Integral representation of some functions related to the Gamma function

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Abstract

We prove that the functions $\Phi(x) = [\Gamma(x+1)]^{1/x}(1+1/x)^x/x$ and $\log\Phi(x)$ are Stieltjes transforms.

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1 Introduction and main results

In [11] the authors introduce a subclass of the completely monotonic functions which they call *logarithmically completely monotonic*, and the main result in [12] is that the function

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x \tag{1}$$

is logarithmically completely monotonic.

We characterize the class of logarithmically completely monotonic functions as the infinitely divisible completely monotonic functions studied by Horn in [14]. We prove that Stieltjes transforms (see (8) below) belong to this class and that Φ and $\log\Phi$ are both Stieltjes transforms. Each of these statements imply the result of [12]. The following explicit representations are obtained:

$$\log\Phi(x) = \int_0^\infty \frac{\varphi(s)}{s+x}\,ds, \quad x>0, \tag{2}$$

where

$$\varphi(s) = \begin{cases} 1-s & \text{if } 0\le s<1 \\ 1-n/s & \text{if } n\le s<n+1, n=1,2,\ldots \end{cases} \tag{3}$$

and

$$\Phi(x) = 1 + \int_0^\infty \frac{h(s)}{s+x}\,ds, \quad x>0, \tag{4}$$

with

$$h(s) = \frac{1}{\pi} \frac{s^{s-1}}{|1-s|^s |\Gamma(1-s)|^{1/s}} \sin(\pi\varphi(s)), \; s \geq 0. \tag{5}$$

Note that the density $\varphi(s)$ takes its values in the interval $[0,1]$, and this is the clue to the fact that also Φ is a Stieltjes transform. The density h is continuous on $[0,\infty[$ with $h(0) = \exp(-\gamma)$, where γ is Euler's constant, and $h(n) = 0$ for $n \in \mathbb{N}$.

Recall that a function $f :]0,\infty[\to \mathbb{R}$ is said to be completely monotonic, if f has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad \text{for all} \quad x > 0 \quad \text{and} \quad n = 0, 1, 2, ...$$

Bernstein's Theorem, cf. [15, p. 161], states that f is completely monotonic if and only if

$$f(x) = \int_0^\infty e^{-xs} d\mu(s), \tag{6}$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. The set of completely monotonic functions is denoted $\mathcal{C}$.

In [12] the authors call a function $f :]0,\infty[\to]0,\infty[$ *logarithmically completely monotonic* if it is C^∞ and

$$(-1)^k [\log f(x)]^{(k)} \geq 0, \text{ for } k = 1, 2, \ldots. \tag{7}$$

If we denote the class of logarithmically completely monotonic functions by $\mathcal{L}$, we have $f \in \mathcal{L}$ if and only if f is a positive C^∞-function such that $-(\log f)' \in \mathcal{C}$.

The functions of class $\mathcal{L}$ have been implicitly studied in [3], and Lemma 2.4(ii) in that paper can be stated as the inclusion $\mathcal{L} \subset \mathcal{C}$, a fact also established in [11].

The class $\mathcal{L}$ can be characterized in the following way, established by Horn[14, Theorem 4.4]:

Theorem 1.1 *For a function $f :]0,\infty[\to]0,\infty[$ the following are equivalent:*

(i) $f \in \mathcal{L}$

(ii) $f^\alpha \in \mathcal{C}$ *for all* $\alpha > 0$

(iii) $\sqrt[n]{f} \in \mathcal{C}$ *for all* $n = 1, 2, \ldots$.

Another way of expressing the conditions of Theorem 1.1 is that the functions in $\mathcal{L}$ are those completely monotonic functions for which the representing measure μ in (6) is infinitely divisible in the convolution sense: For each $n \in \mathbb{N}$ there exists a positive measure ν on $[0,\infty[$ with n'th convolution power equal to μ, viz. $\nu^{*n} = \mu$. By condition (ii) there exists a convolution semigroup $(\mu_\alpha)_{\alpha>0}$ of positive measures such that the Laplace transform of μ_α is f^α. Note that the convolution of any two positive measures on $[0,\infty[$ is well-defined and we have $\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}$.

In the special case of $f(0+) = 1$ this is very classical: This is the description of infinitely divisible distributions in probability. Since there are probabilities which are not infinitely divisible we have $\mathcal{C} \setminus \mathcal{L} \neq \emptyset$.

In various papers complete monotonicity for special functions has been established by proving the stronger statement that the function is a Stieltjes transform, i.e. is of the form

$$f(x) = a + \int_0^\infty \frac{d\mu(s)}{s+x}, \tag{8}$$

where $a \geq 0$ and μ is a nonnegative measure on $[0,\infty[$ satisfying

$$\int_0^\infty \frac{1}{1+s} d\mu(s) < \infty.$$

See [2],[3],[5],[6],[8],[9].

The set of Stieltjes transforms will be denoted $\mathcal{S}$. We clearly have $\mathcal{S} \subset \mathcal{C}$. For more information about this class see [7].

Theorem 1.2 $\mathcal{S} \setminus \{0\} \subset \mathcal{L}$.

Theorem 1.3 *The functions*

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x$$

and

$$\log \Phi(x) = \frac{\log \Gamma(x+1)}{x} - \log x + x \log \left(1 + \frac{1}{x}\right)$$

are Stieltjes transforms with the representations (4) and (2).

Remark 1.4 The class $\mathcal{S}$ has the following stability properties: If $f \in \mathcal{S}, f \neq 0$ then $1/f(1/x)$ and $1/(xf(x))$ are again Stieltjes transforms, cf. [5]. Therefore the following functions belong to $\mathcal{S}$:

$$\frac{1}{[\Gamma(1+x)]^{1/x}(1+1/x)^x}, \quad [\Gamma(1+1/x)]^x (1+x)^{1/x}, \quad \frac{1}{x[\Gamma(1+1/x)]^x (1+x)^{1/x}}.$$

It was proved in [3] that $[\Gamma(1+1/x)]^x \in \mathcal{S}$, so also the following functions are Stieltjes transforms:

$$\frac{[\Gamma(1+x)]^{1/x}}{x}, \quad \frac{1}{[\Gamma(1+x)]^{1/x}}, \quad \frac{1}{x[\Gamma(1+1/x)]^x}.$$

In [2] it was proved that $(1+1/x)^{-x} \in \mathcal{S}$. Therefore the function Φ given by (1) is a quotient of known Stieltjes transforms, but this does not imply that the function itself is a Stieltjes transform.

2 Proofs

For completeness we include a proof of Theorem 1.1.

"(i) $\Rightarrow$ (ii)" Since $f \in \mathcal{L}$ implies $f^\alpha \in \mathcal{L}$ for all $\alpha > 0$, it is enough to prove the inclusion $\mathcal{L} \subset \mathcal{C}$. Although this is done in [3] and [11] we include the easy proof. By assumption $-(\log f)' = -f'/f \in \mathcal{C}$, so in particular $-f' \geq 0$. Assume now that $(-1)^k f^{(k)} \geq 0$ for $k \leq n$. Then

$$\begin{aligned} & (-1)^{n+1} f^{(n+1)} = (-1)^n ((-\log f)' f)^{(n)} \\ = & \sum_{k=0}^{n} \binom{n}{k} (-1)^k ((-\log f)')^{(k)} (-1)^{n-k} f^{(n-k)} \geq 0, \end{aligned}$$

and (ii) follows by induction.

"(ii) $\Rightarrow$ (iii)" is obvious.

"(iii) $\Rightarrow$ (i)" If $f^{1/n} \in \mathcal{C}$ we have in particular $-(f^{1/n})' = (-1/n) f^{-1+1/n} f' \in \mathcal{C}$. Multiplying by n and letting $n \to \infty$ we see that the limit function $-f'/f$ belongs to $\mathcal{C}$, because $\mathcal{C}$ is closed under pointwise limits, cf. [7]. This establishes (i). $\square$

Proof of Theorem 1.2: Let $f \in \mathcal{S}$ be non-zero, and let $\alpha > 0$. By Theorem 1.1 it is enough to prove that $f^\alpha \in \mathcal{C}$. Writing $\alpha = n + a$ with $n = 0, 1, \ldots$ and $0 \leq a < 1$ we have $f^\alpha = f^n f^a$, and using the stability of $\mathcal{C}$ under multiplication and that $f^a \in \mathcal{S}$, cf. [4], the assertion follows. $\square$

Proof of Theorem 1.3: Using the expression (3) for φ we find

$$\begin{aligned} & \int_0^\infty \frac{\varphi(s)}{s+x}\, ds = \int_0^1 \frac{1-s}{s+x}\, ds + \sum_{k=1}^\infty \int_k^{k+1} \frac{1-k/s}{s+x}\, ds \\ = \; & -1 + (x+1)\log\left(1+\frac{1}{x}\right) + \sum_{k=1}^\infty \left[\left(1+\frac{k}{x}\right)\log\left(1+\frac{1}{x+k}\right) - \frac{k}{x}\log\left(1+\frac{1}{k}\right)\right]. \end{aligned}$$

Therefore,

$$\int_0^\infty \frac{\varphi(s)}{s+x}\, ds = \log \Phi(x)$$

if and only if

$$\log\Gamma(x+1) = x(\log(1+x)-1) + \sum_{k=1}^\infty \left[(k+x)\log\left(1+\frac{1}{x+k}\right) - k\log\left(1+\frac{1}{k}\right)\right] \tag{9}$$

for $x \geq 0$. Both sides vanish for $x = 0$, and they have the same derivative $\psi(x+1)$, where ψ is the digamma function. This follows easily by the classical formula

$$\psi(x) = \log x + \sum_{k=0}^\infty \left[\log\left(1+\frac{1}{x+k}\right) - \frac{1}{x+k}\right]$$

cf. [13, 8.362(2)]. This shows that $\log\Phi$ is a Stieltjes transform with the representation (2). In particular Φ is completely monotonic with the limit 1 at infinity.

To see that a function f is a Stieltjes transform we will use the characterization of these functions via complex analysis, see [1, p. 127] or [5]. It is necessary and sufficient that f has a holomorphic extension to the cut plane $\mathcal{A}=\mathbb{C}\setminus]-\infty,0]$ and satisfies $\operatorname{Im} f(z)\le 0$ for $\operatorname{Im} z>0$ and $f(x)\ge 0$ for $x>0$. For a Stieltjes transform f given by (8) we have $a=\lim_{x\to\infty} f(x)$, and the measure μ is the limit in the vague topology of

$$-\frac{1}{\pi}\operatorname{Im} f(-x+iy)\,dx$$

as $y\to 0^+$.

We clearly have $\Phi(x)>0$ for $x>0$ and the holomorphic extension of Φ is given by $\Phi(z)=\exp(\log\Phi(z))$, where $\log\Phi(z)$ is the holomorphic extension obtained by the representation (2). This can also be described in the following way: For $z\in\mathcal{A}$ we let $\log\Gamma(z)$ be the unique holomorphic branch, which is real for $x>0$, and we let Log denote the principal logarithm. Then

$$\frac{\log\Gamma(z+1)}{z}-\operatorname{Log} z+z\operatorname{Log}\left(1+\frac{1}{z}\right) \tag{10}$$

is a holomorphic branch of $\log\Phi(z)$ in $\mathcal{A}$, and since it agrees with $\log\Phi(x)$ for $x>0$, we have

$$\log\Phi(z)=\frac{\log\Gamma(z+1)}{z}-\operatorname{Log} z+z\operatorname{Log}\left(1+\frac{1}{z}\right)=\int_0^\infty\frac{\varphi(s)}{s+z}\,ds,\quad z\in\mathcal{A}. \tag{11}$$

For $z=x+iy, y>0$ we get

$$\operatorname{Im}\log\Phi(x+iy)=-\int_0^\infty\frac{\varphi(s)y}{(s+x)^2+y^2}\,ds,$$

and since $0\le\varphi(s)\le 1$ for $s\ge 0$ we get

$$\operatorname{Im}\log\Phi(x+iy)\in\,]-\pi,0[\,,$$

hence

$$\operatorname{Im}\Phi(x+iy)=|\Phi(x+iy)|\sin(\operatorname{Im}\log\Phi(x+iy))<0,$$

which shows that Φ is a Stieltjes transform. For $x\ge 0, y\to 0^+$ we further get

$$-\frac{1}{\pi}\operatorname{Im}\Phi(-x+iy)\to h(x):=\frac{1}{\pi}|\Phi(-x)|\sin(\pi\varphi(x)), \tag{12}$$

which is is a continuous nonnegative function on $[0,\infty[$. Therefore the convergence is uniform for x in compact subsets of $[0,\infty[$, so h is the density of the representing measure as a Stieltjes transform.

This shows the following integral representation of Φ:

$$\Phi(z) = 1 + \int_0^\infty \frac{h(s)}{s+z}\,ds,\ z \in \mathcal{A} \tag{13}$$

where

$$h(s) = \frac{1}{\pi}\frac{s^{s-1}}{|1-s|^s|\Gamma(1-s)|^{1/s}}\sin(\pi\varphi(s)),\ s \geq 0.$$

□

Remark 2.1 There is a close relationship between Stieltjes transforms and Pick functions. For the latter see [1] and [10]. It is possible to find the integral representation (2) of $\log\Phi(x)$ using integral representations of the three terms of (10). Here $\operatorname{Log}(z)$ and $z\operatorname{Log}(1+1/z)$ are Pick functions. The author first found the density (3) in this way, but once φ is found the present direct approach seems easier.

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