From discrete to absolutely continuous solutions of indeterminate moment problems

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Abstract

We consider well-known families of discrete solutions to indeterminate moment problems and show how they can be used in a simple way to generate absolutely continuous solutions to the same moment problems. The following cases are considered: log-normal, generalized Stieltjes -Wigert, q-Laguerre and discrete q-Hermite II.

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1 Introduction

If $(s_n)_{n\geq 0}$ is an indeterminate moment sequence, it is well-known that the set V of solutions to the corresponding moment problem contains discrete measures as well as absolutely continuous measures. Since V is convex, we can form convex combinations of known solutions to get new solutions, and since V is compact in the weak topology, also limits will lead to new solutions. In particular we have the following result:

Proposition 1.1 *Let $t \mapsto \mu_t$ be a continuous mapping of an interval I into V. For any probability measure τ on I the vector integral $\kappa = \int \mu_t d\tau(t)$ belongs to V.*

We shall apply this result to various families $(\mu_t)_{t\in I}$ of discrete solutions, and show how this makes it possible to find absolutely continuous solutions.

As a general example we consider the N-extremal solutions $(\nu_t)_{t\in\mathbb{R}}$, determined by their Stieltjes transforms

$$\int \frac{d\nu_t(x)}{x-z} = -\frac{A(z)t - C(z)}{B(z)t - D(z)}\,, \quad z \in \mathbb{C} \setminus \mathbb{R} \tag{1.1}$$

cf. [1], where A, B, C, D are certain entire functions. It is well-known that

$$\nu_t = \sum_{\lambda \in \Lambda_t} \rho(\lambda) \varepsilon_\lambda. \tag{1.2}$$

where $\Lambda_t \subseteq \mathbb{R}$ is the discrete set of zeros of the entire function $Bt - D$, and

$$\rho(x) = \left(\sum_{k=0}^{\infty} p_k^2(x) \right)^{-1} \tag{1.3}$$

with (p_k) being the orthonormal polynomials with respect to any of the measures in V. We denote by ε_λ the unit mass concentrated at the point λ.

Proposition 1.2 *Let τ be the probability on $\mathbb{R}$ with a density given by the Poisson kernel for the upper half-plane*

$$\tau = \tau_{u,v} = \frac{1}{\pi} \frac{v}{(u-t)^2 + v^2}\, dt\,, \qquad u \in \mathbb{R}, v > 0.$$

The solution $\kappa = \int \nu_t d\tau(t) \in V$ is given by

$$\kappa = \frac{v}{\pi} \frac{1}{(uB(x) - D(x))^2 + v^2 B^2(x)}\, dx. \tag{1.4}$$

Proof Fix z in the upper half-plane. It is known that $D(z)/B(z)$ lies in the lower half-plane, since B/D is a Pick function, cf. [1] or [9, p. 177]. The Möbius transformation

$$w \mapsto -\frac{A(z)w - C(z)}{B(z)w - D(z)}$$

is holomorphic in $\mathbb{C}^* \setminus \{D(z)/B(z)\}$, in particular harmonic and is reproduced by the Poisson integral. Therefore we have

$$-\frac{A(z)(u+iv) - C(z)}{B(z)(u+iv) - D(z)} = -\int \frac{A(z)t - C(z)}{B(z)t - D(z)}\, d\tau(t),$$

which shows that

$$\int \frac{d\kappa(x)}{x - z} = -\frac{A(z)(u+iv) - C(z)}{B(z)(u+iv) - D(z)}\,, \quad \operatorname{Im}(z) > 0. \tag{1.5}$$

We find κ by the Perron-Stieltjes inversion procedure. According to this we consider the imaginary part of the right-hand side of (1.5) on the horizontal line $z = x+iy, y > 0$. As a function of x it converges for $y \to 0$ to

$$\frac{v}{(uB(x) - D(x))^2 + v^2 B^2(x)},$$

uniformly for x in compact subsets of $\mathbb{R}$, and the formula (1.4) for κ follows. $\square$

Remark 1.3 The solution (1.4) corresponds in the Nevanlinna parametrization to the Pick function which is constant in the upper half-plane equal to $u + iv$. This was established in [8].

Remark 1.4 The set of solutions

$$\int \nu_t d\tau(t),$$

where τ is an arbitrary probability on $\mathbb{R}^*$, is equal to the closed convex hull of the set of N-extremal solutions. This set is strictly smaller than V, since the set $\mathrm{ex}(V)$ of extreme points of V is dense in V by a theorem of Glazman and Naiman, cf. [1, p.131].

In the rest of the paper we consider different indeterminate moment problems connected to q-special functions. In all of the cases considered: log-normal, generalized Stieltjes-Wigert, q-Laguerre and discrete q-Hermite II, there exists a family $(\mu_t)_{t>0}$ of discrete solutions with the periodicity property $\mu_{tq} = \mu_t$, $t > 0$. These families are known, and it is easy to establish that they are solutions to the moment problem in question, which is incidentally shown to be indeterminate.

In this paper we shall calculate the vector integrals

$$\frac{1}{\log(1/q)} \int_q^1 \mu_t \frac{dt}{t}, \tag{1.6}$$

which lead to absolutely continuous solutions of the moment problems. These solutions are not new, but are obtained here in an elementary way, and we want to underline the unifying point of view.

The families μ_t are given in the form $\sigma_t / m(t)$, where $m(t)$ is the total mass of σ_t. We shall also apply Proposition 1.1 with

$$\tau = \frac{1}{M} m(t)\varphi(t)dt, \text{ where } M = \int_q^1 m(t)\varphi(t)dt \tag{1.7}$$

and φ is a conveniently chosen positive function.

In this case we get

$$\kappa = \int_q^1 \mu_t d\tau(t) = \frac{1}{M} \int_q^1 \sigma_t \varphi(t)dt. \tag{1.8}$$

The q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ constitute the most general of the examples considered in the sense that the other examples follow by specialization of the parameter α and simple transformations of the independent variable. For more precise statements see e.g. [18]. We have chosen to present each case in stead of starting with the most general example. Another reason for this is that we show how solutions to the strong log-normal moment problem can be used to form solutions to the generalized Stieltjes-Wigert problem.

The idea of changing an integral (1.6) to an integral over $]0,\infty[$ using the periodicity $\mu_{tq} = \mu_t$ has been exploited in [16] and is well-known in Fourier analysis. Concerning q-special functions we adopt the terminology from [13].

In Askey's paper [2] it is pointed out that certain formulas of Ramanujan can be considered as q-extensions of the gamma and beta functions, and furthermore that some of these formulas can be interpreted as giving solutions to an indeterminate moment problem. The corresponding orthogonal polynomials are now called q-Laguerre polynomials, and they have been considered by a number of authors, see [3],[14],[17], [21]. Later Askey and Roy, cf. [5], evaluated the following integral

$$\int_0^\infty t^c \frac{(-q^{a+c}t, -q^{b+1-c}/t; q)_\infty}{(-t, -q/t; q)_\infty} \frac{dt}{t} = \frac{\Gamma(c)\Gamma(1-c)}{\Gamma_q(c)\Gamma_q(1-c)} \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}, \tag{1.9}$$

where Γ_q is Jackson's q-extension of the gamma function

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{1.10}$$

Other evaluations of the integral in (1.9) were given in [4],[13], [16]. Letting a, b tend to infinity leads to the formula

$$\int_0^\infty \frac{t^c}{(-t, -q/t; q)_\infty} \frac{dt}{t} = (q;q)_\infty \frac{\Gamma(c)\Gamma(1-c)}{\Gamma_q(c)\Gamma_q(1-c)}(1-q) = \frac{\pi}{\sin(\pi c)} \frac{(q^c, q^{1-c}; q)_\infty}{(q;q)_\infty}, \tag{1.11}$$

cf. [4].

2 The log-normal distribution

Stieltjes discovered that a weight function on an unbounded interval need not be uniquely determined by its moments, cf. [6, Letter 325], and in his monumental work [24] he showed that the log-normal distribution with density on $]0,\infty[$

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right) \tag{2.1}$$

belongs to this category. For $p \in \mathbb{R}$ we have

$$s_p(d_\sigma) = \int_0^\infty x^p d_\sigma(x)\,dx = e^{\frac{1}{2}p^2\sigma^2}, \tag{2.2}$$

and in particular the moment sequence is given as

$$s_n(d_\sigma) = q^{-\frac{1}{2}n^2}, \quad n \geq 0, \tag{2.3}$$

where $0 < q < 1$ is defined by $q = e^{-\sigma^2}$. Stieltjes showed that the moments (2.3) belong to an indeterminate moment problem by pointing out that all the densities ($s \in [-1, 1]$)

$$d_\sigma(x)\left(1 + s\sin(\frac{2\pi}{\sigma^2}\log x)\right)$$

have the same moments (2.3).

Chihara [10] and later Leipnik [19] gave the following family of discrete measures with the moments (2.3). For $a > 0$ define the discrete probability

$$\lambda_a = \frac{1}{L(a)}\sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2}\varepsilon_{aq^k}, \tag{2.4}$$

where

$$L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2}. \tag{2.5}$$

It is easy to calculate the moments of λ_a using the translation invariance of $\sum_{-\infty}^{\infty}$. In fact

$$\begin{aligned} s_n(\lambda_a) &= \frac{1}{L(a)}\sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2}(aq^k)^n = \frac{q^{-\frac{1}{2}n^2}}{L(a)}\sum_{k=-\infty}^{\infty} a^{k+n}q^{\frac{1}{2}(k+n)^2} \\ &= q^{-\frac{1}{2}n^2}. \end{aligned}$$

The translation invariance also yields the formulas

$$L(aq) = \frac{L(a)}{a\sqrt{q}}, \quad \lambda_{aq} = \lambda_a, \quad a > 0, \tag{2.6}$$

which shows that the family $(\lambda_a)_{a>0}$ is completely determined by the values $a \in]q, 1]$. The value of the sum $L(a)$ is known by Jacobi's triple product identity, cf. [13]

$$L(a) = (q, -\sqrt{q}\,a, -\sqrt{q}/a; q)_\infty. \tag{2.7}$$

In [3] Askey found another continuous weight function for the log-normal moments, namely

$$\omega(x) = \frac{1}{\log(1/q)xL(x).} \tag{2.8}$$

We shall derive this density from the discrete family (λ_a) in (2.4) using Proposition 1.1.

Proposition 2.1 *Let τ be the probability on $]q, 1]$ given as*

$$\tau = \frac{da}{\log(1/q)a}.$$

The vector integral $\kappa = \int \lambda_a d\tau(a)$ is the measure with density (2.8).

Proof For any continuous and bounded function $\varphi :]0,\infty[\to \mathbb{C}$ we have by definition of a vector integral

$$\begin{aligned}\int \varphi d\kappa &= \frac{1}{\log(1/q)}\int_q^1 (\int \varphi d\lambda_a)\frac{da}{a} \\ &= \frac{1}{\log(1/q)}\int_q^1 \left(\sum_{k=-\infty}^{\infty} \frac{a^k q^{\frac{1}{2}k^2}}{L(a)}\varphi(aq^k)\right)\frac{da}{a} \\ &= \frac{1}{\log(1/q)}\sum_{k=-\infty}^{\infty} q^{\frac{1}{2}k^2}\int_q^1 \frac{a^k\varphi(aq^k)}{L(a)}\frac{da}{a}.\end{aligned}$$

Iterating the transformation formula (2.6) we get

$$L(aq^k) = a^{-k}q^{-\frac{1}{2}k^2}L(a)\,, \quad k \in \mathbb{Z}. \tag{2.9}$$

Substituting $x = aq^k$ in the inner integral we get

$$\int \varphi d\kappa = \frac{1}{\log(1/q)}\sum_{k=-\infty}^{\infty}\int_{q^{k+1}}^{q^k}\frac{\varphi(x)}{L(x)}\frac{dx}{x} = \frac{1}{\log(1/q)}\int_0^\infty \frac{\varphi(x)}{L(x)}\frac{dx}{x},$$

which proves that κ has the density (2.8). $\square$

The density (2.8) has a one-parameter extension

$$\omega_c(x) = \frac{x^{c-1}}{M_c L(xq^{-c})}\,, \quad M_c = \int_0^\infty \frac{x^{c-1}}{L(xq^{-c})}\,dx, \tag{2.10}$$

where c is a real number. Using the transformation formula (2.6) we see that

$$M_{c+1} = q^{c+\frac{1}{2}}M_c\,, \quad \omega_{c+1}(x) = \omega_c(x), \tag{2.11}$$

so it is enough to consider $c \in [0,1[$.

Proposition 2.2 *The measure $\omega_c(x)dx$ on $]0,\infty[$ has the log-normal moments*

$$\int_0^\infty x^n\omega_c(x)dx = q^{-\frac{1}{2}n^2}. \tag{2.12}$$

Proof Using (2.9) we can write (2.4) as

$$\lambda_a = \frac{L(aq^{-c})}{L(a)}\sum_{k=-\infty}^{\infty}\frac{q^{kc}}{L(aq^{k-c})}\varepsilon_{aq^k},$$

so the moment equation for λ_a reads

$$\sum_{k=-\infty}^{\infty}\frac{(aq^k)^n q^{kc}}{L(aq^{k-c})} = q^{-\frac{1}{2}n^2}\frac{L(a)}{L(aq^{-c})}. \tag{2.13}$$

Multiplying this equation with a^{c-1} and integrating with respect to a over $[q,1]$ we get

$$\sum_{k=-\infty}^{\infty}\int_q^1 \frac{(aq^k)^n(aq^k)^c}{L(aq^kq^{-c})}\frac{da}{a} = q^{-\frac{1}{2}n^2}\int_q^1 \frac{L(a)a^{c-1}}{L(aq^{-c})}\,da,$$

hence

$$\int_0^\infty \frac{t^nt^{c-1}}{L(tq^{-c})}\,dt = q^{-\frac{1}{2}n^2}\int_q^1 \frac{L(t)t^{c-1}}{L(tq^{-c})}\,dt,$$

and in particular for $n=0$

$$M_c = \int_0^\infty \frac{t^{c-1}}{L(tq^{-c})}\,dt = \int_q^1 \frac{L(t)t^{c-1}}{L(tq^{-c})}\,dt. \qquad \square$$

Remark 2.3 In this way we do not get the value of M_c which follows from (1.11):

$$\begin{aligned} M_c &= \int_0^\infty \frac{(tq^{c-\frac{1}{2}})^c}{L(tq^{-\frac{1}{2}})}\frac{dt}{t} = q^{c(c-\frac{1}{2})}\int_0^\infty \frac{t^c}{(q,-t,-q/t;q)_\infty}\frac{dt}{t} \\ &= q^{c(c-\frac{1}{2})}\frac{\Gamma(c)\Gamma(1-c)}{\Gamma_q(c)\Gamma_q(1-c)}(1-q). \end{aligned}$$

Remark 2.4 We see that the densities (2.10) fit into the scheme given in (1.8) with

$$\sigma_t = \sum_{-\infty}^{\infty}\frac{q^{kc}}{L(tq^{k-c})}\varepsilon_{tq^k},$$

$m(t) = L(t)/L(tq^{-c})$ and $\varphi(t) = t^{c-1}$.

3 Generalized Stieltjes-Wigert

Wigert [25] studied the orthogonal polynomials for the log-normal distribution.

The generalized Stieltjes-Wigert moment problem (cf. [11], [12]) has the weight function on $]0,\infty[$

$$w(x;p,q) = (p,-p\sqrt{q}/x;q)_\infty d_\sigma(x), \tag{3.1}$$

where d_σ is the log-normal density (2.1) and $0 \le p < 1$ is the additional parameter which for $p=0$ gives the log-normal density. As before $q=\exp(-\sigma^2)$. The moments of $w(x;p,q)$ are easy to calculate using the power series expansion of the q-exponential function, cf. [13]

$$(-p\sqrt{q}/x;q)_\infty = \sum_{k=0}^{\infty}\frac{q^{\frac{1}{2}k^2}p^k}{(q;q)_k}x^{-k}.$$

We want to emphasize that more is valid. Let $\widetilde{V}$ denote the of measures μ on $]0,\infty[$, which solve the strong Stieltjes moment problem

$$s_n = \int_0^\infty x^n d\mu(x) = q^{-\frac{1}{2}n^2}, \quad n \in \mathbb{Z}.$$

The calculations in section 2 show that d_σ, λ_a and $\omega_c(x)$ from (2.10) all belong to $\widetilde{V}$.

Proposition 3.1 *For arbitrary $\mu \in \widetilde{V}$ the measure*

$$(p, -p\sqrt{q}/x; q)_\infty d\mu(x)$$

has the moments

$$s_n = (p;q)_n q^{-\frac{1}{2}n^2}, \quad n \geq 0. \tag{3.2}$$

Proof Since all the functions below are positive we find for $n \in \mathbb{N}_0$

$$\begin{aligned}\int_0^\infty x^n (p, -p\sqrt{q}/x; q)_\infty d\mu(x) &= (p;q)_\infty \sum_{k=0}^\infty \frac{q^{\frac{1}{2}k^2} p^k}{(q;q)_k} \int_0^\infty x^{n-k} d\mu(x) \\ &= (p;q)_\infty q^{-\frac{1}{2}n^2} \sum_{k=0}^\infty \frac{(pq^n)^k}{(q;q)_k} = (p;q)_n q^{-\frac{1}{2}n^2},\end{aligned}$$

because

$$\frac{1}{(z;q)_\infty} = \sum_{k=0}^\infty \frac{z^k}{(q;q)_k} \text{ for } |z| < 1.$$

This also shows that the measure in question can integrate x^n for negative n as long as $pq^n < 1$. $\square$

Corollary 3.2 *In addition to $w(x;p,q)$ given by (3.1) the following measures have the moments (3.2):*

$$\frac{x^c}{M_c} \frac{(p, -p\sqrt{q}/x; q)_\infty}{(q, -q^{\frac{1}{2}-c}x, -q^{\frac{1}{2}+c}/x; q)_\infty} \frac{dx}{x} \tag{3.3}$$

$$\lambda_{a,p} = \frac{1}{L(a,p)} \sum_{k=-\infty}^\infty p^k (-a\sqrt{q}/p; q)_k \varepsilon_{aq^k}, \quad a > 0 \tag{3.4}$$

where

$$L(a,p) = \frac{L(a)}{(p, -p\sqrt{q}/a; q)_\infty}. \tag{3.5}$$

Proof Formula (3.3) is a direct consequence of Proposition 3.1 and (2.10). To see (3.4), we shall prove that for $k \in \mathbb{Z}$

$$(p, -p\sqrt{q}/(aq^k); q)_\infty a^k q^{\frac{1}{2}k^2} = (p, -p\sqrt{q}/a; q)_\infty p^k (-a\sqrt{q}/p; q)_k,$$

but this follows from the transformation formulas (I.2) and (I.5) in [13].

Remark 3.3 a) The evaluation of the moments of (3.3) is a special case of the Askey-Roy q-beta integral (1.9).

b) The measure $\lambda_{a,p}$ has the periodicity $\lambda_{aq,p} = \lambda_{a,p}$, so it is enough to consider $a \in]q, 1]$. The function (3.5) satisfies

$$L(aq, p) = \frac{L(a, p)}{a\sqrt{q} + p}.$$

c) The vector integral

$$\frac{1}{\log(1/q)} \int_q^1 \lambda_{a,p} \frac{da}{a}$$

is equal to the measure (3.3) in the case $c = 0$.

4 q-Laguerre

We follow the normalization of the q-Laguerre polynomials given in [18], i.e. we remove the factor $(1-q)$ used by Moak in [21]. They belong to an indeterminate moment problem with moments

$$s_n(\alpha; q) = q^{-\alpha n - \binom{n+1}{2}} (q^{\alpha+1}; q)_n, \tag{4.1}$$

when $0 < q < 1$, $\alpha > -1$, as pointed out by Askey in [2]. More information about this indeterminate moment problem can be found in [17]. In [21] one finds the following family $(\tau_c)_{c>0}$ of probabilities with moment sequence (4.1):

$$\tau_c = \frac{1}{B(c)} \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-q^k c; q)_\infty} \varepsilon_{cq^k}, \tag{4.2}$$

where

$$B(c) = \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-q^k c; q)_\infty}$$

has the transformation property $B(cq) = q^{-(\alpha+1)} B(c)$, and it follows that $\tau_{cq} = \tau_c$. The verification that τ_c has the moments (4.1) can be done using Ramanujan's bilateral sum ${}_1\psi_1$, which also yields that

$$B(c) = \frac{(-q^{\alpha+1}c, -q^{-\alpha}/c, q; q)_\infty}{(q^{\alpha+1}, -q/c, -c; q)_\infty}. \tag{4.3}$$

Askey [2] pointed out that the density on $]0,\infty[$

$$k_\alpha \frac{t^\alpha}{(-t;q)_\infty} \tag{4.4}$$

has the moments (4.1), and that this result follows from an integral given by Ramanujan in 1915. The constant k_α can be expressed via the Γ_q-function (1.10) and the ordinary Γ-function:

$$k_\alpha = (1-q)^{-\alpha-1}\frac{\Gamma_q(-\alpha)}{\Gamma(-\alpha)\Gamma(\alpha+1)} = -\frac{\sin(\pi\alpha)}{\pi}\,\frac{(q;q)_\infty}{(q^{-\alpha};q)_\infty}. \tag{4.5}$$

We want to point out that the density (4.4) can be constructed from the family (4.2) in the following way: The fact that (4.2) has the moments (4.1) can be expressed

$$\sum_{k=-\infty}^{\infty} (cq^k)^n \frac{q^{k(\alpha+1)}}{(-q^k c;q)_\infty} = B(c)s_n(\alpha;q).$$

If we multiply this equation with c^α and integrate over $[q,1]$, we get

$$\sum_{k=-\infty}^{\infty} \int_q^1 (cq^k)^n \frac{(cq^k)^\alpha}{(-q^k c;q)_\infty} d(cq^k) = s_n(\alpha;q) \sum_{k=-\infty}^{\infty} \int_c^1 \frac{(cq^k)^\alpha}{(-q^k c;q)_\infty} d(cq^k),$$

or

$$\int_0^\infty t^n \frac{t^\alpha}{(-t;q)_\infty}\,dt = s_n(\alpha;q)\int_0^\infty \frac{t^\alpha dt}{(-t;q)_\infty}.$$

On the other hand the vector integral

$$\frac{1}{\log(1/q)}\int_q^1 \tau_c \frac{dc}{c}$$

yields another density with moments (4.1). Using the transformation formula for $B(c)$ we get

$$\tau_c = \sum_{k=-\infty}^{\infty} \frac{\varepsilon_{cq^k}}{(-q^k c;q)_\infty B(cq^k)}$$

and hence

$$\frac{1}{\log(1/q)}\int_q^1 \tau_c \frac{dc}{c} = \frac{1}{\log(1/q)}\int_0^\infty \frac{\varepsilon_y}{(-y;q)_\infty B(y)}\,\frac{dy}{y}.$$

This shows that

$$\omega(y) = \frac{1}{\log(1/q)}\,\frac{(q^{\alpha+1},-q/y;q)_\infty}{y(-q^{\alpha+1}y,-q^{-\alpha}/y,q;q)_\infty} \tag{4.6}$$

has the moments (4.1). The result can be verified directly using the Askey-Roy q-beta-integral (1.9).

The generalized Stieltjes-Wigert moment problem and the q-Laguerre moment problem are closely related. If a measure μ has the moments s_n, then the image measure $\tau_a(\mu)$ of μ under the transformation $\tau_a(x) = ax$ has the moments $a^n s_n$.

It follows that if μ is an arbitrary solution to the generalized Stieltjes-Wigert problem with $p = q^{\alpha+1}, \alpha > -1$ and if $a = q^{-\alpha-\frac{1}{2}}$, then $\tau_a(\mu)$ is a solution to the q-Laguerre problem, and all solutions arise in this way. Applying this to the family (3.4) we get (4.2), and applied to (3.3) we get the following family of densities

$$\frac{q^{c(\alpha+\frac{1}{2})}}{M_c} y^c \frac{(q^{\alpha+1}, -q/y; q)_\infty}{(q, -q^{\alpha+1-c}y, -q^{c-\alpha}/y; q)_\infty} \frac{dy}{y} \tag{4.7}$$

for the q-Laguerre problem. The family (4.7) is periodic in c with period 1. Putting $c = 0$ we get (4.6).

5 Discrete q-Hermite II

For $0 < q < 1$ the discrete q-Hermite polynomials II are the orthogonal polynomials associated with the moment sequence

$$s_n = \begin{cases} 0\,, & n = 2m+1 \\ (q;q^2)q^{-m^2}\,, & n = 2m \end{cases} \tag{5.1}$$

where $0 < q < 1$. The moments give rize to a symmetric indeterminate Hamburger moment problem. The orthogonal polynomials have appeared in another normalization in a study of a q-harmonic oscillator, cf. [15], [20], [22], [23]. The following family of discrete solutions to (5.1) is given in [18].

$$\mu_c = \frac{1}{2A(c)} \sum_{k=-\infty}^{\infty} \frac{q^k}{(-c^2q^{2k}; q^2)_\infty} (\varepsilon_{cq^k} + \varepsilon_{-cq^k}), \tag{5.2}$$

where

$$A(c) = \sum_{k=-\infty}^{\infty} \frac{q^k}{(-c^2q^{2k}; q^2)_\infty} = \frac{(-qc^2, -q/c^2, q^2; q^2)_\infty}{(-c^2, -q^2/c^2; q; q^2)_\infty}. \tag{5.3}$$

The paper [7] contains a proof that all the measures μ_c have the moments (5.1), based on the ${}_1\psi_1$-sum of Ramanujan. It is easily seen that

$$A(cq) = \frac{1}{q}A(c)\,, \quad \mu_{cq} = \mu_c. \tag{5.4}$$

In [7] it was proved that the lattice

$$\Lambda = \{q^{k+\frac{1}{2}} \mid k \in \mathbb{Z}\}$$

supports infinity many different solutions to (5.1), namely the convex combinations of the two measures

$$\mu_+ = \frac{1}{A(\sqrt{q})} \sum_{k=-\infty}^{\infty} \frac{q^{2k}}{(-q^{4k+1};q^2)_\infty} (\varepsilon_{q^{2k+\frac{1}{2}}} + \varepsilon_{-q^{2k+\frac{1}{2}}})$$

$$\mu_- = \frac{1}{A(\sqrt{q})} \sum_{k=-\infty}^{\infty} \frac{q^{2k+1}}{(-q^{4k+3};q^2)_\infty} (\varepsilon_{q^{2k+\frac{3}{2}}} + \varepsilon_{-q^{2k+\frac{3}{2}}})$$

concentrated on respectively the even and odd part of Λ.

Using the discrete family $(\mu_c)_{c\in\,]q,1]}$ we shall find two absolutely continuous solutions to (5.1).

Proposition 5.1 *The moment problem (5.1) has the following symmetric solutions:*

$$\frac{1}{\pi}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{1}{(-x^2;q^2)_\infty} \tag{5.5}$$

$$\frac{1}{2\log(1/q)}\frac{(-q^2/x^2,q;q^2)_\infty}{|x|(-qx^2,-q/x^2,q^2;q^2)_\infty}. \tag{5.6}$$

Proof We have

$$\sum_{k=-\infty}^{\infty} \frac{(cq^k)^{2n}}{(-c^2q^{2k};q^2)_\infty} q^k = A(c)(q;q^2)_n q^{-n^2}, \quad n \geq 0,$$

which expresses that the $2n$'th moment of μ_c is given by (5.1). Integrating this formula with respect to c over $[q,1]$ we get after the substitution $x = cq^k$

$$\int_0^\infty \frac{x^{2n}\,dx}{(-x^2;q^2)_\infty} = \int_0^\infty \frac{dx}{(-x^2;q^2)_\infty}(q;q^2)_n q^{-n^2}$$

which yields (5.5) except for the normalization

$$\int_0^\infty \frac{dx}{(-x^2;q^2)_\infty} = \frac{\pi}{2}\frac{(q;q^2)_\infty}{(q^2,q^2)_\infty},$$

which follows from Ramanujan's integral, cf. (4.5) with $\alpha = -\frac{1}{2}$.

If instead we consider the vector integral

$$\frac{1}{\log(1/q)}\int_q^1 \mu_c \frac{dc}{c}$$

we obtain (5.6). $\square$

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