

The Chen-Rubin conjecture in a continuous setting

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Abstract

We study the median $m(x)$ in the gamma distribution with parameter x and show that $0 < m'(x) < 1$ for all $x > 0$. This proves a generalization of a conjecture of Chen and Rubin from 1986: The sequence $m(n) - n$ decreases for $n \geq 1$. We also describe the asymptotic behaviour of m and m' at zero and at infinity.

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1 Introduction

The gamma distribution with parameter x has density with respect to Lebesgue measure on $(0, \infty)$ given by $e^{-t}t^{x-1}/\Gamma(x)$. We consider the median $m(x)$ of this distribution which is defined implicitly as

$$\int_0^{m(x)} \frac{e^{-t}t^{x-1}}{\Gamma(x)} dt = \frac{1}{2},$$

or

$$\int_0^{m(x)} e^{-t}t^{x-1} dt = \frac{1}{2} \int_0^\infty e^{-t}t^{x-1} dt. \quad (1)$$

We of course also have

$$\int_{m(x)}^\infty e^{-t}t^{x-1} dt = \frac{1}{2} \int_0^\infty e^{-t}t^{x-1} dt. \quad (2)$$

We show that m is continuous and increasing. This is a consequence of a result about general convolution semigroups of probabilities on the positive half-line, that we give in Section 2. There we also show that m is real analytic and that m satisfies a certain differential equation.

We shall mainly study m through the function

$$\varphi(x) \equiv \log \frac{x}{m(x)}, \quad x > 0. \quad (3)$$

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This function appears if we make the substitution $u = \log(x/t)$ in the relation (2). We get:

$$\int_{-\infty}^{\varphi(x)} e^{-x(e^{-u}+u)} du = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x(e^{-u}+u)} du. \quad (4)$$

Chen and Rubin (see [5]) studied the median of the gamma distribution and proved that $x - 1/3 < m(x) < x$ for $x > 0$. The relation (4) was also used in [5] to establish that $\varphi(x) > 0$, or equivalently $m(x) < x$. (It follows by observing that $\int_{-\infty}^0 e^{-x(e^{-u}+u)} du < \int_0^{\infty} e^{-x(e^{-u}+u)} du$, which is true because $\sinh u > u$ for $u > 0$.)

Chen and Rubin furthermore conjectured that the sequence $m(n) - n$ decreases. This conjecture has recently been verified by Alzer (see [1]). He proved that $m(n+1) - \alpha n$ decreases for all $n \geq 0$ exactly when $\alpha \geq 1$ and increases exactly when $\alpha \leq m(2) - \log 2$. Note that $m(1) = \log 2$.

In this paper we investigate the properties of m as a function on $(0, \infty)$. Chen and Rubin's conjecture follows from the relation

$$m'(x) < 1 \quad \text{for all } x > 0$$

and it is our main goal to verify this relation. (See Theorem 3.3.)

We have drawn the graph of m in Figure 1. (The dotted line is given by $x - 1/3$.)

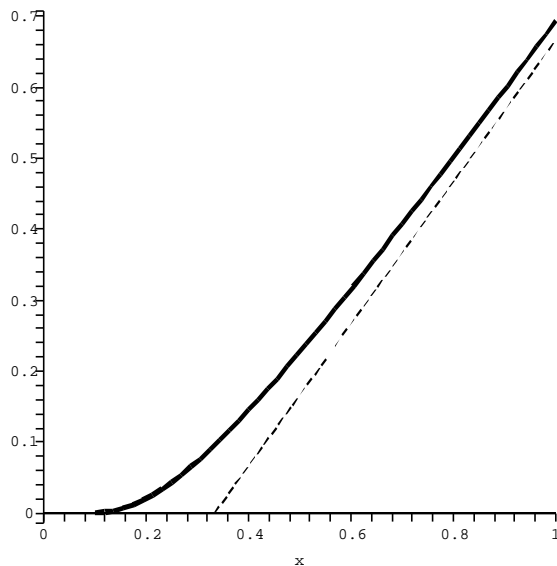


Figure 1: The graph of m

In terms of φ Chen and Rubins conjecture takes the form $1 - x\varphi'(x) < e^{\varphi(x)}$. As a crucial step towards this result we show that $x\varphi(x)$ decreases and that

$$\frac{1}{3} < x\varphi(x) < \log 2$$

for all $x > 0$ (see Proposition 3.5). In terms of m this relation can be rewritten as

$$xe^{-\log 2/x} < m(x) < xe^{-1/3x}.$$

If we use that $e^{-a} < 1 - a + a^2/2$ for $a > 0$ then we see that in fact

$$m(x) < x \left(1 - \frac{1}{3x} + \frac{1}{18x^2} \right) = x - \frac{1}{3} + \frac{1}{18x}$$

for all $x > 0$. This result improves a result of Choi about $m(n+1)$, see [6, Theorem 1], even though it is there claimed to be best possible.

The asymptotic behaviour of $m(x)$ for $x \rightarrow 0$ and for $x \rightarrow \infty$ is given in Section 4. From this we can deduce that $m^{(k)}(x) > 0$ for x close to 0 for each k while $(-1)^k m^{(k)}(x) > 0$ for x sufficiently large and $k \geq 2$. It is reasonable to believe that $m''(x) > 0$ for all $x > 0$, i.e. m is convex, but the higher derivatives of odd order change sign.

We also relate our work to papers of Ramanujan, Watson and others on Ramanujan's rational approximation to e^x , see Section 6.

2 Medians of convolution semigroups on the half-line

A family $\{\mu_x\}_{x>0}$ of probabilities concentrated on $[0, \infty)$ is called a convolution semigroup if it has the properties

- (i) $\mu_x([0, \infty)) = 1$ for all $x > 0$;
- (ii) $\mu_x * \mu_y = \mu_{x+y}$ for all $x, y > 0$;
- (iii) $\mu_x \rightarrow \delta_0$ for $x \rightarrow 0$ in the vague topology (here δ_0 denotes the Dirac mass at zero).

The gamma distributions $\{e^{-t}t^{x-1}/\Gamma(x) dt\}_{x>0}$ is an example of such a convolution semigroup.

A probability measure μ on $[0, \infty)$ has median m if

$$\mu([0, m]) = \frac{1}{2}.$$

Of course a probability measure may not have a median and if it exists it may not be unique. However, if the measure has density w.r.t. Lebesgue measure on $[0, \infty)$ then the median exists: this follows from the fact that $M \rightarrow \mu([0, M])$ is continuous and increases from 0 to 1. If the density is strictly positive almost everywhere on $[0, \infty)$ then the median is unique.

Proposition 2.1 *Let $\{\mu_x\}_{x>0}$ be a convolution semigroup of probabilities on $[0, \infty)$ having a.e. strictly positive densities w.r.t. Lebesgue measure on $[0, \infty)$. Then the median $m(x)$ of μ_x is a continuous and strictly increasing function on $(0, \infty)$.*

Proof. We write $d\mu_x(t) = g_x(t) dt$. As we have seen above, m exists as a function on the positive half-line. We now show that m is increasing. Let $x > 0$ and $h > 0$. Then

$$\begin{aligned}
\int_0^{m(x)} g_{x+h}(t) dt &= \int_0^{m(x)} g_x * g_h(t) dt \\
&= \int_0^{m(x)} \int_0^t g_h(t-s) g_x(s) ds dt \\
&= \int_0^{m(x)} \int_s^{m(x)} g_h(t-s) dt g_x(s) ds \\
&= \int_0^{m(x)} \int_0^{m(x)-s} g_h(t) dt g_x(s) ds \\
&< \int_0^{m(x)} \int_0^\infty g_h(t) dt g_x(s) ds \\
&= \int_0^{m(x)} g_x(s) ds = \frac{1}{2}.
\end{aligned}$$

Since $m(x+h)$ by definition satisfies

$$\int_0^{m(x+h)} g_{x+h}(t) dt = \frac{1}{2},$$

we must therefore have $m(x) < m(x+h)$. This shows that m is strictly increasing.

Concerning the continuity we first notice that for any $A > 0$,

$$\int_0^A g_{x_0+h}(t) dt \rightarrow \int_0^A g_{x_0}(t) dt$$

as $h \rightarrow 0$. In fact, using the same computation as above we get

$$\begin{aligned}
\int_0^A g_{x_0+h}(t) dt &= \int_0^A g_{x_0} * g_h(t) dt \\
&= \int_0^A \int_0^{A-s} g_h(t) dt g_{x_0}(s) ds \\
&\rightarrow \int_0^A \int_0^{A-s} d\delta_0(t) g_{x_0}(s) ds \\
&= \int_0^A g_{x_0}(s) ds,
\end{aligned}$$

as $h \rightarrow 0$, because of the vague convergence. (All measures have the same total mass, so vague convergence implies weak convergence.) Now let $x_0 > 0$ and let $\epsilon > 0$ be given. For $A = m(x_0) + \epsilon$ we have, as $h \rightarrow 0_+$,

$$\int_0^{m(x_0)+\epsilon} g_{x_0+h}(t) dt \rightarrow \int_0^{m(x_0)+\epsilon} g_{x_0}(t) dt > \frac{1}{2},$$

so that $m(x_0 + h) \leq m(x_0) + \epsilon$ for all sufficiently small and positive h . Similarly we find $m(x_0 - h) \geq m(x_0) - \epsilon$, and this shows that m is continuous. \square

We now specialize to consider the medians $m(x)$ of the gamma distributions. We notice

Proposition 2.2 *The median $m(x)$ of the gamma distributions is real analytic.*

Proof. We consider the C^1 -function

$$F(x, y) = \int_0^y e^{-t} t^{x-1} \frac{1}{\Gamma(x)} dt, \quad x, y > 0.$$

The median is implicitly defined as $F(x, m(x)) = 1/2$. The fact that the continuous function m is C^1 follows from the implicit function theorem, which yields the following differential equation for m :

$$e^{-m(x)} m(x)^{x-1} m'(x) = \frac{1}{2} \Gamma'(x) - \int_0^{m(x)} (\log t) e^{-t} t^{x-1} dt, \quad (5)$$

which shows that m satisfies a differential equation of the form

$$m'(x) = G(x, m(x)),$$

with $G(x, y)$ being real analytic for $x, y > 0$. Therefore m is real analytic. \square

Remark 2.3 *From (5) it seems difficult to deduce monotonicity properties of m ; e.g. it is not at all clear that $m'(x) > 0$, which we know from Proposition 2.1. For another derivation of $m'(x) > 0$ see Proposition 3.5.*

3 Uniform results

In this section we prove the generalized version of Chen and Rubin's conjecture (Theorem 3.3).

Proposition 3.2 below is the key to our results. Before stating it we need some notation.

We consider the function $f(x) = e^{-x} + x$. It is easily seen that $f(x)$ decreases for $x < 0$ and then increases for $x > 0$. Hence f has an inverse on $(-\infty, 0]$, which we call u , and an inverse on $[0, \infty)$, which we call v . The function u is defined for $t \geq 1$ as $u(t) \leq 0$ and

$$e^{-u(t)} + u(t) = t, \quad t \geq 1;$$

the function v as $v(t) \geq 0$ and

$$e^{-v(t)} + v(t) = t, \quad t \geq 1.$$

The following function ξ plays an important role:

$$\xi(t) = \frac{1}{1 - e^{-v(t)}} + \frac{1}{1 - e^{-u(t)}} = u'(t) + v'(t). \quad (6)$$

The function $\xi(t)$ is defined for $t > 1$ by (6), but for $t = 1$ this expression yields $\infty - \infty$ and a closer study is necessary. The following result holds and will be proved in Section 5.

Proposition 3.1 *The function $\xi(t)$ defined for $t > 1$ by (6) has a holomorphic extension to the cut plane $\mathbb{C} \setminus \{x \pm 2\pi i \mid x \geq 1\}$. The following holds:*

- (i) $\lim_{t \rightarrow -\infty} \xi(t) = 0$, $\lim_{t \rightarrow \infty} \xi(t) = 1$, $\xi(t)$ is increasing;
- (ii) $\lim_{t \rightarrow \infty} \xi^{(n)}(t) = 0$ for $n \geq 1$;
- (iii) $\xi(t)$ is concave for $t \in [1, \infty)$;
- (iv) $\xi(1) = 2/3$, $\xi'(1) = 8/135$ and $\xi''(1) = -16/2835$.
- (v) All points on $\{x \pm 2\pi i \mid x \geq 1\}$ are singular points for ξ and the series

$$\sum_{k=0}^{\infty} \frac{\xi^{(k)}(1)}{x^{k+1}}$$

diverges for all x .

In Figure 2 we have shown the graph of $\xi(t)$ for $t \geq 1$.

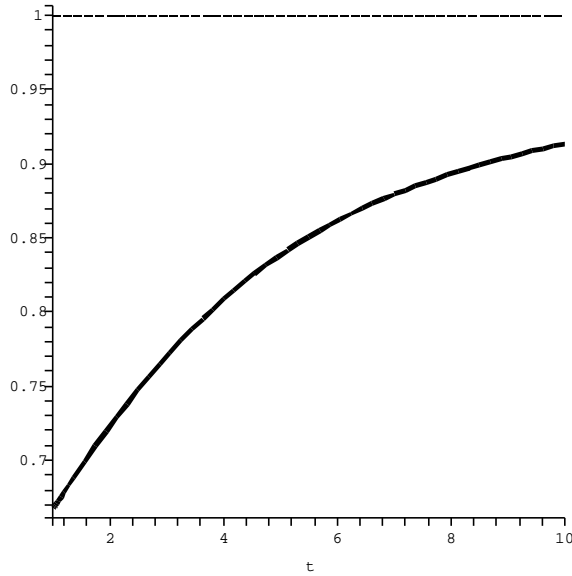


Figure 2: The graph of ξ

Proposition 3.2 *We have*

$$2 \int_0^{\varphi(x)} e^{-x(e^{-u}+u)} du = \int_1^{\infty} \xi(t)e^{-xt} dt, \quad (7)$$

where the function ξ is defined in (6).

Proof. From (4) we get

$$2 \int_0^{\varphi(x)} e^{-x(e^{-u}+u)} du = \int_0^{\infty} e^{-x(e^{-u}+u)} du - \int_{-\infty}^0 e^{-x(e^{-u}+u)} du.$$

In the first integral on the right hand side we make the substitution $u = v(t)$ and in the second the substitution $u = u(t)$, where $u(t)$ and $v(t)$ denote the functions above. In this way we get

$$\begin{aligned} 2 \int_0^{\varphi(x)} e^{-x(e^{-u}+u)} du &= \int_1^{\infty} e^{-xt} \left(\frac{1}{1 - e^{-v(t)}} + \frac{1}{1 - e^{-u(t)}} \right) dt \\ &= \int_1^{\infty} \xi(t) e^{-xt} dt. \end{aligned}$$

□

It is easy to see that in terms of the function φ the relation $m'(x) < 1$ takes the form

$$1 - x\varphi'(x) < e^{\varphi(x)}.$$

Theorem 3.3 *We have*

$$1 - x\varphi'(x) < e^{\varphi(x)}$$

for all $x > 0$.

Before proving this result, we give the following Lemma.

Lemma 3.4 *The relation*

$$2 \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} ds = \frac{2}{3} + \int_1^{\infty} \xi'(t) e^{-x(t-1)} dt$$

holds for all $x > 0$.

Proof. In the left hand side of the relation in Proposition 3.2 we make the substitution $s = xu$ and get in this way

$$\frac{2}{x} \int_0^{x\varphi(x)} e^{-s-xe^{-s/x}} ds = \int_1^{\infty} \xi(t) e^{-xt} dt.$$

On the right hand side we perform integration by parts, and this gives

$$\begin{aligned} 2 \int_0^{x\varphi(x)} e^{-s-xe^{-s/x}} ds &= \int_1^{\infty} \xi(t) x e^{-xt} dt \\ &= e^{-x} \xi(1) + \int_1^{\infty} \xi'(t) e^{-xt} dt. \end{aligned}$$

Here $\xi(1) = 2/3$ (see Proposition 3.1) and the assertion of the lemma now follows by multiplication by e^x . □

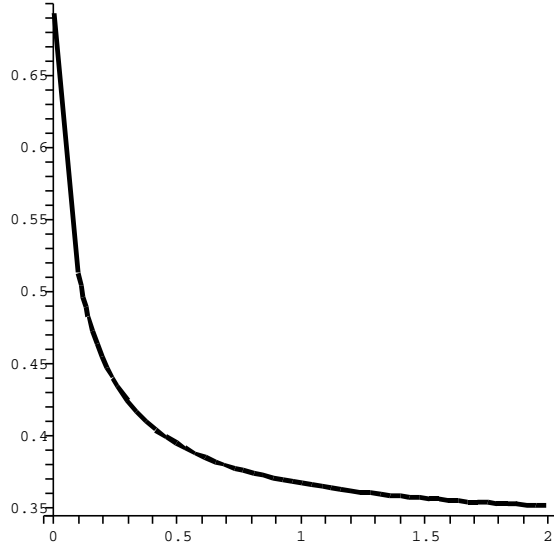


Figure 3: The graph of $x\varphi(x)$

Proposition 3.5 *The function $x \rightarrow x\varphi(x)$ decreases for $x > 0$ and we have*

$$\begin{aligned} \lim_{x \rightarrow 0^+} x\varphi(x) &= \log 2, \\ \lim_{x \rightarrow \infty} x\varphi(x) &= \frac{1}{3}. \end{aligned}$$

In particular φ is decreasing from ∞ to 0 (and m is increasing). (The graph of $x\varphi(x)$ is shown in Figure 3.)

Proof. From Lemma 3.4 we find by differentiation, that

$$\begin{aligned} e^{-x(\varphi(x)-1+e^{-\varphi(x)})} (x\varphi(x))' &= \\ &= - \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} \left(1 - \left(1 + \frac{s}{x}\right) e^{-s/x}\right) ds \\ &\quad - \frac{1}{2} \int_0^\infty t e^{-xt} \xi'(t+1) dt. \end{aligned}$$

Since $1 - (1+a)e^{-a} > 0$ for $a > 0$ and $\xi'(t+1) > 0$ for $t > 0$ (see Proposition 3.1) we have

$$A(x) \equiv \int_0^{x\varphi(x)} e^{-s} e^{x(1-e^{-s/x})} \left(1 - \left(1 + \frac{s}{x}\right) e^{-s/x}\right) ds > 0$$

and

$$B(x) \equiv \frac{1}{2} \int_0^\infty t e^{-xt} \xi'(t+1) dt > 0.$$

Therefore we have

$$(x\varphi(x))' = -e^{x(\varphi(x)-1+e^{-\varphi(x)})}(A(x) + B(x)), \quad (8)$$

which is a negative quantity, so $x\varphi(x)$ decreases.

Furthermore, since $e^{-s}e^{x(1-e^{-s/x})} \geq e^{-s}$, we get from Lemma 3.4,

$$2 \int_0^{x\varphi(x)} e^{-s} ds \leq \frac{2}{3} + \int_1^\infty \xi'(t) dt = \frac{2}{3} + \lim_{x \rightarrow \infty} \xi(x) - \xi(1) = 1,$$

since $\lim_{x \rightarrow \infty} \xi(x) = 1$. (See Proposition 3.1.) From this we obtain that $x\varphi(x)$ is bounded by $\log 2$ for all $x > 0$.

If we let $l = \lim_{x \rightarrow 0^+} x\varphi(x)$ then by the dominated convergence theorem we find

$$2 \int_0^l e^{-s} ds = \frac{2}{3} + \int_1^\infty \xi'(t) dt = 1,$$

so that $l = \log 2$.

We let finally $L = \lim_{x \rightarrow \infty} x\varphi(x)$. We get in the same way as before

$$2 \int_0^L 1 ds = \frac{2}{3},$$

so that $L = 1/3$. □

Proof of Theorem 3.3. We use the expressions $A(x)$ and $B(x)$ and the relation (8) from the proof of Proposition 3.5. We get

$$\begin{aligned} 1 - x\varphi'(x) &= e^{x(\varphi(x)-1+e^{-\varphi(x)})}(A(x) + B(x)) + \varphi(x) + 1 \\ &\leq e^{x\varphi(x)}(A(x) + B(x)) + \varphi(x) + 1. \end{aligned}$$

Now, using that $1 - e^{-a} < a$ and $1 - (1+a)e^{-a} < a^2/2$ for $a > 0$, we find

$$A(x) \leq \int_0^{x\varphi(x)} e^{-s} e^{x(s/x)} \left(\frac{s^2}{2x^2} \right) ds = \frac{x\varphi(x)^3}{6}.$$

Furthermore, since $\xi'(t+1) \leq \xi'(1) = 8/135$ for $t \geq 0$ (see again Proposition 3.1),

$$B(x) \leq \frac{1}{2} \frac{8}{135} \int_0^\infty t e^{-xt} dt = \frac{4}{135x^2}.$$

This gives

$$1 - x\varphi'(x) \leq 1 + \varphi(x) + e^{x\varphi(x)} \left(\frac{x\varphi(x)}{6} + \frac{4}{135(x\varphi(x))^2} \right) \varphi(x)^2.$$

It is easily seen that the function

$$\rho(u) = e^u \left(\frac{u}{6} + \frac{4}{135u^2} \right)$$

attains its maximum on $[1/3, \log 2]$ at $u = 1/3$ with value

$$\rho\left(\frac{1}{3}\right) = \frac{29}{90} e^{1/3} < \frac{1}{2}.$$

We obtain from this the relation

$$1 - x\varphi'(x) < 1 + \varphi(x) + \frac{1}{2}\varphi(x)^2 < e^{\varphi(x)}.$$

□

Remark 3.6 *The difference between $e^{\varphi(x)}$ and $1 + \varphi(x) + \frac{1}{2}\varphi(x)^2$ is $O(\varphi(x)^3)$, that is (by Proposition 3.5) the difference is $O(x^{-3})$. Hence for large x the difference is very small, reflecting the fact that $m'(x)$ is very close to 1. For x close to 0, the difference is large, reflecting the fact that $m'(x)$ approaches 0 rapidly. See Proposition 4.1.*

4 Asymptotic results

Here we describe the behaviour of m and m' at zero and at infinity. We first investigate m near zero. We use $f(x) \sim g(x)$ for $x \rightarrow 0$ to denote that

$$\lim_{x \rightarrow 0} f(x)/g(x) = 1.$$

Proposition 4.1 *We have $m(x)^x \rightarrow 1/2$ as $x \rightarrow 0$ and*

$$m(x) \sim e^{-\gamma} 2^{-1/x} \quad \text{as } x \rightarrow 0.$$

Furthermore,

$$m'(x) \sim (\log 2) e^{-\gamma} \frac{2^{-1/x}}{x^2} \quad \text{as } x \rightarrow 0.$$

Proof. For the function $l(x) \equiv \log(m(x)^x)$ we have $l(x) = -x\varphi(x) + x \log x$. Since $x \log x \rightarrow 0$ as $x \rightarrow 0$ then $l(x) \rightarrow -\log 2$ as $x \rightarrow 0$. Therefore $m(x)^x \rightarrow 1/2$ as $x \rightarrow 0$.

By definition of $m(x)$ and by the functional equation of the gamma function we have

$$\int_0^{m(x)} e^{-t} x t^{x-1} dt = \frac{1}{2} \Gamma(x+1).$$

We perform integration by parts on the integral on the left hand side in this relation and we thus get

$$m(x)^x e^{-m(x)} + \int_0^{m(x)} e^{-t} t^x dt = \frac{1}{2} \Gamma(x+1).$$

We next differentiate this relation and get after some manipulation

$$\log m(x) + \frac{x}{m(x)} m'(x) = e^{m(x)} m(x)^{-x} \left(\frac{1}{2} \Gamma'(x+1) - \int_0^{m(x)} (\log t) e^{-t} t^x dt \right).$$

We now use that $m(x)^x \rightarrow 1/2$ as $x \rightarrow 0$ and we get in this way

$$\lim_{x \rightarrow 0} \left(\log m(x) + \frac{x}{m(x)} m'(x) \right) = \Gamma'(1) = -\gamma. \quad (9)$$

This is the same as $l'(x) = (\log(m(x)^x))' \rightarrow -\gamma$ as $x \rightarrow 0$. Using l'Hospital's rule we get

$$\frac{l(x) + \log 2}{x} \rightarrow -\gamma$$

for $x \rightarrow 0$. Therefore $m(x) \sim 2^{-1/x} e^{-\gamma}$ as $x \rightarrow 0$.

By (9) we get

$$x \log m(x) + \frac{x^2}{m(x)} m'(x) \rightarrow 0$$

for $x \rightarrow 0$ and, since (as used before) $x \log m(x) \rightarrow -\log 2$, we get

$$m'(x) \sim \frac{\log 2}{x^2} 2^{-1/x} e^{-\gamma}$$

as $x \rightarrow 0$. □

At infinity we have

Proposition 4.2 *The functions m and m' have asymptotic expansions at infinity. We have in particular*

$$m'(x) = 1 - \frac{8}{405x^2} - \frac{368}{25515x^3} + o(x^{-3}) \quad \text{as } x \rightarrow \infty$$

and

$$m(x) = x - \frac{1}{3} + \frac{8}{405x} + \frac{184}{25515x^2} + o(x^{-2}) \quad \text{as } x \rightarrow \infty.$$

Remark 4.3 *Choi ([6]) found the asymptotic expansion of $m(n+1)$ up to order $o(n^{-3})$. Higher order expansions of $m(n+1)$ were found in [9]. In the appendix we have included higher order expansions of $m(x)$ and $\varphi(x)$ (and higher order derivatives of the function ξ at 1). Because of the complexity, the computations behind the expansions in the appendix were made using "Maple 9", using the same method as described in Lemma 4.4.*

Lemma 4.4 *The functions φ and φ' have asymptotic expansions at infinity. For the function φ we have in particular*

$$\varphi(x) = \frac{1}{3x} + \frac{29}{810x^2} - \frac{37}{25515x^3} + o(x^{-3})$$

as $x \rightarrow \infty$.

Proof. We have already seen in Proposition 3.5 that $\varphi(x) = O(x^{-1})$. Therefore

$$2 \int_0^{\varphi(x)} e^{-x(u+e^{-u}-1)} du = 2 \sum_{k=0}^n \frac{(-1)^k x^k}{k!} \int_0^{\varphi(x)} (u+e^{-u}-1)^k du + o(x^{-(n+1)}).$$

On the other hand, by partial integration,

$$\int_0^\infty \xi(t+1)e^{-xt} dt = \sum_{k=0}^n \frac{\xi^{(k)}(1)}{x^{k+1}} + o(x^{-(n+1)}). \quad (10)$$

(We remark here that the sum involving the derivatives of ξ at 1 describes the asymptotic behaviour of Ramanujans function, see Section 6.) From Proposition 3.2 we thus get

$$\varphi(x) = \sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^{k+1}} - \sum_{k=1}^n \frac{(-1)^k x^k}{k!} \int_0^{\varphi(x)} (u + e^{-u} - 1)^k du + o(x^{-(n+1)}). \quad (11)$$

From this relation it is possible to deduce that $\varphi(x)$ has an asymptotic expansion of the form

$$\varphi(x) = \sum_{k=1}^n \frac{c_k}{x^k} + o(x^{-n}) \quad (12)$$

as $x \rightarrow \infty$ and for any n . (See below.)

Let us first find the coefficients c_1, c_2 and c_3 in the expansion. If $n = 0$ we get, using $\xi(1) = 2/3$,

$$\varphi(x) = \frac{1}{3x} + o(x^{-1}).$$

(This we have already found in Proposition 3.5.) For $n = 1$ we get, using $\xi'(1) = 8/135$,

$$\varphi(x) = \frac{1}{3x} + \frac{4}{135x^2} + x \int_0^{\varphi(x)} (u + e^{-u} - 1) du + o(x^{-2}).$$

Here we use $u + e^{-u} - 1 = u^2/2 + o(u^2)$ for $u \rightarrow 0$ and we get in this way

$$\varphi(x) = \frac{1}{3x} + \frac{4}{135x^2} + x \left(\frac{\varphi(x)^3}{6} + o(x^{-3}) \right) + o(x^{-2}).$$

Then we use the result for $n = 0$ to obtain that $x\varphi(x)^3 = 1/(27x^2) + o(x^{-2})$, and if we insert this into the relation above we get

$$\varphi(x) = \frac{1}{3x} + \left(\frac{4}{135} + \frac{1}{162} \right) \frac{1}{x^2} + o(x^{-2}) = \frac{1}{3x} + \frac{29}{810x^2} + o(x^{-2}).$$

We repeat the argument to obtain the following for $n = 2$:

$$\begin{aligned} \varphi(x) &= \frac{1}{3x} + \frac{4}{135x^2} - \frac{8}{2835x^3} \\ &\quad + x \int_0^{\varphi(x)} (u + e^{-u} - 1) du - \frac{x^2}{2} \int_0^{\varphi(x)} (u + e^{-u} - 1)^2 du + o(x^{-3}), \end{aligned}$$

where we use $u + e^{-u} - 1 = u^2/2 - u^3/6 + o(u^3)$ and $(u + e^{-u} - 1)^2 = u^4/4 + o(u^4)$ for $u \rightarrow 0$ and we get

$$\begin{aligned} \varphi(x) &= \frac{1}{3x} + \frac{4}{135x^2} - \frac{8}{2835x^3} \\ &\quad + x \left(\frac{\varphi(x)^3}{6} - \frac{\varphi(x)^4}{24} + o(x^{-4}) \right) \\ &\quad - \frac{x^2}{2} \left(\frac{\varphi(x)^5}{20} + o(x^{-5}) \right) + o(x^{-3}). \end{aligned}$$

In the term $x\varphi(x)^3/6$ we substitute the expansion of $\varphi(x)$ for $n = 1$ and in the terms $x\varphi(x)^4/24$ and $x^2\varphi(x)^5/40$ the expansion of $\varphi(x)$ for $n = 0$. In this way we get

$$\begin{aligned}\varphi(x) &= \frac{1}{3x} + \frac{29}{810x^2} + \left(-\frac{8}{2835} + \frac{1}{18} \frac{29}{810} - \frac{1}{24} \frac{1}{81} - \frac{1}{40} \frac{1}{243} \right) \frac{1}{x^3} + o(x^{-3}) \\ &= \frac{1}{3x} + \frac{29}{810x^2} - \frac{37}{25515x^3} + o(x^{-3}).\end{aligned}$$

These computations also indicate how to show that there is an asymptotic expansion (12) for every $n \geq 1$. One could use an inductive argument based on the relation (11): First of all, the sum

$$\sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^{k+1}}$$

contains a term of order $1/x^{n+1}$. Next the integrand $(u + e^{-u} - 1)^k$ is approximated by its Taylor polynomial of order $n + k$,

$$(u + e^{-u} - 1)^k = \sum_{l=2k}^{n+k} \alpha_{k,l} u^l + o(u^{n+k}),$$

and the expansion $\varphi(x) = \sum_{k=1}^n c_k/x^{-k} + o(x^{-n})$ is then used in the upper limit of the integrals in the sum

$$\sum_{k=1}^n \frac{(-1)^k x^k}{k!} \int_0^{\varphi(x)} (u + e^{-u} - 1)^k du.$$

Using these approximations it is possible to obtain

$$\varphi(x) = \sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^{k+1}} - \sum_{k=1}^n \frac{(-1)^k x^k}{k!} \sum_{l=2k}^{n+k} \frac{\alpha_{k,l}}{l+1} \left(\sum_{j=1}^n \frac{c_j}{x^j} \right)^{l+1} + o(x^{-(n+1)}).$$

This is an expansion of $\varphi(x)$ of order $n + 1$.

To see that also $\varphi'(x)$ has an asymptotic expansion we differentiate (7) and get

$$2\varphi'(x)e^{-x(\varphi(x)+e^{-\varphi(x)})} - 2 \int_0^{\varphi(x)} e^{-x(u+e^{-u})} (u + e^{-u}) du = - \int_1^\infty t\xi(t)e^{-xt} dt.$$

Adding (7) to the relation above we get after multiplication by e^x and a change of variable $s = t - 1$,

$$\begin{aligned}2\varphi'(x)e^{-x(\varphi(x)+e^{-\varphi(x)}-1)} \\ = 2 \int_0^{\varphi(x)} e^{-x(u+e^{-u}-1)} (u + e^{-u} - 1) du - \int_0^\infty s\xi(1+s)e^{-xs} ds.\end{aligned}$$

In the first integral we consider again Taylor approximation and in the second integral we perform integration by parts. Using also the asymptotic expansion of

$e^{x(\varphi(x)+e^{-\varphi(x)}-1)}$ we are finally able to see that there is an asymptotic expansion of $\varphi'(x)$. The coefficients in this expansion can be identified by integrating the expansion and using the known expansion of $\varphi(x)$, cf [2][Appendix C]. \square

Proof of Proposition 4.2. Since $\varphi(x)$ has an asymptotic expansion,

$$m(x) = xe^{-\varphi(x)} = x \sum_{k=0}^{\infty} \frac{(-1)^k \varphi(x)^k}{k!}$$

also has an asymptotic expansion. We have in particular

$$m(x) = xe^{-\varphi(x)} = x \left(1 - \varphi(x) + \frac{\varphi(x)^2}{2} - \frac{\varphi(x)^3}{6} + o(x^{-3}) \right).$$

We insert in this relation the asymptotic expansion of $\varphi(x)$ from the lemma above. We get, after some computation,

$$\begin{aligned} m(x) &= x \left(1 - \frac{1}{3x} + \frac{8}{405x^2} + \frac{184}{25515x^3} + o(x^{-3}) \right) \\ &= x - \frac{1}{3} + \frac{8}{405x} + \frac{184}{25515x^2} + o(x^{-2}). \end{aligned}$$

Since $\varphi'(x)$ has an asymptotic expansion, the same is true for $m'(x)$, since $m'(x) = (1 - x\varphi'(x))e^{-\varphi(x)}$. The expansion of $m(x)$ can be found by integrating the expansion of $m'(x)$, and this gives the desired expansion of $m'(x)$. \square

Remark 4.5 *It is clear that the methods above can be continued so the asymptotic behaviour of $m^{(k)}(x)$, $k \geq 2$ for $x \rightarrow 0$ and $x \rightarrow \infty$ can be determined by differentiation of the asymptotic formulas for $m'(x)$.*

5 Properties of the auxiliary function ξ

In this section we derive the properties of the function ξ , stated in Proposition 3.1. Most of these properties we found surprisingly difficult to establish. Much of the difficulty lies in the fact that ξ is given as a sum of two terms, where it is necessary to control the cancellation between these two terms.

Throughout this section u and v denote the functions defined in (6). We begin by considering them separately. The investigation is based on a simple lemma of independent interest. We recall that a C^∞ -function $\nu(t)$ defined on the positive half-line is called *completely monotonic* if

$$(-1)^p \nu^{(p)}(t) \geq 0, \quad \text{for all integers } p \geq 0.$$

It is called a *Bernstein function* if

$$\nu(t) \geq 0 \quad \text{and} \quad (-1)^p \nu^{(p)}(t) \leq 0, \quad \text{for all integers } p \geq 1.$$

The last conditions can also be expressed that $\nu'(t)$ is completely monotonic. For details about these classes of functions see e.g. [3].

Lemma 5.1 *Let F be completely monotonic, let $\varsigma(t)$ be a positive C^∞ -function for $t > 0$ and assume that $\varsigma'(t) = F(\varsigma(t))$. Then we have:*

For each $n \geq 1$ there exists a completely monotonic function F_n such that

$$(-1)^{n-1}\varsigma^{(n)}(t) = F_n(\varsigma(t)).$$

In particular ς is a Bernstein function.

Proof. For $n = 1$ we can use $F_1 = F$. If $(-1)^{n-1}\varsigma^{(n)}(t) = F_n(\varsigma(t))$ for some completely monotonic function F_n then

$$(-1)^n\varsigma^{(n+1)}(t) = -F_n'(\varsigma(t))\varsigma'(t) = -F_n'(\varsigma(t))F(\varsigma(t)).$$

Here $F_{n+1} \equiv -F_n'F$ is completely monotonic as a product of two completely monotonic functions.

Furthermore we have

$$(-1)^n\varsigma^{(n)}(t) = -F_n(\varsigma(t)) \leq 0,$$

so ς is a Bernstein function. □

Proposition 5.2 *The functions $-u(t+1)$ and $v(t+1)$ are Bernstein functions and*

$$\lim_{t \rightarrow \infty} v(t)/t = 1, \quad \lim_{t \rightarrow \infty} u(t)/t = 0.$$

Proof. We have $v'(t) = F(v(t))$ where

$$F(x) = \frac{1}{1 - e^{-x}} = \sum_{k=0}^{\infty} e^{-kx}$$

is completely monotonic. Since $v(t) \rightarrow \infty$ as $t \rightarrow \infty$ we have $e^{-v(t)} \rightarrow 0$ and hence $v(t)/t = 1 - e^{-v(t)}/t \rightarrow 1$ as $t \rightarrow \infty$.

For the function $w \equiv -u$ we have $w'(t) = G(w(t))$ where

$$G(x) = \frac{1}{e^x - 1} = \sum_{k=1}^{\infty} e^{-kx}$$

is completely monotonic. We have $t = e^{-u(t)} + u(t) > 1 + u(t)^2/2$ (since $u(t) < 0$) so that $-u(t) < \sqrt{2(t-1)}$ and thus $u(t)/t \rightarrow 0$ as $t \rightarrow \infty$. □

Bernstein functions admit integral representations (see e.g. [3, p. 64]) and thus we have

$$v(t+1) = t + \int_0^\infty (1 - e^{-xt}) d\lambda(x)$$

and

$$-u(t+1) = \int_0^\infty (1 - e^{-xt}) d\sigma(x),$$

for some positive measures λ and σ on $(0, \infty)$. Since $t+1 - v(t+1) = e^{-v(t+1)} \rightarrow 0$ for $t \rightarrow \infty$ we conclude that $\lambda((0, \infty)) = 1$, hence

$$v(t+1) = t + 1 - \int_0^\infty e^{-xt} d\lambda(x).$$

The measure σ has infinite total mass because $u(t) \rightarrow -\infty$ for $t \rightarrow \infty$.

From these representations we see that $\xi(t) = u'(t) + v'(t) \rightarrow 1$ as $t \rightarrow \infty$ and also that $u^{(n+1)}(t)$ and $v^{(n+1)}(t)$ both tend to zero as t tends to infinity for any $n \geq 1$. This shows that $\xi^{(n)}(t) = u^{(n+1)}(t) + v^{(n+1)}(t) \rightarrow 0$ as $t \rightarrow \infty$ for any $n \geq 1$.

One could hope to deduce further properties of ξ by using these integral representations. We have not succeeded in doing this, since much cancellation between u and v takes place and we do not know λ and σ explicitly. Furthermore, it is not even true that $\xi(t+1)$ is a Bernstein function even though it is increasing and concave.

We have found an approach using complex analysis and we shall make extensive use of the theory of the so-called Pick functions, see [7]. A holomorphic function p defined in the upper half plane is a Pick function if it maps the upper half plane into the closed upper half plane, or put in another way, if $\Im p$ is a non-negative harmonic function. A Pick function has an integral representation,

$$p(z) = az + b + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t), \quad (13)$$

where $a \geq 0$, b is real and μ is a positive measure on the real line. Furthermore,

$$a = \lim_{y \rightarrow \infty} \frac{\Im p(iy)}{y}, \quad b = \Re p(i)$$

and

$$\mu = \lim_{y \rightarrow 0^+} \frac{\Im p(x+iy)}{\pi} dx$$

in the vague topology. Note that a Pick function can be extended to a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$ by (13). If $A \subseteq \mathbb{R}$ is closed we recall that p has a holomorphic extension to $\mathbb{C} \setminus A$ if and only if the support of μ is contained in A .

The proofs of the results to follow are based on an investigation of the holomorphic function

$$f(z) = e^{-z} + z. \quad (14)$$

Concerning the values of the function ξ and its derivatives at $z = 1$ we have the following proposition.

Proposition 5.3 *The function $\xi(z)$ is holomorphic in a neighbourhood of $z = 1$ and we have $\xi(1) = 2/3$ and $\xi'(1) = 8/135$.*

Proof of Proposition 5.3. The function $f(z) - 1$ has a zero of multiplicity 2 at $z = 0$. Hence there exists a holomorphic function h in a neighbourhood of 0 such that $f(z) - 1 = h(z)^2$ there. Since $h'(0)^2 = 1/2 \neq 0$, h is one-to-one near $z = 0$. We choose $h'(0) = 1/\sqrt{2}$ and with this choice h is uniquely determined.

We have thus a radius $r > 0$ such that for any w with $|w| < r$, there are exactly two solutions to the equation $f(z) = w$, namely $z = h^{-1}(\pm\sqrt{w-1})$. In particular, for $t > 1$ and close to 1 we have

$$u(t) = h^{-1}(-\sqrt{t-1}) \quad \text{and} \quad v(t) = h^{-1}(\sqrt{t-1}),$$

where u and v are the functions appearing in (6). We denote by $\sum_{n=1}^{\infty} a_n w^n$ the Taylor series of $h^{-1}(w)$. Then

$$u(t) + v(t) = \sum_{k=1}^{\infty} 2a_{2k}(t-1)^k,$$

and, since $\xi(t) = u'(t) + v'(t)$, we find $\xi(1) = 2a_2$ and $\xi'(1) = 4a_4$. The numbers a_2 and a_4 can be found as follows. If $h(z) = \sum_{n=1}^{\infty} b_n z^n$ then

$$\left(\sum_{n=1}^{\infty} b_n z^n \right)^2 = e^{-z} + z - 1 = \frac{1}{2}z^2 - \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots$$

and this gives after some manipulation $b_1 = 1/\sqrt{2}$, $b_2 = -\sqrt{2}/12$, $b_3 = \sqrt{2}/72$ and $b_4 = -\sqrt{2}/540$. The numbers a_2 and a_4 can now be identified if we differentiate four times the relation $(h^{-1})(h(z)) = z$ and put $z = 0$ (so that expressions involving e.g. $(h^{-1})^{(4)}(0)$ appear). We find, again after some manipulation, $(h^{-1})^{(2)}(0) = 2/3$ and $(h^{-1})^{(4)}(0) = 16/45$, so that $\xi(1) = 2a_2 = 2/3$ and $\xi'(1) = 4a_4 = 4 \cdot (16/45) \cdot (1/24) = 8/135$. \square

Remark 5.4 *One can in principle find any derivative of ξ at 1 in this way. Computations show that $\xi''(1) = -16/2835$. The higher order derivatives of ξ at 1 can be found in the appendix.*

The monotonicity properties of ξ follows from the next proposition.

Proposition 5.5 *The function ξ increases on the real line and it is concave on $[1, \infty)$.*

The key to the proof of this proposition is an integral representation of ξ that we give in Proposition 5.12.

Lemma 5.6 *The function $f(z) = e^{-z} + z$ is a conformal mapping of the strip*

$$\mathcal{S}_{2\pi} = \{0 < \Im z < 2\pi\}$$

onto the domain

$$\mathcal{T} = \mathbb{C} \setminus (\{\Im w = 2\pi i, \Re w \geq 1\} \cup \{\Im w = 0, \Re w \geq 1\}).$$

In Figure 4 we have indicated the image of some horizontal lines in $\mathcal{S}_{2\pi}$ under the function f .

Proof. We have $f(x + iy) = \sigma + i\tau$ if and only if

$$\begin{aligned} e^{-x} \cos y + x &= \sigma \\ -e^{-x} \sin y + y &= \tau. \end{aligned}$$

For $y = \pi$ we get $\tau = \pi$ and $\sigma = x - e^{-x}$ so f maps the horizontal line $\mathbb{R} + i\pi$ onto itself and is one-to-one there.

Note that for $0 < y < \pi$ we have $\tau = -e^{-x} \sin y + y < y$ so $f(\mathcal{S}) \subseteq \{\Im w < \pi\}$.

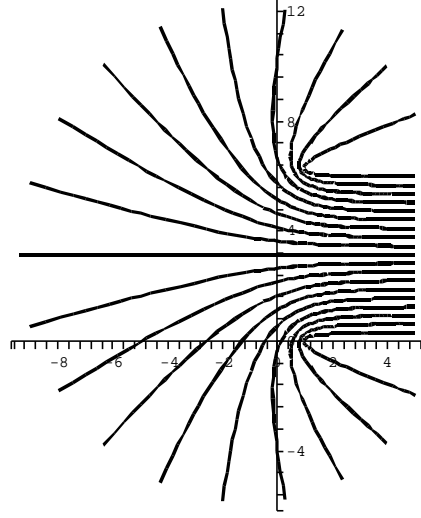


Figure 4: The image of horizontal lines under f

For $\tau < \pi$ we get $e^x = (\sin y)/(y - \tau)$ and since $\tau < y$ we find $x = \log(\sin y/(y - \tau))$ and finally we shall look for a solution y to the equation

$$\log \frac{\sin y}{y - \tau} + \frac{(y - \tau) \cos y}{\sin y} = \sigma.$$

For $\tau < \pi$ we put

$$F_\tau(y) = \log \frac{\sin y}{y - \tau} + \frac{(y - \tau) \cos y}{\sin y}. \quad (15)$$

We claim the following:

1. For $0 < \tau < \pi$, $F_\tau(y)$ decreases for $y \in (\tau, \pi)$ from ∞ to $-\infty$.
2. For $\tau = 0$, $F_\tau(y)$ decreases for $y \in (\tau, \pi)$ from 1 to $-\infty$.
3. For $\tau < 0$, $F_\tau(y)$ decreases for $y \in (0, \pi)$ from ∞ to $-\infty$.

It is easy to see that $\lim_{y \rightarrow \pi^-} F_\tau(y) = -\infty$, $\lim_{y \rightarrow \tau^+} F_\tau(y) = \infty$ for $\tau > 0$ and $\lim_{y \rightarrow 0^+} F_0(y) = 1$. We also have $\lim_{y \rightarrow 0^+} F_\tau(y) = \infty$ for $\tau < 0$ since the first term in $F_\tau(y)$ has a logarithmic singularity. To verify 1., 2. and 3. we need to show that $F'_\tau(y) < 0$. We find, after some computation,

$$F'_\tau(y) = \frac{1}{\sin y} \left(2 \cos y - \frac{\sin y}{y - \tau} - \frac{y - \tau}{\sin y} \right).$$

If we put $\kappa = (\sin y)/(y - \tau)$ then $\kappa > 0$ and hence $\kappa + 1/\kappa \geq 2$. Since $\cos y < 1$ we see that indeed $F'_\tau(y) < 0$.

From 1. and 3. it follows that for given $\sigma \in \mathbb{R}$ and $\tau < \pi$, $\tau \neq 0$ there is a unique solution y to the equation $F_\tau(y) = \sigma$ and therefore there is a unique solution to $f(x + iy) = \sigma + i\tau$. If $\tau = 0$ there is by 2. a unique solution y to the equation $F_\tau(y) = \sigma$ when $\sigma < 1$ and none when $\sigma \geq 1$.

For $\pi < y < 2\pi$ we have $\sin y < 0$ so $\tau > y$ and therefore

$$f(\{\pi < \Im z < 2\pi\}) \subseteq \{\pi < \Im w\}.$$

For $\tau > \pi$ we put

$$F_\tau(y) = \log \frac{-\sin y}{\tau - y} + \frac{(\tau - y) \cos y}{-\sin y}.$$

It follows that $F_\tau(y)$ increases with $y \in (\pi, 2\pi)$, and the conformality follows in the same way as for the case $\tau < \pi$. \square

Lemma 5.7 *The function $w \mapsto f^{-1}(1 - w)$ is a Pick function with the representation*

$$f^{-1}(1 - w) = \Re f^{-1}(1 - i) + \int_0^\infty \left(\frac{1}{t - w} - \frac{t}{t^2 + 1} \right) \eta(t) dt,$$

where

$$\eta(t) = \frac{1}{\pi} \Im f^{-1}(1 - t)$$

is increasing on $(0, \infty)$ from 0 to 1.

Proof. First of all, $1 - w$ belongs to the lower half plane when w belongs to the upper half plane. Since f^{-1} maps all of \mathcal{T} , and hence in particular the lower half plane, into the strip $0 < \Im z < 2\pi$, $f^{-1}(1 - w)$ is certainly a Pick function. We next derive its integral representation. Since its imaginary part is bounded the number a in the representation (13) must be zero.

It remains to identify the measure μ . Since the function $f^{-1}(1 - t - iy)$ is continuous on e.g. $\mathbb{R} \times [0, 1]$ and is real for $y = 0$ and $t < 0$ we find

$$\eta(t) = \lim_{y \rightarrow 0_+} \frac{\Im f^{-1}(1 - t - iy)}{\pi} = \frac{\Im f^{-1}(1 - t)}{\pi}$$

for $t > 0$ and $\eta(t) = 0$ for $t < 0$.

By definition the function $Y(t) = \pi\eta(t)$ satisfies the equation

$$F_0(Y(t)) = \log \frac{\sin Y(t)}{Y(t)} + \frac{Y(t) \cos Y(t)}{\sin Y(t)} = 1 - t$$

(with the notation of Lemma 5.6). We find from this

$$Y'(t)F_0'(Y(t)) = -1,$$

and since F_0' is negative (see again Lemma 5.6), $Y(t)$ and hence $\eta(t)$ must be increasing. Since $Y(t)$ tends to π as t tends to ∞ (this is because $F_0(Y(t)) \rightarrow -\infty$), $\eta(t) \rightarrow 1$ as $t \rightarrow \infty$. \square

Lemma 5.8 *The function*

$$g(w) = f^{-1}(w) + \overline{f^{-1}(\overline{w})}$$

has a holomorphic extension to $\mathbb{C} \setminus \{\Im w = \pm 2\pi i, \Re w \geq 1\}$.

For $t > 1$ we have $g(t) = u(t) + v(t)$ and $g'(t) = \xi(t)$. The function g' is thus a holomorphic extension of ξ .

All points on the lines $\{\Im w = \pm 2\pi i, \Re w \geq 1\}$ are singular points for g .

Proof. Since f^{-1} is a conformal mapping of the region

$$\mathbb{C} \setminus (\{\Im w = 2\pi i, \Re w \geq 1\} \cup \{\Im w = 0, \Re w \geq 1\}),$$

the function g is holomorphic in

$$\mathbb{C} \setminus (\{\Im w = \pm 2\pi i, \Re w \geq 1\} \cup \{\Im w = 0, \Re w \geq 1\}).$$

However, g has a continuous extension to $\{\Im w = 0, \Re w \geq 1\}$ with boundary values $u + v$, so from Moreras theorem we conclude that g is holomorphic across this half-line. Concerning the boundary values on the half-lines $\{\Im w = \pm 2\pi i, \Re w \geq 1\}$ we have ($\sigma \geq 1$)

$$\begin{aligned} \lim_{\tau \rightarrow 2\pi_-} g(\sigma + i\tau) &= \lim_{\tau \rightarrow 2\pi_-} f^{-1}(\sigma + i\tau) + \overline{f^{-1}(\sigma - 2\pi i)} \\ &= v(\sigma) + 2\pi i + \overline{f^{-1}(\sigma - 2\pi i)} \end{aligned}$$

and similarly

$$\lim_{\tau \rightarrow 2\pi_+} g(\sigma + i\tau) = u(\sigma) + 2\pi i + \overline{f^{-1}(\sigma - 2\pi i)}.$$

Therefore g is not continuous across any segment of the half-line

$$\{\Im w = 2\pi i, \Re w \geq 1\}$$

and so all these points are singular points. Since $g(w) = \overline{g(\overline{w})}$ the same conclusion holds for the points on the other half-line. \square

Proposition 5.9 *The Taylor series for $\xi(z)$ centered at 1 has radius of convergence equal to 2π and the asymptotic series*

$$\sum_{k=0}^{\infty} \frac{\xi^{(k)}(1)}{z^k}$$

diverges for any z in \mathbb{C} .

Proof. We have $\xi = g'$, where g is the function in Lemma 5.8. If the radius of convergence of ξ at 1 were larger than 2π then the primitive g would also have a holomorphic extension to this larger disk, and this contradicts the fact that g has singular points in that disk.

Concerning the divergence of the asymptotic series we use that the radius of convergence of the Taylor series is finite. It means that

$$\limsup_{k \rightarrow \infty} \left(\frac{|\xi^{(k)}(1)|}{k!} \right)^{1/k} > 0,$$

and hence that for some $\epsilon > 0$, $|\xi^{(k)}(1)|^{1/k}/(k!)^{1/k} > \epsilon$ for infinitely many k . From Stirlings formula we have $(k!)^{1/k} \sim k/e$ and therefore

$$\limsup_{k \rightarrow \infty} |\xi^{(k)}(1)|^{1/k} = \infty.$$

This shows on the other hand that the asymptotic series diverges for any complex number z . \square

In the following \log denotes the principal logarithm defined in the cut plane $\mathbb{C} \setminus (-\infty, 0]$.

Lemma 5.10 *The function*

$$\Psi(w) = g(\log w) = f^{-1}(\log w) + \overline{f^{-1}(\log \bar{w})}$$

is a Pick function. It has the representation

$$\Psi(w) = \Re \Psi(i) - \int_0^\infty \left(\frac{1}{t+w} - \frac{t}{t^2+1} \right) h(t) dt,$$

where the function h is given as

$$h(t) = 1 - \frac{\Im f^{-1}(\log t - i\pi)}{\pi}.$$

Furthermore, h increases on $(0, \infty)$ from 0 to 1.

For $\tau \in [e, \infty)$ we have

$$\Psi(\tau) = u(\log \tau) + v(\log \tau).$$

Proof. We consider the holomorphic function g from Lemma 5.8 in the strip $\mathcal{S} = \{0 < \Im w < \pi\}$. Hence the function $V(w) = \Im g(w)$ is a harmonic function in \mathcal{S} . Furthermore, since f^{-1} maps all of $\{\Im w < \pi\} \setminus \{x | x \geq 1\}$ into \mathcal{S} , V is also bounded there, with the apriori bound $-\pi < V(w) < \pi$. We claim that indeed $0 < V(w) < \pi$ for all $w \in \mathcal{S}$.

We consider the boundary values of V . The horizontal line $\{t + i\pi, |t \in \mathbb{R}\}$ is mapped by f^{-1} to itself, whereas $\{t - i\pi, |t \in \mathbb{R}\}$ is mapped by f^{-1} to some curve inside the strip \mathcal{S} . Therefore $\overline{f^{-1}(t - i\pi)}$ has imaginary part greater than $-\pi$ and hence the boundary values $V(t + i\pi), t \in \mathbb{R}$, are all non negative. Since $g(\bar{w}) = \overline{g(w)}$, the boundary values $V(t), t \in \mathbb{R}$, are all zero. We conclude that V has non-negative boundary values. Since it is bounded, the maximum principle in an unbounded region (see e.g. [10]) yields that $V(w) > 0$ for all $w \in \mathcal{S}$.

Since \log maps the upper half plane onto the strip \mathcal{S} , the function $\Psi(w) = g(\log w)$ maps the upper half plane into itself, and is hence a Pick function. Since $\Im \Psi$ is bounded, Ψ has an integral representation of the form

$$\Psi(w) = \Re \Psi(i) + \int_{-\infty}^\infty \left(\frac{1}{t-w} - \frac{t}{t^2+1} \right) d\mu(t),$$

for some positive measure μ . Here $\Im\Psi(t) = 0$ for $t > 0$ since V is zero on the real line. We thus find the measure μ to be supported on the negative half-line with density

$$1 - \frac{\Im f^{-1}(\log(-t) - i\pi)}{\pi}, \quad t < 0.$$

After making a change of variable ($t \mapsto -t$) in the integral we conclude that

$$\Psi(w) = \Re\Psi(i) - \int_0^\infty \left(\frac{1}{t+w} - \frac{t}{t^2+1} \right) h(t) dt, \quad (16)$$

where h is the function in the statement of the lemma.

By definition of h , $\pi(1 - h(t)) = \Im f^{-1}(\log t - i\pi)$, so h is increasing if the solution $Y(t)$ to the equation $F_{-\pi}(Y(t)) = \log t$ is decreasing. This is indeed the case, since

$$F'_{-\pi}(Y(t))Y'(t) = \frac{1}{t} > 0,$$

and $F'_{-\pi} < 0$. □

Remark 5.11 *Since it is easily verified that $V(w) = \Im g(w)$ is positive for $\pi < \Im w < 2\pi$ it follows that $g(2 \log w)$ is also a Pick function.*

Proposition 5.12 *The function $\xi(\log w)$ is holomorphic in the cut plane $\mathbb{C} \setminus (-\infty, 0]$ with the representation*

$$\xi(\log w) = 1 - \int_0^\infty \frac{t}{t+w} h'(t) dt,$$

where $h(t)$ is given in Lemma 5.10.

If we let $w \rightarrow 0_+$ we see that

$$\xi(\log w) \rightarrow 1 - \int_0^\infty h'(t) dt = 1 - \lim_{t \rightarrow \infty} h(t) + h(0) = 0.$$

Hence $\xi(t) \rightarrow 0$ as $t \rightarrow -\infty$.

Proof of Proposition 5.12. As we have noticed above, $\Psi(w) = u(\log w) + v(\log w)$, for $w \geq e$, where Ψ is the function in Lemma 5.10. Hence $\xi(\log w) = w\Psi'(w)$ and so we get from (16),

$$\xi(\log w) = \int_0^\infty \frac{wh(t)}{(t+w)^2} dt.$$

If we perform integration by parts on the right-hand side of this relation then we get

$$\xi(\log w) = \int_0^\infty \frac{w}{t+w} h'(t) dt,$$

or

$$\xi(\log w) = 1 - \int_0^\infty \frac{t}{t+w} h'(t) dt.$$

□

Proof of Proposition 5.5. We get by differentiation of the formula in Proposition 5.12 that

$$\frac{\xi'(\log w)}{w} = \int_0^\infty \frac{t}{(t+w)^2} h'(t) dt,$$

and since we know that $h'(t) > 0$ for all $t \in \mathbb{R}$, we see that $\xi'(t) > 0$ for all $t \in \mathbb{R}$.

It is more technical to establish that $\xi(t)$ is concave for $t \geq 1$. We differentiate both sides of the relation above once more and we get in this way

$$\xi''(\log w) - \xi'(\log w) = -2w^2 \int_0^\infty \frac{t}{(t+w)^3} h'(t) dt,$$

so that, again using the integral representation of $\xi'(\log w)$,

$$\begin{aligned} \xi''(\log w) &= \int_0^\infty \left(\frac{-2w^2}{(t+w)^3} + \frac{w}{(t+w)^2} \right) th'(t) dt \\ &= \int_0^\infty \frac{t-w}{(t+w)^3} wth'(t) dt \\ &= w \int_0^\infty \frac{s-1}{(s+1)^3} sh'(sw) ds. \end{aligned}$$

This last integral we split into two, one for $s < 1$ and another for $s > 1$. We thus get, after making the substitution $s \rightarrow 1/s$ in the latter one,

$$\begin{aligned} \frac{\xi''(\log w)}{w} &= \int_0^1 \frac{s-1}{(s+1)^3} sh'(sw) ds + \int_0^1 \frac{1/s-1}{(1/s+1)^3} \frac{1}{s} h'\left(\frac{w}{s}\right) \frac{ds}{s^2} \\ &= \int_0^1 \frac{1-s}{(1+s)^3} \left(\frac{1}{s} h'\left(\frac{w}{s}\right) - sh'(sw) \right) ds. \end{aligned}$$

From this relation we see that ξ is concave on $[\log w_0, \infty)$ if

$$\frac{1}{s} h'\left(\frac{w}{s}\right) - sh'(sw) \leq 0 \quad \text{for } s \in (0, 1) \quad \text{and } w \geq w_0. \quad (17)$$

Now, as noted in the proof of Lemma 5.10, $h(s) = 1 - Y_{-\pi}(\log s)/\pi$, where $Y_{-\pi}(t) \in (0, \pi)$ is the solution to the equation $F_{-\pi}(Y_{-\pi}(t)) = t$, see (15). Therefore $h'(s) = -Y'_{-\pi}(\log s)/(\pi s)$ and hence

$$\frac{1}{s} h'\left(\frac{w}{s}\right) = -\frac{Y'_{-\pi}(\log w - \log s)}{\pi w}$$

and

$$sh'(sw) = -\frac{Y'_{-\pi}(\log w + \log s)}{\pi w}.$$

We see that (17) holds provided

$$Y'_{-\pi}(\log w - \log s) \geq Y'_{-\pi}(\log w + \log s).$$

Since $\log s$ runs through $(-\infty, 0)$ when $s \in (0, 1)$, this condition is the same as

$$Y'_{-\pi}(\log w + t) \geq Y'_{-\pi}(\log w - t)$$

for all $t \geq 0$ and all $w \geq w_0$. We aim at proving this for $w_0 = e$, so what we really should prove is the following:

$$Y'_{-\pi}(\alpha + t) \geq Y'_{-\pi}(\alpha - t) \quad (18)$$

for all $t \geq 0$ and all $\alpha \geq 1$. We verify this inequality by using Lemma 5.7. According to that lemma, with $w = 1 - \alpha - t + i\pi$, we have

$$\begin{aligned} Y_{-\pi}(\alpha + t) &= \Im f^{-1}(\alpha + t - i\pi) \\ &= \Im f^{-1}(1 - (1 - \alpha - t + i\pi)) \\ &= \int_0^\infty \frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \eta(s) ds \end{aligned}$$

so that

$$\begin{aligned} Y'_{-\pi}(\alpha + t) &= \int_0^\infty \left(\frac{\partial}{\partial t} \frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \right) \eta(s) ds \\ &= \int_0^\infty \left(\frac{\partial}{\partial s} \frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \right) \eta(s) ds \\ &= \left[\frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \eta(s) \right]_0^\infty \\ &\quad - \int_0^\infty \frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \eta'(s) ds \\ &= - \int_0^\infty \frac{\pi}{(s + t + \alpha - 1)^2 + \pi^2} \eta'(s) ds. \end{aligned}$$

If we replace t by $-t$ in this formula we get

$$Y'_{-\pi}(\alpha - t) = - \int_0^\infty \frac{\pi}{(s - t + \alpha - 1)^2 + \pi^2} \eta'(s) ds,$$

so that

$$\begin{aligned} &Y'_{-\pi}(\alpha + t) - Y'_{-\pi}(\alpha - t) \\ &= 4\pi t \int_0^\infty \frac{s + \alpha - 1}{((s + t + \alpha - 1)^2 + \pi^2)((s - t + \alpha - 1)^2 + \pi^2)} \eta'(s) ds. \end{aligned}$$

Since $\eta'(s) > 0$ we see from this expression that $Y'_{-\pi}(\alpha + t) - Y'_{-\pi}(\alpha - t)$ has the same sign as t for $\alpha \geq 1$ and hence is positive for $t > 0$. \square

6 Relation to work of Ramanujan

Defining $\theta(n)$ for natural numbers n by

$$\frac{e^n}{2} = \sum_{k=0}^{n-1} \frac{n^k}{k!} + \theta(n) \frac{n^n}{n!}, \quad (19)$$

Ramanujan [11] claimed that $\frac{1}{3} < \theta(n) < \frac{1}{2}$. This was later proved independently by Szegő [12] and Watson [13]. Further details about Ramanujan's

problem can be found in [4]. See also [8]. Choi noticed the relation between this problem and the median in the gamma distribution, see [6]. By n partial integrations we get the formula

$$\frac{1}{n!} \int_0^n t^n e^{-t} dt = 1 - e^{-n} \sum_{k=0}^n \frac{n^k}{k!},$$

hence by (19)

$$\frac{1}{n!} \int_0^n t^n e^{-t} dt = \frac{1}{2} - \frac{n^n}{n!} (1 - \theta(n)) e^{-n},$$

and finally

$$1 - \theta(n) = \left(\frac{e}{n}\right)^n \int_n^{m(n+1)} t^n e^{-t} dt. \quad (20)$$

Following Watson [13] we can write

$$\begin{aligned} \theta(n) &= 1 + \frac{e^n}{2n^n} n! - \frac{n!}{n^n} \sum_{k=0}^n \frac{n^k}{k!} \\ &= 1 + \frac{e^n}{2n^n} \int_0^\infty t^n e^{-t} dt - \frac{1}{n^n} \int_0^\infty (n+t)^n e^{-t} dt \\ &= 1 + \frac{n}{2} \int_0^\infty t^n e^{n(1-t)} dt - \int_0^\infty (1+t/n)^n e^{-t} dt. \end{aligned}$$

We now make the substitution $s = 1 + t/n$ in the last integral and split the first integral in two by integrating over $(0, 1)$ and $(1, \infty)$. This gives

$$\theta(n) = 1 + \frac{n}{2} \left(\int_0^1 (te^{1-t})^n dt - \int_1^\infty (te^{1-t})^n dt \right). \quad (21)$$

Watson notices that the right-hand side of this equation makes perfect sense when n is replaced by a positive real variable x and the corresponding function he denotes $y(x)$. We keep the notation $\theta(x)$ and make the substitution $t = e^{-u}$ in the integrals to get

$$\theta(x) = 1 + \frac{x}{2} e^x \left(\int_0^\infty e^{-x(u+e^{-u})-u} du - \int_{-\infty}^0 e^{-x(u+e^{-u})-u} du \right).$$

Introducing the functions u, v from Section 3 we get

$$\theta(x) = 1 + \frac{x}{2} e^x \int_1^\infty e^{-xt} \left(e^{-v(t)} v'(t) + e^{-u(t)} u'(t) \right) dt$$

and using

$$e^{-v(t)} v'(t) + e^{-u(t)} u'(t) = \xi(t) - 2,$$

we get after some calculation

$$\theta(x) = \frac{x}{2} \int_0^\infty e^{-xt} \xi(t+1) dt = \frac{1}{3} + \frac{1}{2} \int_0^\infty e^{-xt} \xi'(t+1) dt. \quad (22)$$

From (10) we get

$$\theta(x) = \sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^k} + o(x^{-n}).$$

We further see that the function $\phi(t)$ from [13, p.300] is given as

$$\phi(t) = \frac{135}{8}\xi'(t+1).$$

In [13] Watson discusses the behaviour of $-t^{-1}\log\phi(t)$ for $t > 0$. He reaches a conclusion using a combination of rigorous analysis and numerical tabulation. The present analysis does not contribute in making his conclusion rigorous.

7 Appendix: Higher order expansions and Maple code

In this section we give the derivatives of ξ at 1 of order up to 10 and higher order asymptotic expansions of $\varphi(x)$ and $m(x)$. These results we have found using the ‘‘Maple 9’’ system. We have also included the Maple code, with a short description.

Computation of the derivatives of ξ

With the notation of Proposition 5.3 we have $\xi^{(k)}(1) = 2a_{2(k+1)}(k+1)!$. Here the numbers a_k are defined by

$$h^{-1}(w) = \sum_{k=1}^{\infty} a_k w^k,$$

where h^{-1} is the inverse to

$$h(z) = \sum_{k=1}^{\infty} b_k z^k,$$

which satisfies $(h(z))^2 = e^{-z} + z - 1$ and $h'(0) > 0$.

The procedure `h_polynomial(n)` returns the list of numbers $[b_1, \dots, b_n]$. These numbers are computed by equating the coefficients in the relation

$$\left(\sum_{k=1}^{\infty} b_k z^k \right)^2 = \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} z^k.$$

We get

$$\sum_{l=1}^{k-1} b_l b_{k-l} = \frac{(-1)^k}{k!}$$

for $k \geq 2$, from which we determine the number b_{k+1} from b_1, \dots, b_k . The numbers b_1, \dots, b_n are just the coefficients in the Taylor expansion of $h(z)$ at $z = 0$ up to order n . We could therefore also have used the command `taylor` but we prefer the more direct and efficient method outlined in the procedure below.

```

h_polynomial:=proc(n)
  local b, j:
  b[1]:=1/sqrt(2):
  for j from 1 to (n-1) do
    b[j+1]:=(-1)^j/(j+2)!
    -sum(b[1+1]*b[j-1+1], l=1..(j-1))/2/b[1]:
  end do: j:='j':
  return [seq(b[j], j=1..(n))];
end proc;

```

The procedure `xi_derivatives(n)` computes the numbers $[\xi(1), \dots, \xi^{(n-1)}(1)]$ and it uses the procedure `h_polynomial`. To find these numbers it is enough to determine the numbers a_1, \dots, a_{2n} and this we do by equating coefficients in the relation $h^{-1}(h(z)) = z$. We put

$$\sum_{l=k}^{\infty} b_{k,l} z^l = \left(\sum_{l=1}^{\infty} b_l z^l \right)^k.$$

Hence $b_{1,l} = b_l$ for $l \geq 1$ and by Cauchy multiplication,

$$b_{k+1,l} = \sum_{m=k}^{l-1} b_{k,m} b_{l-m},$$

for $l \geq k + 1$. Since $h^{-1}(h(z)) = z$ we must furthermore have

$$z = \sum_{k=1}^{\infty} a_k \left(\sum_{l=k}^{\infty} b_{k,l} z^l \right) = \sum_{l=1}^{\infty} \left(\sum_{k=1}^l a_k b_{k,l} \right) z^l.$$

Therefore $a_1 b_{1,1} = 1$ and

$$\sum_{k=1}^l a_k b_{k,l} = 0$$

for $l \geq 2$. Hence $a_1 = 1/b_{1,1} = 1/b_1$. The point is now that a_l can be computed when we know a_1, \dots, a_{l-1} as well as $b_{1,l}, \dots, b_{l,l}$. This is the main idea behind the procedure below.

```

xi_derivatives:=proc(N)
  local i,p,j,k,l,temp,a,h,b:
  h:=h_polynomial(2*N):
  for i from 1 to 2*N do
    for p from 1 to 2*N do
      b[i,p]:=0:
    end do: p:='p':
  end do: i:='i':
  for j from 1 to 2*N do b[1,j]:=h[j]: end do: j:='j':
  a[1]:=sqrt(2):

```

```

for k from 1 to (2*N-1) do
  for l from (k+1) to 2*N do
    temp:=0:
    for j from 1 to (l-1) do
      temp:=temp+b[k,j]*h[l-j]:
    end do: j:='j':
    b[k+1,l]:=temp:
  end do: l:='l':
  a[k+1]:=-sum(a[m]*b[m,k+1],m=1..k)/b[k+1,k+1]:
end do: k:='k':
return [seq(2*a[2*j]*(j!),j=1..N)];
end proc;

```

We could also have used the command `taylor` to compute the coefficients in the expansion of $h^{-1}(h(z))$, but it runs very slowly, so a more efficient program is needed. We have found the derivatives of ξ to be:

$$\begin{aligned}
\xi(1) &= \frac{2}{3} \\
\xi'(1) &= \frac{2^3}{3^3 \cdot 5} \\
\xi^{(2)}(1) &= -\frac{2^4}{3^4 \cdot 5 \cdot 7} \\
\xi^{(3)}(1) &= -\frac{2^5}{3^5 \cdot 5 \cdot 7} \\
\xi^{(4)}(1) &= \frac{2^6 \cdot 281}{3^8 \cdot 5^2 \cdot 7 \cdot 11} \\
\xi^{(5)}(1) &= \frac{2^7 \cdot 23 \cdot 227}{3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} \\
\xi^{(6)}(1) &= -\frac{2^8 \cdot 53 \cdot 103}{3^{10} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} \\
\xi^{(7)}(1) &= -\frac{2^9 \cdot 373 \cdot 4439 \cdot 557}{3^{12} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} \\
\xi^{(8)}(1) &= \frac{2^{10} \cdot 2650986803}{3^{13} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19} \\
\xi^{(9)}(1) &= \frac{2^{11} \cdot 6171801683}{3^{14} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19} \\
\xi^{(10)}(1) &= -\frac{2^{12} \cdot 1117 \cdot 3835213201}{3^{16} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23}.
\end{aligned}$$

The asymptotic expansion of φ

The asymptotic expansion of $\varphi(x)$ can be found from relation (11), that is

$$\varphi(x) = \sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^{k+1}} - \sum_{k=1}^n \frac{(-1)^k x^k}{k!} \int_0^{\varphi(x)} (u + e^{-u} - 1)^k du + o(x^{-(n+1)}).$$

We use two procedures `sum_of_int(n,PHI)` and `xi_sum(n)`. The procedure `sum_of_int(n,PHI)` takes an expression of the form

$$\text{PHI} = \sum_{k=1}^m \frac{c_k}{x^k}$$

and computes the sum of integrals

$$\sum_{k=1}^n \frac{(-1)^k x^k}{k!} \int_0^{\text{PHI}} (u + e^{-u} - 1)^k du$$

up to $o(x^{-(n+1)})$. This is done in the same way as in Lemma 4.4. We put

$$(e^{-u} + u - 1)^k = \sum_{l=2k}^{\infty} t_{k,l} u^l,$$

and observe that by Cauchy multiplication

$$t_{k+1,m} = \sum_{l=2}^{m-2k} \frac{(-1)^l}{l!} t_{k,m-l},$$

for $m \geq 2(k+1)$. Since $t_{1,m} = (-1)^m/m!$ we can compute the numbers $t_{k,\cdot}$ from the numbers $t_{k-1,\cdot}$ for $k \geq 2$. The numbers $t_{k,\cdot}$ are then used to approximate the integrand $(u + e^{-u} - 1)^k$ by its Taylor polynomial centered at $u = 0$ of order $n + k$.

```

sum_of_int:=proc(n,PHI)
  local l,t,T,s,k,m,j,temp,temp1:
  for l from 2 to (n+1) do
    t[1,l]:=(-1)^l/l!:
  end do: l:='1':
  T:=sum(t[1,l]*PHI^(l+1)/(l+1), l=2..(n+1)):
  s:=-x*T:
  for k from 2 to n do
    T:=0:
    for m from 2*k to (n+k) do
      t[k,m]:=
      sum((-1)^l/l!*t[k-1,m-l], l=2..(m-2*k+2)):
      T:=T+t[k,m]*PHI^(m+1)/(m+1):
    end do: m:='m':
    s:=s+(-1)^k*x^k/k!*T:
  end do: k:='k':
  temp:=subs(x=1/x,expand(s)):
  temp1:=0:
  for j from 0 to (n+1) do
    temp1:=temp1+coeff(temp,x,j)*x^(-j):
  end do: j:='j':
  return temp1;
end proc;

```

We notice that the variable `s` in the procedure `sum_of_int(n,PHI)` contains many terms of order x^{-l} with $l > k$. These terms are not needed and they are therefore deleted. That happens in the `for` loop containing `temp1`. (It improves program efficiency.)

The second procedure `xi_sum(n+1)` simply computes the asymptotic expansion

$$\sum_{k=0}^n \frac{\xi^{(k)}(1)}{2x^{k+1}}$$

by calling the procedure `xi_derivatives(n+1)`.

```
xi_sum:=proc(n)
    local temp;
    temp:=xi_derivatives(n);
    return sum(temp[k]/2/x^k, k=1..n);
end proc;
```

To find the expansion of $\varphi(x)$ we combine these two procedures in the procedure `asyp_phi(n)`. This procedure computes the asymptotic expansion of $\varphi(x)$ up to order $o(x^{-n-1})$.

```
asyp_phi:=proc(n)
    local phi,k;
    phi:=1/3/x;
    for k from 1 to n do
        phi:=xi_sum(k+1)-sum_of_int(k,phi);
    end do;
    return phi;
end proc;
```

We have

$$\varphi(x) = \sum_{k=1}^{10} \frac{c_k}{x^k} + o(x^{-10}),$$

where

$$\begin{aligned}
c_1 &= \frac{1}{3} \\
c_2 &= \frac{29}{3^4 \cdot 5 \cdot 2} \\
c_3 &= -\frac{37}{3^6 \cdot 5 \cdot 7} \\
c_4 &= -\frac{3877}{3^9 \cdot 5^2 \cdot 2^2} \\
c_5 &= \frac{8957413}{3^{13} \cdot 5^3 \cdot 7 \cdot 11} \\
c_6 &= \frac{401 \cdot 8842279}{2 \cdot 3^{15} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} \\
c_7 &= -\frac{356146891 \cdot 2039}{3^{18} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} \\
c_8 &= -\frac{216607304027 \cdot 3077479}{2^3 \cdot 3^{21} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17} \\
c_9 &= \frac{31 \cdot 743 \cdot 4569027042343}{3^{23} \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19} \\
c_{10} &= \frac{71 \cdot 282699240672481 \cdot 1949 \cdot 5113}{2 \cdot 3^{27} \cdot 5^7 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19}.
\end{aligned}$$

The asymptotic expansion of m

Since $m(x) = xe^{-\varphi(x)}$ it is easy to find the asymptotic expansion of $m(x)$ when we have the asymptotic expansion of $\varphi(x)$. It is done in the procedure `asyp_m(n)`, that gives the expansion of $m(x)$ up to order x^{1-n} .

```

asyp_m:=proc(n)
  local temp:
  temp:=subs(x=1/x, asyp_phi(n)):
  temp:=convert(taylor(exp(-temp), x=0, n+2), polynom):
  temp:=subs(x=1/x,temp):
  return expand(x*temp);
end proc;

```

We have

$$m(x) = x - \frac{1}{3} + \sum_{k=1}^9 \frac{m_k}{x^k} + o(x^{-9}),$$

where

$$\begin{aligned}
m_1 &= \frac{2^3}{3^4 \cdot 5} \\
m_2 &= \frac{2^3 \cdot 23}{3^6 \cdot 5 \cdot 7} \\
m_3 &= \frac{2^3 \cdot 281}{3^9 \cdot 5^2 \cdot 7} \\
m_4 &= -\frac{2^3 \cdot 17 \cdot 139753}{3^{13} \cdot 5^3 \cdot 7 \cdot 11} \\
m_5 &= -\frac{2^3 \cdot 708494947}{3^{15} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} \\
m_6 &= \frac{2^3 \cdot 140814348739}{3^{18} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13} \\
m_7 &= \frac{2^3 \cdot 7663181003289047}{3^{21} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17} \\
m_8 &= -\frac{2^3 \cdot 653 \cdot 1359581 \cdot 759929 \cdot 3307}{3^{23} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19} \\
m_9 &= -\frac{2^3 \cdot 29 \cdot 1376560394479059407}{3^{27} \cdot 5^7 \cdot 7^3 \cdot 11^2 \cdot 17}.
\end{aligned}$$

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