On infinitely divisible solutions to indeterminate moment problems

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Abstract

For a convolution semigroup of measures with moments of any order we prove that the n'th moment is a polynomial of degree at most n in the time parameter. Special focus is on the case where the measures are indeterminate, in particular the log-normal and the q-Laguerre case.

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1 Introduction and results

This paper treats only the second part of the lecture given at the "International Workshop on Special Functions, Asymptotics, Harmonic Analysis and Mathematical Physics", Hong Kong June 21-25, 1999.[1]

In the first part of the lecture we considered a fixed indeterminate moment sequence (s_n) and the set V of solutions μ to the moment equations

$$\int x^n \, d\mu(x) = s_n, \quad n = 0, 1, \ldots . \tag{1.1}$$

The n'th moment of a solution μ will be denoted $s_n(\mu)$.

Let $(\mu_t)_{t \in I}$ be a one-parameter family of solutions in V. The basic assumptions are that I is an interval and the mapping $t \mapsto \mu_t$ of I into V is continuous, when V is equipped with the weak topology. It is well-known that V is a compact convex set in this topology, so for any probability measure τ on I the vector integral $\kappa = \int \mu_t \, d\tau(t)$ belongs to V. In the lecture we showed how this simple result can be used to calculate solutions to the following moment problems: log-normal, generalized

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Stieltjes-Wigert or q-Laguerre and discrete q-Hermite II. The results have appeared in [7]. See also [6] for some calculation of moments and [1] for the general theory.

In the second part of the lecture we considered probabilities with moments of any order and which are infinitely divisible. A probability μ with these properties can be imbedded in a convolution semigroup $(\mu_t)_{t>0}$ such that $\mu_1 = \mu$. For the precise conditions assumed for a convolution semigroup see section 2 below. We shall prove that all the measures in the convolution semigroup have moments of any order, and the n'th moment $s_n(\mu_t)$ of μ_t is a polynomial in t of degree at most n without constant term if $n \geq 1$. The coefficients of this polynomial can be expressed in terms of the values at zero of the derivatives of the negative definite function of the convolution semigroup. These derivatives are closely related to the moments of the Lévy measure for the convolution semigroup. For convolution semigroups supported by the half-line and not equal to the trivial semigroup $(\varepsilon_{bt})_{t>0}$ of uniform motion the n'th moment is a polynomial of degree n with positive coefficients. Here and in the following ε_a denotes the unit mass at the point a.

The set V of solutions to an indeterminate moment problem can contain no, one or infinitely many infinitely divisible distributions, other possibilities do not happen. The occurence of the two last cases depends on the determinacy/indeterminacy of the Lévy measure ([16]).

The log-normal distribution with parameter $\sigma > 0$ has density on $]0, \infty[$ given by

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right). \tag{1.2}$$

It is indeterminate and infinitely divisible. As far as the author knows the convolution roots of this distribution are not explicitely known. We show that they are also indeterminate probabilities as a consequence of the fact that there are more than one infinitely divisible distribution in the set V. This is because the densities

$$\omega_c(x) = \frac{x^{c-1}}{M_c(q, -q^{\frac{1}{2}-c}x, -q^{\frac{1}{2}+c}/x; q)_\infty} \tag{1.3}$$

are infinitely divisible, as we shall see below, and they have the same moments

$$s_n = q^{-\frac{1}{2}n^2}$$

as (1.2) when $q = \exp(-\sigma^2)$. The latter follows from the Askey-Roy beta-integral, [4], [3]. See also [7]. For the notation above concerning q-special functions we refer to [15], [17]. The parameter c is real, but since $\omega_{c+1}(x) = \omega_c(x)$ we may restrict c to the interval $[0, 1[$. The constant M_c normalizing ω_c to a probability can be expressed using the Γ-function and Jackson's Γ_q-function and is given in [7]. The density (1.3) for $c = \frac{1}{2}$ was given by Askey in [2].

Another important class of indeterminate and infinitely divisible distributions is connected to the q-Laguerre moment problem with moments

$$s_n(\alpha; q) = q^{-\alpha n - \binom{n+1}{2}} (q^{\alpha+1}; q)_n, \tag{1.4}$$

where $0 < q < 1$, $\alpha > -1$.

We shall see below that the density

$$k_\alpha \frac{x^\alpha}{(-x;q)_\infty}, \tag{1.5}$$

which has the moments (1.4), is infinitely divisible. The measure (1.5) corresponds to $c = \alpha + 1$ in the one-parameter family of densities

$$\frac{q^{c(\alpha+\frac{1}{2})}}{M_c} x^{c-1} \frac{(q^{\alpha+1}, -q/x;q)_\infty}{(q, -q^{\alpha+1-c}x, -q^{c-\alpha}/x;q)_\infty} \tag{1.6}$$

all having the q-Laguerre moments, cf. [7]. This family is also periodic in c with period 1. We have only been able to prove that the special case (1.5) is infinitely divisible.

It is well-known that the log-normal case is a limiting case of q-Laguerre for α tending to infinity, cf. [2].

At the end of the paper we give evidence to the following conjecture about convolution semigroups $(\mu_t)_{t>0}$ with moments of any order: *Either all the measures* μ_t *are determinate or all the measures are indeterminate.*

2 Convolution semigroups of measures with moments of any order

In 1977 O. Thorin, cf. [19], established that the log-normal distribution (1.2) is infinitely divisible. This is probably the first known explicit example of an indeterminate and infinitely divisible probability measure. The existence of such measures was noticed in Heyde [16]. We recall that a probability measure μ on $\mathbb{R}$ is called *infinitely divisible*, if μ has n'th probability roots under convolution for any n, i.e. for any $n \in \mathbb{N}$ there shall exist a probability σ such that $\sigma^{*n} = \mu$. It is well-known that such an n'th root is uniquely determined and that μ has arbitrary positive convolution roots. This means that there exists a (uniquely determined) family $(\mu_t)_{t>0}$ of probabilities with $\mu = \mu_1$ such that $(\mu_t)_{t>0}$ is a *convolution semigroup*, i.e. has the properties

$$\mu_t * \mu_s = \mu_{t+s}, \tag{2.1}$$

$$t \mapsto \mu_t \text{ is weakly continuous for } t > 0, \tag{2.2}$$

$$\lim_{t \to 0} \mu_t = \varepsilon_0 \text{ weakly.} \tag{2.3}$$

We recall the Lévy-Khinchine formula for the Fourier transform of a convolution semigroup, cf. Lukacs [18],

$$\hat{\mu}_t(x) = \int_{-\infty}^{\infty} e^{-ixy} d\mu_t(y) = e^{-t\psi(x)}, \quad t > 0, x \in \mathbb{R}, \tag{2.4}$$

$$\psi(x) = \alpha x^2 + i\beta x + \int_{-\infty}^{\infty} (1 - e^{-ixy} - \frac{ixy}{1+y^2}) d\lambda(y), \quad x \in \mathbb{R}, \tag{2.5}$$

where $\alpha \geq 0, \beta \in \mathbb{R}$, and λ is a non-negative measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{-\infty}^{\infty} \frac{y^2}{1+y^2} d\lambda(y) < \infty. \tag{2.6}$$

Condition (2.6) implies that $\lambda(\mathbb{R} \setminus]-1,1[) < \infty$. The function ψ and the measure λ are called the *negative definite function* and the *Lévy measure* for the convolution semigroup.

Conversely, given $\alpha \geq 0, \beta \in \mathbb{R}$ and a non-negative measure λ on $\mathbb{R} \setminus \{0\}$ satisfying (2.6), there exists a uniquely determined convolution semigroup $(\mu_t)_{t>0}$ such that (2.4) and (2.5) hold. Any non-negative measure λ on $\mathbb{R} \setminus \{0\}$ satisfying (2.6) will be called a Lévy measure.

Concerning the existence of moments of all orders for the measures in a convolution semigroup we prove:

Theorem 2.1 *For a convolution semigroup $(\mu_t)_{t>0}$ as above the following conditions are equivalent:*

(i) μ_{t_0} *has moments of all orders for some* $t_0 > 0$,

(ii) μ_t *has moments of all orders for all* $t > 0$,

(iii) $\psi \in C^\infty(\mathbb{R})$,

(iv) $\int_{-\infty}^{\infty} y^{2n}\, d\lambda(y) < \infty$ *for* $n = 1, 2, \ldots$.

In case (i)-(iv) hold, then $s_n(\mu_t)$ *is a polynomial of degree* $\leq n$ *in* t *without constant term (for* $n \geq 1$*), the coefficients of which are uniquely determined by the sequence* $(\psi^{(n)}(0))$.

Proof: We shall use the fact that a probability has moments of every order if and only if its Fourier transform is a C^∞-function, cf. e.g. [18]. From this the equivalence of (i)-(iii) is straight-forward.

Suppose next that (i)-(iii) hold. We establish by induction that $s_n(\mu_t)$ for $n \geq 1$ is a polynomial of degree $\leq n$ in t without constant term. Differentiating (2.4) at $x = 0$ we clearly get

$$s_1(\mu_t) = -it\psi'(0),$$

so the result holds for $n=1$. Differentiating (2.4) $n+1$ times we find

$$\begin{aligned}
(-i)^{n+1}s_{n+1}(\mu_t) &= \frac{d^{n+1}}{dx^{n+1}}\{e^{-t\psi(x)}\}_{x=0} = \frac{d^n}{dx^n}\{-t\psi'(x)e^{-t\psi(x)}\}_{x=0} \\
&= -t\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}\{e^{-t\psi(x)}\}_{x=0}\psi^{(n-k+1)}(0) \\
&= -t\sum_{k=0}^{n}\binom{n}{k}(-i)^k s_k(\mu_t)\psi^{(n-k+1)}(0),
\end{aligned}$$

which is a polynomial as required, if we assume the result up to order n.

Since $t^{-1}\mu_t|(\mathbb{R}\setminus\{0\})$ converges vaguely on $\mathbb{R}\setminus\{0\}$ to the Lévy measure λ for $t\to 0$, cf. [9], we get for $n\geq 1$

$$\int_{-\infty}^{\infty} y^{2n}\,d\lambda(y) \leq \liminf_{t\to 0} t^{-1}\int_{-\infty}^{\infty} y^{2n}\,d\mu_t(y) = \frac{d}{dt}\{s_{2n}(\mu_t)\}_{t=0} < \infty,$$

which shows that (iv) holds.

Suppose finally that (iv) holds. The assumption $\int y^2\,d\lambda(y)<\infty$ implies that ψ is C^2 with derivatives

$$\psi'(x) = \int_{-\infty}^{\infty} iy(e^{-ixy}-\frac{1}{1+y^2})\,d\lambda(y)+2\alpha x+i\beta$$

$$\psi''(x) = \int_{-\infty}^{\infty} y^2e^{-ixy}\,d\lambda(y)+2\alpha,$$

and since the finite measure $y^2\lambda$ has moments of all even orders and hence of all orders, we see that ψ is C^∞ and

$$\psi'(0) = i\int_{-\infty}^{\infty}\frac{y^3}{1+y^2}\,d\lambda(y)+i\beta,$$

$$\psi''(0) = \int_{-\infty}^{\infty} y^2\,d\lambda(y)+2\alpha,$$

$$\psi^{(n)}(0) = (-i)^{n-2}\int_{-\infty}^{\infty} y^n\,d\lambda(y),\quad n\geq 3.$$

□

Remark 2.2 (a) Defining

$$\sigma_n = -i^{n+1}\psi^{(n+1)}(0),\quad n\geq 0, \tag{2.7}$$

we see from above that σ_n is real and

$$\sigma_n = s_{n-1}(2\alpha\varepsilon_0+y^2\lambda),\;\; n\geq 1,\quad \sigma_0 = \int_{-\infty}^{\infty}\frac{y^3}{1+y^2}\,d\lambda(y)+\beta. \tag{2.8}$$

The first moments are

$$s_1(\mu_t) = \sigma_0 t$$
$$s_2(\mu_t) = \sigma_1 t + \sigma_0^2 t^2$$
$$s_3(\mu_t) = \sigma_2 t + 3\sigma_0\sigma_1 t^2 + \sigma_0^3 t^3.$$

Writing

$$s_n(\mu_t) = \sum_{k=0}^{n} c_{n,k} t^k, \quad n \geq 0, \tag{2.9}$$

we have $c_{0,0} = 1, c_{n,0} = 0$ for $n \geq 1$ and it is easy to see that

$$c_{n,n} = \sigma_0^n, \quad c_{n,1} = \sigma_{n-1}. \tag{2.10}$$

In general the coefficients $c_{n,k}$ satisfy the recurrence equation

$$c_{n+1,l+1} = \sum_{k=l}^{n} c_{k,l} \binom{n}{k} \sigma_{n-k}, \quad n, l \geq 0. \tag{2.11}$$

(b) The assumption (iv) for $n = 1$ is of course stronger than (2.6). On the other hand, if (2.6) holds, condition (iv) can be replaced by

$$(iv') \qquad \int_{|y|\geq 1} y^{2n}\, d\lambda(y) < \infty, \quad n \geq 1.$$

Example 2.3 The degree of $s_n(\mu_t)$ can be strictly less than n as the first of the following examples show.

(a) The Gaussian semigroup

$$\mu_t = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}) dx,$$

$$s_{2n}(\mu_t) = \frac{(2n)!}{n!} t^n, \quad s_{2n+1}(\mu_t) = 0.$$

(b) The drift semigroup $\mu_t = \varepsilon_{\beta t}, \quad s_n(\mu_t) = \beta^n t^n, \quad \beta \in \mathbb{R}$.

(c) The Γ-semigroup

$$\mu_t = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} 1_{]0,\infty[}(x) dx,$$

$$s_n(\mu_t) = \frac{\Gamma(n+t)}{\Gamma(t)} = (t)_n.$$

(d) The Poisson semigroup

$$\mu_t = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \varepsilon_k,$$

$$s_n(\mu_t) = e^{-t}\sum_{k=0}^{\infty}\frac{k^n t^k}{k!} = e^{-t}L^n(e^t),$$

where L is the differential operator $Lf(x) = xf'(x)$.

For symmetric semigroups like (a) it is clear that $s_{2n+1}(\mu_t) = 0$, and it can be proved by induction that $s_{2n}(\mu_t)$ is of degree n unless μ_t is reduced to ε_0.

For convolution semigroups $(\mu_t)_{t>0}$ supported by the half-line $[0,\infty[$ we have also more information as indicated by the examples (b)-(d).

Proposition 2.4 *Let $(\mu_t)_{t>0}$ be a convolution semigroup of measures supported by $[0,\infty[$ and with moments of all orders. Then $s_n(\mu_t)$ is a polynomial (2.9) of degree n with $c_{n,k} > 0$ for $k = 1,\dots,n$, unless $\mu_t = \varepsilon_{bt}$ for some $b \geq 0$.*

Proof: The result can be deduced from (2.8) and (2.11), but it is perhaps more instructive to make use of the Laplace transformation. The Laplace transform of an arbitrary convolution semigroup $(\mu_t)_{t>0}$ on $[0,\infty[$ has the form

$$\mathcal{L}\mu_t(s) = \int_0^{\infty} e^{-sy}\, d\mu_t(y) = e^{-tf(s)}, \quad t > 0, \tag{2.12}$$

where the *Bernstein function* f associated with $(\mu_t)_{t>0}$ is given as

$$f(s) = bs + \int_0^{\infty}(1 - e^{-sy})\, d\lambda(y). \tag{2.13}$$

Here $b \geq 0$ and the Lévy measure λ is concentrated on $]0,\infty[$ and satisfies the integrability condition

$$\int_0^{\infty}\frac{y}{1+y}\, d\lambda(y) < \infty,$$

cf. [9]. Note that the negative definite function is $\psi(x) = f(ix)$. In the present case, where all the measures have moments of any order, the Lévy measure satisfies

$$\int_0^{\infty} y^n\, d\lambda(y) < \infty, \quad n \geq 1.$$

Therefore $f \in C^{\infty}([0,\infty[)$ and

$$\sigma_n = (-1)^n f^{(n+1)}(0) = s_n(b\varepsilon_0 + y d\lambda(y)) < \infty$$

for $n \geq 0$. It follows that $\sigma_n > 0$ for all $n \geq 0$, unless $\lambda = 0$ in which case $\mu_t = \varepsilon_{bt}$. The assertion now follows from the recursion formula (2.11). $\square$

To discuss the indeterminacy of the measures in a convolution semigroup with moments of any order, we shall say that a Lévy measure λ on $\mathbb{R}\setminus\{0\}$ is *indeterminate*, if there exists a Lévy measure $\tilde{\lambda}$ on $\mathbb{R}\setminus\{0\}$ with $\lambda \neq \tilde{\lambda}$ and such that

$$\int_{-\infty}^{\infty} y^n\, d\lambda(y) = \int_{-\infty}^{\infty} y^n\, d\tilde{\lambda}(y) \text{ for } n \geq 2. \tag{2.14}$$

Theorem 2.5 (Heyde [16]). *Let $(\mu_t)_{t>0}$ be a convolution semigroup with moments of any order and indeterminate Lévy measure. Then the semigroup is indeterminate in the sense that all the measures μ_t are indeterminate.*

Proof: Let ψ and λ denote the negative definite function and the Lévy measure for $(\mu_t)_{t>0}$ and let $\tilde{\lambda} \neq \lambda$ satisfy (2.14). Let $\tilde{\psi}$ be defined by (2.5) with λ replaced by $\tilde{\lambda}$ and β replaced by $\tilde{\beta}$, the latter defined so that $\psi'(0) = \tilde{\psi}'(0)$. Then $\psi \neq \tilde{\psi}$ and $\psi^{(n)}(0) = \tilde{\psi}^{(n)}(0)$ for $n \geq 1$. Letting $(\tilde{\mu}_t)_{t>0}$ be the convolution semigroup associated with $\tilde{\psi}$, it follows from (2.8) and (2.11) that $s_n(\mu_t) = s_n(\tilde{\mu}_t)$ but $\mu_t \neq \tilde{\mu}_t$ for $t > 0$. $\square$

Remark 2.6 (a) The theorem shows that there exist as many indeterminate convolution semigroups as indeterminate measures. In fact, if ν is any indeterminate measure with moments of all orders and with no mass at zero, then $\lambda = y^{-2}\nu$ is an indeterminate Lévy measure.

(b) The converse of Theorem 2.5 is not true, although it is claimed in [16, Theorem 2]. Lemma 2.2 in [5] can be used to give a counterexample. In fact, the Lemma asserts the existence of a determinate probability λ such that $y^2\lambda$ is determinate and λ^{*3} is indeterminate. Since the measure λ is constructed to have no mass at 0, it is a determinate Lévy measure. The convolution semigroup

$$\mu_t = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda^{*n}, \quad t > 0$$

is indeterminate since each μ_t majorizes a multiple of λ^{*3}.

(c) Let $(\mu_t)_{t>0}$ be an indeterminate convolution semigroup with a determinate Lévy measure, and let V_t be the set of measures with the same moments as μ_t. Then for each t the set V_t contains only one infinitely divisible measure.

(d) In [16] Heyde gives an example of an indeterminate measure μ such that V contains no infinitely divisible measures.

Theorem 2.7 *The densities (1.2) and (1.3) with log-normal moments $s_n = q^{-n^2/2}$ are all infinitely divisible. In particular all positive convolution roots of these measures are infinitely divisible.*

Proof: The infinite divisibility of (1.2) was proved first by Thorin in [19]. He introduced the subclass of infinitely divisible distributions called generalized gamma convolutions and proved that the log-normal distribution belongs to this subclass $\mathcal{T}$. Bondesson found in [10] a subclass $\mathcal{B} \subset \mathcal{T}$ given as all the probability densities on $]0, \infty[$ which are pointwise limits of densities of the form

$$Cx^{\beta-1} \prod_{i=1}^{N} (1 + c_i x)^{-\gamma_i}, \quad \beta, c_i, \gamma_i > 0. \tag{2.15}$$

In later papers [11],[12] Bondesson found an extremely elegant characterization of $\mathcal{B}$ as the probability densities f on $]0,\infty[$ such that, for each u, $f(uv)f(u/v)$ is completely monotone as a function of $w = v + v^{-1}$. Using this result Bondesson gave a simple proof that (1.2) belongs to the class $\mathcal{B}$, see [11],[12]. The density (1.3) also belongs to the class $\mathcal{B}$ and is therefore a generalized gamma convolution and in particular infinitely divisible. In fact the probability density

$$C(n,c)x^{c-1}\{(-q^{\frac{1}{2}-c}x, -q^{\frac{1}{2}+c}/x; q)_n\}^{-1} = C(n,c)x^{c+n-1}\{(-q^{\frac{1}{2}-c}x, -q^{\frac{1}{2}-c-n}x; q)_n\}^{-1}$$

has the right form and converges pointwise to $\omega_c(x)$.

The set V of measures with the log-normal moments therefore contains infinitely many infinitely divisible measures, and the assertion follows. $\square$

By the same method as above we get:

Theorem 2.8 *The q-Laguerre density (1.5) belongs to the class $\mathcal{B}$ and is in particular infinitely divisible.*

Since the same method does not apply to the densities (1.6), except when $c = \alpha + 1$, we do not know if the set V of solutions with q-Laguerre moments (1.4) contains one or infinitely may infinitely divisible distributions.

3 A conjecture

If μ and ν are measures with moments of any order then the convolution $\mu * \nu$ has also moments of any order and

$$s_n(\mu * \nu) = \sum_{k=0}^{n} \binom{n}{k} s_k(\mu) s_{n-k}(\nu). \tag{3.1}$$

It was proved by Devinatz, cf.[14],[5], that if $\nu \neq 0$ and μ is indeterminate, then $\mu * \nu$ is indeterminate. On the other hand it is possible that μ is determinate but $\mu * \mu$ is indeterminate, cf. [5],[13].

Let $(\mu_t)_{t>0}$ be a convolution semigroup with moments of any order. If it is known that μ_t is indeterminate it follows from the result of Devinatz that $\mu_t * \mu_s$ is indeterminate for any $s > 0$.

However, we do not need the full result of Devinatz to prove this, since $\hat{\mu}_s$ does not vanish because of (2.4). We can simplify the argument as follows: If σ has the same moments as μ_t, then $\sigma * \mu_s$ has the same moments as $\mu_t * \mu_s$ by (3.1), and if $\sigma * \mu_s = \mu_t * \mu_s$, we get by Fourier transformation and cancelation that $\sigma = \mu_t$.

The question is if it can happen that μ_t is determinate for $0 < t < t_0$ and indeterminate for $t_0 < t$ for some number $0 < t_0 < \infty$. By Theorem 2.5 this would require the Lévy measure to be determinate. As noticed in [5] the Carleman condition cannot be satisfied for the measures $\mu_t, t < t_0$.

Determinacy of a measure μ depends on the behaviour of the smallest eigenvalue λ_n of the Hankel matrix

$$(s_{i+j}(\mu))_{0\le i,j\le n}. \tag{3.2}$$

It was proved in [8] that μ is determinate if and only if $\lim \lambda_n = 0$. In the special case of $\mu = \mu_t$ the Hankel matrix consists of polynomials in t of degree at most $2n$, and since the smallest eigenvalue is the minimum of the Hankel form on the unit sphere in $\mathbb{C}^{n+1}$, it is easily seen that the smallest eigenvalue $\lambda_n(t)$ is a strictly positive continuous function.

The existence of a number t_0 as above would require that $\lambda_n(t)$ converges to 0 for $t < t_0$ and to a strictly positive limit for $t > t_0$. Because of the polynomial behaviour of the moments we do not think that this is possible and hence we formulate:

Conjecture 3.1 *For a convolution semigroup $(\mu_t)_{t>0}$ of measures with moments of any order there are only the possibilities:*

1. *All the measures μ_t are determinate.*
2. *All the measures μ_t are indeterminate.*

References

[1] N. I. Akhiezer, The Classical Moment Problem and Some Related Questions in Analysis. English translation, Oliver and Boyd, Edinburgh, 1965.

[2] R. Askey, *Limits of some q-Laguerre polynomials,* J. Approx. Theory **46** (1986), 213-216.

[3] R. Askey, *Orthogonal polynomials and theta functions.* In: Theta functions Bowdoin 1987, ed. L. Ehrenpreis, R. C. Gunning. Proc. Symp. Pure Math. **49**-part 2 (1989), 299-321.

[4] R. Askey and R. Roy, *More q-beta integrals,* Rocky Mountain J. Math. **16** (1986), 365-372.

[5] C. Berg, *On the preservation of determinacy under convolution*, Proc. Amer. Math. Soc. **93** (1985), 351–357.

[6] C. Berg, *On some indeterminate moment problems for measures on a geometric progression*, J. Comput. Appl. Math. **99** (1998), 67–75.

[7] C. Berg, *From discrete to absolutely continuous solutions of indeterminate moment problems*, Arab J. Math. Sc. **4**, No. 2 (1998), 1–18.

[8] C. Berg, Y. Chen, M.E.H. Ismail, *Small eigenvalues of large Hankel matrices: The indeterminate case*, to appear in Math. Scand.

[9] C. Berg, G. Forst, *Potential theory on locally compact abelian groups*, Springer, Berlin-Heidelberg-New York, 1975.

[10] L. Bondesson, *A general result on infinite divisibility*, Ann. Probab. **7** (1979), 965–979.

[11] L. Bondesson, *Generalized gamma convolutions and complete monotonicity*, Probab. Th. Rel. Fields **85** (1990), 181–194.

[12] L. Bondesson, *Generalized gamma convolutions and related classes of distributions and densities.* Lecture Notes in Statistics **76**, Springer, New York, 1992.

[13] H. Buchwalter, G. Cassier, *Mesures canoniques dans le problème classique des moments*, Ann. Inst. Fourier (Grenoble) **34** (1984), 45–52.

[14] A. Devinatz, *On a theorem of Lévy-Raikov*, Ann. Math. Statist. **30** (1959), 583–586.

[15] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[16] C. C. Heyde, *Some remarks on the moment problem (II)*, Quart. J. Math. Oxford Ser. (2) **14** (1963), 97–105.

[17] R. Koekoek and R.F. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report 94-05, TU-Delft, 1994.

[18] E. Lukacs, *Characteristic functions, Second Edition*, Griffin, London 1970.

[19] O. Thorin, *On the infinite divisibility of the lognormal distribution*, Scand. Actuarial. J. (1977), 121–148.

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