On some indeterminate moment problems for measures on a geometric progression

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Abstract

We consider some indeterminate moment problems which all have a discrete solution concentrated on geometric progressions of the form $cq^k, k \in \mathbb{Z}$ or $\pm cq^k, k \in \mathbb{Z}$. It turns out to be possible that such a moment problem has infinitely many solutions concentrated on the geometric progression.

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1 Introduction

If $s = (s_n)_{n\geq 0}$ is an indeterminate moment sequence, it is well-known that the set V of solutions to the corresponding moment problem contains measures of a very different nature: There are measures $\mu \in V$ with a C^∞-density, discrete measures and measures which are continuous singular. It is even so that there are many solutions of each of the three types in the sense that each of these classes of measures is a dense subset of V, cf.[3].

It is true however that in concrete cases only a few families of solutions are explicitly known.

Recently Andreas Ruffing asked the author if the solution associated with the discrete q-Hermite polynomials II is uniquely determined by being restricted to the countable set $\{\pm q^{k+\frac{1}{2}} | k \in \mathbb{Z}\}$. We shall answer this question in the negative. We also treat the same question for the Stieltjes-Wigert polynomials [21], which are associated with the log-normal distribution, and for the q-Laguerre polynomials, cf. [17]. The question by Ruffing was motivated by the study of a q-deformation of the harmonic oscillator, see [16], [18],[19],[10].

We shall make use of the following elementary result, which is a special case of a theorem of Naimark, cf.[1, p.47], characterizing the extreme points of the convex set V as those $\mu \in V$ for which the polynomials are dense in $L^1(\mu)$. We recall that a point in a convex set is called *extreme*, if it is not the midpoint of any segment contained in the convex set.

Proposition 1.1 *Let $\Lambda \subset \mathbb{R}$ be an infinite countable set and let*

$$\mu = \sum_{\lambda \in \Lambda} a_\lambda \varepsilon_\lambda, \;\; a_\lambda > 0 \tag{1.1}$$

be a solution to an indeterminate moment problem. Then μ is not an extreme point of V if and only if there exists a non-zero sequence $\varphi : \Lambda \to [-1, 1]$ such that

$$\sum_{\lambda \in \Lambda} \lambda^n \varphi(\lambda) a_\lambda = 0 \text{ for } n \geq 0. \tag{1.2}$$

If the conditions are satisfied, there are infinitely many measures in V concentrated on Λ.

We include the proof, since it shows how to construct solutions on Λ different from (1.1), using the sequence φ from (1.2). For $s \in [-1, 1]$ we define

$$\mu_s = \sum_{\lambda \in \Lambda} a_\lambda (1 + s\varphi(\lambda)) \varepsilon_\lambda, \tag{1.3}$$

which is a family of measures concentrated on Λ with the same moments as $\mu = \mu_0$, $\mu_s \neq \mu_t$ for $s \neq t$ and $\mu = (\mu_1 + \mu_{-1})/2$. This shows that μ is not an extreme point of V.

Conversely, if μ is not an extreme point, it is of the form $\mu = (\nu + \sigma)/2$, where ν, σ are different measures from V. Since $\mu(\mathbb{R} \setminus \Lambda) = 0$, we get that ν and σ are both concentrated on Λ, and defining

$$\varphi(\lambda) = \frac{\nu(\{\lambda\}) - \sigma(\{\lambda\})}{2a_\lambda}, \;\; \lambda \in \Lambda,$$

we get a non-zero sequence $\varphi : \Lambda \to [-1, 1]$ satisfying (1.2).

2 The log-normal distribution

The log-normal distribution with parameter $\sigma > 0$ has the following density on $]0, \infty[$

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right). \tag{2.1}$$

For $p \in \mathbb{R}$ one has

$$s_p(d_\sigma) = \int_0^\infty x^p d_\sigma(x)\,dx = e^{\frac{1}{2}p^2\sigma^2}, \tag{2.2}$$

and in particular the moment sequence is given as

$$s_n(d_\sigma) = q^{-n^2}, \quad n \geq 0, \tag{2.3}$$

where $0 < q < 1$ is defined by $q = e^{-\frac{1}{2}\sigma^2}$. Note that $d_\sigma(q^2 x) = (x/q) d_\sigma(x)$, so a suitable change of scale removes the factor x^{-1} in (2.1).

Stieltjes showed the indeterminacy of (2.3) by remarking that all the densities ($s \in [-1,1]$)

$$d_\sigma(x)\left(1 + s\sin(\frac{2\pi}{\sigma^2}\log x)\right)$$

have the same moments (2.3), cf.[20]. The crucial point in the calculation is the $\mathbb{Z}$-periodicity of $\sin(2\pi x)$.

Chihara [6] and later Leipnik [15] gave the following family of solutions to (2.3) concentrated on countable sets. For $a > 0$ define the measure

$$H_a = \frac{1}{c(a)} \sum_{k=-\infty}^{\infty} a^k q^{k^2} \varepsilon_{aq^{2k}} \tag{2.4}$$

concentrated on $\Lambda(a) = \{aq^{2k} \mid k \in \mathbb{Z}\}$, where

$$c(a) = \sum_{k=-\infty}^{\infty} a^k q^{k^2}. \tag{2.5}$$

Note that $c(a) = (q^2, -aq, -q/a; q^2)_\infty$ by Jacobi's triple product identity, cf. [9].

The moments of H_a can be calculated using the translation invariance of $\sum_{-\infty}^{\infty}$:

$$\begin{aligned} s_n(H_a) &= \frac{1}{c(a)} \sum_{k=-\infty}^{\infty} a^k q^{k^2} (aq^{2k})^n = \frac{q^{-n^2}}{c(a)} \sum_{k=-\infty}^{\infty} a^{k+n} q^{(k+n)^2} \\ &= q^{-n^2}. \end{aligned}$$

Note that this calculation is valid for $n \in \mathbb{Z}$, so that d_σ and H_a have the same positive and negative moments. The calculation in [15] involved characteristic functions.

We now put $a = q$ and show that H_q is not an extreme point of V.

The following formula holds

$$\sum_{k=-\infty}^{\infty} q^{k^2+k}(-1)^k = 0. \tag{2.6}$$

In fact, the terms corresponding to k and $-k-1$ cancel.

The bounded sequence $\varphi : \Lambda(q) \to [-1,1]$ given by $\varphi(q^{2k+1}) = (-1)^k$ satisfies (1.2), since for $n \in \mathbb{Z}$

$$\frac{1}{c(q)} \sum_{k=-\infty}^{\infty} q^{(2k+1)n}(-1)^k q^{k^2+k} = (-1)^n \frac{q^{-n^2}}{c(q)} \sum_{k=-\infty}^{\infty} q^{(k+n)^2+k+n}(-1)^{n+k} = 0. \tag{2.7}$$

Proposition 2.1 *The measure H_q is not an extreme point of V. All the measures on $\Lambda(q)$ of the form*

$$\mu_s = \frac{1}{c(q)} \sum_{k=-\infty}^{\infty} q^{k^2+k}(1+s(-1)^k)\varepsilon_{q^{2k+1}},\ s \in [-1,1]$$

have the same moments (2.3).

Using the translation invariance it follows that $H_{aq^2} = H_a$ and in particular $H_{q^{2m+1}} = H_q$ for all $m \in \mathbb{Z}$. One may ask for which values of $a > 0$ it is the case that H_a is an extreme point of V. We cannot answer this question, but shall answer the corresponding question for the strong (or two-sided) Stieltjes moment problem. Concerning this problem see [4],[12]. We notice that H_a and d_σ are solutions of a strong Stieltjes moment problem since

$$s_n(H_a) = s_n(d_\sigma) = q^{-n^2} \text{ for } n \in \mathbb{Z}. \tag{2.8}$$

Proposition 2.2 *Let $\tilde{V}$ be the convex set of solutions to the strong Stieltjes moment problem with moments* (2.8).

Then $H_a, a > 0$ is an extreme point of $\tilde{V}$ if and only if $a \neq q^{2m+1}$ for all $m \in \mathbb{Z}$.

Proof: Since (2.7) holds for all $n \in \mathbb{Z}$, it follows by an obvious extension of Proposition 1.1 that H_q is not an extreme point of $\tilde{V}$. We shall next prove that if a is a positive number not equal to an odd power of q, then H_a is an extreme point of $\tilde{V}$. Let $\varphi : \mathbb{Z} \to [-1,1]$ annihilate x^n for all integers n with respect to H_a. By showing that φ is identically zero, we obtain the assertion.

Putting $f_a(k) = a^k q^{k^2}$, the assumption on φ is equivalent to the convolution equation $f_a * \check{\varphi} = 0$ on the group $\mathbb{Z}$. By Wiener's Tauberian Theorem we get $\varphi \equiv 0$ provided that

$$\hat{f}_a(\theta) = \sum_{k=-\infty}^{\infty} a^k q^{k^2} e^{ik\theta} \neq 0 \text{ for all } \theta \in \mathbb{R}.$$

Using [5, Theorem 1, p. 59], we see that the above function vanishes precisely if a is an odd power of q and θ is an odd multiple of π. □

3 The discrete q-Hermite polynomials II

According to Koekoek and Swarttouw [13] the monic discrete q-Hermite polynomials II are given by the recurrence relation

$$x\tilde{h}_n(x;q) = \tilde{h}_{n+1}(x;q) + q^{-2n+1}(1-q^n)\tilde{h}_{n-1}(x;q) \tag{3.1}$$

with the initial conditions $\tilde{h}_{-1}(x;q) = 0$, $\tilde{h}_0(x;q) = 1$. For more information about these polynomials see [14]. They appear in another normalization as the analogues of the classical Hermite polynomials in the study of a q-deformation of the harmonic oscillator, see [16], [18],[19],[10].

By Favard's theorem it is easy to see that $(\tilde{h}_n(x;q))$ are orthogonal with respect to a positive measure if and only if

$$q^{-2n+1}(1-q^n) > 0 \text{ for } n \geq 1,$$

which is satisfied if and only if $0 < q < 1$. The corresponding orthonormal polynomials (P_n) with positive leading coefficients satisfy

$$xP_n(x) = b_n P_{n+1}(x) + b_{n-1}P_{n-1}(x) \tag{3.2}$$

with

$$b_n = q^{-n-\frac{1}{2}}\sqrt{1-q^{n+1}} \tag{3.3}$$

and

$$P_n(x) = \left(\frac{q^{n^2}}{(q;q)_n}\right)^{\frac{1}{2}} \tilde{h}_n(x;q), \tag{3.4}$$

where we use the notation from [9] for q-shifted factorials

$$(z;q)_n = \prod_{k=1}^{n}(1-zq^{k-1}), \quad n \in \mathbb{N}\cup\{\infty\},\ z \in \mathbb{C}, \tag{3.5}$$

and extended to negative n by

$$(z;q)_{-n} = \frac{(z;q)_\infty}{(zq^{-n};q)_\infty}, \quad n \in \mathbb{N}. \tag{3.6}$$

The polynomials $\tilde{h}_n(x;q)$ correspond to a symmetric Hamburger moment problem with vanishing odd moments. For $c > 0$ we consider the family (μ_c) of symmetric probabilities

$$\mu_c = \frac{1}{2A(c)}\sum_{k=-\infty}^{\infty}\frac{q^k}{(-c^2q^{2k};q^2)_\infty}(\varepsilon_{cq^k}+\varepsilon_{-cq^k}), \tag{3.7}$$

where the normalizing constant

$$A(c) = \sum_{k=-\infty}^{\infty}\frac{q^k}{(-c^2q^{2k};q^2)_\infty}$$

can be found from Ramanujans ${}_1\psi_1$-sum ($|b/a| < |x| < 1$)

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}x^k = \frac{(ax,q/ax,q,b/a;q)_\infty}{(x,b/ax,b,q/a;q)_\infty}, \tag{3.8}$$

cf.[9],[2].

We find

$$A(c)=\frac{1}{(-c^2;q^2)_\infty}\sum_{k=-\infty}^{\infty}(-c^2;q^2)_k q^k=\frac{(-qc^2,-q/c^2,q^2;q^2)_\infty}{(-c^2,-q^2/c^2,q;q^2)_\infty}. \tag{3.9}$$

The even moments of μ_c can also be evaluated by (3.8)

$$\begin{aligned}
s_{2n}(\mu_c) &= \frac{1}{A(c)}\sum_{k=-\infty}^{\infty}\frac{(cq^k)^{2n}}{(-c^2q^{2k};q^2)_\infty}q^k\\
&= \frac{c^{2n}}{A(c)(-c^2;q^2)_\infty}\sum_{k=-\infty}^{\infty}(-c^2;q^2)_k q^{(2n+1)k}\\
&= c^{2n}\frac{(-c^2q^{2n+1},-q^{1-2n}/c^2,q;q^2)_\infty}{(q^{2n+1},-qc^2,-q/c^2;q^2)_\infty}=c^{2n}\frac{(-q^{1-2n}/c^2,q;q^2)_n}{(-qc^2;q^2)_n}\\
&= (q;q^2)_n q^{-n^2},
\end{aligned}$$

so

$$s_n(\mu_c)=\begin{cases}0, & n=2m+1\\ (q;q^2)_m q^{-m^2}, & n=2m\end{cases} \tag{3.10}$$

This shows that all the measures μ_c have the same moments (3.10). From the moments (s_n) we can calculate the Hankel determinants and find

$$D_n=\det(s_{i+j})_{0\le i,j\le n}=q^{-\sum_{k=1}^n k^2}\prod_{k=1}^{n}(1-q^k)^{n-k+1}, \tag{3.11}$$

and we then get

$$b_n=\frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}=q^{-n-\frac{1}{2}}\sqrt{1-q^{n+1}},$$

which shows that (P_n) given by (3.2) are orthonormal with respect to μ_c for any $c>0$.

We now put $c=\sqrt{q}$ and show that $\mu_{\sqrt{q}}$ is non-extreme and that there are different measures with the moments (3.10) concentrated on

$$\Lambda=\{\pm q^{k+\frac{1}{2}}\mid k\in\mathbb{Z}\}.$$

The bounded even sequence $\varphi:\Lambda\to[-1,1]$ given by $\varphi(\pm q^{k+\frac{1}{2}})=(-1)^k$ satisfies

$$\int x^n\varphi(x)d\mu_{\sqrt{q}}(x)=0 \text{ for } n\ge 0.$$

This is clear for n odd, and for even n we find by (3.8)

$$\begin{aligned}
\int x^{2n}\varphi(x)d\mu_{\sqrt{q}}(x) &= \frac{1}{A(\sqrt{q})}\sum_{k=-\infty}^{\infty}q^{(k+\frac{1}{2})2n}(-1)^k\frac{q^k}{(-q^{2k+1};q^2)_\infty}\\
&= \frac{q^n}{A(\sqrt{q})(-q;q^2)_\infty}\sum_{k=-\infty}^{\infty}(-q;q^2)_k(-q^{2n+1})^k,
\end{aligned}$$

which vanishes because of the factor $(q^{-2n};q^2)_\infty$. Summing up we have:

Proposition 3.1 *The family of measures* ($s \in [-1, 1]$)

$$\frac{1}{2A(\sqrt{q})} \sum_{k=-\infty}^{\infty} \frac{q^k}{(-q^{2k+1}; q^2)_\infty} \left(1 + s(-1)^k\right) (\varepsilon_{q^{k+\frac{1}{2}}} + \varepsilon_{-q^{k+\frac{1}{2}}})$$

on Λ *has the moments* (3.10).

Taking the image measure of μ_c under the mapping $x \mapsto x^2$, we obtain an indeterminate Stieltjes moment problem with moment sequence

$$s_n = (q; q^2)_n q^{-n^2}.$$

Replacing q by $\sqrt{q}$ and c by $\sqrt{c}$ we get the following family of measures on $]0, \infty[$

$$\nu(q, c) = \frac{1}{\tilde{A}(c)} \sum_{k=-\infty}^{\infty} \frac{q^{k/2}}{(-cq^k; q)_\infty} \varepsilon_{cq^k} \tag{3.12}$$

for $c > 0, 0 < q < 1$ and

$$\tilde{A}(c) = \sum_{k=-\infty}^{\infty} \frac{q^{k/2}}{(-cq^k; q)_\infty}.$$

They all have the moments

$$s_n(\nu(q, c)) = (\sqrt{q}; q)_n q^{-\frac{1}{2}n^2}. \tag{3.13}$$

4 The q-Laguerre polynomials

The q-Laguerre polynomials $L_n^{(\alpha)}(x; q)$ are studied in [17]. We restrict the parameters to $0 < q < 1, \alpha > -1$, and in this case they are associated with an indeterminate Stieltjes moment problem with the moment sequence

$$s_n(\alpha; q) = (1 - q)^{-n} q^{-\alpha n - \binom{n+1}{2}} (q^{\alpha+1}; q)_n. \tag{4.1}$$

See [11] for a calculation of the corresponding Nevanlinna matrix. In [2] and in [17] one finds the following family $(\tau(c, \alpha))_{c>0}$ of probabilities with moment sequence (4.1):

$$\tau(c, \alpha) = \frac{1}{B(c)} \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-q^k c(1-q); q)_\infty} \varepsilon_{cq^k}, \tag{4.2}$$

where

$$B(c) = \sum_{k=-\infty}^{\infty} \frac{q^{k(\alpha+1)}}{(-q^k c(1-q); q)_\infty} = \frac{(-q^{\alpha+1}c(1-q), -q^{-\alpha}/c(1-q), q; q)_\infty}{(q^{\alpha+1}, -q/c(1-q), -c(1-q); q)_\infty}.$$

The sum above is evaluated using (3.8), and by the same formula it is easy to show that the n'th moment of $\tau(c,\alpha)$ is given by (4.1). We finally use (3.8) to establish (1.2) for a certain value of c. For $\varphi(cq^k)=(-1)^k$ we get

$$\begin{aligned}\int x^n\varphi(x)\,d\tau(c,\alpha)(x) &= \frac{c^n}{(-c(1-q);q)_\infty B(c)}\sum_{k=-\infty}^{\infty}(-c(1-q);q)_k(-q^{n+\alpha+1})^k\\ &= c^n\frac{(q^{\alpha+1},q^{n+\alpha+1}c(1-q),q^{-n-\alpha}/c(1-q);q)_\infty}{(-q^{\alpha+1}c(1-q),-q^{-\alpha}/c(1-q),-q^{n+\alpha+1};q)_\infty},\end{aligned}$$

which vanishes for $c=q^{m-\alpha}/(1-q),\, m=0,1,\ldots$.
We have established

Proposition 4.1 *The moments of the measures* ($s\in[-1,1]$)

$$\frac{1}{B(q^{-\alpha}/(1-q))}\sum_{k=-\infty}^{\infty}q^{k(\alpha+1)}/(-q^{k-\alpha};q)_\infty(1+s(-1)^k)\varepsilon_{q^{k-\alpha}/(1-q)}$$

are given by (4.1).

For a measure σ on $[0,\infty[$ and $a>0$ we denote by $h_a(\sigma)$ the image measure of σ under the mapping $h_a(x)=ax$. Note that

$$h_{1-q}(\tau(c/(1-q),-\frac{1}{2}))=\nu(q,c).$$

The q-Laguerre polynomials are related to the generalized Stieltjes-Wigert polynomials of Chihara, cf. [7],[8]. They are obtained from the Stieltjes-Wigert polynomials by introducing a parameter $0\le p<1$ and forming the density function on $]0,\infty[$

$$\omega(x)=(p,-pq/x;q^2)_\infty d_\sigma(x),$$

where d_σ is given by (2.1). In order to find the moments of ω we use the well-known formula for the q-exponential function

$$(-pq/x;q^2)_\infty=\sum_{k=0}^{\infty}\frac{q^{k^2}p^k}{(q^2;q^2)_k}x^{-k},$$

and integrating the corresponding series term by term, it is easy by (2.3) to get

$$s_n(\omega)=(p;q^2)_n q^{-n^2},\ n\ge 0. \tag{4.3}$$

The family (H_a) from (2.4) can be extended in the following way to yield a family of discrete measures with the moments (4.3). Define

$$H_{a,p}=\frac{1}{c(a,p)}\sum_{k=-\infty}^{\infty}\frac{p^k}{(-q^{2k+1}a/p;q^2)_\infty}\varepsilon_{aq^{2k}},\ a>0, 0<p<1, \tag{4.4}$$

where $c(a,p)$ normalizes $H_{a,p}$ to have mass one. The mass at aq^{2k} is by (3.8) given as

$$\frac{p^k}{c(a)}(-aq/p;q^2)_k(p,-pq/a;q^2)_\infty,$$

which has limit $a^k q^{k^2}/c(a)$ for $p \to 0$. This shows that $\lim_{p\to 0} H_{a,p} = H_a$. The moments of $H_{a,p}$ are indeed given by (4.3). This follows easily from (3.8) like the proof of (3.10). The family $H_{a,p}$ is given in [8] in a slightly different normalization. Note that the image measure of μ_c (cf. (3.7)) under $x \mapsto x^2$ is equal to $H_{c^2,q}$.

Putting $a = q$ we shall see that $H_{q,p}$ is not an extreme point in the set of solutions to the moment problem with moments (4.3). This follows as above calculating that

$$\int x^n \varphi(x)\, dH_{q,p}(x) = 0 \text{ for } n \geq 0,$$

where as before $\varphi(q^{2k+1}) = (-1)^k$.

We shall finally see how the families $H_{a,p}$ and $\tau(c,\alpha)$ are related. If in $H_{a,p}$ we put $p = q^{2(\alpha+1)}$, where $\alpha > -1$, replace q^2 by q, put $a = c(1-q)q^{\alpha+\frac{1}{2}}$ and take the image measure under $x \mapsto xq^{-\alpha-\frac{1}{2}}/(1-q)$, we get the family $\tau(c,\alpha)$ from (4.2).

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