# On a fixed point in the metric space of normalized Hausdorff moment sequences

Christian Berg* and Maryam Beygmohammadi

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**Abstract**

We show that the transformation $(x_n)_{n\geq 1} \to (1/(1+x_1+\ldots+x_n))_{n\geq 1}$ of the compact set of sequences $(x_n)_{n\geq 1}$ of numbers from the unit interval $[0,1]$ has a unique fixed point, which is attractive. The fixed point turns out to be a Hausdorff moment sequence studied in [3].

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## 1 Introduction

Let $\mathcal{K} = [0,1]^{\mathbb{N}}$ denote the product space of sequences $(x_n) = (x_n)_{n\geq 1}$ of numbers from the unit interval $[0,1]$.

We consider a transformation $T : \mathcal{K} \to \mathcal{K}$ defined by

$$(T(x_n))_n = \frac{1}{1+x_1+\ldots+x_n}, \quad n \geq 1. \tag{1}$$

Since $T$ is a continuous transformation of the compact convex set $\mathcal{K}$ in the space $\mathbb{R}^{\mathbb{N}}$ of real sequences equipped with the product topology, it has a fixed point $(m_n)$ by Tychonoff's extension of Brouwer's fixed point theorem. Furthermore, it is clear by (1) that the fixed point $(m_n)$ is uniquely determined by the equations

$$(1+m_1+\ldots+m_n)m_n = 1, \quad n \geq 1. \tag{2}$$

Therefore

$$m_{n+1}^2 + \frac{m_{n+1}}{m_n} - 1 = 0, \tag{3}$$

*Corresponding author

giving

$$m_1 = \frac{-1+\sqrt{5}}{2}, \quad m_2 = \frac{\sqrt{22+2\sqrt{5}}-\sqrt{5}-1}{4}, \ldots .$$

Berg and Durán studied this fixed point in [3], and it was proved that $m_0 = 1, m_1, m_2, \ldots$ is a normalized Hausdorff moment sequence, i.e., of the form

$$m_n = \int_0^1 x^n \, d\tau(x), \tag{4}$$

where $\tau$ is a probability measure on the interval $[0,1]$. For details about the measure $\tau$, see [3] and [4]. We mention that $\tau$ has an increasing and convex density with respect to Lebesgue measure.

There is no reason a priori that the fixed point $(m_n)$ should be a Hausdorff moment sequence, but the motivation for the study of $T$ came from the theory of moment sequences because of the following theorem from [2]:

**Theorem 1.1** *Let $(a_n)_{n\geq 0}$ be a Hausdorff moment sequence of a measure $\mu \neq 0$ on $[0,1]$. Then the sequence $(b_n)_{n\geq 0}$ defined by $b_n = 1/(a_0 + \ldots + a_n)$ is again a Hausdorff moment sequence, and its associated measure $\nu = \widehat{T}(\mu)$ has the properties $\nu(\{0\}) = 0$ and*

$$\int_0^1 \frac{1-t^{z+1}}{1-t} \, d\mu(t) \int_0^1 t^z \, d\nu(t) = 1 \quad \textit{for} \quad \Re z \geq 0. \tag{5}$$

Let $\mathcal{H}$ denote the set of normalized Hausdorff moment sequences, where we throw away the zero'th moment which is always 1, i.e.,

$$\mathcal{H} = \{(a_n)_{n\geq 1} \mid a_n = \int_0^1 x^n \, d\mu(x), \ \mu([0,1]) = 1\}. \tag{6}$$

Clearly, $\mathcal{H} \subset \mathcal{K}$ and by Theorem 1.1 we have $T(\mathcal{H}) \subseteq \mathcal{H}$. It is easy to see that $\mathcal{H}$ is a compact convex subset of $\mathcal{K}$, e.g., by using Hausdorff's 1921 characterization of Hausdorff moment sequences as completely monotonic sequences, i.e., sequences $(a_n)_{n\geq 0}$ satisfying

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} \geq 0 \quad \text{for} \quad m, n \geq 0. \tag{7}$$

See [1] for details.

It was proved in Theorem 2.3 in [3] that $(m_n)$ is an attractive fixed point of the restriction of $T$ to $\mathcal{H}$. This proof was a direct proof not building on any classical results on attracting fixed points. For classical fixed point theory see [5].

The purpose of the present paper is to prove the following extension of this:

**Theorem 1.2** *The unique fixed point $(m_n)$ of the transformation $T : \mathcal{K} \to \mathcal{K}$ given by* (1) *is attractive.*

## 2 Proofs and complements

The product topology on the vector space $\mathbb{R}^{\mathbb{N}}$ is induced by the metric

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} 2^{-n} \min\{|a_n - b_n|, 1\} \quad \text{for} \quad (a_n),(b_n) \in \mathbb{R}^{\mathbb{N}},$$

which makes it a Fréchet space.

On the compact subset $\mathcal{K} = [0,1]^{\mathbb{N}}$ the expression for the metric is simplified to

$$d((a_n),(b_n)) = \sum_{n=1}^{\infty} 2^{-n} |a_n - b_n| \quad \text{for} \quad (a_n),(b_n) \in \mathcal{K}. \tag{8}$$

Before we find the best Lipschitz constant for $T$, let us introduce some notation. For $0 \le a \le 1$ let $\underline{a} = (a^n)_{n\ge1}$, so $\underline{0} = 0, 0, \ldots$ and $\underline{1} = 1, 1, \ldots$. Clearly, $T(\underline{0}) = \underline{1}$, $T(\underline{1}) = 1/2, 1/3, \ldots$, while

$$\left(T\left(\underline{\frac{1}{n+1}}\right)\right)_n = \frac{1}{H_{n+1}}, \quad \text{where} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

The numbers $H_n$ are called the harmonic numbers. In [2] it is proved that

$$\frac{1}{H_{n+1}} = \int_0^1 x^n \left( \sum_{p=0}^{\infty} \alpha_p x^{-\xi_p} \right) dx,$$

where $0 = \xi_0 > \xi_1 > \xi_2 > \ldots$ satisfy $-p-1 < \xi_p < -p$ for $p = 1, 2, \ldots$ and $\alpha_p > 0, p = 0, 1, \ldots$. More precisely, it is proved that $\xi_p$ is the unique solution $x \in \,]-p-1, -p[$ of the equation $\Psi(1+x) = -\gamma$, and $\alpha_p = 1/\Psi'(1+\xi_p)$. Here $\Psi(x) = \Gamma'(x)/\Gamma(x)$ and $\gamma$ is Euler's constant.

In [3] it is proved directly that $T^n(\underline{0})$ converges to the fixed point $(m_n)$, and this was used to derive that the same holds independent of where in $\mathcal{H}$ the iteration starts.

We cannot apply Banach's fixed point theorem directly because of the following Lemma.

**Lemma 2.1** *The best Lipschitz constant $c$ in*

$$d(T(a_n), T(b_n)) \le c\, d((a_n),(b_n)) \quad \textit{for} \quad (a_n),(b_n) \in \mathcal{K}$$

*is $c = 2$.*

*Proof.* For $(a_n),(b_n)$ we find

$$d(T(a_n),T(b_n)) = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\sum_{j=1}^{k}(b_j-a_j)\right|}{(1+a_1+\ldots+a_k)(1+b_1+\ldots+b_k)}$$
$$\le \sum_{k=1}^{\infty} 2^{-k}\sum_{j=1}^{k}|a_j-b_j| = \sum_{j=1}^{\infty}|a_j-b_j|\sum_{k=j}^{\infty}2^{-k} = 2d((a_n),(b_n)), \tag{9}$$

which shows that $T$ is Lipschitz with constant $c=2$.

Assume next that $T$ satisfies a Lipschitz condition with constant $c$.

We note that

$$d(\underline{0},\underline{a}) = \sum_{n=1}^{\infty}(a/2)^n = \frac{a}{2-a}.$$

Furthermore, $T(\underline{0}) = \underline{1}$ and for $0 \le a < 1$ we have $T(\underline{a})_n = (1-a)/(1-a^{n+1})$, so finally

$$d(T(\underline{0}),T(\underline{a})) = a\sum_{n=1}^{\infty}2^{-n}\frac{1-a^n}{1-a^{n+1}}.$$

This gives

$$\sum_{n=1}^{\infty}2^{-n}\frac{1-a^n}{1-a^{n+1}} \le c\frac{1}{2-a}, \quad 0 < a \le 1,$$

and letting $a \to 0$ we find $c \ge 2$. $\square$

*Proof of Theorem 1.2.*

Let us now introduce the set

$$\mathcal{C} = \left\{(a_n) \in \mathcal{K} \mid a_1 \ge \tfrac{1}{2}\right\},$$

which is a compact convex subset of $\mathcal{K}$. We first note that $T(\mathcal{K}) \subseteq \mathcal{C}$ because for any $(a_n) \in \mathcal{K}$ we have $a_1 \le 1$, hence

$$T(a_n)_1 = \frac{1}{1+a_1} \ge \frac{1}{2}.$$

By (9) we always have

$$d(T(a_n),T(b_n)) = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\sum_{j=1}^{k}(b_j-a_j)\right|}{(1+a_1+\ldots+a_k)(1+b_1+\ldots+b_k)}$$
$$\le \frac{1}{(1+a_1)(1+b_1)}\sum_{k=1}^{\infty}2^{-k}\sum_{j=1}^{k}|a_j-b_j| = \frac{1}{(1+a_1)(1+b_1)}\sum_{j=1}^{\infty}|a_j-b_j|\sum_{k=j}^{\infty}2^{-k}$$
$$= \frac{2}{(1+a_1)(1+b_1)}d((a_n),(b_n)). \tag{10}$$

If now $(a_n), (b_n) \in \mathcal{C}$, we have $a_1, b_1 \geq \frac{1}{2}$, and hence $\frac{2}{(1+a_1)(1+b_1)} \leq \frac{8}{9}$ so that

$$d(T(a_n), T(b_n)) \leq \frac{8}{9} d((a_n), (b_n)) \quad \text{for} \quad (a_n), (b_n) \in \mathcal{C},$$

showing that $T$ is a contraction on $\mathcal{C}$. Since $T$ maps $\mathcal{K}$ into $\mathcal{C}$ any fixed point of $T$ on $\mathcal{K}$ must belong to $\mathcal{C}$, and $T : \mathcal{C} \to \mathcal{C}$ has a unique fixed point by Banach's fixed point theorem, and this must be the sequence $(m_n)$ determined by (3). We also see that for any $\xi = (a_n) \in \mathcal{K}$ the iterates $T^{n+1}(\xi) = T^n(T(\xi))$ converge to the fixed point $(m_n)$, which finishes the proof of Theorem 1.2. □

**Proposition 2.2** *The compact set $\mathcal{H}$ defined in* (6) *has diameter* $\mathrm{diam}(\mathcal{H}) = 1$ *and the only two points $a, b \in \mathcal{H}$ for which $d(a, b) = 1$ are $\{a, b\} = \{\underline{0}, \underline{1}\}$.*

*Proof.* It is easy to see that

$$1 = d(\underline{0}, \underline{1}) \leq \mathrm{diam}(\mathcal{H}) \leq 1,$$

so we only have to prove that if $d(a, b) = 1$ for some $a = (a_n), b = (b_n)$ from $\mathcal{H}$, then $\{a, b\} = \{\underline{0}, \underline{1}\}$. Supposing that

$$\sum_{n=1}^{\infty} 2^{-n} |a_n - b_n| = 1,$$

then necessarily $|a_n - b_n| = 1$ for all $n \geq 1$. In particular, for each $n$ necessarily $a_n$ and $b_n$ are either 0 and 1 or these numbers reversed. Assume now that $a_1 = 0, b_1 = 1$. Since $a_n = \int_0^1 x^n \, d\mu(x)$ for a probability measure $\mu$, the condition $a_1 = 0$ forces $\mu$ to be the Dirac measure $\delta_0$ with unit mass concentrated at 0, and hence $a_n = 0$ for all $n \geq 1$. This shows that $a = \underline{0}$ and $b = \underline{1}$. □

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Christian Berg
Institute of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 København Ø, Denmark
E-mail berg@math.ku.dk

Maryam Beygmohammadi
Department of Mathematics
Islamic Azad University-Kermanshah branch
Kermanshah, Iran
E-mail maryambmohamadi@mathdep.iust.ac.ir