RECENT RESULTS ABOUT MOMENT PROBLEMS[1]

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Abstract. We give a survey of the moment problem in one and several dimensions with focus on the question of determinacy and the relation to polynomial density. In addition the paper contains a new characterization of the measures of finite density index and an example of a rotation invariant determinate measure whos marginal distributions are indeterminate.

1. Introduction

The moment problem was introduced in a seminal paper by T.J. Stieltjes [28]. In this paper he also introduced the Stieltjes integral, which is necessary in the formulation of the problem: Given a sequence (s_n) of nonnegative numbers, find necessary and sufficient conditions in order that

$$s_n = \int_0^\infty t^n d\sigma(t), \quad n = 0, 1, \cdots \tag{1.1}$$

for a nonnegative measure σ on $[0, \infty[$.

He also discovered that the problem can have infinitely many solutions σ. In this case we say that the moment problem is *indeterminate*, otherwise it is called *determinate*. He showed that

$$\int_0^\infty x^{n-\log x} \sin(2\pi \log x)\, dx = 0 \;\text{ for } n = 0, 1, \cdots, \tag{1.2}$$

which implies that all the densities on the half-line

$$d_\lambda(x) = x^{-\log x} \left(1 + \lambda \sin(2\pi \log x)\right), \quad \lambda \in [-1, 1] \tag{1.3}$$

have the same moments.

In the fundamental work of Hamburger [19] the theory of the moment problem was extended from the half-line to the real line. The problem is now to characterize real sequences (s_n) which can be represented as

$$s_n = \int_{-\infty}^{\infty} t^n \, d\mu(t), \quad n = 0, 1, \cdots, \tag{1.4}$$

where μ is considered in the class $\mathcal{M}^*(\mathbb{R})$ of measures for which t^n is integrable for every $n \geq 0$.

Shortly after Hamburger new treatments of the moment problem were given by M. Riesz [25] and Nevanlinna [23] using tools from functional analysis and function theory. An account of the history of the moment problem from Stieltjes to Nevanlinna can be found in a recent paper by Kjeldsen [22]. Nevanlinna gave a parametrization of the convex set of all the solutions to an indeterminate moment problem, and he exhibited a subfamily of solutions, which we now call N-extremal.

Hamburger discovered that there are moment sequences which admit a representation (1.1) for exactly one measure on $[0, \infty[$, but there are different measures $\mu \in \mathcal{M}^*(\mathbb{R})$ for which (1.4) holds. To be able to speak in an unambiguous way about such problems we say that a moment sequence or a measure on the half-line is *determinate in the sense of Stieltjes*, if there is only one measure on the half-line with the given moments, and it will be called *indeterminate in the sense of Stieltjes*, if there are infinitely many measures on the half-line with the given moments. We shall use the short notation *det(S)* and *indet(S)* for the two cases. A moment sequence of the type discovered by Hamburger is indeterminate but det(S). We denote by $\mathcal{M}^*([0, \infty[)$ the set of measures $\sigma \in \mathcal{M}^*(\mathbb{R})$ which are supported by $[0, \infty[$. We refer to [1],[17],[27] for information about moment problems and orthogonal polynomials.

2. The moment problem in euclidean space

We shall consider the set $\mathcal{M}^*(\mathbb{R}^k)$ of nonnegative measures on $\mathbb{R}^k$ with moments of any order, i.e. for which

$$\int |x^n| d\mu(x) < \infty \text{ for } n \in \mathbb{N}^k,$$

where we use the multiindex notation $x^n = x_1^{n_1} \cdots x_k^{n_k}$. The moment sequence of $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ is given by

$$s_n = s_n(\mu) = \int x^n \, d\mu(x), \quad n \in \mathbb{N}^k. \tag{2.1}$$

For $k > 1$ there is no characterization of moment sequences, which is similar to the simple results of Stieltjes and Hamburger, see [3],[7],[18]. We denote by V_μ the set of measures $\nu \in \mathcal{M}^*(\mathbb{R}^k)$ for which $s_n(\mu) = s_n(\nu)$ for all n. Like before we say that μ is determinate if $V_\mu = \{\mu\}$, otherwise it is called indeterminate.

Let us consider the following example. For $\alpha > 0$ we let μ_α have density $\exp(-||x||^\alpha)$ with respect to Lebesgue measure. One can prove that μ_α is indeterminate for $\alpha < 1$ and determinate for $\alpha \geq 1$. Using the formula ($\alpha < 1$)

$$\int_{\mathbb{R}^k} x^n \exp(-||x||^\alpha) \begin{bmatrix} \cos \\ \sin \end{bmatrix} (||x||^\alpha \tan \frac{\alpha\pi}{2})\, dx = 0, \tag{2.2}$$

where cosinus is used when k is odd, and sinus is used when k is even, the indeterminacy of μ_α follows in the same way as (1.2) implies the indeterminacy expressed in (1.3).

We shall review the main results about the following problem:

For which $p \geq 1$ *and* $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ *is it true that the set of polynomials* $\mathbb{C}[x_1, \ldots, x_k]$ *is dense in* $L^p(\mu)$?

For $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ we denote by marginal distributions the k one-dimensional measures $\pi_j(\mu)$, which are the image measures of μ under the projections $\pi_j(x) = x_j, j = 1, \ldots, k$.

1°. If μ has compact support then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for all p. (Weierstrass).

2°. If $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for some $p > 2$, then μ is determinate.

This was proved in [5] for $k = 1$ and in [18] for $k > 1$.

3°. Let $\nu \in V_\mu$. Then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^1(\nu)$ if and only if ν is an extreme point of the convex set V_μ. In particular $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^1(\mu)$ if μ is determinate.

This was proved by Naimark, cf. [1], p. 46.

4°. If $\mathbb{C}[t]$ is dense in L^p with respect to each of the k marginal distributions of $\mu \in \mathcal{M}^*(\mathbb{R}^k)$, then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^r(\mu)$ for $1 \leq r < p$.

This was proved in [24], and was used to establish the following: If the marginal distributions of μ are determinate, then μ is determinate. The paper [24] also contains an example showing that the converse does not hold. An example of a rotation invariant determinate measure with indeterminate marginals will be given below.

5°. For $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ define

$$\lambda_n = \int (x_1^n + \cdots + x_k^n)\, d\mu(x), \quad n \geq 0.$$

If $\sum \lambda_{2n}^{-1/2n} = \infty$, then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for any $p \geq 1$, and μ is determinate.

The infinite series condition is due to Carleman, who (for $k = 1$) proved that the measure is determinate under this condition. A proof of the density result can be found in [6].

6°. Suppose $k = 1$. If μ is determinate, then $\mathbb{C}[t]$ is dense in $L^2(\mu)$. If μ is indeterminate and $\nu \in V_\mu$, then $\mathbb{C}[t]$ is dense in $L^2(\nu)$ if and only if ν is N-extremal.

This was proved by M. Riesz in [26]. It is natural to ask, if the result can be extended to dimension $k > 1$, and this was explicitly posed as the Challifour-Devinatz problem, cf. [20]. The problem was solved in the negative in 1990 by Thill and the author, see [10],[11],[4]. We shall discuss the solution in section 3 below.

7°. Suppose $\sigma \in \mathcal{M}^*([0,\infty[)$ is det(S). Then $\mathbb{C}[t]$ is dense in $L^2(\sigma)$.

This follows from 6° since σ is either determinate, or indeterminate and N-extremal, cf. [10], Corollary 1.2.

8°. Suppose $\mu \in \mathcal{M}^*(\mathbb{R})$, $a \notin \mathbb{R} \setminus \operatorname{supp}(\mu)$. Then μ is determinate if and only if $\mathbb{C}[t]$ is dense in $L^2(\mu + \varepsilon_a)$.

See [10], Corollary 1.5.

3. Rotation invariant moment problems

We denote by $\mathcal{M}^*_{\mathrm{rot}}(\mathbb{R}^k)$ the set of rotation invariant measures on $\mathbb{R}^k$ with moments of every order.

There is a one-to-one correspondence between $\mu \in \mathcal{M}^*_{\mathrm{rot}}(\mathbb{R}^k)$ and $\sigma \in \mathcal{M}^*([0,\infty[)$ given by $\sigma = \psi(\mu)$, where $\psi(x) = \|x\|^2$. This means that

$$\int_0^\infty f(t)d\,\sigma(t) = \int_{\mathbb{R}^k} f(\|x\|^2)d\,\mu(x) \tag{3.1}$$

for functions f on $[0,\infty[$. The inverse mapping is given by $\mu = j(\sigma \otimes \omega)$, where

$$j : [0,\infty[\times\Omega_k \to \mathbb{R}^k, \quad j(t,\xi) = \sqrt{t}\xi\ . \tag{3.2}$$

Here Ω_k denotes the unit sphere in $\mathbb{R}^k$ and ω denotes the normalized surface measure of Ω_k.

Using (3.2) we get

$$\int x^n d\mu(x) = \int_0^\infty t^{|n|/2} d\sigma(t) \int_{\Omega_k} \xi^n d\omega(\xi)\ . \tag{3.3}$$

It is clear that $\int \xi^n d\omega(\xi) = 0$ if at least one n_i is odd, and it is easy to see that

$$\int_{\Omega_k} \xi^{2n} d\omega(\xi) = \frac{\Gamma(n_1+\frac{1}{2})\cdots\Gamma(n_k+\frac{1}{2})}{\Gamma(n_1+\cdots+n_k+\frac{k}{2})}\Gamma(\frac{k}{2})\pi^{-\frac{k}{2}}\ .$$

This shows that the non-trivial moments $s_n(\mu)$ (all n_i are even) are expressed by (3.3) in terms of the moments of the corresponding measure σ on $[0,\infty[$.

The following two theorems about properties of $\mu \in \mathcal{M}^*_{\mathrm{rot}}(\mathbb{R}^k)$ in terms of properties of the corresponding $\sigma \in \mathcal{M}^*([0,\infty[)$ were derived by Thill and the author in [10].

Theorem 3.1. *The measure μ is determinate on $\mathbb{R}^k$ if and only if σ is det(S).*

Theorem 3.2. $1°$. *Assume $k > 1$. Then $\mathbb{C}[x_1,\ldots,x_k]$ is dense in $L^2(\mu)$ if and only if $\mathbb{C}[t]$ is dense in $L^2(t^p d\,\sigma(t))$ for $p = 0,1,\ldots$.*

$2°$. *Assume $k = 1$. Then $\mathbb{C}[x]$ is dense in $L^2(\mu)$ if and only if $\mathbb{C}[t]$ is dense in $L^2(t^p d\,\sigma(t))$ for $p = 0,1$.*

An explanation for the difference in nature between the results in dimension one and higher dimensions can be found in the theory of spherical harmonics. Let $\mathcal{H}_p(k)$ denote the space of homogeneous harmonic polynomials of degree p i $\mathbb{R}^k$. Then $\dim \mathcal{H}_p(k) > 0$ for $k = 1$ if and only if $p = 0,1$, but for $k > 1$ we have $\dim \mathcal{H}_p(k) > 0$ for $p = 0,1,\ldots$.

For $k = 1$ a rotation invariant measure on $\mathbb{R}^k$ is the same as a symmetric measure, and it is easy to prove Theorem 3.2 in this case. In fact, the mapping

$$T : L^2(\sigma) \times L^2(td\,\sigma(t)) \to L^2(\mu)$$

given by $T(f,g)(t) = f(t^2) + tg(t^2)$ is a unitary transformation between Hilbert spaces mapping $\mathbb{C}[t]\times\mathbb{C}[t]$ onto $\mathbb{C}[t]$, and the result follows. In particular, if $(P_{0,n})_n$ and $(P_{1,n})_n$ denote the orthonormal polynomials corresponding to σ and $td\,\sigma(t)$ respectively, then the system $\{P_{0,n}(t^2), tP_{1,n}(t^2), n \geq 0\}$ is the orthonormal polynomials corresponding to μ.

A similar but more complicated method involving tensor products can be applied in dimension $k > 1$, cf. [4]. Here we shall contend ourselves with giving an orthonormal basis for $\mathbb{C}[x_1,\ldots,x_k]$ in $L^2(\mu)$ in terms of the orthonormal polynomial systems $(P_{p,n})_n$ for the measures $t^p d\,\sigma(t), p = 0,1,\ldots$. We choose an orthonormal basis $S_{p,j}$ of spherical harmonics from $\mathcal{H}_p(k)$, $j = 1,\ldots,\dim \mathcal{H}_p(k)$ with respect to the inner product in $L^2(\Omega_k,\omega)$. Then the system of polynomials

$$\{P_{p,n}(\|x\|^2)S_{p,j}(x) \mid j = 1,\ldots,\dim \mathcal{H}_p(k), p = 0,1,\ldots,n = 0,1,\ldots\}$$

is an orthonormal basis for $\mathbb{C}[x_1,\ldots,x_k]$ in $L^2(\mu)$.

In order to construct a measure $\mu \in \mathcal{M}^*_{\mathrm{rot}}(\mathbb{R}^k)$ which is determinate and such that $\mathbb{C}[x_1,\ldots,x_k]$ is not dense in $L^2(\mu)$, we have to find $\sigma \in \mathcal{M}^*([0,\infty[)$ which is det(S) and such that $\mathbb{C}[t]$ is not dense in $L^2(t^p d\sigma(t)$ for p sufficiently big.

For each $\sigma \in \mathcal{M}^*([0,\infty[)$ such that $\mathbb{C}[t]$ is dense in $L^2(\sigma)$, there is a density index $\delta(\sigma)$ defined by

$$\delta(\sigma) = \sup\{p \in \mathbb{N} \mid \mathbb{C}[t] \text{ is dense in } L^2(t^p d\,\sigma(t))\}, \tag{3.4}$$

cf. [11]. It turns out that any number $0, 1, \ldots, \infty$ can be the density index for some σ. If σ is indet(S) then $\delta(\sigma) \le 2$, and if σ is det(S) then $\delta(\sigma) \ge 2$, and any number ≥ 2 is $\delta(\sigma)$ for some σ which is det(S). We return to this question in the next section, where we shall give a complete characterization of the measures of finite density index. Let us just mention that they are discrete, and their mass points are the zeros of certain entire functions of minimal exponential type. We therefore have the following:

Theorem 3.3. *The determinate measures $\mu \in \mathcal{M}^*_{\mathrm{rot}}(\mathbb{R}^k)$ for which $\mathbb{C}[x_1, \ldots, x_k]$ is not dense in $L^2(\mu)$ are given as*

$$\mu = \sum_{n=0}^{\infty} \alpha_n \omega_{\sqrt{r_n}} \, ,$$

where $\alpha_n > 0$ and $0 \le r_0 < r_1 < \cdots \to \infty$ are such that $\sigma = \sum \alpha_n \varepsilon_{r_n}$ is det(S) of finite density index, and ω_r denotes the normalized uniform distribution on the sphere $\|x\| = r$.

In particular, $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^2(\mu)$, if the rotation invariant measure has a density with respect to Lebesgue measure.

Also in the case of non-rotation invariant measures $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ we can get information about denseness of $\mathbb{C}[x_1, \ldots, x_k]$ in various spaces from corresponding results about $\sigma := \psi(\mu)$, the image measure under $\psi(x) = \|x\|^2$.

Theorem 3.4. *Let $\mu \in \mathcal{M}^*(\mathbb{R}^k)$ where $k > 1$, define $\sigma = \psi(\mu)$ and let $r \in [1, \infty[$.*

If $\mathbb{C}[t]$ is dense in $L^r(t^{p\frac{r}{2}} d\sigma(t))$ for every integer $p \ge 0$, then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^r(\mu)$.

If $\mathbb{C}[t]$ is dense in $L^r(\sigma)$ for some $r \in]1, \infty[$, then $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^{r'}(\mu)$ for every $r' \in [1, r[$.

Theorem 3.4 can be applied to prove Theorem 3.1 in the following way:

Assume that σ is det(S). For $\nu \in V_\mu$ we see that $\psi(\nu)$ and σ have the same moments, hence $\psi(\nu) = \sigma$. The polynomials are dense in $L^2(\sigma)$, cf. 6° in section 2, and it follows by Theorem 3.4 that $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^1(\nu)$, so that ν is an extreme point of V_μ by 3°. The measure $\nu \in V_\mu$ being arbitrary, this is only possible if μ is determinate.

It is more elementary to prove that if σ is indet(S), then μ is indeterminate, because if $\tau \in \mathcal{M}^*([0, \infty[)$ has the same moments as σ and is different form σ, then the corresponding rotation invariant measure is different from μ and has the same moments as μ.

As another application of Theorem 3.4 it should be noticed that $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for $1 \le p < 2$ for the measures μ of Theorem 3.3. Furthermore, the orthogonal complement of $\mathbb{C}[x_1, \ldots, x_k]$ in $L^2(\mu)$ is of infinite dimension, [10].

4. Indeterminate Hamburger moment problems on the real line

Let (s_n) be an indeterminate Hamburger moment sequence and let V be the set of positive Borel measures μ on $\mathbb{R}$ satisfying $\int x^n d\mu(x) = s_n\,,\ n \geq 0$. It is well-known that V is a compact convex set in the weak topology from the duality with continuous functions vanishing at infinity. Let (P_n) be the corresponding orthonormal polynomials and let (Q_n) be the polynomials of the second kind given by

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y}\, d\mu(y)\,, \quad \mu \in V\,. \tag{4.1}$$

It is well-known that the series $\sum |P_n(z)|^2$, $\sum |Q_n(z)|^2$ converge uniformly on compact subsets of $\mathbb{C}$, which makes it possible to define four entire functions on $\mathbb{C}^2$

$$\begin{aligned}
\mathcal{A}(u,v) &= (v-u)\sum_{n=0}^{\infty} Q_n(u)Q_n(v)\\
\mathcal{B}(u,v) &= -1 + (v-u)\sum_{n=0}^{\infty} Q_n(u)P_n(v)\\
\mathcal{C}(u,v) &= 1 + (v-u)\sum_{n=0}^{\infty} P_n(u)Q_n(v)\\
\mathcal{D}(u,v) &= (v-u)\sum_{n=0}^{\infty} P_n(u)P_n(v)\,,
\end{aligned} \tag{4.2}$$

where we follow [15]. Note that

$$\mathcal{A}(u,v) = -\mathcal{A}(v,u)\,;\ \mathcal{B}(u,v) = -\mathcal{C}(v,u)\,;\ \mathcal{D}(u,v) = -\mathcal{D}(v,u)\,. \tag{4.3}$$

The functions verify the fundamental identity

$$\begin{vmatrix} \mathcal{A}(u,v) & \mathcal{C}(u,v) \\ \mathcal{B}(u,v) & \mathcal{D}(u,v) \end{vmatrix} \equiv 1\,, \tag{4.4}$$

cf. [15]. If the first variable is put equal to zero we get the functions

$$A(z) = \mathcal{A}(0,z)\,,\ B(z) = \mathcal{B}(0,z)\,,\ C(z) = \mathcal{C}(0,z)\,,\ D(z) = \mathcal{D}(0,z)\,, \tag{4.5}$$

which form the *Nevanlinna matrix*

$$M(z) = \begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix} \tag{4.6}$$

for the indeterminate problem.

The above functions of two variables seem more natural than the functions in the Nevanlinna matrix because there is no reason why 0 should play a special role. On the other hand the functions $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ can be expressed in a simple way using the functions in the Nevanlinna matrix, cf. [15]

$$\begin{aligned}\mathcal{A}(u,v)&=A(v)C(u)-A(u)C(v)\\ \mathcal{B}(u,v)&=B(v)C(u)-A(u)D(v)\\ \mathcal{D}(u,v)&=B(v)D(u)-B(u)D(v)\;.\end{aligned} \tag{4.7}$$

The set V of solutions μ to the indeterminate moment problem was parametrized by Nevanlinna in 1922 using the matrix (4.6), cf. [23]. The parameter space is the one-point compactification of the set $\mathcal{P}$ of Pick functions, which are holomorphic functions in the upper half-plane $\mathbb{H}$ with non-negative imaginary part. The Pick functions are given by the following representation

$$\varphi(z)=az+b+\int\frac{tz+1}{t-z}\,d\sigma(t)\;, \tag{4.8}$$

where $a\ge 0\,,\ b\in\mathbb{R},\ \sigma\in\mathbb{M}_+(\mathbb{R})$. Formula (4.8) gives a holomorphic extension of φ to $\mathbb{C}\setminus\operatorname{supp}(\sigma)$ and in particular to the lower half-plane. Note that $\varphi(\overline{z})=\overline{\varphi(z)}$.

The *Nevanlinna parametrization* is the homeomorphism $\varphi\mapsto\nu_\varphi$ of $\mathcal{P}\cup\{\infty\}$ onto V given by

$$\int\frac{d\nu_\varphi(u)}{u-z}=-\frac{A(z)\varphi(z)-C(z)}{B(z)\varphi(z)-D(z)}\;,\quad z\in\mathbb{C}\setminus\mathbb{R}\;, \tag{4.9}$$

which expresses that the Stieltjes transform of any solution $\nu\in V$ is given by (4.9) for a unique $\varphi\in\mathcal{P}\cup\{\infty\}$.

Riesz [26] characterized the solutions $\nu_t\,,\ t\in\mathbb{R}\cup\{\infty\}$ as those $\mu\in V$ for which the set of polynomials $\mathbb{C}[x]$ is dense in $L^2(\mu)$. These measures are called N-extremal.

By (4.9) the Stieltjes transform of the N-extremal measure ν_t is the meromorphic function

$$-\frac{A(z)t-C(z)}{B(z)t-D(z)}\quad\left(=-\frac{A(z)}{B(z)}\text{ if }t=\infty\right)\;, \tag{4.10}$$

which implies that ν_t is a discrete measure

$$\nu_t=\sum_{\lambda\in\Lambda_t}c_\lambda\varepsilon_\lambda\;, \tag{4.11}$$

where Λ_t is the zero set of the entire function $Bt-D$ (B if $t=\infty$) and

$$c_\lambda=\frac{A(\lambda)t-C(\lambda)}{B'(\lambda)t-D'(\lambda)}\text{ for }\lambda\in\Lambda_t\;. \tag{4.12}$$

The functions $Bt - D$ are real entire functions with only real and simple zeros, and they are of minimal exponential type.

Let $\mathcal{S}$ denote the set of Stieltjes moment sequences. For $(s_n) \in \mathcal{S}$ all zeros of D are non-negative, and since $D(0) = 0$ they can be enumerated $r_0 = 0 < r_1 < r_2 < \cdots$. Therefore ν_0 is a solution to the moment problem in $\mathcal{M}^*([0,\infty[)$. If the problem is det(S), then ν_0 is the only solution in V supported by $[0,\infty[$. If the problem is indet(S), then there exists a constant $\alpha < 0$ such that $\mathrm{supp}(\nu_t) \subseteq [0,\infty[$ for $t \in [\alpha, 0]$, and $\mathrm{supp}(\nu_t) \cap]-\infty, 0[\neq \emptyset$ for $t \in \mathbb{R} \cup \{\infty\} \setminus [\alpha, 0]$, cf. [11]. The density index of the N-extremal solutions has the following values:

Lemma 4.1. *Let $(s_n) \in \mathcal{S}$ be an indeterminate Hamburger moment sequence.*

(i) *If (s_n) is indet(S) then*

$$\delta(\nu_t) = 0 \text{ for } t \in]\alpha, 0[\,, \delta(\nu_\alpha) = 1, \delta(\nu_0) = 2.$$

(ii) *If (s_n) is det(S) then $\delta(\nu_0) = 3$ and $t d\nu_0(t)$ is det(S).*

If one of the mass points of an N-extremal measure is removed, the new measure becomes determinate ([1],[5]), and if further k masses are removed, we get a determinate measure μ of index of determinacy $\mathrm{ind}(\mu) = k$, a concept which has been introduced and studied in [8],[9]. For $\mu \in \mathcal{M}^*(\mathbb{R})$ the index of determinacy of μ with respect to a point $z \in \mathbb{C}$ is defined as

$$\mathrm{ind}_z(\mu) = \sup\{k \in \mathbb{N} \mid |t - z|^{2k}\mu \quad \text{is determinate}\}.$$

It was shown that if $\mathrm{ind}_z(\mu) < \infty$ for at least one point z then

$$\mathrm{ind}_z(\mu) = \begin{cases} k, & \text{for } z \notin \mathrm{supp}(\mu) \\ k+1, & \text{for } z \in \mathrm{supp}(\mu), \end{cases}$$

and we define $\mathrm{ind}(\mu) = k$ in this case. Otherwise $\mathrm{ind}_z(\mu) = \infty$ for all $z \in \mathbb{C}$ in the case of which we put $\mathrm{ind}(\mu) = \infty$.

In the opposite direction, if we add k new mass points to an N-extremal measure, it becomes k-canonical, meaning that it is indeterminate and the closure of the polynomials in L^2 has codimension k. This characterization of k-canonical measures was proved in [14].

Let us consider the following sets of measures

$$\mathcal{I} = \{\nu_0 - \nu_0(\{0\})\varepsilon_0 \mid (s_n) \in \mathcal{S} \text{ is indet(S)}\}, \tag{4.13}$$

$$\mathcal{D} = \{\nu_0 - \nu_0(\{0\})\varepsilon_0 \mid (s_n) \in \mathcal{S} \text{ is indeterminate and det(S)}\}, \tag{4.14}$$

where in both cases ν_0 is the N-extremal measure for the sequence (s_n) corresponding to the parameter value 0 in the Nevanlinna parametrization. The measures in $\mathcal{I}$ and $\mathcal{D}$ are determinate by the remark above, and they have no mass at zero.

The following result holds:

Theorem 4.2. *(i) The class $\mathcal{I}$ consists of the measures $\sigma \in \mathcal{M}^*([0,\infty[)$ which are determinate with $\delta(\sigma) = 2$.*

(ii) The class $\mathcal{D}$ consists of the measures $\sigma \in \mathcal{M}^*([0,\infty[)$ which are determinate with $\delta(\sigma) = 3$.

(iii) The measures $\sigma \in \mathcal{M}^*([0,\infty[)$ with $\delta(\sigma) = 2 + j, j \geq 0$ are given as

$$\sigma = \alpha\varepsilon_0 + t^{-j} d\tau(t),\ \alpha \geq 0, \tau \in \mathcal{I}.$$

(iv) $\sigma \in \mathcal{M}^*([0,\infty[)$ is determinate and of finite density index if and only if it can be derived from a measure $\tau \in \mathcal{I} \cup \mathcal{D}$ by removing finitely many mass points.

Proof. The properties (i)-(iii) are included in [10],[11].

If $\tau \in \mathcal{I} \cup \mathcal{D}$ and $\Lambda \subset \mathrm{supp}(\tau)$ has k points, then

$$\sigma = \tau - \sum_{a\in\Lambda} \tau(\{a\})\varepsilon_a$$

has the property that $\mathrm{ind}_0(\sigma) = k$, i.e. $t^{2j} d\sigma(t)$ is determinate for $j = 0, 1, \ldots, k$ but $t^{2k+2} d\sigma(t)$ is indeterminate, cf. [8]. Therefore $t^{2k+2} d\sigma(t)$ is N-extremal, and $\delta(t^{2k+2} d\sigma(t)) \in \{0, 1\}$ by Lemma 4.1 since there is no mass at the origin. This shows that $\delta(\sigma) \leq 2k + 3$.

Conversely, if σ is determinate with $\delta(\sigma) = j$, where $(2 \leq) j < \infty$, then if $k = [j/2]$ we have that $t^{2k} d\sigma(t)$ is indeterminate, hence $\mathrm{ind}(\sigma) < k$. If we add masses at $\mathrm{ind}(\sigma) + 1$ different points from $[0,\infty[\setminus \mathrm{supp}(\sigma)$ we get an N-extremal measure. By a result of [5] we can always arrange it such that 0 belongs to the support of this N-extremal measure, and the result follows.

Example 4.3. We shall give an example of a determinate rotation invariant measure $\tilde{\mu}$ in $\mathbb{R}^k, k > 1$ for which the marginal distributions are indeterminate and for which $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for $1 \leq p < 2$, but not dense for $p = 2$.

Let $\nu_0 \in \mathcal{M}^*([0,\infty[)$ be indet(S) and N-extremal with $\alpha_0 := \nu_0(\{0\}) > 0$. Then $\sigma = \nu_0 - \alpha_0\varepsilon_0 \in \mathcal{I}$ is determinate and in particular det(S). We know that

$$\sigma = \sum_{n=1}^{\infty} \alpha_n \varepsilon_{r_n},$$

where $0 < r_1 < r_2 < \cdots$. The corresponding rotation invariant measure on $\mathbb{R}^k, k >$ 1

$$\mu = \sum_{n=1}^{\infty} \alpha_n \omega_{\sqrt{r_n}}$$

is determinate, but $\tilde{\mu} = \mu + \alpha_0\varepsilon_0$ is indeterminate. Furthermore $\mathbb{C}[x_1, \ldots, x_k]$ is dense in $L^p(\mu)$ for $1 \leq p < 2$, and not dense for $p = 2$. This follows from the theorems 3.2 and 3.4 since $\delta(\sigma) = 2$. The marginal distributions of $\tilde{\mu}$ are all equal to $\tilde{\tau} = \tau + \alpha_0\varepsilon_0$ where τ is a measure on $\mathbb{R}$ with the following symmetric density with respect to Lebesgue measure

$$\frac{\Gamma(\frac{k}{2})}{\sqrt{\pi}\Gamma(\frac{k-1}{2})}\sum_{n=1}^{\infty} r_n^{2-k}(r_n^2 - t^2)^{\frac{k-3}{2}} 1_{]-r_n, r_n[}(t).$$

By the result of Petersen [24], cf. the remark after 4°, we see that $\tilde{\tau}$ is indeterminate, but since it is non-discrete, it is not N-extremal. By removing the mass at the origin we get τ which is again indeterminate, for otherwise $\mathbb{C}[t]$ would be dense in $L^2(\tilde{\tau})$ by the result of 8°.

5. Examples of N-extremal measures

As far as we know the first concrete measure recognized as being N-extremal is an orthogonality measure for the q-polynomials of Al-Salam and Carlitz [2], namely

$$\beta^{(a)} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(aq;q)_n (q;q)_n} \varepsilon_{q^{-n}} \ , \tag{5.1}$$

where $0 < q < 1, q < a < q^{-1}$ and

$$(z;q)_n = \prod_{k=1}^{n} (1 - zq^{k-1}) \ , \quad z \in \mathbb{C}, \ n = 0, 1, \cdots, \infty \ . \tag{5.2}$$

The observation that $\beta^{(a)}$ is N-extremal (when $a \leq 1$) is due to Chihara [16]. For a different proof of this result see [12].

There is another explicit example of an N-extremal measure derived in a study by Valent [29] of birth and death processes with quartic rates. The D-function is particularly simple, viz.

$$D(z) = \frac{4}{\pi}\sqrt{z}\sin\left(\frac{z^{\frac{1}{4}}}{2}K_0\right)\sinh\left(\frac{z^{\frac{1}{4}}}{2}K_0\right) = \frac{K_0^2}{\pi} z \prod_{n=1}^{\infty}\left(1 - \frac{z}{x_n}\right) \ ,$$

where $x_n = (2\pi n/K_0)^4$. Here K_0 is a constant from the theory of elliptic functions in the lemniscatic case corresponding to the modulus $k = 1/\sqrt{2}$.

The N-extremal measure supported by the zeros of D is

$$\nu_0 = \frac{\pi}{K_0^2}\left(\delta_0 + 4\sum_{n=1}^{\infty}\frac{2n\pi}{\sinh(2n\pi)}\varepsilon_{x_n}\right) \ . \tag{5.3.}$$

For a complete calculation of the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in the two cases see [13]. For still another example of N-extremal measures see [21].

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