if and only if $\rho_0 < \infty$. The result of Theorem 1.1 can be reformulated to say that boundedness implies trace class for this family of operators.

3 Examples

We shall follow the notation and terminology for q -special functions as those in Gasper and Rahman [7].

Example 1. The Stieltjes-Wigert Polynomials.

These polynomials are orthonormal with respect to the weight function

$$(3.1) \quad \omega(x) = \frac{k}{\sqrt{\pi}} \exp(-k^2(\log x)^2), \quad x > 0,$$

where $k > 0$ is a positive parameter, cf. [4],[12]. They are given by

$$(3.2) \quad p_n(x) = (-1)^n q^{\frac{n}{2} + \frac{1}{4}} (q; q)_n^{-\frac{1}{2}} \sum_{k=0}^n \binom{n}{k}_q q^{k^2} (-q^{\frac{1}{2}} x)^k,$$

where we have defined $q = \exp\{-(2k^2)^{-1}\}$.

It follows by (1.18) that

$$(3.3) \quad \begin{aligned} \rho_0 &= \sum_{n=0}^{\infty} \frac{q^{n+\frac{1}{2}}}{(q; q)_n} \sum_{k=0}^n q^{k(2k+1)} \binom{n}{k}_q^2 \\ &= \sum_{k=0}^{\infty} q^{2k^2+k+\frac{1}{2}} \sum_{n=k}^{\infty} \frac{q^n}{(q; q)_n} \binom{n}{k}_q^2. \end{aligned}$$

Putting $n = k + j$, the inner sum is

$$\sum_{j=0}^{\infty} \frac{q^{k+j}}{(q; q)_k^2} \frac{(q; q)_{k+j}}{(q; q)_j^2} = \frac{q^k}{(q; q)_k} {}_2\phi_1(q^{k+1}, 0; q; q)$$

and hence

$$(3.4) \quad \rho_0 = \sum_{k=0}^{\infty} \frac{q^{2(k+\frac{1}{2})^2}}{(q; q)_k} {}_2\phi_1(0, q^{k+1}; q; q).$$

We can obtain another expression for ρ_0 . We apply the transformation [7, (III.5)]

$$(3.5) \quad {}_2\phi_1(a, b; c; q; z) = \frac{(abz/c; q)_{\infty}}{(bz/c; q)_{\infty}} {}_3\phi_2(a, c/b, 0; c, cq/bz; q, q)$$

to see that

$$(3.6) \quad \sum_{n=k}^{\infty} \frac{q^n}{(q; q)_n} \binom{n}{k}_q^2 = \frac{1}{(q; q)_{\infty}} \sum_{j=0}^k \frac{q^{k+j}}{(q; q)_j^2}.$$

We then find

$$(3.7) \quad \rho_0 = \frac{1}{(q; q)_{\infty}} \sum_{k=0}^{\infty} q^{2(k+\frac{1}{2})^2} \sum_{j=0}^k \frac{q^j}{(q; q)_j^2}.$$

A formula more general than (3.6) is

$$\sum_{n=k}^{\infty} \frac{\omega^n}{(q; q)_n} \binom{n}{k}_q^2 = \frac{1}{(\omega; q)_{\infty}} \sum_{j=0}^k \frac{(\omega; q)_j \omega^{2k-j}}{(q; q)_j (q; q)_{k-j}^2}$$

and is stated in [2]. This more general identity also follows from (3.5) and the simple observation

$$\frac{(q^{-k}; q)_j}{(q^{1-k}/\omega; q)_j} = \frac{(q; q)_k (\omega; q)_{k-j}}{(\omega; q)_k (q; q)_{k-j}} (\omega/q)^j.$$

We have numerically computed the smallest eigenvalue of the Hankel matrix of various dimensions with the Stieltjes–Wigert weight from which we extrapolate to determine the smallest eigenvalue $s = \lim_{N \rightarrow \infty} \lambda_N$ of the infinite Hankel matrix for different values of q . This is then compared with the numerically computed lower bound $l = 1/\rho_0$. For $q = \frac{1}{2}$ we have $s = 0.3605\dots$, $l = 0.3435\dots$. The percentage error $100(s - l)/s$ is plotted for various values of q and is shown in figure 1.

Example 2. Al-Salam–Carlitz polynomials.

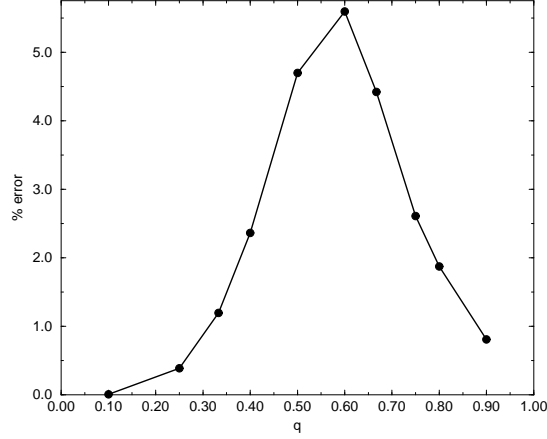


Figure 1: Percentage error plotted for various values of q .

The Al-Salam-Carlitz polynomials were introduced in [2]. We consider the indeterminate polynomials $V_n^{(a)}(x; q)$, where $0 < q < 1$ and $q < a < 1/q$, cf. [3]. For the corresponding orthonormal polynomials $\{p_k\}$ we have by [3, (4.24)]

$$(3.8) \quad \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 = \frac{(qe^{i\theta}, qe^{-i\theta}; q)_{\infty}}{(aq, q, q; q)_{\infty}} {}_3\phi_2(e^{i\theta}, e^{-i\theta}, aq; qe^{i\theta}, qe^{-i\theta}; q, q/a).$$

Therefore

$$(3.9) \quad \rho_0 = \int_0^{2\pi} \sum_{k=0}^{\infty} |p_k(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \frac{1}{(aq, q, q; q)_{\infty}} \sum_{n=0}^{\infty} I_n \frac{(aq; q)_n}{(q; q)_n} \left(\frac{q}{a}\right)^n,$$

where

$$(3.10) \quad \begin{aligned} I_n &= \int_0^{2\pi} \frac{(e^{i\theta}, e^{-i\theta}; q)_{\infty}}{(1 - q^n e^{i\theta})(1 - q^n e^{-i\theta})} \frac{d\theta}{2\pi} \\ &= \int_{|z|=1} \frac{(z, 1/z; q)_{\infty}}{(1 - q^n z)(1 - q^n/z)} \frac{dz}{2\pi i z}. \end{aligned}$$

Recall the Jacobi triple product identity [7],

$$(3.11) \quad j(z) := (q, z, 1/z; q)_{\infty} = \sum_{k=-\infty}^{\infty} c_k z^k,$$

with

$$(3.12) \quad c_k = (-1)^k \left[q^{k(k+1)/2} + q^{k(k-1)/2} \right].$$

Note that $c_k = c_{-k}$.

Using the partial fraction decomposition

$$\frac{q^n}{1 - q^n z} - \frac{q^{-n}}{1 - q^{-n} z} = \frac{1 - q^{2n}}{(1 - q^n z)(z - q^n)}$$

we find by the residue theorem and the Jacobi triple product identity (3.11) that for $n \geq 1$, I_n is given by

$$\begin{aligned} (1 - q^{2n})(q; q)_\infty I_n &= q^n \operatorname{Res} \left(\frac{j(z)}{1 - q^n z}, z = 0 \right) - q^{-n} \operatorname{Res} \left(\frac{j(z)}{1 - q^{-n} z}, z = 0 \right) \\ &= q^n \sum_{k=0}^{\infty} q^{nk} c_{-k-1} - q^{-n} \sum_{k=0}^{\infty} q^{-nk} c_{-k-1} \\ &= \sum_{k=1}^{\infty} (q^{nk} - q^{-nk}) c_k, \end{aligned}$$

while for $n = 0$, I_0 is

$$\begin{aligned} (q; q)_\infty I_0 &= \int_{|z|=1} \frac{j(z)}{(1-z)(z-1)} \frac{dz}{2\pi i} = -\operatorname{Res} \left(\frac{j(z)}{(1-z)^2}, z = 0 \right) \\ &= -\sum_{k=0}^{\infty} (k+1) c_{-k-1} = \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2}. \end{aligned}$$

The conclusion is

$$(3.13) \quad \begin{aligned} I_0 &= \frac{1}{(q; q)_\infty} \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2}, \\ I_n &= \frac{1}{(1 - q^{2n})(q; q)_\infty} \sum_{k=1}^{\infty} c_k (q^{nk} - q^{-nk}), \quad n \geq 1. \end{aligned}$$

The above formulas can be further simplified. Using the Jacobi triple product identity (3.11) we find for integer values of n

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{nk} q^{\binom{k}{2}} = 0,$$

hence

$$(3.14) \quad \sum_{k=0}^{\infty} (-1)^k q^{nk} q^{\binom{k}{2}} = -\sum_{k=1}^{\infty} (-1)^k q^{-nk} q^{\binom{k+1}{2}}, \quad n = 0, \pm 1, \dots$$

This analysis implies

$$(3.15) \quad (q; q)_\infty (1 - q^{2n}) I_n = 2 \sum_{k=1}^{\infty} (-1)^k q^{\binom{k}{2}} [q^{nk} - q^{-nk}].$$

Thus we have established the representation for $n \geq 1$

$$(3.16) \quad I_n = \frac{2q^{-n}}{(q; q)_\infty} \sum_{k=1}^{\infty} (-1)^{k-1} q^{\binom{k}{2}} \frac{\sin(nk\tau)}{\sin(n\tau)}, \quad q = e^{-i\tau}.$$

It is clear that I_0 is the limiting case of I_n as $n \rightarrow 0$. The representation (3.16) indicates that I_n is a theta function evaluated at the special point $n\tau$, hence we do not expect to find a closed form expression for I_n .

Example 3. Freud-like weight.

In [3] Berg-Valent found the Nevanlinna matrix in the case of the indeterminate moment problem corresponding to a birth and death process with quartic rates. Later Chen and Ismail, cf. [5], considered the corresponding symmetrized moment problem, found the Nevanlinna matrix and observed that there are solutions which behave as the Freud weight $\exp(-\sqrt{|x|})$. In particular they found the entire functions

$$(3.17) \quad B(z) = -\delta_0(K_0\sqrt{z/2}), \quad D(z) = \frac{4}{\pi}\delta_2(K_0\sqrt{z/2}),$$

where

$$(3.18) \quad \delta_l(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+l)!} z^{4n+l}, \quad l = 0, 1, 2, 3,$$

$$(3.19) \quad K_0 = \frac{\Gamma(1/4)\Gamma(5/4)}{\sqrt{\pi}}.$$

Note that

$$(3.20) \quad \delta_0(z) = \frac{1}{2} [\cosh(z\sqrt{i}) + \cos(z\sqrt{i})],$$

$$(3.21) \quad \delta_2(z) = \frac{1}{2i} [\cosh(z\sqrt{i}) - \cos(z\sqrt{i})].$$

If $\omega := \exp(i\pi/4) = (1+i)/\sqrt{2}$, then a simple calculation shows that

$$(3.22) \quad \begin{aligned} & B(x)D(y) - D(x)B(y) \\ &= \frac{-2i}{\pi} \left[\cos(\omega^3 K_0 \sqrt{x/2}) \cos(\omega K_0 \sqrt{y/2}) - \cos(\omega^3 K_0 \sqrt{y/2}) \cos(\omega K_0 \sqrt{x/2}) \right]. \end{aligned}$$

If $x = e^{i\theta}$, and $y = e^{-i\theta}$, then we linearize the products of cosines and find that the right-hand side of (3.22) is

$$\begin{aligned} & \frac{-i}{\pi} \left\{ \cos[K_0(\omega^3 e^{i\theta/2} + \omega e^{-i\theta/2})/\sqrt{2}] + \cos[K_0(\omega^3 e^{i\theta/2} - \omega e^{-i\theta/2})/\sqrt{2}] \right. \\ & \quad \left. - \cos[K_0(\omega^3 e^{-i\theta/2} + \omega e^{i\theta/2})/\sqrt{2}] - \cos[K_0(\omega^3 e^{-i\theta/2} - \omega e^{i\theta/2})/\sqrt{2}] \right\} \end{aligned}$$

We now combine the first and third terms, then combine the second and fourth terms and apply the addition theorem for trigonometric functions. We then see that the above is

$$\frac{2i}{\pi} \{ \sinh[K_0 \cos(\theta/2)] \sinh[K_0 \sin(\theta/2)] + \sin[K_0 \cos(\theta/2)] \sin[K_0 \sin(\theta/2)] \}.$$

Thus we have proved that

$$(3.23) \quad \frac{B(e^{i\theta})D(e^{-i\theta}) - B(e^{-i\theta})D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} \\ = \frac{1}{\pi \sin \theta} \{ \sinh[K_0 \cos(\theta/2)] \sinh[K_0 \sin(\theta/2)] + \sin[K_0 \cos(\theta/2)] \sin[K_0 \sin(\theta/2)] \}.$$

Thus in the case under consideration, after some straightforward calculations and the evaluation of a beta integral, we obtain

$$(3.24) \quad \rho_0 = \int_0^{2\pi} \sum_{n=0}^{\infty} |p_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \frac{K_0^2}{\pi} \sum_{m,n \geq 0, m+n \text{ even}} \frac{(K_0/2)^{2m+2n}}{(2m+1)(2n+1)m!n!(m+n)!}.$$

Example 4. q^{-1} -Hermite polynomials.

Ismail and Masson [9] proved that for this moment problem the functions B and D are given by

$$(3.25) \quad B(\sinh \xi) = -\frac{(qe^{2\xi}, qe^{-2\xi}; q^2)_{\infty}}{(q, q; q^2)_{\infty}}, \quad D(\sinh \xi) = \frac{\sinh \xi}{(q; q)_{\infty}} (q^2 e^{2\xi}, q^2 e^{-2\xi}; q^2)_{\infty},$$

[9, (5.32)], [9, (5.36)]; respectively. Ismail and Masson also showed that [9, (6.25)]

$$(3.26) \quad B(\sinh \xi)D(\sinh \eta) - B(\sinh \eta)D(\sinh \xi) \\ = \frac{-e^{\eta}}{2(q; q)_{\infty}} \prod_{n=0}^{\infty} [1 - 2e^{-\eta}q^n \sinh \xi - e^{-2\eta}q^{2n}] [1 + 2e^{\eta}q^{n+1} \sinh \xi - e^{2\eta}q^{2n+2}].$$

We rewrite the infinite product as

$$\prod_{n=0}^{\infty} a_n b_n = a_0 \prod_{n=1}^{\infty} a_n b_{n-1},$$

and with $\sinh \xi = e^{i\theta}$ and $\sinh \eta = e^{-i\theta}$ we get the following representation

$$(3.27) \quad \frac{B(e^{i\theta})D(e^{-i\theta}) - B(e^{-i\theta})D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} \\ = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} [1 + 4q^n - 2q^{2n} + 4q^{3n} + q^{4n} - 8q^{2n} \cos(2\theta)] \\ = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} [(1 + q^n)^4 - 16q^{2n} \cos^2 \theta].$$

Writing the infinite product as a power series in $\cos^2 \theta$ and using

$$\int_{-\pi}^{\pi} \cos^{2k} \theta \frac{d\theta}{2\pi} = 2^{-2k} \binom{2k}{k},$$

we evaluate the integral of (3.27) with respect to $d\theta/2\pi$ as

$$(3.28) \quad \rho_0 = \frac{(-q; q)_{\infty}^4}{(q; q)_{\infty}} \sum_{k=0}^{\infty} \binom{2k}{k} \sum_{1 \leq n_1 < \dots < n_k} \frac{(-2)^{2k} q^{2(n_1 + \dots + n_k)}}{[(1 + q^{n_1}) \dots (1 + q^{n_k})]^4}.$$

The formula (3.27) can be transformed further by putting $\cos^2 \psi = -\cos \theta$ and $p^2 = q$, because then

$$\prod_{n=1}^{\infty} [(1 + q^n)^2 + 4q^n \cos \theta] = \prod_{n=1}^{\infty} [1 + p^{4n} - 2p^{2n} \cos(2\psi)]$$

can be expressed by means of the theta function $\vartheta_1(p; \psi)$. We find

$$(3.29) \quad \prod_{n=1}^{\infty} [(1 + q^n)^2 + 4q^n \cos \theta] = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} U_{2n}(\cos \psi),$$

where

$$U_{2n}(\cos \psi) = \frac{\sin(2n+1)\psi}{\sin \psi}$$

is the Chebyshev polynomial of the second kind given by

$$(3.30) \quad U_{2n}(x) = \sum_{k=0}^n \binom{2n+1}{2k+1} (-1)^k x^{2(n-k)} (1-x^2)^k.$$

Similarly putting $\cos^2 \varphi = \cos \theta$ we find

$$(3.31) \quad \prod_{n=1}^{\infty} [(1 + q^n)^2 - 4q^n \cos \theta] = \frac{1}{(q; q)_{\infty}} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n+1}{2}} U_{2n}(\cos \varphi).$$

If we let U_n^* be the polynomial of degree n such that $U_{2n}(x) = U_n^*(x^2)$, we get

$$(3.32) \quad \frac{B(e^{i\theta})D(e^{-i\theta}) - B(e^{-i\theta})D(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} \\ = \frac{1}{(q; q)_{\infty}^2} \sum_{n,m=0}^{\infty} (-1)^m q^{\binom{n+1}{2} + \binom{m+1}{2}} U_n^*(-\cos \theta) U_m^*(\cos \theta).$$

For non-negative integers k, l, r we have

$$(3.33) \quad \begin{aligned} C(k, l, r) &:= \frac{1}{2\pi} \int_0^{2\pi} (1 + \cos \theta)^k (1 - \cos \theta)^l \cos^r \theta \, d\theta \\ &= \frac{2^{k+l}}{\pi} (-1)^r B\left(k + \frac{1}{2}, l + \frac{1}{2}\right) {}_2F_1\left(k + \frac{1}{2}, -r; k + l + 1; 2\right), \end{aligned}$$

which gives

$$(3.34) \quad \begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} U_n^*(-\cos \theta) U_m^*(\cos \theta) \, d\theta \\ &= \sum_{k=0}^n \sum_{l=0}^m \binom{2n+1}{2k+1} \binom{2m+1}{2l+1} (-1)^{n+l} C(k, l, n+m-k-l). \end{aligned}$$

Putting these formulas together we get a 5-fold sum for ρ_0 .

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