Linearization coefficients of Bessel polynomials and properties of Student-t distributions

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Abstract

We prove positivity results about linearization and connection coefficients for Bessel polynomials. The proof is based on a recursion formula and explicit formulas for the coefficients in special cases. The result implies that the distribution of a convex combination of independent Student-t random variables with arbitrary odd degrees of freedom has a density which is a convex combination of certain Student-t densities with odd degrees of freedom.

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1 Introduction

In this paper we consider the Bessel polynomials q_n of degree n

$$q_n\left(u\right) = \sum_{k=0}^{n} \alpha_k^{(n)} u^k, \tag{1}$$

where

$$\alpha_k^{(n)} = \frac{\binom{n}{k}}{\binom{2n}{k}} \frac{2^k}{k!} = \frac{n!\,(2n-k)!\,2^k}{(2n)!\,(n-k)!\,k!}. \tag{2}$$

The first examples of these polynomials are

$$q_0\left(u\right) = 1, q_1\left(u\right) = 1 + u, q_2\left(u\right) = 1 + u + \frac{u^2}{3}.$$

They are normalized according to

$$q_n\left(0\right) = 1,$$

and thus differ from the polynomials $\theta_n\left(u\right)$ in Grosswald's monograph [12] by the constant factor $\frac{(2n)!}{n!2^n}$, i.e.

$$\theta_n(u) = \frac{(2n)!}{n!2^n} q_n(u).$$

The polynomials θ_n are sometimes called the reverse Bessel polynomials and $y_n(u) = u^n\theta_n(1/u)$ the ordinary Bessel polynomials. Two-parameter extensions of these polynomials are studied in [12], and we refer to this work concerning references to the vast literature and the history about Bessel polynomials. For a study of the zeros of the Bessel polynomials we refer to [5].

For $\nu > 0$ we recall that the probability density on $\mathbb{R}$

$$f_\nu(x) = \frac{A_\nu}{(1+x^2)^{\nu+\frac{1}{2}}}, \quad A_\nu = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\nu)} \tag{3}$$

has the characteristic function

$$\int_{-\infty}^{\infty} e^{ixy} f_\nu(x)\,dx = k_\nu(|y|), \quad y \in \mathbb{R}, \tag{4}$$

where

$$k_\nu(u) = \frac{2^{1-\nu}}{\Gamma(\nu)} u^\nu K_\nu(u), \quad u \geq 0, \tag{5}$$

and K_ν is the modified Bessel function of the third kind. If $\nu = n + \frac{1}{2}$ with $n = 0, 1, 2, \ldots$ then

$$k_\nu(u) = e^{-u} q_n(u), \quad u \geq 0, \tag{6}$$

and f_ν is called a Student-t density with $2\nu = 2n+1$ degrees of freedom. For $\nu = \frac{1}{2}$ then f_ν is density of a Cauchy distribution. Note that for simplicity we have avoided the usual scaling of the Student-t distribution.

In this paper, we provide the solutions of the three following problems:

1. Explicit values of the connection coefficients $c_k^{(n)}(a)$ and their positivity for $a \in [0, 1]$ in the expansion

$$q_n(au) = \sum_{k=0}^{n} c_k^{(n)}(a)\, q_k(u). \tag{7}$$

2. Explicit value of the linearization coefficients $\beta_i^{(n)}(a)$ and their positivity for $a \in [0, 1]$ in the expansion

$$q_n(au)q_n((1-a)u) = \sum_{i=0}^{n} \beta_i^{(n)}(a) q_{n+i}(u). \tag{8}$$

3. Positivity of the linearization coefficients $\beta_k^{(n,m)}(a)$ for $a \in [0, 1]$ in the expansion

$$q_n(au)q_m((1-a)u) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) q_k(u). \tag{9}$$

Note that $\beta_i^{(n)}(a) = \beta_{n+i}^{(n,n)}(a)$ and that (7) is a special case of (9) corresponding to $m = 0$ with $c_k^{(n)}(a) = \beta_k^{(n,0)}(a)$. Note also that $u = 0$ in (9) yields

$$\sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) = 1,$$

so (9) is a convex combination. As polynomial identities, (7)-(9) of course hold for all complex a, u, but as we will see later, the positivity of the coefficients holds only for $0 \le a \le 1$.

Because of (4) and (6) formula (9) is equivalent with the following identity between Student-t densities

$$\frac{1}{a} f_{n+\frac{1}{2}}\left(\frac{x}{a}\right) * \frac{1}{1-a} f_{m+\frac{1}{2}}\left(\frac{x}{1-a}\right) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) f_{k+\frac{1}{2}}(x) \tag{10}$$

for $0 < a < 1$ and $*$ is the ordinary convolution of densities.

Although (9) is more general than (7),(8), we stress that we give explicit formulas below for $c_k^{(n)}(a)$ and $\beta_i^{(n)}(a)$ from which the positivity is clear. The positivity of $\beta_k^{(n,m)}(a)$ for the general case can be deduced from the special cases via a recursion formula, see Lemma 3.4 below.

Our use of the words “linearization coefficients” is not agreeing completely with the terminology of [2], which defines the linearization coefficients for a polynomial system $\{q_n\}$ as the coefficients $a(k, n, m)$ such that

$$q_n(u)q_m(u) = \sum_{k=0}^{n+m} a(k, n, m) q_k(u).$$

Since, in the Bessel case

$$q_1(u)^2 = -q_0(u) - q_1(u) + 3q_2(u)$$

the linearization coefficients in the proper sense are not non-negative.

It is interesting to note that in [13] Koornwinder proved that the Laguerre polynomials

$$L_n^{(\alpha)}(u) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-u)^k)}{k!}$$

satisfy a positivity property like (9), i.e.

$$L_n^{(\alpha)}(au) L_n^{(\alpha)}((1-a)u) = \sum_{k=0}^{n+m} \kappa_k^{(n,m)}(a) L_k^{(\alpha)}(u),$$

with $\kappa_k^{(n,m)}(a) \ge 0$ for $a \in [0, 1]$ provided $\alpha \ge 0$.

In this connection it is worth pointing out that there is an easy established relationship between q_n and the Laguerre polynomials with α outside the range of orthogonality for the Laguerre polynomials, namely

$$q_n(u) = \frac{(-1)^n}{\binom{2n}{n}} L_n^{(-2n-1)}(2u).$$

The problems discussed have an important application in statistics: the Behrens-Fisher problem consists in testing the equality of the means of two normal populations. Fisher [7][1] has shown that this test can be performed using the $d-$statistics defined as

$$d_{f_1,f_2,\theta} = t_1 \sin\theta - t_2 \cos\theta,$$

where t_1 and t_2 are two independent Student-t random variables with respective degrees of freedom f_1 and f_2 and $\theta \in \left[0, \frac{\pi}{2}\right]$. Many different results have been obtained on the behaviour of the $d-$statistics. Tables of the distribution of $d_{f_1,f_2,\theta}$ have been provided in 1938 by Sukhatme [15] at Fisher's suggestion. In 1956, Fisher and Healy explicited the distribution of $d_{f_1,f_2,\theta}$ as a mixture of Student-t distributions (Student-t distribution with a random, discrete number of degrees of freedom) for small, odd values of f_1 and f_2. This work was extended by Walker and Saw [16] who provided, still in the case of odd numbers of degrees of freedom, an explicit way of computing the coefficients of the Student-t mixture as solutions of a linear system; however, they did not prove the positivity of these coefficients, claiming only

> "Extensive numerical investigation indicates also that $\eta_i \geq 0$ for all i; however, an analytic proof has not been found."

This conjecture is proved in Theorem 2 and 3 below. Section 2 of this paper gives the explicit solutions to problems 1, 2 and 3, whereas section 3 is dedicated to their proofs. The last section gives an extension of Theorem 2 in terms of inverse Gamma distributions.

To relate the Behrens-Fisher problem to our discussion we note that due to symmetry $d_{f_1,f_2,\theta}$ has the same distribution as $t_1 \sin\theta + t_2 \cos\theta$, which for $\theta \in]0, \frac{\pi}{2}[$ is a scaling of a convex combination of independent Student-t variables.

Using the fact that the Student-t distribution is a scale mixture of normal distributions by an inverse Gamma distribution our positivity result is equivalent to an analogous positivity result for inverse Gamma distributions. This result has been observed for small values of the degrees of freedom in [18]. In [9] the coefficients are claimed to be non-negative but the paper does not contain any arguments to prove it.

2 Results

2.1 Solution of problem 1 and a stochastic interpretation

Theorem 2.1 *The coefficients $c_k^{(n)}(a)$ in (7) are expressed as follows*

$$c_k^{(n)}(a) = a^k \frac{\binom{n}{k}}{\binom{2n}{2k}} \sum_{r=1}^{(n-k)\wedge(k+1)} \binom{n+1}{k+1-r} \binom{n-k-1}{r-1} (1-a)^r$$

for $0 \leq k \leq n-1$ while $c_n^{(n)}(a) = a^n$. Hence they are positive for $0 \leq a \leq 1$.

[1] the collected papers of R.A. Fisher are available at the following address http://www.library.adelaide.edu.au/digitised/fisher/

A stochastic interpretation of Theorem 2.1 is obtained as follows: replacing u by $|u|$ and multiplying equation (7) by $\exp(-|u|)$, we get

$$e^{-(1-a)|u|}e^{-a|u|}q_n(a|u|) = \sum_{k=0}^{n} c_k^{(n)}(a)\, q_k(|u|)e^{-|u|}. \tag{11}$$

Equation (11) can be expressed that the convex combination of an independent Cauchy variable C and a Student-t variable X_n with $2n+1$ degrees of freedom follows a Student-t distribution with random number $2K(\omega)+1$ of degrees of freedom:

$$(1-a)\,C + aX_n \overset{d}{=} X_{K(\omega)},$$

where $K(w) \in [0,n]$ is a discrete random variable such that

$$\Pr\{K(w) = k\} = c_k^{(n)}(a), \quad 0 \le k \le n.$$

2.2 Solution of problem 2 and a probabilistic interpretation

Theorem 2.2 *The coefficients $\beta_i^{(n)}(a)$ in (8) are expressed as follows*

$$\begin{aligned}\beta_i^{(n)}(a) &= (4a(1-a))^i \left(\frac{n!}{(2n)!}\right)^2 2^{-2n}\frac{(2n-2i)!(2n+2i)!}{(n-i)!(n+i)!} \\ &\times \sum_{j=0}^{n-i}\binom{2n+1}{2j}\binom{n-j}{i}(2a-1)^{2j}.\end{aligned}$$

Hence they are positive for $0 \le a \le 1$.

A probabilistic interpretation of this result can be formulated as follows.

Corollary 2.3 *Let X, Y be independent Student-t variables with $2n+1$ degrees of freedom, then the distribution of $aX+(1-a)Y$ follows a Student-t distribution with a random number of degrees of freedom $F(\omega)$ distributed according to*

$$\Pr\{F(\omega) = 2n+2i+1\} = \beta_i^{(n)}(a), \quad 0 \le i \le n.$$

2.3 Problem 3

Theorem 2.4 *The coefficients $\beta_k^{(n,m)}(a)$ in (9) are positive for $0 \le a \le 1$.*

We are unable to derive the explicit values of the coefficients $\beta_k^{(n,m)}(a)$. However, their positivity allows us to claim the following Corollary.

Corollary 2.5 *Let X, Y be independent Student-t variables with resp. $2n+1, 2m+1$ degrees of freedom, then the distribution of $aX+(1-a)Y$ follows a Student-t distribution with a random number of degrees of freedom $F(\omega)$ distributed according to*

$$\Pr\{F(\omega) = 2k+1\} = \beta_k^{(n,m)}(a), \quad n \wedge m \le k \le n+m.$$

Theorem 2.6 *For $k \geq 2$ let $n_1, \ldots, n_k$ be nonnegative integers and let $a_1, \ldots, a_k$ be positive real numbers with sum 1. Then*

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) \cdots q_{n_k}(a_k u) = \sum_{j=l}^{L} \beta_j q_j(u), \quad u \in \mathbb{R}, \tag{12}$$

where the coefficients β_j are nonnegative with sum 1, $l = \min(n_1, \ldots, n_k)$ and $L = n_1 + \cdots + n_k$.

3 Proofs

3.1 Generalities about Bessel polynomials

As a preparation to the proofs we give some recursion formulas for q_n. They follow from corresponding formulas for θ_n from [12], but they can also be proved directly from the definitions (1) and (2). The formulas are

$$q_{n+1}(u) = q_n(u) + \frac{u^2}{4n^2 - 1} q_{n-1}(u), \quad n \geq 1, \tag{13}$$

$$q_n'(u) = q_n(u) - \frac{u}{2n - 1} q_{n-1}(u), \quad n \geq 1. \tag{14}$$

We can write

$$u^n = \sum_{i=0}^{n} \delta_i^{(n)} q_i(u), \quad n = 0, 1, \ldots \tag{15}$$

and $\delta_i^{(n)}$ is given by a formula due to Carlitz [6], see [12, p. 73] or [16]:

$$\delta_i^{(n)} = \begin{cases} \frac{(n+1)!}{2^n} \frac{(-1)^{n-i}(2i)!}{(n-i)!i!(2i+1-n)!} & \text{for } \frac{n-1}{2} \leq i \leq n \\ 0 & \text{for } 0 \leq i < \frac{n-1}{2} \end{cases} . \tag{16}$$

Later we need the following extension of (13) which we formulate using the Pochhammer symbol $(z)_n := z(z+1)\cdots(z+n-1)$ for $z \in \mathbb{C}, n = 0, 1, \ldots$.

Lemma 3.1 *For $0 \leq k \leq n$ we have*

$$u^{2k} q_{n-k}(u) = \sum_{i=0}^{k} \gamma_i^{(n,k)} q_{n+i}(u)$$

where

$$\gamma_i^{(n,k)} = 2^{2k} \binom{k}{i} (n - k + \tfrac{1}{2})_{k+i} (-n - \tfrac{1}{2})_{k-i}. \tag{17}$$

Proof: The Lemma is trivial for $k = 0$ and reduces to the recursion (13) for $k = 1$ written as

$$u^2 q_{n-1}(u) = 2^2 (n - \tfrac{1}{2})_2 \left(q_{n+1}(u) - q_n(u) \right). \tag{18}$$

We will prove the formula (17) by induction in n, so assume it holds for some n and all $0 \leq k \leq n$. Multiplying the formula of the lemma by u^2 we get

$$u^{2k+2} q_{n-k}(u) = \sum_{i=0}^{k} \gamma_i^{(n,k)} u^2 q_{n+i}(u),$$

hence by (18)

$$\begin{aligned} u^{2(k+1)} q_{n+1-(k+1)}(u) &= \sum_{i=0}^{k} \gamma_i^{(n,k)} 2^2 (n+i+\tfrac{1}{2})_2 \left[q_{n+i+2}(u) - q_{n+i+1}(u) \right] \\ = & \ \gamma_k^{(n,k)} 2^2 (n+k+\tfrac{1}{2})_2 \, q_{n+k+2}(u) \\ + & \ \sum_{i=1}^{k} 2^2 (n+i+\tfrac{1}{2}) \left[\gamma_{i-1}^{(n,k)} (n+i-\tfrac{1}{2}) - \gamma_i^{(n,k)} (n+i+\tfrac{3}{2}) \right] q_{n+1+i}(u) \\ - & \ \gamma_0^{(n,k)} 2^2 (n+\tfrac{1}{2})_2 \, q_{n+1}(u). \end{aligned}$$

Using the induction hypothesis we easily get

$$\gamma_k^{(n,k)} 2^2 (n+k+\tfrac{1}{2})_2 = 2^{2k+2} (n-k+\tfrac{1}{2})_{2k+2} = \gamma_{k+1}^{(n+1,k+1)},$$

and

$$-\gamma_0^{(n,k)} 2^2 (n+\tfrac{3}{2})(n+\tfrac{1}{2}) = 2^{2k+2} (n-k+\tfrac{1}{2})_{k+1} (-n-\tfrac{3}{2})_{k+1} = \gamma_0^{(n+1,k+1)}.$$

Concerning the coefficient C to $q_{n+1+i}(u)$ above we have

$$\begin{aligned} C = 2^{2k+2}(n+i+\tfrac{1}{2}) & \left[\binom{k}{i-1} (n-k+\tfrac{1}{2})_{k+i-1} (-n-\tfrac{1}{2})_{k-i+1} (n+i-\tfrac{1}{2}) \right. \\ & \left. - \binom{k}{i} (n-k+\tfrac{1}{2})_{k+i} (-n-\tfrac{1}{2})_{k-i} (n+i+\tfrac{3}{2}) \right] \\ = & \ 2^{2k+2} (n-k+\tfrac{1}{2})_{k+1+i} (-n-\tfrac{1}{2})_{k-i} \\ \times & \ \left[\binom{k}{i-1} (-n-\tfrac{1}{2}+k-i) - \binom{k}{i} (n+i+\tfrac{3}{2}) \right] \\ = & \ 2^{2k+2} (n-k+\tfrac{1}{2})_{k+1+i} (-n-\tfrac{1}{2})_{k-i} \left[\binom{k+1}{i} (-n-\tfrac{3}{2}) \right] \\ = & \ 2^{2k+2} \binom{k+1}{i} (n-k+\tfrac{1}{2})_{k+1+i} (-n-\tfrac{3}{2})_{k+1-i} = \gamma_i^{(n+1,k+1)}. \end{aligned}$$

□

We stress that Lemma 3.1 is the special case $\nu = n + \frac{1}{2}$ of the following recursion for modified Bessel functions of the third kind.

Lemma 3.2 *For all $\nu > 0$ and all nonnegative integers $j < \nu$ we have for $u > 0$*

$$u^{\nu+j} K_{\nu-j}(u) = \sum_{i=0}^{j} (-2)^{j-i} \binom{j}{i} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-(j-i))} u^{\nu+i} K_{\nu+i}(u)$$

and

$$u^{2j}k_{\nu-j}\left(u\right)=\sum_{i=0}^{j}\left(-1\right)^{j-i}2^{2j}\binom{j}{i}\frac{\Gamma\left(\nu+1\right)\Gamma\left(\nu+i\right)}{\Gamma\left(\nu+1-\left(j-i\right)\right)\Gamma\left(\nu-j\right)}k_{\nu+i}\left(u\right).$$

Proof: The second formula follows from the first using formula (5), and the first can be proved by induction using the following recursion formula for modified Bessel functions of the third kind, cf. [17, p. 79]

$$K_{\nu-1}(u)=K_{\nu+1}(u)-\frac{2\nu}{u}K_{\nu}(u).$$

We skip the details. $\square$

3.2 Proof of Theorem 2.1

From (1) and (15) we get

$$q_n\left(au\right)=\sum_{j=0}^{n}\alpha_j^{(n)}a^j\sum_{i=0}^{j}\delta_i^{(j)}q_i\left(u\right)=\sum_{k=0}^{n}c_k^{(n)}(a)q_k\left(u\right)$$

with

$$\begin{aligned}c_k^{(n)}\left(a\right) &= \sum_{j=k}^{n}a^j\alpha_j^{(n)}\delta_k^{(j)}\\ &= a^k\frac{n!}{(2n)!}\frac{(2k)!}{k!}\sum_{j=k,j\leq 2k+1}^{n}\left(-a\right)^{j-k}\frac{\left(2n-j\right)!\left(j+1\right)}{\left(n-j\right)!\left(j-k\right)!\left(2k+1-j\right)!}\end{aligned}$$

In particular $c_n^{(n)}\left(a\right)=a^n$ and for $0\leq k\leq n-1$

$$c_k^{(n)}\left(a\right)=a^k\frac{n!}{(2n)!}\frac{(2k)!}{k!}p(a), \tag{19}$$

where

$$p(a)=\sum_{i=0}^{(n-k)\wedge(k+1)}(-a)^i\frac{(2n-k-i)!(k+i+1)}{(n-k-i)!i!(k+1-i)!}.$$

We clearly have

$$p(a)=\sum_{r=0}^{(n-k)\wedge(k+1)}(-1)^r\frac{p^{(r)}(1)}{r!}(1-a)^r$$

with

$$p^{(r)}(1)=\sum_{i=r}^{(n-k)\wedge(k+1)}(-1)^i\frac{(2n-k-i)!(k+i+1)}{(n-k-i)!(i-r)!(k+1-i)!}$$

and we only consider $0\leq r\leq(n-k)\wedge(k+1)$. To sum this we shift the summation by r. For simplicity we define $T:=(n-k-r)\wedge(k+1-r)$ and get

$$(-1)^r p^{(r)}(1)=\sum_{i=0}^{T}(-1)^i\frac{(2n-k-r-i)!\,(k+r+i+1)}{(n-k-r-i)!\,i!\,(k+1-r-i)!}.$$

We write $k+r+1+i=(2k+2)-(k+1-r-i)$ and split the above sum accordingly

$$\begin{aligned}(-1)^r p^{(r)}(1) &= (2k+2)\sum_{i=0}^{T}(-1)^i\frac{(2n-k-r-i)!}{(n-k-r-i)!\,i!\,(k+1-r-i)!}\\ &- \sum_{i=0}^{T}(-1)^i\frac{(2n-k-r-i)!}{(n-k-r-i)!\,i!\,(k-r-i)!}.\end{aligned}$$

Note that for nonnegative integers a,b,c with $b,c\le a$ we have

$$\sum_{i=0}^{b\wedge c}(-1)^i\frac{(a-i)!}{(b-i)!(c-i)!i!}=\frac{a!}{b!c!}\sum_{i=0}^{b\wedge c}\frac{(-b)_i(-c)_i}{(-a)_i i!}=\frac{a!}{b!c!}\,{}_2F_1(-b,-c;-a;1),$$

where we use that the sum is an ${}_2F_1$ evaluated at 1. Its value is given by the Chu-Vandermonde formula, see [1], hence

$$\sum_{i=0}^{b\wedge c}(-1)^i\frac{(a-i)!}{(b-i)!(c-i)!i!}=\frac{a!(c-a)_b}{(-a)_b b!c!}.$$

The two sums above are of this form and we get

$$(-1)^r p^{(r)}(1)=\frac{(2n-k-r)!}{(n-k-r)!(k+1-r)!}Q,$$

where

$$Q=(2k+2)\frac{(2k-2n+1)_{n-k-r}}{(k+r-2n)_{n-k-r}}-(k+1-r)\frac{(2k-2n)_{n-k-r}}{(k+r-2n)_{n-k-r}}$$

$$=\frac{(2k-2n+1)_{n-k-r-1}}{(k+r-2n)_{n-k-r}}[(2k+2)(k-r-n)-(k+1-r)(2k-2n)]$$

$$=2r(n+1)\frac{(n+r+1-k)_{n-k-r-1}}{(n+1)_{n-k-r}},$$

where we used $(a)_n=(-1)^n(1-a-n)_n$ twice. This gives

$$(-1)^r p^{(r)}(1)=2r\binom{n+1}{k+1-r}\frac{(2n-2k-1)!}{(n-k-r)!}$$

and finally

$$p(a)=\sum_{r=0}^{(n-k)\wedge(k+1)}(1-a)^r\frac{2r}{r!}\binom{n+1}{k+1-r}\frac{(2n-2k-1)!}{(n-k-r)!}.$$

Note that the term corresponding to $r=0$ is zero. If we insert this expression for $p(a)$ in (19), we get the formula of Theorem 2.1. $\square$

Remark 3.3 The evaluation above of $(-1)^r p^{(r)}(1)$ can be done using generating functions like in [16]. The authors want to thank Mogens Esrom Larsen for the idea to use the Chu-Vandermonde identity twice.

3.3 Proof of Theorem 2.2

The starting point is the following formula of Macdonald, see [17]

$$K_\nu(z)K_\nu(X) = \frac{1}{2}\int_0^\infty \exp[-\frac{s}{2} - \frac{z^2+X^2}{2s}]K_\nu(\frac{zX}{s})\frac{ds}{s}, \tag{20}$$

which we will use for $\nu = n + \frac{1}{2}, z = au, X = (1-a)u$. Multiplying (20) by

$$\left(\frac{2^{1-\nu}}{\Gamma(\nu)}\right)^2 (a(1-a)u^2)^\nu$$

and using (5) we find

$$k_\nu(au)k_\nu((1-a)u) =$$

$$\frac{1}{2^\nu\Gamma(\nu)}\int_0^\infty \exp\left[-\frac{s}{2} - u^2\frac{a^2+(1-a)^2}{2s}\right] s^{\nu-1}k_\nu\left(\frac{a(1-a)u^2}{s}\right)\, ds.$$

We now insert that with $\nu = n+\frac{1}{2}$ we have $k_\nu(|u|) = e^{-|u|}q_n(|u|)$ and hence after some simplification

$$e^{-|u|}q_n(a|u|)q_n((1-a)|u|) =$$

$$\frac{1}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}\int_0^\infty \exp\left[-\frac{s}{2} - \frac{u^2}{2s}\right] s^{n-\frac{1}{2}}q_n\left(\frac{a(1-a)u^2}{s}\right)\, ds.$$

We next insert the expression (1) for q_n under the integral sign. This gives

$$e^{-|u|}q_n(a|u|)q_n((1-a)|u|) =$$

$$\sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k u^{2k}\frac{1}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}\int_0^\infty \exp\left[-\frac{s}{2} - \frac{u^2}{2s}\right] s^{n-k-\frac{1}{2}}\, ds.$$

Using the following formula from [10, 3.471(9)]

$$\int_0^\infty x^{\nu-1}\exp\left(-\frac{\beta}{x} - \gamma x\right)\, dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\beta\gamma}\right) \tag{21}$$

and again (5) the above is equal to

$$= \sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k\frac{2^{n-k+\frac{1}{2}}\Gamma(n-k+\frac{1}{2})}{2^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}e^{-|u|}u^{2k}q_{n-k}(|u|).$$

Finally, using Pochhammer symbols and skipping absolute values since we are now dealing with a polynomial identity, we get

$$q_n(au)q_n((1-a)u) = \sum_{k=0}^n \alpha_k^{(n)}(a(1-a))^k\frac{(\frac{1}{2})_{n-k}}{2^k(\frac{1}{2})_n}u^{2k}q_{n-k}(u). \tag{22}$$

Using the expression for $u^{2k}q_{n-k}(u)$ from Lemma 3.1 and the expression for $\alpha_k^{(n)}$ in (22) we then get

$$\begin{aligned} q_n(au)q_n((1-a)u) &= \sum_{k=0}^{n} \frac{\binom{n}{k}(\frac{1}{2})_{n-k}}{\binom{2n}{k}(\frac{1}{2})_n k!}(a(1-a))^k \sum_{i=0}^{k} \gamma_i^{(n,k)} q_{n+i}(u) \\ &= \sum_{i=0}^{n} q_{n+i}(u) \sum_{k=i}^{n} (a(1-a))^k \frac{\binom{n}{k}(\frac{1}{2})_{n-k}}{\binom{2n}{k}(\frac{1}{2})_n k!} \gamma_i^{(n,k)} \end{aligned}$$

hence

$$q_n(au)q_n((1-a)u) = \sum_{i=0}^{n} \beta_i^{(n)}(a) q_{n+i}(u)$$

with

$$\begin{aligned} \beta_i^{(n)}(a) &= \sum_{k=i}^{n} (a(1-a))^k \frac{\binom{n}{k}(\frac{1}{2})_{n-k}}{\binom{2n}{k}(\frac{1}{2})_n k!} 2^{2k} \binom{k}{i} (n-k+\frac{1}{2})_{k+i}(-n-\frac{1}{2})_{k-i} \\ &= (a(1-a))^i \sum_{l=0}^{n-i} (a(1-a))^l \frac{\binom{n}{i+l}(\frac{1}{2})_{n-i-l}}{\binom{2n}{i+l}(\frac{1}{2})_n (i+l)!} 2^{2i+2l} \binom{i+l}{i} \\ &\times \; (n-i-l+\frac{1}{2})_{l+2i}(-n-\frac{1}{2})_l \end{aligned}$$

Collecting

$$(\frac{1}{2})_{n-i-l}(n-i-l+\frac{1}{2})_{l+2i} = (\frac{1}{2})_{n+i}$$

we get

$$\beta_i^{(n)}(a) = (a(1-a))^i \sum_{l=0}^{n-i} (a(1-a))^l \frac{\binom{n}{i+l}(\frac{1}{2})_{n+i}}{\binom{2n}{i+l}(\frac{1}{2})_n} \frac{2^{2i+2l}}{i!\,l!}(-n-\frac{1}{2})_l$$

$$= (a(1-a))^i \frac{n!(\frac{1}{2})_{n+i}2^{2i}}{(2n)!\,(\frac{1}{2})_n\, i!} \sum_{l=0}^{n-i} (4a(1-a))^l \frac{(2n-i-l)!\,(-n-\frac{1}{2})_l}{(n-i-l)!\,l!}$$

$$= (a(1-a))^i \left(\frac{n!}{(2n)!}\right)^2 \frac{(2n+2i)!}{(n+i)!i!} \sum_{l=0}^{n-i} \left(1-(2a-1)^2\right)^l \frac{(2n-i-l)!\,(-n-\frac{1}{2})_l}{(n-i-l)!\,l!}.$$

Expanding $(1-(2a-1)^2)^l$ using the binomial formula and interchanging the sums we get

$$\sum_{l=0}^{n-i} (1-(2a-1)^2)^l \frac{(2n-i-l)!(-n-\frac{1}{2})_l}{(n-i-l)!l!}$$

$$= \sum_{j=0}^{n-i} (-1)^j \frac{(2a-1)^{2j}}{j!} \sum_{l=j}^{n-i} \frac{(2n-i-l)!(-n-\frac{1}{2})_l}{(n-i-l)!(l-j)!}.$$

We claim that

$$\begin{aligned} S: &= \sum_{l=j}^{n-i} \frac{(2n-i-l)!(-n-\frac{1}{2})_l}{(n-i-l)!(l-j)!} \\ &= \frac{(2n-2i)!}{(n-i)!} 2^{2i-2n}(-1)^j j! i! \binom{2n+1}{2j}\binom{n-j}{i}. \end{aligned} \tag{23}$$

and we have obtained the final formula

$$\begin{aligned} \beta_i^{(n)}(a) = (4a(1-a))^i \left(\frac{n!}{(2n)!}\right)^2 2^{-2n} \frac{(2n-2i)!(2n+2i)!}{(n-i)!(n+i)!} \\ \times \sum_{j=0}^{n-i} \binom{2n+1}{2j}\binom{n-j}{i}(2a-1)^{2j}. \end{aligned}$$

We will see that (23) is a Chu-Vandermonde formula. In fact, shifting the summation index putting $l = j + m$ we get

$$S = (-n-\frac{1}{2})_j \sum_{m=0}^{n-i-j} \frac{(2n-i-j-m)!(-n+j-\frac{1}{2})_m}{m!(n-i-j-m)!},$$

and calling the general term in this sum c_m we get

$$\frac{c_{m+1}}{c_m} = \frac{(m+i+j-n)(m+j-n-\frac{1}{2})}{(m+1)(m+i+j-2n)},$$

which shows that the sum is a ${}_2F_1$. We have

$$S = (-n-\tfrac{1}{2})_j\, c_0 \times {}_2F_1(-(n-i-j), j-n-\frac{1}{2}; i+j-2n; 1),$$

and using the Chu-Vandermonde identity, cf. [1]:

$${}_2F_1(-n, a; c; 1) = \frac{(c-a)_n}{(c)_n},$$

we get

$$\begin{aligned} S &= (-n-\frac{1}{2})_j \frac{(2n-i-j)!}{(n-i-j)!} \frac{(-n+i+\frac{1}{2})_{n-i-j}}{(i+j-2n)_{n-i-j}} \\ &= (-n-\frac{1}{2})_j \frac{(2n-i-j)!}{(n-i-j)!} \frac{(j+\frac{1}{2})_{n-i-j}}{(n+1)_{n-i-j}}, \end{aligned}$$

where we used $(a)_n = (-1)^n(1-a-n)_n$ twice. We can now simplify to get

$$S = (-n-\frac{1}{2})_j \frac{n!}{(n-i-j)!} \frac{(\frac{1}{2})_{n-i}}{(\frac{1}{2})_j},$$

and applying the formula

$$(\frac{1}{2})_p = \frac{(2p)!}{p! 2^{2p}}$$

twice we get

$$S=\frac{(2n-2i)!}{(n-i)!}2^{2i-2n}(-n-\frac{1}{2})_j\frac{n!}{(n-i-j)!}\frac{j!2^{2j}}{(2j)!}.$$

Now we can write

$$(-n-\frac{1}{2})_j 2^{2j}=(-1)^j\frac{(2n+1)!(n-j)!}{n!(2n-2j+1)!},$$

and hence

$$S=\frac{(2n-2i)!}{(n-i)!}2^{2i-2n}(-1)^j j!i!\binom{2n+1}{2j}\binom{n-j}{i}.$$

$\square$

3.4 Proof of Theorem 2.4

For $n,m\geq 0$ and $a\in\mathbb{R}$, we can write

$$q_n(au)\,q_m((1-a)\,u)=\sum_{k=0}^{m+n}\beta_k^{(n,m)}(a)\,q_k(u) \tag{24}$$

for some uniquely determined coefficients since the left-hand side is a polynomial in u of degree $\leq n+m$. Clearly $\beta_k^{(n,m)}(a)$ is a polynomial in a satisfying

$$\beta_k^{(n,m)}(a)=\beta_k^{(m,n)}(1-a). \tag{25}$$

We shall prove that $\beta_k^{(n,m)}(a)\geq 0$ for $0\leq a\leq 1$ and that $\beta_k^{(n,m)}(a)=0$ if $k<n\wedge m$, which will be a consequence of the following recursion formula.

Lemma 3.4 *For $n,m\geq 1$, we have*

$$\frac{1}{2k+1}\beta_{k+1}^{(n,m)}(a)=\frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a)+\frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a), \tag{26}$$

where $k=0,1,\ldots,m+n-1$.

Furthermore $\beta_0^{(n,m)}(a)=0$.

Proof: Differentiating (24) with respect to u gives

$$aq_n'(au)\,q_m((1-a)\,u)+(1-a)\,q_n(au)\,q_m'((1-a)\,u)=\sum_{k=1}^{m+n}\beta_k^{(n,m)}(a)\,q_k'(u)$$

and using the formula (14) we find

$$\begin{aligned} & a\left(q_n(au)-\tfrac{au}{2n-1}q_{n-1}(au)\right)q_m((1-a)\,u)\\ + \quad & (1-a)\,q_n(au)\left(q_m((1-a)\,u)-\tfrac{(1-a)u}{2m-1}q_{m-1}((1-a)\,u)\right)\\ = \quad & \textstyle\sum_{k=1}^{m+n}\beta_k^{(n,m)}(a)\left(q_k(u)-\tfrac{u}{2k-1}q_{k-1}(u)\right)\end{aligned}$$

and using (24) once more we get

$$\begin{aligned} & -\tfrac{a^2u}{2n-1} q_{n-1}(au)\, q_m((1-a)\,u) - \tfrac{(1-a)^2u}{2m-1} q_n(au)\, q_{m-1}((1-a)\,u) \\ = & \qquad -\beta_0^{(n,m)}(a) - u \textstyle\sum_{k=0}^{n+m-1} \beta_{k+1}^{(n,m)}(a)\,(2k+1)^{-1} q_k(u). \end{aligned}$$

For $u=0$ this gives $\beta_0^{(n,m)}(a)=0$ and dividing by $-u$ and equating the coefficients of $q_k(u)$, we get the desired formula. □

Now the proof of Theorem 2.4 is easy by induction in k and by the symmetry formula (25), we can assume $n \geq m$. Let $0 \leq a \leq 1$. We prove that $\beta_k^{(n,m)}(a) \geq 0$ for $k \leq n+m$ and that it is zero for $k < m$ (under the assumption $n \geq m$). This is true for $k=0$ by Lemma 3.4 when $m \geq 1$, and for $m=0$ it follows by Theorem 2.1. Assume now that the results hold for $k=k_0$ and assume $k_0+1 \leq n+m$. The nonnegativity for $k=k_0+1$ now follows by Lemma 3.4, and likewise if $k_0+1 < m \leq n$ the coefficient is 0 since $k_0 < (n-1) \wedge (m-1)$. □

3.5 Proof of Theorem 2.6

By Theorem 2.4 the result holds for $k=2$. Assuming it holds for $k-1 \geq 2$ we have

$$q_{n_1}(a_1u)\cdots q_{n_{k-1}}(a_{k-1}u) = \sum_{j=l'}^{L'} \gamma_j q_j((1-a_k)u), \quad u \in \mathbb{R} \tag{27}$$

with $l' = \min(n_1,\ldots,n_{k-1}), L' = n_1+\cdots+n_{k-1}$ and $\gamma_j \geq 0$ because we can write

$$a_j u = \frac{a_j}{1-a_k}(1-a_k)u, \quad j=1,\ldots,k-1.$$

If we multiply (18) with $q_{n_k}(a_ku)$ we get

$$\sum_{j=l'}^{L'} \gamma_j q_{n_k}(a_ku) q_j((1-a_k)u) = \sum_{j=l'}^{L'} \gamma_j \sum_{i=n_k\wedge j}^{n_k+j} \beta_i^{(n_k,j)}(a_k) q_i(u),$$

and the assertion follows. □

4 Inverse Gamma distribution

Grosswald proved [11] that the Student-t distribution is infinitely divisible. This is a consequence of the infinite divisibility of the inverse Gamma distribution because of subordination. It was proved later that the inverse Gamma distribution is a generalized Gamma convolution in the sense of Thorin, which is stronger than self-decomposability and in particular stronger than infinite divisibility, see e.g. Bondesson [4] and the recent book by Steutel and van Harn [14].

The following density on the half-line is an inverse Gamma density with scale parameter $\frac{1}{4}$ and shape parameter $\nu > 0$:

$$C_\nu \exp(-\frac{1}{4t})t^{-\nu-1}, \quad t>0, \quad C_\nu = \frac{1}{2^{2\nu}\Gamma(\nu)}. \tag{28}$$

Let the corresponding probability measure be denoted $\tilde{\gamma}_\nu$ and let further

$$g_t(x) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}), \quad t>0, x \in \mathbb{R}$$

denote the Gaussian semigroup of normal densities (in the normalization of [3]). Then the mixture

$$f_\nu(x) = \int_0^\infty g_t(x)\, d\tilde{\gamma}_\nu(t) \tag{29}$$

is the Student-t density (3) with 2ν degrees of freedom. The corresponding probability measure is denoted by σ_ν. This formula says that σ_ν is subordinated to the Gaussian semigroup by an inverse Gamma distribution, and it implies the infinite divisibility of Student-t from the infinite divisibility of inverse Gamma. Since the Laplace transformation is one-to-one, it is clear that if two probabilities γ_1, γ_2 on $]0,\infty[$ lead to the same subordinated density

$$\int_0^\infty g_t(x)\, d\gamma_1(t) = \int_0^\infty g_t(x)\, d\gamma_2(t), \quad x \in \mathbb{R},$$

then $\gamma_1 = \gamma_2$.

If we denote $\tau_a(x) = ax$, the distribution $\tau_a(\sigma_{n+\frac{1}{2}}) * \tau_{1-a}(\sigma_{m+\frac{1}{2}})$ is given in (10). However note that $\tau_a(g_t(x)dx) = g_{ta^2}(x)dx$, so

$$\tau_a(\sigma_\nu) = \int_0^\infty g_{ta^2}(x) d\tilde{\gamma}_\nu(t)\ dx, \tag{30}$$

hence

$$\begin{aligned}
\tau_a(\sigma_{\nu_1}) * \tau_{1-a}(\sigma_{\nu_2}) &= \int_0^\infty \int_0^\infty (g_{ta^2}dx) * (g_{s(1-a)^2}dx)\, d\tilde{\gamma}_{\nu_1}(t) d\tilde{\gamma}_{\nu_2}(s) \\
&= \int_0^\infty \int_0^\infty (g_{ta^2+s(1-a)^2}dx)\, d\tilde{\gamma}_{\nu_1}(t) d\tilde{\gamma}_{\nu_2}(s) \\
&= \int_0^\infty g_u(x)\ d\tau_{a^2}(\tilde{\gamma}_{\nu_1}) * \tau_{(1-a)^2}(\tilde{\gamma}_{\nu_2})(u)\ dx.
\end{aligned}$$

Therefore, using (30) we see that for $\nu_1 = n + \frac{1}{2}, \nu_2 = m + \frac{1}{2}$ with $n, m = 0, 1, \ldots$ the formula (10) rewritten as

$$\tau_a(\sigma_{n+\frac{1}{2}}) * \tau_{1-a}(\sigma_{m+\frac{1}{2}}) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) \sigma_{k+\frac{1}{2}}$$

is equivalent to

$$\tau_{a^2}(\tilde{\gamma}_{n+\frac{1}{2}}) * \tau_{(1-a)^2}(\tilde{\gamma}_{m+\frac{1}{2}}) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a) \tilde{\gamma}_{k+\frac{1}{2}}. \tag{31}$$

This shows that Theorem 2.4 is equivalent to the following result about inverse Gamma distributions:

The distribution of $a^2 Z_n + (1-a)^2 Z_m$*, where* Z_n, Z_m *are independent inverse Gamma random variables with distribution (28) for* $\nu = n+\frac{1}{2}, m+\frac{1}{2}$ *respectively, has a density which is a convex combination of inverse Gamma densities.*

This result can be extended to the multivariate Student-t distributions as follows. A rotation invariant $N-$variate Student-t probability density is given for $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ by

$$f_{N,\nu}(\mathbf{x}) = A_{N,\nu}\left(1+|\mathbf{x}|^2\right)^{-\nu-\frac{N}{2}}, \quad A_{N,\nu} = \frac{\Gamma\left(\nu+\frac{N}{2}\right)}{\Gamma(\nu)\left(\Gamma(\frac{1}{2})\right)^N},$$

where

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{N} x_i y_i, \quad |\mathbf{x}| = \left(\langle \mathbf{x}, \mathbf{x} \rangle\right)^{\frac{1}{2}}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N.$$

It is easy to verify that $f_{N,\nu}(\mathbf{x})$ is subordinated to the N-variate Gaussian semigroup

$$g_{N,t}(\mathbf{x}) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right), \quad t > 0, \mathbf{x} \in \mathbb{R}^N$$

by the inverse Gamma density (28), i.e.

$$f_{N,\nu}(\mathbf{x}) = \int_0^\infty g_{N,t}(\mathbf{x})\, d\tilde{\gamma}_\nu(t). \tag{32}$$

Therefore the characteristic function is given by

$$\int_{\mathbb{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} f_{N,\nu}(\mathbf{x})\, d\mathbf{x} = k_\nu(|\mathbf{y}|) \tag{33}$$

generalizing (4). In fact

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} f_{N,\nu}(\mathbf{x})\, d\mathbf{x} &= \int_0^\infty \left(\int_{\mathbb{R}^N} e^{i\langle \mathbf{x}, \mathbf{y} \rangle} g_{N,t}(\mathbf{x})\, d\mathbf{x} \right) d\tilde{\gamma}_\nu(t) \\
&= \int_0^\infty e^{-t|\mathbf{y}|^2}\, d\tilde{\gamma}_\nu(t)
\end{aligned}$$

and the result follows by (21).

As a conclusion, the Theorems 2.1, 2.2 and 2.4 apply in the multivariate case. For example, an equivalent form of (10) writes as follows: with $0 < a < 1$,

$$\frac{1}{a^N} f_{N,n+\frac{1}{2}}\left(a^{-1}\mathbf{x}\right) * \frac{1}{(1-a)^N} f_{N,m+\frac{1}{2}}\left((1-a)^{-1}\mathbf{x}\right) = \sum_{k=n\wedge m}^{n+m} \beta_k^{(n,m)}(a)\, f_{N,k+\frac{1}{2}}(\mathbf{x}).$$

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