A completely monotone function related to the Gamma function

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Abstract

We show that the reciprocal of the function

$$f(z) = \frac{\log \Gamma(z+1)}{z \log z}, \qquad z \in \mathbb{C} \setminus]-\infty, 0],$$

is a Stieltjes transform. As a corollary we obtain that the derivative of f is completely monotone, in the sense that $(-1)^{n-1} f^{(n)}(x) \geq 0$ for all $n \geq 1$ and all $x > 0$. This answers a question raised by Dimitar Dimitrov at the Fifth International Symposium on Orthogonal Polynomials, Special Functions and Applications held in Patras in September 1999. To prove the result we examine the imaginary part of $1/f$ in the upper half-plane, in particular close to the negative real axis, where Stirling's formula is not valid.

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1 Introduction and results

Monotonicity properties of the function

$$f(x) = \frac{\log \Gamma(x+1)}{x \log x}, \quad x > 0$$

has attracted the attention of several authors. A similar function (where $\Gamma(x+1)$ is replaced by $\Gamma(1 + x/2)$) occurred in the paper of Anderson, Vamanamurthy and Vuorinen ([4]). In [3], Anderson and Qiu showed that f increases on the interval $[1, \infty[$ and they conjectured that f is concave on the interval $[1, \infty[$. The concavity of f on $[1, \infty[$ was established by Elbert and Laforgia ([8]). At the conference in Patras in September 1999, the following conjecture about f was made: for every $n \geq 1$, the inequality

$$(-1)^{n-1} f^{(n)}(x) \geq 0$$

holds for $x \in [1, \infty[$. The purpose of this paper is to turn the conjecture into a theorem. We prove:

Theorem 1.1 *We have, for $n \geq 1$,*

$$(-1)^{n-1} \left(\frac{\log \Gamma(x+1)}{x \log x} \right)^{(n)} > 0$$

for $x > 0$.

We actually prove a stronger statement, namely that the reciprocal function $x \log x / \log \Gamma(x+1)$ is a Stieltjes transform, i.e. belongs to the Stieltjes cone $\mathcal{S}$ of functions of the form

$$g(x) = a + \int_0^\infty \frac{d\mu(t)}{x+t}, \quad x > 0, \tag{1}$$

where $a \geq 0$ and μ is a non-negative measure on $[0, \infty[$ satisfying

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

At the end of the paper we find a and μ for the function in question, see (6) and (7). We note that both f and its reciprocal have removable singularities at $x = 1$. We obtain our results on the entire half-line $]0, \infty[$.

A Stieltjes transform g is easily seen to be completely monotone, i.e. satisfies

$$(-1)^n g^{(n)}(x) \geq 0, \quad x > 0, \quad n \geq 0. \tag{2}$$

Note that strict inequality always holds in (2) unless g is constant.

In concrete cases it is often easier to establish that a function is a Stieltjes transform than to verify complete monotonicity. This is because the Stieltjes cone can be described via complex analysis due to its relationship with the class of Nevanlinna-Pick functions.

In the following result given in the Addenda and Problems in Akhiezer's monograph [2, p.127], we denote the cut plane by

$$\mathcal{A} = \mathbb{C} \setminus]-\infty, 0].$$

Proposition 1.2 *Let $\tilde{\mathcal{S}}$ denote the set of holomorphic functions $G : \mathcal{A} \to \mathbb{C}$ satisfying* *(i)* $\Im G(z) \le 0$ *for* $\Im z > 0$,

(ii) $G(x) \ge 0$ *for* $x > 0$.

Then $\{G|]0,\infty[: \ G \in \tilde{\mathcal{S}}\} = \mathcal{S}$.

A proof is written out in [5], which also contains a list of stability properties of the cone $\mathcal{S}$.

The constant a in (1) is clearly given as

$$a = \lim_{x\to\infty} g(x).$$

The measure μ can be found from the holomorphic extension of (1) to $\mathcal{A}$ given by

$$G(z) = a + \int_0^\infty \frac{d\mu(t)}{z+t}, \quad z \in \mathcal{A}.$$

In fact,

$$-\frac{1}{\pi}\Im G(-t+iy) = P_y * \mu(t), \quad t \in \mathbb{R}, \quad y > 0,$$

where

$$P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

is the Poisson kernel for the upper half-plane $\mathbb{H} = \{z = x+iy : y > 0\}$. It follows that μ is the vague limit of the sequence of measures $-(1/\pi)\Im G(-t+i/n)dt$ as $n \to \infty$ in the sense that

$$-(1/\pi)\int_{-\infty}^{\infty} \varphi(t)\Im G(-t+i/n)dt \longrightarrow \int_{-\infty}^{\infty} \varphi(t)d\mu(t)$$

for all continuous functions φ of compact support.

Non-negative functions on the half-line $]0,\infty[$ with a completely monotone derivative appears in the literature under the names of completely monotone mappings, cf. [7] and Bernstein functions, cf. [6]. Theorem 1.1 can be rephrased that $\log\Gamma(x+1)/x\log x$ is a Bernstein function.

There is an important relation between the class $\mathcal{S}$ and the class $\mathcal{B}$ of Bernstein functions. We state this relation as a proposition, and indicate the proof. The relation can be interpreted as a result about potential kernels, cf. [5], [6].

Proposition 1.3 *For any function* $g \in \mathcal{S} \setminus \{0\}$ *we have* $1/g \in \mathcal{B}$.

Proof: Suppose that g is a non-zero Stieltjes transform. Using Proposition 1.2, it is easy to see that $x \to 1/g(1/x)$ is again a Stieltjes transform. It therefore has an integral representation

$$1/g(1/x) = a + \int_{[0,\infty[} \frac{d\mu(t)}{x+t},$$

but this means that $1/g(x)$ has the representation

$$1/g(x) = a + \mu(\{0\})x + \int_{]0,\infty[} (1 - \frac{t}{x+t})t\, d\mu(1/t),$$

from which we deduce that $1/g$ is a Bernstein function. In fact

$$(1/g(x))' = \mu(\{0\}) + \int_{]0,\infty[} \frac{t^2}{(x+t)^2}\, d\mu(1/t),$$

which is completely monotone. □

Theorem 1.1 will be obtained as a combination of Proposition 1.3 and Theorem 1.4 below.

First we fix some notation. Throughout the paper, Log denotes the principal logarithm, holomorphic in the cut plane $\mathcal{A}$ and defined in terms of the principal argument Arg. The function $\log \Gamma$ denotes the holomorphic branch that is real on the positive real axis. Such a branch exists, since Γ is holomorphic in the simply connected domain $\mathcal{A}$ and has no zeros there.

Theorem 1.4 *The function*

$$g_0(z) = \frac{z \operatorname{Log} z}{\log \Gamma(z+1)}, \quad z \in \mathcal{A}$$

is a Stieltjes transform.

To see that g_0 is holomorphic in $\mathcal{A}$ we need that $\log \Gamma(z+1) = 0$ for $z \in \mathcal{A}$ only for $z = 1$, and this is proved in the Appendix. We note that $z = 1$ is a removable singularity for g_0 with value $1/(1-\gamma)$, where γ is Euler's constant. It is easy to see that g_0 is positive on the positive axis.

Defining

$$V(z) = \Im g_0(z),$$

we have a harmonic function in $\mathcal{A}$. We shall show $V \leq 0$ in the upper half-plane. This will follow, if we prove that V has non-positive boundary values (from above) on the real line, and prove that the growth of V is controlled in the upper half-plane. Indeed, we note the following result, that can be found in [9, p.27].

Theorem 1.5 *Let $u(z)$ be subharmonic for $\Im z > 0$ with $u(z) \leq A|z| + o(|z|)$ when $|z|$ is large. Suppose that, for each real t,*

$$\limsup_{z \to t, \Im z > 0} u(z) \leq 0.$$

Then $u(z) \leq A \Im z$ for $\Im z > 0$.

We shall use this result for $A = 0$.

2 Proofs

In this section we prove the difficult part of Theorem 1.4, namely $V(z) \leq 0$ in the upper half-plane $\mathbb{H}$, and we shall find the integral representation (1) for $x\log x/\log\Gamma(x+1)$.

The relation

$$\log\Gamma(z) = \log\Gamma(z+k) - \sum_{l=0}^{k-1}\mathrm{Log}(z+l) \tag{3}$$

for $z \in \mathcal{A}$ and for any $k \geq 1$ is going to be very useful for us. It follows from the fact that the functions on both sides of the relation are holomorphic functions in $\mathcal{A}$, and they agree on the positive half-line by repeated applications of the functional equation for the Gamma function.

Lemma 2.1 *We have, for any $k \geq 1$,*

$$\lim_{z\to t, \Im z>0} \log\Gamma(z) = \log|\Gamma(t)| - i\pi k$$

for $t \in]-k, -k+1[$ and

$$\lim_{z\to t, \Im z>0} |\log\Gamma(z)| = \infty$$

for $t = 0, -1, -2, \ldots$.

Proof: Suppose that $t \in]-k, -k+1[$. By the relation (3) we find

$$\begin{aligned}
\lim_{z\to t, \Im z>0} \log\Gamma(z) &= \lim_{z\to t, \Im z>0} \log\Gamma(z+k) - \lim_{z\to t, \Im z>0} \sum_{l=0}^{k-1}\mathrm{Log}(z+l) \\
&= \log\Gamma(t+k) - \sum_{l=0}^{k-1}(\log|t+l| + i\pi) \\
&= \log\Gamma(t+k) - \sum_{l=0}^{k-1}\log|t+l| - ik\pi \\
&= \log\left|\frac{\Gamma(t+k)}{t(t+1)\cdots(t+k-1)}\right| - ik\pi \\
&= \log|\Gamma(t)| - ik\pi.
\end{aligned}$$

The other assertion follows from the fact that $|\log\Gamma(z)| \geq \log|\Gamma(z)|$. $\square$.

Remark 2.2 Let h be a meromorphic function in $\mathbb{C}$ all of whos zeros and poles are on $]-\infty, 0]$. Suppose further that h is real and positive on the positive half-line. Then an analogous conclusion holds for the holomorphic branch of $\log h$ that is real on the positive half-line: The limit of $\log h$ at a regular point $t < 0$ from above is $i\pi$ multiplied by the number of zeros minus the number of poles in $]t, 0]$ counted with multiplicity.

Proposition 2.3 *We have*

$$\limsup_{z\to t, \Im z>0} V(z) \le 0.$$

for all real t.

Proof: Suppose that $t \in \,]-k,-k+1[$ for some $k \ge 1$. By the lemma just stated,

$$\log\Gamma(z+1) \longrightarrow \log|\Gamma(t+1)| - (k-1)\pi i$$

as $z \to t$ within $\mathbb{H}$, the upper half-plane. Therefore,

$$\frac{z \operatorname{Log} z}{\log\Gamma(z+1)} \longrightarrow \frac{t(\log|t| + i\pi)(\log|\Gamma(t+1)| + (k-1)\pi i)}{|\log|\Gamma(t+1)||^2 + (k-1)^2\pi^2},$$

so that

$$V(z) \longrightarrow \frac{\pi t(\log|\Gamma(t+1)| + (k-1)\log|t|)}{|\log|\Gamma(t+1)||^2 + (k-1)^2\pi^2}$$

as $z \to t$ within $\mathbb{H}$. This expression is negative. For $k = 1$ it follows from the fact that $t+1 \in \,]0,1[$ so that $\Gamma(t+1) > 1$. For $k \ge 2$ we use

$$\begin{aligned} \log|\Gamma(t+1)| + (k-1)\log|t| &= \log|\Gamma(t+k)| + (k-1)\log|t| - \sum_{l=1}^{k-1}\log|t+l| \\ &= \log|\Gamma(t+k)| + \sum_{l=1}^{k-1}\log\frac{|t|}{|t+l|}, \end{aligned}$$

a positive quantity (recall that $|t+l| < |t|$ for $l = 1,\dots,k-1$).

For $t = -1,-2,\dots$ we have $|\log\Gamma(z+1)| \to \infty$ so that $V(z) \to 0$ as $z \to t$ within $\mathbb{H}$. For positive t we evidently have $V(z) \to 0$; the function g_0 is holomorphic in $\mathcal{A}$ and is real-valued on $]0,\infty[$.

The case $t = 0$ requires a more refined analysis. Since $\log\Gamma(z+1)$ is holomorphic at $z = 0$ and is zero at that point,

$$\log\Gamma(z+1) = \sum_{n=1}^{\infty} a_n z^n$$

for $|z| < 1$. The number a_1 is real and negative (it is in fact equal to $-\gamma$). We thus get

$$g_0(z) = \frac{\operatorname{Log} z}{\sum_{n=1}^{\infty} a_n z^{n-1}} = \operatorname{Log} z \sum_{n=0}^{\infty} b_n z^n,$$

for $|z| < 1$ and with $b_0 = 1/a_1$. For $n \ge 1$, $|z^n \operatorname{Log} z| \to 0$ as $z \to 0$. Therefore

$$\limsup_{z\to 0, \Im z>0} V(z) = \limsup_{z\to 0, \Im z>0} \frac{\operatorname{Arg} z}{a_1} = 0.$$

□

Proposition 2.4 *There is a constant $C > 0$, such that $|V(z)| \le C \log |z|$ holds for all $z \in \mathbb{H}$ of large absolute value.*

Proof: Stieltjes ([11, formula 20]) found the following formula for $\log \Gamma(z)$ for z in the cut plane $\mathcal{A}$

$$\log \Gamma(z) = \log \sqrt{2\pi} + (z - 1/2) \operatorname{Log} z - z + \mathcal{J}(z).$$

Here

$$\mathcal{J}(z) = \sum_{n=0}^{\infty} h(z+n) = \int_0^\infty \frac{P(t)}{z+t} dt,$$

where $h(z) = (z + 1/2) \operatorname{Log}(1 + 1/z) - 1$ and P is periodic with period 1 and $P(t) = 1/2 - t$ for $t \in [0, 1[$. The integral above is improper, and integration by parts yields

$$\mathcal{J}(z) = \frac{1}{2} \int_0^\infty \frac{Q(t)}{(z+t)^2} dt,$$

where Q is periodic with period 1 and $Q(t) = t - t^2$ for $t \in [0, 1[$. Using that $\mathcal{J}(z+1) + h(z) = \mathcal{J}(z)$ it follows that

$$\log \Gamma(z+1) = \log \sqrt{2\pi} + (z + 1/2) \operatorname{Log} z - z + \mathcal{J}(z).$$

We put $R_k = \{z = x + iy \in \mathbb{C} : -k \le x < -k+1,\ 0 < y \le 1\}$ for $k \in \mathbb{Z}$ and $R = \cup_{k=0}^{\infty} R_k$ and claim that

$$|\mathcal{J}(z)| \le \frac{\pi}{8}$$

for $z \in \mathbb{H} \setminus R$. In fact, since $0 \le Q(t) \le 1/4$, we get for $z = x + iy \in \mathbb{H} \setminus R$

$$|\mathcal{J}(z)| \le \frac{1}{8} \int_0^\infty \frac{dt}{(t+x)^2 + y^2}.$$

For $x \ge 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} \le \int_0^\infty \frac{dt}{(t+1)^2} = 1,$$

and for $x < 1, y \ge 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} = \frac{1}{y} \left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right) \le \pi.$$

This gives us

$$\frac{\log \Gamma(z+1)}{z \operatorname{Log} z} = 1 + \frac{\log \sqrt{2\pi} + 1/2 \operatorname{Log} z - z + \mathcal{J}(z)}{z \operatorname{Log} z},$$

for $z \in \mathbb{H} \setminus R$. In particular we see that there is a constant $c > 0$ such that

$$\frac{|\log \Gamma(z+1)|}{|z \operatorname{Log} z|} \geq c \tag{4}$$

for all $z \in \mathbb{H} \setminus R$ of large absolute value.

If $z \in R_k$ for some $k \geq 1$, we use the relation (3). We find

$$|\log \Gamma(z+1)| \geq \left| \sum_{l=1}^{k} \operatorname{Log}(z+l) \right| - |\log \Gamma(z+k+1)|.$$

Here, $x + l < 0$ for $l \leq k-1$, so that

$$\left| \sum_{l=1}^{k} \operatorname{Log}(z+l) \right| \geq \sum_{l=1}^{k} \operatorname{Arg}(z+l) \geq (k-1)\pi/2.$$

Furthermore, $z + k + 1 \in R_{-1}$ and on $\overline{R_{-1}}$, $\log \Gamma$ is holomorphic and is therefore bounded by some constant M independent of k. This implies that

$$|\log \Gamma(z+1)| \geq (k-1)\pi/2 - M. \tag{5}$$

From this relation and $|z| \leq \sqrt{k^2+1}$ we deduce that $|\log \Gamma(z+1)|/|z| \geq \text{const} > 0$ for all $z \in R$ of large absolute value. Combined with (4) we conclude that

$$|\log \Gamma(z+1)| \geq \text{const}|z|$$

for all $z \in \mathbb{H}$ of large absolute value. This implies in turn

$$|V(z)| \leq |g_0(z)| \leq \frac{|z|(\log|z| + \pi)}{\text{const}|z|} \leq \text{const} \log|z|$$

as $|z| \to \infty$ within $\mathbb{H}$. $\square$

As mentioned earlier, Theorem 1.4 follows from the above results. It may be of interest to know the exact integral representation of g_0. We have

$$\frac{z \operatorname{Log} z}{\log \Gamma(z+1)} = 1 + \int_0^\infty \frac{d(t)\, dt}{t+z}, \tag{6}$$

where $d(t) = -(1/\pi)V(-t)$, and $V(-t)$ is defined for $t > 0$ as the limit of $V(z)$ as z tends to $-t$ from above. Indeed, in Proposition 2.3 we actually showed that V has a continuous extension down to the negative real axis. This means that $V(t + i/n) \to V(t)$ uniformly on compact subsets of $]-\infty, 0[$ as $n \to \infty$ and hence that the measure μ in (1) has the continuous density $d(t)$ given by

$$d(t) = t \frac{\log|\Gamma(1-t)| + (k-1)\log t}{(\log|\Gamma(1-t)|)^2 + (k-1)^2\pi^2}, \quad t \in]k-1, k[, \ k = 1, 2, \ldots, \tag{7}$$

and $d(t) = 0$ for $t = 1, 2, \ldots$. It is easily seen that $d(t)$ tends to $1/\gamma$ for t tending to zero. It remains to be proved that μ has no mass at zero. From (1) we get

$$\lim_{x \to 0^+} xg(x) = \mu(\{0\}),$$

but from the analysis in Proposition 2.3 concerning the behaviour at $t = 0$, we get that the above limit is 0 for $g = g_0$. The constant a in (1) is 1 by Stirling's formula.

We studied the function g_0 through its imaginary part V. We shall now find the exact logarithmic growth of V in the upper half-plane.

Proposition 2.5 *We have*

$$\limsup_{|z| \to \infty, \Im z > 0} \frac{|V(z)|}{\log |z|} = \frac{1}{\pi}.$$

Proof: In the proof of Proposition 2.4 we obtained $|g_0(z)| \to 1$, and hence $|V(z)| / \log |z| \to 0$ for z tending to infinity in $\mathbb{H} \setminus R$.

For $z \in R_k$ we found

$$|\log \Gamma(z+1)| \geq \sum_{l=1}^{k} \operatorname{Arg}(z+l) - M,$$

where M is the maximum of $|\log \Gamma|$ on $\overline{R_{-1}}$. We claim that for every $a < \pi$ there is k_0 such that

$$|\log \Gamma(z+1)| \geq a|z|$$

for all $k \geq k_0$ and $z \in R_k$. Indeed, let a be any given number less than π. We choose $\varepsilon > 0$ such that $(1+\varepsilon)(a+\varepsilon) < \pi$ and then find m_0 such that $\operatorname{Arg} w \geq (1+\varepsilon)(a+\varepsilon)$ for all $m \geq m_0$ and all $w \in R_m$.

For $k > m_0$ we have

$$\sum_{l=1}^{k} \operatorname{Arg}(z+l) \geq \sum_{l=1}^{k-m_0} \operatorname{Arg}(z+l) \geq (1+\varepsilon)(a+\varepsilon)(k-m_0),$$

because $x + l < -m_0 + 1$ for $z \in R_k$ and $l = 1, \ldots, k - m_0$. This gives us

$$\frac{|\log \Gamma(z+1)|}{|z|} \geq \frac{(1+\varepsilon)(a+\varepsilon)(k-m_0)}{|z|} - \frac{M}{|z|}$$

for $k > m_0$ and all $z \in R_k$.

We choose now $k_0 > m_0$ such that $(k - m_0)/|z| \geq 1/(1+\varepsilon)$ *and* $M/|z| \leq \varepsilon$ for all $k \geq k_0$ and all $z \in R_k$. For these values of k and z we thus find

$$\frac{|\log \Gamma(z+1)|}{|z|} \geq a,$$

and the claim follows. It implies

$$\frac{|V(z)|}{\log|z|} \leq \frac{1}{a} + \frac{\pi}{a\log|z|},$$

for $k \geq k_0$ and all $z \in R_k$. Therefore,

$$\limsup_{|z|\to\infty, \Im z>0} \frac{|V(z)|}{\log|z|} \leq \frac{1}{a}.$$

The number a could, however, be chosen as close to π as we want and thus

$$\limsup_{|z|\to\infty, \Im z>0} \frac{|V(z)|}{\log|z|} \leq \frac{1}{\pi}.$$

On the other hand, the maximum of $d(t)$ on the interval $[k-1, k]$ tends to infinity as $k \to \infty$. In fact, it is known that the minimum of $|\Gamma(1-t)|$ on $]k-1, k[$ is less than 1 for $k \geq 4$. For these values of k there exists $\xi_k \in \,]k-1, k[$ such that $\log|\Gamma(1-\xi_k)| = 0$ and hence

$$d(\xi_k) = \frac{\xi_k \log \xi_k}{(k-1)\pi^2} > \frac{\log(k-1)}{\pi^2}.$$

Therefore $|V(-\xi_k)| = \pi d(\xi_k) > (\log(k-1))/\pi$. Since V, as mentioned, is continuous down to the negative real axis, we must therefore have (for each $k \geq 4$) a number $z_k \in R_k$ such that $|V(z_k) - V(-\xi_k)| \leq 1$ and hence we obtain

$$\limsup_{k} \frac{|V(z_k)|}{\log|z_k|} \geq \frac{1}{\pi}.$$

□

Finally, we remark that the above proposition actually states that

$$\liminf_{|z|\to\infty, \Im z>0} \frac{V(z)}{\log|z|} = -\frac{1}{\pi},$$

because V is negative.

3 Appendix

We recall that $\log\Gamma$ was defined to be the holomorphic branch of the logarithm of Γ that is real on the positive real axis. We now verify that this function has no zeros in $\mathcal{A} \setminus \{1, 2\}$.

We notice that the equation $\Gamma(z) = 1$ has non-real solutions in addition to the obvious solutions on the real axis. They are found as the intersection of the level set $\{|\Gamma(z)| = 1\}$ and the curves where $\arg\Gamma(z) = 2\pi p$, p being a non-zero integer.

Theorem 3.1 *The only zeros of the holomorphic function* $\log\Gamma$ *defined in* $\mathcal{A}$ *are* 1 *and* 2.

Proof: Clearly, the only *real* zeros of $\log\Gamma$ are at 1 and 2. To show that there cannot be any non-real zeros amounts to showing that $\log\Gamma(z+1)$ has no non-real zeros. This we proceed to verify.

From the Weierstrass product for the Gamma function we get for $z \in \mathcal{A}$

$$-\log\Gamma(z+1) = \gamma z + \sum_{n=1}^{\infty}\left(\mathrm{Log}\left(1+\frac{z}{n}\right) - \frac{z}{n}\right). \tag{8}$$

Indeed, the negative of the right-hand side *is* a holomorphic branch of the logarithm of $\Gamma(z+1)$, and it is real on the positive real axis.

Taking real parts on both sides of (8) we obtain

$$-\log|\Gamma(z+1)| = \gamma x + \sum_{n=1}^{\infty}\left(\log\left|1+\frac{z}{n}\right| - \frac{x}{n}\right). \tag{9}$$

In particular, we see that $y \to |\Gamma(x+iy)|$ is strictly decreasing for $y \geq 0$.

Taking imaginary parts on both sides of (8) we get

$$-\arg\Gamma(z+1) = \gamma y + \sum_{n=1}^{\infty}\left(\mathrm{Arg}\left(1+\frac{z}{n}\right) - \frac{y}{n}\right),$$

and in particular, for $x > -1$,

$$P_x(y) \equiv \gamma y + \sum_{n=1}^{\infty}\left(\arctan\left(\frac{y}{n+x}\right) - \frac{y}{n}\right) = -\arg\Gamma(z+1). \tag{10}$$

It is known that the convex function $\log\Gamma(x+1)$ has its minimum on $[0,\infty[$ at a point $x_0 \approx 0.461$, c.f. [10]. We have $\log\Gamma(x_0+1) \approx -0.121 > -1/5$. Since $\log\Gamma(\overline{z}+1)$ is the complex conjugate of $\log\Gamma(z+1)$ for $z \in \mathcal{A}$, it is enough to prove that

$$\log\Gamma(z+1) \neq 0, \quad \text{for } z = x+iy,\ y > 0. \tag{11}$$

This will be done in five steps:

(i) The function $\log\Gamma(z+1)$ is univalent in the half-plane $\{\Re z > x_0\}$, cf. [1], and since it vanishes at $z=1$, it does not vanish elsewhere in $\{\Re z > x_0\}$.

(ii) In the strip $0 < \Re z \leq x_0$ we know that $|\Gamma(z+1)| \leq \Gamma(x+1) < 1$, and hence $\Re\log\Gamma(z+1) = \log|\Gamma(z+1)| < 0$, so (11) holds.

(iii) Suppose $y \geq 1$ and $-k < x \leq -k+1$, where $k = 1, 2, \ldots$. By the functional equation for $\Gamma(z)$ we get

$$|\Gamma(z+k+1)| = |(z+k) \cdot \ldots \cdot (z+1)||\Gamma(z+1)| \geq y^k |\Gamma(z+1)| \geq |\Gamma(z+1)|,$$

but since $\Re(z+k+1) \in]1,2]$, the left-hand side in the relation above is strictly less than 1 and again $\log|\Gamma(z+1)| < 0$.

(iv) Suppose $0 < y < 1$ and $-k < x \leq -k+1$, where $k = 2, 3, \ldots$. From (5) we have

$$|\log\Gamma(z+1)| \geq (k-1)\frac{\pi}{2} - M \geq \frac{\pi}{2} - M,$$

where

$$M = \max\{|\log\Gamma(z+1)| \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}. \tag{12}$$

In Lemma 3.2 below we prove that $M < \pi/2$, and therefore $|\log\Gamma(z+1)| > 0$.

(v) Suppose $0 < y < 1, -1 < x \leq 0$. From (10) we see that $P_x(y) \geq P_0(y)$ for $-1 < x \leq 0, y > 0$. For $0 < y < 1$ we can insert the power series for the function arctan. After reversing the resulting double sum we get

$$P_0(y) = y\left(\gamma - \frac{\zeta(3)}{3}y^2 + \frac{\zeta(5)}{5}y^4 - \cdots\right).$$

Because $\zeta(x)/x$ is decreasing for $x > 1$ we therefore get

$$P_0(y) > y\left(\gamma - \frac{\zeta(3)}{3}\right) > 0, \quad 0 < y < 1,$$

which shows that $-\arg\Gamma(z+1) > 0$ for $z = x + iy$, where $0 < y < 1, -1 < x \leq 0$, so (11) holds. $\square$

Lemma 3.2 *We have $M < \pi/2$, where M is given by (12).*

Proof: The constant M can be evaluated using Maple and the fact that the maximum is attained on the boundary of the square. We find $M \approx 0.72$. We shall independent of this show that $M < \pi/2$.

For $0 \leq x \leq 1$ and $0 \leq y \leq 1$ we have, by (10),

$$|\log\Gamma(z+1)|^2 = (\log|\Gamma(z+1)|)^2 + (P_x(y))^2.$$

The relation (9) gives us the inequality

$$\log\frac{1}{|\Gamma(x+i+1)|} \leq \log\frac{1}{\Gamma(x+1)} + \frac{1}{2}\sum_{n=1}^{\infty}\log\left(1 + \frac{1}{n^2}\right).$$

Indeed, from the elementary fact that $\log(a+b) \le \log a + \log(1+b)$ for $a \ge 1$ and $b \ge 0$, we obtain (putting $a = (1+x/n)^2$ and $b = 1/n^2$)

$$\begin{aligned}
-\log|\Gamma(x+i+1)| &= \gamma x + \sum_{n=1}^{\infty}\left(\frac{1}{2}\log\left(\left(1+\frac{x}{n}\right)^2 + \frac{1}{n^2}\right) - \frac{x}{n}\right) \\
&\le \gamma x + \sum_{n=1}^{\infty}\left(\log\left(1+\frac{x}{n}\right) + \frac{1}{2}\log\left(1+\frac{1}{n^2}\right) - \frac{x}{n}\right) \\
&= -\log\Gamma(x+1) + \frac{1}{2}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right).
\end{aligned}$$

As noted before, the function $|\Gamma(x+iy+1)|$ is decreasing for positive y. Its values are all ≤ 1 since $1+x \in [1,2]$. Therefore

$$0 \le \log\frac{1}{|\Gamma(z+1)|} \le \log\frac{1}{|\Gamma(x+i+1)|} \le \log\frac{1}{\Gamma(x+1)} + \frac{1}{2}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right)$$

$$\le \frac{1}{5} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{5} + \frac{\pi^2}{12}.$$

Here we have also used that $\log\Gamma(1+x) \ge -1/5$ for $0 \le x \le 1$.

We further get for $0 \le y \le 1$

$$P_1(y) \le P_x(y) \le P_0(y) \le \gamma$$

and

$$P_1(y) = P_0(y) - \arctan(y) \ge -\frac{\pi}{4},$$

so

$$(P_x(y))^2 \le \left(\frac{\pi}{4}\right)^2.$$

This finally shows that $M < \pi/2$. $\square$

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