On a generalized Gamma convolution related to the q-calculus

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Abstract

We discuss a probability distribution I_q depending on a parameter $0 < q < 1$ and determined by its moments $n!/(q;q)_n$. The treatment is purely analytical. The distribution has been discussed recently by Bertoin, Biane and Yor in connection with a study of exponential functionals of Lévy processes.

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1 Introduction

In [13] Bertoin et al. studied the distribution I_q of the exponential functional

$$\mathcal{I}_q = \int_0^\infty q^{N_t}\, dt, \tag{1}$$

where $0 < q < 1$ is fixed and $(N_t, t \geq 0)$ is a standard Poisson process. They found the density $i_q(x), x > 0$ and its Laplace and Mellin transforms. They also showed that a simple construction from I_q leads to the density

$$\lambda_q(x) = \frac{1}{\log(1/q)(q, -x, -q/x; q)_\infty}, \tag{2}$$

found by Askey, cf. [2], and having log-normal moments. The notation in (2) is the standard notation from [21], see below.

The distribution I_q has also appeared in recent work of Cowan and Chiu [19], Dumas et al. [20] and Pakes [24].

The proofs in [13] rely on earlier work on exponential functionals which use quite involved notions from the theory of stochastic processes, see [17],[18].

The purpose of this note is to give a self-contained analytic treatment of the distribution I_q and its properties.

In Section 2 we define a convolution semigroup $(I_{q,t})_{t>0}$ of probabilities supported by $[0,\infty[$, and it is given in terms of the corresponding Bernstein function $f(s)=\log(-s;q)_\infty$ with Lévy measure ν on $]0,\infty[$ having the density

$$\frac{d\nu}{dx}=\frac{1}{x}\sum_{n=0}^{\infty}\exp(-xq^{-n}). \tag{3}$$

The function $1/\log(-s;q)_\infty$ is a Stieltjes transform of a positive measure which is given explicitly, and this permits us to determine the potential kernel of $(I_{q,t})_{t>0}$.

The measure $I_q := I_{q,1}$ is a *generalized Gamma convolution* in the sense of Thorin, cf. [27], [28]. The moment sequence of I_q is shown to be $n!/(q;q)_n$, and the n'th moment of $I_{q,t}$ is a polynomial of degree n in t. We give a recursion formula for the coefficients of these polynomials. We establish that I_q has the density

$$i_q(x)=\sum_{n=0}^{\infty}\exp(-xq^{-n})\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n(q;q)_\infty}.$$

A treatment of the theory of generalized Gamma convolutions can be found in Bondesson's monograph [16]. The recent paper [15] contains several examples of generalized Gamma convolutions which are also distributions of exponential functionals of Lévy processes.

We shall use the notation and terminology from the theory of basic hypergeometric functions for which we refer the reader to the monograph by Gasper and Rahman [21]. We recall the q-shifted factorials

$$(z;q)_n=\prod_{k=0}^{n-1}(1-zq^k),\, z\in\mathbb{C}, 0<q<1, n=1,2,\ldots,\infty$$

and $(z;q)_0=1$. Note that $(z;q)_\infty$ is an entire function of z.

For finitely many complex numbers $z_1,z_2,\ldots,z_p$ we use the abbreviation

$$(z_1,z_2,\ldots,z_p;q)_n=(z_1;q)_n(z_2;q)_n\ldots(z_p;q)_n.$$

The q-shifted factorial is defined for arbitrary complex index λ by

$$(z;q)_\lambda=\frac{(z;q)_\infty}{(zq^\lambda;q)_\infty},$$

and this is related to Jackson's function Γ_q defined by

$$\Gamma_q(z)=\frac{(q;q)_{z-1}}{(1-q)^{z-1}}=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}. \tag{4}$$

In Section 3 we introduce the entire function $h(z) = \Gamma(z)(q^z;q)_\infty$ and use it to express the Mellin transform of I_q. We finally show that the density λ_q given in (2) can be written as the product convolution of I_q and another related distribution, see Theorem 3.2 below. The Mellin transform of the density λ_q can be evaluated as a special case of the Askey-Roy beta-integral given in [4] and in particular we have, see also [3]:

$$\int_0^\infty \frac{t^c}{(-t,-q/t;q)_\infty}\frac{dt}{t} = (q;q)_\infty \frac{\Gamma(c)\Gamma(1-c)}{\Gamma_q(c)\Gamma_q(1-c)}(1-q), \quad c \in \mathbb{C}\setminus\mathbb{Z}. \tag{5}$$

The value of (5) is an entire function of c and equals $h(c)h(1-c)/(q;q)_\infty$.

The following formulas about the q-exponential functions, cf. [21], are important in the following:

$$e_q(z) = \sum_{n=0}^\infty \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad |z| < 1, \tag{6}$$

$$E_q(z) = \sum_{n=0}^\infty \frac{q^{n(n-1)/2} z^n}{(q;q)_n} = (-z;q)_\infty, \quad z \in \mathbb{C}. \tag{7}$$

2 The analytic method

We recall that a function $\varphi :]0,\infty[\mapsto [0,\infty[$ is called *completely monotonic*, if it is C^∞ and $(-1)^k\varphi^{(k)}(s) \geq 0$ for $s > 0, k = 0, 1, \ldots$. By the Theorem of Bernstein completely monotonic functions have the form

$$\varphi(s) = \int_0^\infty e^{-sx}\, d\alpha(x), \tag{8}$$

where α a non-negative measure on $[0,\infty[$. Clearly $\varphi(0+) = \alpha([0,\infty[)$. The equation (8) expresses that φ is the Laplace transform of the measure α.

To establish that a probability η on $[0,\infty[$ is infinitely divisible, one shall prove that its Laplace transform can be written

$$\int_0^\infty e^{-sx}\, d\eta(x) = \exp(-f(s)), \quad s \geq 0,$$

where the non-negative function f has a completely monotonic derivative. If η is infinitely divisible, there exists a convolution semigroup $(\eta_t)_{t>0}$ of probabilities on $[0,\infty[$ such that $\eta_1 = \eta$ and it is uniquely determined by

$$\int_0^\infty e^{-sx}\, d\eta_t(x) = e^{-tf(s)}, \quad s > 0,$$

cf. [11],[12]. The function f is called the *Laplace exponent* or *Bernstein function* of the semigroup. It has the integral representation

$$f(s) = as + \int_0^\infty (1 - e^{-sx})\, d\nu(x), \tag{9}$$

where $a \geq 0$ and the Lévy measure ν on $]0,\infty[$ satisfies the integrability condition $\int x/(1+x)\, d\nu(x) < \infty$. If f is not identically zero the convolution semigroup is transient with potential kernel $\kappa = \int_0^\infty \eta_t\, dt$, and the Laplace transform of κ is $1/f$ since

$$\int_0^\infty e^{-sx}\, d\kappa(x) = \int_0^\infty (\int_0^\infty e^{-sx}\, d\eta_t(x))dt = \int_0^\infty e^{-tf(s)}\, dt = \frac{1}{f(s)}. \tag{10}$$

The generalized Gamma convolutions η are characterized among the infinitely divisible distributions by the following property of the corresponding Bernstein function f, namely by f' being a Stieltjes transform, i.e. of the form

$$f'(s) = a + \int_0^\infty \frac{d\mu(x)}{s+x}, \quad s > 0,$$

where $a \geq 0$ and μ is a non-negative measure on $[0,\infty[$. The relation between μ and ν is that

$$\frac{d\nu}{dx} = \frac{1}{x}\int_0^\infty e^{-xy}\, d\mu(y).$$

This result was used in [6] to simplify the proof of a theorem of Thorin ([27]), stating that the Pareto distribution is a generalized Gamma convolution.

Theorem 2.1 *Let $0 < q < 1$ be fixed. The function*

$$f(s) = \log(-s;q)_\infty = \sum_{n=0}^\infty \log(1+sq^n), \quad s \geq 0 \tag{11}$$

is a Bernstein function. The corresponding convolution semigroup $((I_{q,t})_{t>0})$ consists of generalized Gamma convolutions and we have

$$\int_0^\infty e^{-sx}\, dI_{q,t}(x) = e^{-tf(s)} = \frac{1}{(-s;q)_\infty^t}, \quad s > 0. \tag{12}$$

The potential kernel $\kappa_q = \int_0^\infty I_{q,t}\, dt$ has the following completely monotonic density

$$k_q(x) = 1 - q + \int_1^\infty e^{-xy}\varphi(y)\, dy, \tag{13}$$

where φ is the continuous function

$$\varphi(x) = \begin{cases} n\left(\log^2|(x;q)_\infty| + n^2\pi^2\right)^{-1} & \text{if } q^{-(n-1)} < x < q^{-n},\ n = 1,2,\ldots \\ 0 & \text{if } x = q^{-n},\ n = 0,1,\ldots. \end{cases} \tag{14}$$

Proof: The function f defined by (11) has the derivative

$$f'(s) = \sum_{n=0}^{\infty} \frac{1}{s+q^{-n}},$$

showing that f' is a Stieltjes transform with $a=0$, and μ is the discrete measure with mass 1 in each of the points $q^{-n}, n \geq 0$. In particular f is a Bernstein function with $a=0$ and Lévy measure given by (3).

Since

$$\frac{\log(1+s)}{s} = \int_1^{\infty} \frac{1}{x+s}\frac{dx}{x}$$

we get ($[x]$ denoting the integer part of x)

$$\frac{f(s)}{s} = \int_1^{\infty} \frac{[\log x/\log(1/q)]+1}{(x+s)x}\,dx$$

showing that $f(s)/s$ is a Stieltjes transform. It follows by the Reuter-Itô Theorem, cf.[22],[25],[5], that $1/f(s)$ is a Stieltjes transform. Since f is an increasing function mapping $]-1,\infty[$ onto the real line with $f(0)=0$ and $f'(0)=1/(1-q)$ we get

$$\frac{1}{f(s)} = \frac{1-q}{s} + \int_1^{\infty} \frac{d\mu(x)}{x+s},$$

where

$$d\mu(x) = \lim_{y\to 0^+} \frac{-1}{\pi}\,\mathrm{Im}\{\frac{1}{f(-x+iy)}\}\,dx$$

in the vague topology.

For $x \in \left]q^{-(n-1)}, q^{-n}\right[$, $n=1,2,\ldots$ we find

$$\begin{aligned}\lim_{y\to 0^+} \frac{1}{f(-x+iy)} &= \left(\sum_{k=0}^{n-1} \log|1-q^k x| + in\pi + \sum_{k=n}^{\infty} \log(1-q^k x)\right)^{-1} \\ &= \left(\log|(x;q)_\infty| + in\pi\right)^{-1}.\end{aligned}$$

These expressions define in fact a continuous function on $[1,\infty[$, vanishing at the points $q^{-n}, n\geq 0$, so the measure μ has the density φ given by (14). Using that the Stieltjes transformation is the second iteration of the Laplace transformation, the assertion about the potential kernel κ_q follows. □

Denoting by $\mathcal{E}_a, a>0$ the exponential distribution with density $a\exp(-ax)$ on the positive half-line, we have

$$\int_0^{\infty} e^{-sx}\,d\mathcal{E}_a(x) = e^{-\log(1+s/a)}, \quad s \geq 0,$$

so we can write $I_q := I_{q,1}$ as the infinite convolution

$$I_q = *_{n=0}^{\infty}\, \mathcal{E}_{q^{-n}}.$$

If we let $\Gamma_{a,t}$ denote the Gamma distribution with density

$$x \mapsto \frac{a^t}{\Gamma(t)} x^{t-1} e^{-ax}, \quad x > 0,$$

then we similarly have

$$I_{q,t} = *_{n=0}^{\infty} \Gamma_{q^{-n},t}.$$

Specializing (12) to $t = 1$ we have

$$\int_0^\infty e^{-sx} \, dI_q(x) = \frac{1}{(-s;q)_\infty}, \quad s > 0, \tag{15}$$

and since the right-hand side of (15) is meromorphic in $\mathbb{C}$ with poles at $s = -q^{-n}, n \geq 0$ and in particular holomorphic for $|s| < 1$, we know that I_q has moments of any order with

$$s_n(I_q) = (-1)^n D^n \{\frac{1}{(-s;q)_\infty}\}_{s=0}, \quad n = 0, 1, \ldots ,$$

cf. [23, p.136]. Here and in the following we denote by $s_n(\mu)$ the n'th moment of the measure μ. However by (6) we have

$$\frac{1}{(-s;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-s)^n}{(q;q)_n}, \tag{16}$$

hence

$$s_n(I_q) = \frac{n!}{(q;q)_n}. \tag{17}$$

Since I_q has an analytic characteristic function, the corresponding Hamburger moment problem is determinate. By Stirling's formula we have

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_{2n}(I_q)}} = \infty,$$

so also Carleman's criterion shows the determinacy, cf. [1]. By [7, Cor. 3.3] follows that $I_{q,t}$ is determinate for all $t > 0$ and by [9] the n'th moment $s_n(I_{q,t})$ is a polynomial of degree n in t given by

$$s_n(I_{q,t}) = \sum_{k=1}^{n} c_{n,k} t^k, \quad n \geq 1, \tag{18}$$

where the coefficients $c_{n,k}$ satisfy the recurrence equation

$$c_{n+1,l+1} = \sum_{k=l}^{n} c_{k,l} \binom{n}{k} \sigma_{n-k}.$$

Here $\sigma_n = (-1)^n f^{(n+1)}(0)$, where f is given by (11), cf. [9, Prop. 2.4], so σ_n is easily calculated to be

$$\sigma_n = \frac{n!}{1-q^{n+1}}, \quad n \geq 0.$$

It follows also by [9] that

$$c_{n,n} = \sigma_0^n = (1-q)^{-n}, \quad c_{n,1} = \sigma_{n-1} = (n-1)!/(1-q^n).$$

Defining $d_{n,k} = (1-q)^k c_{n,k}$ we have

$$s_n(I_{q,t}) = \sum_{k=1}^{n} d_{n,k} \left(\frac{t}{1-q}\right)^k, \quad n \geq 1, \tag{19}$$

and

$$d_{n+1,l+1} = n! \sum_{k=l}^{n} \frac{d_{k,l}}{k!} \left(\sum_{j=0}^{n-k} q^j\right)^{-1}, \; l = 0, 1, \ldots, n. \tag{20}$$

In particular

$$d_{n,n} = 1, \quad d_{n,n-1} = \frac{\binom{n}{2}}{1+q}, \quad d_{n,1} = (n-1)! \left(\sum_{j=0}^{n-1} q^j\right)^{-1}.$$

We give the first coefficients

$$\begin{aligned} d_{1,1} &= 1 \\ d_{2,2} &= 1, \quad d_{2,1} = \frac{1}{1+q} \\ d_{3,3} &= 1, \quad d_{3,2} = \frac{3}{1+q}, \quad d_{3,1} = \frac{2}{1+q+q^2}. \end{aligned}$$

It follows by induction using (20) that $d_{n,k}$ as a function of q has a finite limit for $q \to 1^-$.

The image measures $\mu_t = \tau_{1-q}(I_{q,t})$ under $\tau_{1-q}(x) = x(1-q)$ form a convolution semigroup $(\mu_t)_{t>0}$ with

$$\int_0^\infty e^{-sx}\, d\mu_t(x) = \frac{1}{(-s(1-q);q)_\infty^{t}}, \quad s > 0,$$

and

$$s_n(\mu_t) = \sum_{k=1}^{n} d_{n,k}(1-q)^{n-k} t^k.$$

It follows that $s_n(\mu_t) \to t^n$ for $q \to 1^-$, so $\lim_{q\to 1^-} \mu_t = \delta_t$ weakly by the method of moments. This is also in accordance with

$$\lim_{q\to 1^-} \frac{1}{(-s(1-q);q)_\infty^t} = e^{-st},$$

because the q-exponential function E_q given in (7) converges to the exponential function in the following sense

$$\lim_{q\to 1^-} E_q(z(1-q)) = \exp(z),$$

cf. [21].

Remark 2.2 Consider a non-zero Bernstein function f. In [17], [18] it was proved by probabilistic methods that the sequence

$$s_n = \frac{n!}{f(1)\cdot\ldots\cdot f(n)}$$

is a determinate Stieltjes moment sequence, meaning that it is the moment sequence of a unique probability on $[0,\infty[$. The special case $f(s) = 1 - q^s$ gives the moment sequence (17). In [10] the above result of [17], [18] is obtained as a special case of the following result:

Let (a_n) be a non-vanishing Hausdorff moment sequence. Then (s_n) defined by $s_0 = 1$ and $s_n = 1/(a_1\cdot\ldots\cdot a_n)$ for $n \ge 1$ is a normalized Stieltjes moment sequence.

In order to find an expression for I_q we consider the discrete signed measure

$$\mu_q = \sum_{k=0}^\infty \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k (q;q)_\infty} \delta_{q^k} \tag{21}$$

with moments

$$s_n(\mu_q) = \sum_{k=0}^\infty \frac{(-1)^k q^{nk} q^{k(k+1)/2}}{(q;q)_k (q;q)_\infty} = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_n},$$

where we have used (7). In particular, the signed measure μ_q has mass 1.

For measures ν, τ on $]0,\infty[$ we define the product convolution $\nu \diamond \tau$ as the image of the product measure $\nu \otimes \tau$ under $x, y \mapsto xy$. The product convolution is the ordinary convolution of measures on the locally compact abelian group $]0,\infty[$ with multiplication as group operation. In particular we have

$$\int f \, d\nu \diamond \tau = \iint f(xy) \, d\nu(x) \, d\tau(y).$$

From this equation we get the moment equation

$$s_n(\nu \diamond \tau) = s_n(\nu)s_n(\tau),$$

hence $s_n(\mu_q \diamond \mathcal{E}_1) = n!/(q;q)_n$, which shows that $\mu_q \diamond \mathcal{E}_1$ has the same moments as I_q. Since the first measure is not known to be non-negative, we cannot conclude right-away that the two measures are equal, although I_q is Stieltjes determinate. We shall show that $\mu_q \diamond \mathcal{E}_1$ has a density $i_q(x)$, which is non-negative. Since $\delta_a \diamond \mathcal{E}_1 = \mathcal{E}_{1/a}$ for $a > 0$, it is easy to see that

$$i_q(x) = \sum_{n=0}^{\infty} \exp(-xq^{-n}) \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n (q;q)_\infty}, \tag{22}$$

but it is not obvious that $i_q(x) \geq 0$.

Proposition 2.3 *The function $i_q(x)$ given by* (22) *is non-negative for $x > 0$. Therefore $I_q = \mu_q \diamond \mathcal{E}_1 = i_q(x) 1_{]0,\infty[}(x)\, dx$.*

Proof: The Laplace transform of the function i_q is

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(s+q^{-n})(q;q)_n (q;q)_\infty}, \tag{23}$$

which is the partial fraction expansion of $1/(-s;q)_\infty$, since the residue of this function at the pole $s = q^{-n}$ is

$$\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n (q;q)_\infty}.$$

We claim that

$$\frac{1}{(-s;q)_\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(s+q^{-n})(q;q)_n (q;q)_\infty}, \quad s \neq q^{-n}, n = 0, 1, \ldots\,, \tag{24}$$

which shows that I_q and i_q have the same Laplace transform, so i_q is the density of I_q and hence non-negative.

To see the equation (24) we note that the left-hand side minus the right-hand side of the equation is an entire function ϕ, and by (16) we get

$$\frac{\phi^{(n)}(0)}{n!} = \frac{(-1)^n}{(q;q)_n} - \frac{(-1)^n}{(q;q)_\infty} \sum_{k=0}^{\infty} q^{(n+1)k} \frac{(-1)^k q^{k(k-1)/2}}{(q;q)_k},$$

but by (7) the sum above equals $(q^{n+1};q)_\infty$, and we get $\phi^{(n)}(0)/n! = 0$, which shows that ϕ is identically zero.

We call the attention to the fact that the identity (24) was also used in the work [20] of Dumas et al., but it is in fact a special case of Jackson's transformations, see (III 4) in [21] with $b = 1, a = -s, z = q$. $\square$

Let R_q denote the following positive discrete measure

$$R_q = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \delta_{q^k} \tag{25}$$

with moments

$$s_n(R_q) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{(q^{n+1})^k}{(q;q)_k} = (q;q)_n, \tag{26}$$

by (6). We claim that μ_q given by (21) and R_q are the inverse of each other under the product convolution, i.e.

$$\delta_1 = \mu_q \diamond R_q. \tag{27}$$

This amounts to proving that

$$\sum_{k=0}^{n} \frac{(-1)^k q^{k(k+1)/2} q^{n-k}}{(q;q)_k (q;q)_{n-k}} = \delta_{n0},\ n \geq 0,$$

but this follows by Cauchy multiplication of the power series (6), (7). Combining Proposition 2.3 with (27) we get:

Corollary 2.4 *The following factorization hold*

$$\mathcal{E}_1 = I_q \diamond R_q,$$

which corresponds to the factorization of the moments of $\mathcal{E}_1$ *as*

$$n! = \frac{n!}{(q;q)_n} \cdot (q;q)_n.$$

Remark 2.5 The factorization of Corollary 2.4 is a special case of a general factorization in [14]:

$$\mathcal{E}_1 = I_f \diamond R_f, \quad n! = \frac{n!}{f(1) \cdot \ldots \cdot f(n)} \cdot (f(1) \cdot \ldots \cdot f(n)),$$

where f is a non-zero Bernstein function (9), and I_f, R_f are determined by their moments

$$s_n(I_f) = \frac{n!}{f(1) \cdot \ldots \cdot f(n)}, \quad s_n(R_f) = f(1) \cdot \ldots \cdot f(n).$$

3 The entire function $h(z) := \Gamma(z)(q^z;q)_\infty$

Since the Gamma function has simple poles at $z = -n$, $n = 0, 1, \ldots,$ where $(q^z;q)_\infty$ has simple zeros, it is clear that the product $h(z) := \Gamma(z)(q^z;q)_\infty$ is entire. We have

$$h(0) = \lim_{z\to 0}\Gamma(z)(q^z;q)_\infty = \lim_{z\to 0}\Gamma(z+1)(q^{z+1};q)_\infty\frac{1-q^z}{z} = (q;q)_\infty\log(1/q),$$

and from this it is easy to see that

$$h(-n) = \frac{(q;q)_n}{n!}q^{-n(n+1)/2}(q;q)_\infty\log(1/q). \tag{28}$$

Proposition 3.1 *For $z \in \mathbb{C}$ we have*

$$\int_0^\infty x^z\,dI_q(x) = \frac{h(z+1)}{(q;q)_\infty}. \tag{29}$$

Proof: For $\Re z > -1$ the following calculation holds by (22) and (7):

$$\int_0^\infty x^z\,dI_q(x) = \sum_{n=0}^\infty\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n(q;q)_\infty}\int_0^\infty x^z e^{-xq^{-n}}\,dx = (q;q)_\infty^{-1}\Gamma(z+1)(q^{z+1};q)_\infty.$$

Since the right-hand side is entire and I_q is a positive measure, we get by a classical result (going back to Landau for Dirichlet series, see [29, p.58]) that the integral in (29) must converge for all $z \in \mathbb{C}$, and therefore the equation holds for all $z \in \mathbb{C}$. □

When discussing measures on $]0,\infty[$ it is useful to consider this set as a locally compact group under multiplication. The Haar measure is then $dm(x) = (1/x)\,dx$, and it is useful to consider the density of a measure with respect to the Haar measure m. The Mellin transformation is the Fourier tranformation of the locally compact abelian group $(]0,\infty[,\cdot)$, and when the dual group is realized as the additive group $\mathbb{R}$, the Mellin transformation of a finite measure μ on $]0,\infty[$ is defined as

$$\mathcal{M}(\mu)(\xi) = \int_0^\infty x^{-i\xi}\,d\mu(x), \quad \xi\in\mathbb{R}.$$

We get from (29) that

$$\mathcal{M}(I_q)(\xi) = \frac{h(1-i\xi)}{(q;q)_\infty}. \tag{30}$$

From Proposition 3.1 it follows that I_q has negative moments of any order, and from (28) we get in particular that

$$J_q := \frac{1}{x\log(1/q)}dI_q(x) \tag{31}$$

is a probability.

The image of J_q under the reflection $x \mapsto 1/x$ is denoted $\check{J}_q$.

Theorem 3.2 *The product convolution $L_q := I_q \diamond \check{J}_q$ has the density (2)*

$$\lambda_q(x) = \frac{1}{\log(1/q)(q, -x, -q/x; q)_\infty}$$

with respect to Lebesgue measure on the half-line.

Proof: For $z \in \mathbb{C}$ we clearly have

$$\int_0^\infty x^z \, dL_q(x) = \int_0^\infty x^z \, dI_q(x) \int_0^\infty x^{-z} \, dJ_q(x),$$

and by (29) we get

$$\int_0^\infty x^z \, dL_q(x) = \frac{h(z+1)h(-z)}{\log(1/q)(q;q)_\infty^2}.$$

By (5) it follows that for $z \in \mathbb{C}$

$$\int_0^\infty x^z \, dL_q(x) = \int_0^\infty x^z \lambda_q(x) \, dx,$$

so $L_q = \lambda_q(x) \, dx$. $\square$

Remark 3.3 In [13] the authors prove Theorem 3.2 by showing that

$$\int_0^\infty x^z \, dL_q(x) = \frac{1}{(q;q)_\infty^3 \log(1/q)} \int_0^\infty x^z \left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{x+q^n} \right) dx$$

for $-1 < \Re z < 0$, and then they prove the partial fraction expansion of the meromorphic density $\lambda_q(x)$

$$\lambda_q(x) = \frac{1}{(q;q)_\infty^3 \log(1/q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{x+q^n}.$$

Remark 3.4 The moments of $\check{J}_q$ are given by

$$s_n(\check{J}_q) = \frac{(q;q)_n}{n!} q^{-(n+1)n/2}, \tag{32}$$

so

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_n(\check{J}_q)}} < \infty.$$

Therefore Carleman's criterion gives no information about determinacy of $\check{J}_q$. By the Krein criterion, cf. [8],[26], we can conclude that

$$\int_0^\infty \frac{\log i_q(x)}{\sqrt{x}(1+x)}\,dx = -\infty,$$

because I_q is determinate. The substitution $x = 1/y$ in this integral leads to

$$\int_0^\infty \frac{\log i_q(1/y)}{\sqrt{y}(1+y)}\,dy = -\infty, \tag{33}$$

but since

$$\check{J}_q = \frac{i_q(1/y)\,dy}{y\log(1/q)},$$

we see that (33) gives no information about indeterminacy of $\check{J}_q$. We do not know if $\check{J}_q$ is determinate or indeterminate, and as a factor of an indeterminate distribution L_q none of these possibilities can be excluded.

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