On powers of Stieltjes moment sequences, II

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Abstract

We consider the set of Stieltjes moment sequences, for which every positive power is again a Stieltjes moment sequence, we and prove an integral representation of the logarithm of the moment sequence in analogy to the Lévy-Khintchine representation. We use the result to construct product convolution semigroups with moments of all orders and to calculate their Mellin transforms. As an application we construct a positive generating function for the orthonormal Hermite polynomials.

Key words: moment sequence, infinite divisibility, convolution semigroup, q-series, Hermite polynomials

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1 Introduction

The present paper treats the same circle of ideas as [2], but here we focus on other aspects of the theory. This means that there is no overlap with the main results of [2], but the latter contains more introductory material on previous results.

In his fundamental memoir [15] Stieltjes characterized sequences of the form

$$s_n = \int_0^\infty x^n \, d\mu(x), \; n = 0, 1, \ldots, \tag{1}$$

where μ is a non-negative measure on $[0, \infty[$, by certain quadratic forms being non-negative. These sequences are now called Stieltjes moment sequences. They are called normalized if $s_0 = 1$. A Stieltjes moment sequence is called *S-determinate*, if there is only one measure μ on $[0, \infty[$ such that (1) holds;

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otherwise it is called *S-indeterminate*. It is to be noticed that in the S-indeterminate case there are also solutions μ to (1), which are not supported by $[0,\infty[$, i.e. solutions to the corresponding Hamburger moment problem.

From (1) follows that a Stieltjes moment sequence is either non-vanishing (i.e. $s_n > 0$ for all n) or of the form $s_n = c\delta_{0n}$ with $c \geq 0$, where (δ_{0n}) is the sequence $(1, 0, 0, \ldots)$. The latter corresponds to the Dirac measure δ_0 with mass 1 concentrated at 0.

It is a classical result that the integral powers $(s_n^p), p = 2, 3, \ldots$ of a Stieltjes moment sequence are again Stieltjes moment sequences, but non-integral powers (s_n^c) are not necessarily Stieltjes moment sequences. In this paper we study a certain class of Stieltjes moment sequences which is stable under the formation of powers, so in this respect it is a continuation of [2]. We will however go further by characterizing the full set $\mathcal{I}$ of normalized Stieltjes moment sequences (s_n) with the property that (s_n^c) is a Stieltjes moment sequence for each $c > 0$. The result is given in Theorem 2.4, which contains 3 equivalent conditions. One of them is a kind of Lévy-Khintchine representation of $\log s_n$, and this result is very useful for deciding if a given sequence belongs to $\mathcal{I}$. We study several examples of sequences from $\mathcal{I}$:

$$n!,\ (a)_n,\ (a)_n/(b)_n, 0 < a < b,\ (a;q)_n/(b;q)_n, 0 < q < 1, 0 \leq b < a < 1.$$

Concerning S-determinacy of (s_n^c) when $(s_n) \in \mathcal{I}$, we shall see that the following three cases can occur:

- (s_n^c) is S-determinate for all $c > 0$.
- There exists $c_0, 0 < c_0 < \infty$ such that (s_n^c) is S-determinate for $0 < c < c_0$ and S-indeterminate for $c > c_0$.
- (s_n^c) is S-indeterminate for all $c > 0$.

The moment sequences $(s_n) \in \mathcal{I}$ are closely related to the study of product convolution semigroups (ρ_c) of probabilities with moments of all orders, i.e. convolution semigroups on $]0,\infty[$ considered as a multiplicative group. If (s_n) is the moment sequence of ρ_1, then the following holds

$$s_n^c = \int_0^\infty x^n \, d\rho_c(x),\ c > 0,\ n = 0, 1, \ldots.$$

We discuss these questions and find the Mellin transform of the measures ρ_c.

In Section 5 we use the Stieltjes moment sequence $(\sqrt{n!})$ to prove non-negativity of a generating function for the orthonormal Hermite polynomials. The probability measure with the moment sequence $(\sqrt{n!})$ does not seem to be explicitly known.

During the preparation of this paper Richard Askey kindly drew my attention to the Ph.d.-thesis [19] of Shu-gwei Tyan. It contains a chapter on infinitely divisible moment sequences, and $\mathcal{I}$ is the set of infinitely divisible Stieltjes moment sequences in the sense of Tyan. Theorem 4.2 in [19] is a representation of $\log s_n$ similar to condition (ii) in Theorem 2.4. As far as we know these results of [19] have not been published elsewhere, so we discuss his results in Section 4.

2 Main results

The present paper is a continuation of [2] and is motivated by work of Durán and the author, see [4], which provides a unification of recent work of Bertoin, Carmona, Petit and Yor, see [7],[8], [9]. They associate certain Stieltjes moment sequences with any positive Lévy process.

To formulate these results we need the concept of a Bernstein function.

Let $(\eta_t)_{t>0}$ be a convolution semigroup of sub-probabilities on $[0,\infty[$ with *Laplace exponent* or *Bernstein function* f given by

$$\int_0^\infty e^{-sx}\,d\eta_t(x) = e^{-tf(s)},\quad s>0,$$

cf. [5],[6]. We recall that f has the integral representation

$$f(s) = a + bs + \int_0^\infty (1-e^{-sx})\,d\nu(x), \tag{2}$$

where $a,b \geq 0$ and the Lévy measure ν on $]0,\infty[$ satisfies the integrability condition $\int x/(1+x)\,d\nu(x) < \infty$. Note that $\eta_t([0,\infty[) = \exp(-at)$, so that $(\eta_t)_{t>0}$ consists of probabilities if and only if $a=0$.

In the following we shall exclude the Bernstein function identically equal to zero, which corresponds to the convolution semigroup $\eta_t = \delta_0, t>0$.

Let $\mathcal{B}$ denote the set of Bernstein functions which are not identically zero. For $f \in \mathcal{B}$ we note that f'/f is completely monotonic as product of the completely monotonic functions f' and $1/f$. By Bernstein's Theorem, cf. [20], there exists a non-negative measure κ on $[0,\infty[$ such that

$$\frac{f'(s)}{f(s)} = \int_0^\infty e^{-sx}\,d\kappa(x). \tag{3}$$

It is easy to see that $\kappa(\{0\}) = 0$ using (2) and $f'(s) \geq \kappa(\{0\})f(s)$.

In [7] Bertoin and Yor proved that for any $f \in \mathcal{B}$ with $f(0) = 0$ the sequence (s_n) defined by

$$s_0 = 1,\ s_n = f(1)f(2) \cdot \ldots \cdot f(n), n \geq 1$$

is a Stieltjes moment sequence. The following extension holds:

Theorem 2.1 *Let $\alpha \geq 0, \beta > 0$ and let $f \in \mathcal{B}$ be such that $f(\alpha) > 0$. Then the sequence (s_n) defined by*

$$s_0 = 1, s_n = f(\alpha)f(\alpha + \beta) \cdot \ldots \cdot f(\alpha + (n-1)\beta), \quad n \geq 1$$

belongs to $\mathcal{I}$. Furthermore (s_n^c) is S-determinate for $c \leq 2$.

In most applications of the theorem we put $\alpha = \beta = 1$ or $\alpha = 0, \beta = 1$, the latter provided $f(0) > 0$. (Of course the result is trivially true if $f(\alpha) = 0$.) The case $\alpha = \beta = 1$ is Corollary 1.9 of [4], and the case $\alpha = 0, \beta = 1$ follows from Remark 1.2 in [2].

The moment sequence (s_n^c) of Theorem 2.1 can be S-indeterminate for $c > 2$. This is shown in [2] for the moment sequences

$$s_n^c = (n!)^c \quad \text{and} \quad s_n^c = (n+1)^{c(n+1)} \tag{4}$$

derived from the Bernstein functions $f(s) = s$ and $f(s) = s(1 + 1/s)^{s+1}$. For the Bernstein function $f(s) = s/(s+1)$ the moment sequence $s_n^c = (n+1)^{-c}$ is a Hausdorff moment sequence since

$$\frac{1}{(n+1)^c} = \frac{1}{\Gamma(c)} \int_0^1 x^n (\log(1/x))^{c-1}\, dx,$$

and in particular it is S-determinate for all $c > 0$.

The sequence $(a)_n := a(a+1) \cdot \ldots \cdot (a+n-1), a > 0$ belongs to $\mathcal{I}$ and is a one parameter extension of $n!$. For $0 < a < b$ also $(a)_n/(b)_n$ belongs to $\mathcal{I}$. These examples are studied in Section 6. Finally, in Section 7 we study a q-extension $(a;q)_n/(b;q)_n \in \mathcal{I}$ for $0 < q < 1, 0 \leq b < a < 1$. In Section 8 we give some complementary examples.

Any normalized Stieltjes moment sequence (s_n) has the form $s_n = (1-\varepsilon)\delta_{0n} + \varepsilon t_n$, where $\varepsilon \in [0,1]$ and (t_n) is a normalized Stieltjes moment sequence satisfying $t_n > 0$.

Although the moment sequence (s_n^c) of Theorem 2.1 can be S-indeterminate for $c > 2$, there is a “canonical” solution ρ_c to the moment problem defined by “infinite divisibility”.

The notion of an infinitely divisible probability measure has been studied for arbitrary locally compact groups, cf. [12].

We need the product convolution $\mu \diamond \nu$ of two measures μ and ν on $[0,\infty[$: It is defined as the image of the product measure $\mu \otimes \nu$ under the product mapping $(s,t) \mapsto st$. For measures concentrated on $]0,\infty[$ it is the convolution with respect to the multiplicative group structure on the interval. It is clear that the n'th moment of the product convolution is the product of the n'th moments of the factors.

In accordance with the general definition we say that a probability ρ on $]0,\infty[$ is infinitely divisible on the multiplicative group of positive real numbers, if it has p'th product convolution roots for any natural number p, i.e. if there exists a probability $\tau(p)$ on $]0,\infty[$ such that $(\tau(p))^{\diamond p} = \rho$. This condition implies the existence of a unique family $(\rho_c)_{c>0}$ of probabilities on $]0,\infty[$ such that $\rho_c \diamond \rho_d = \rho_{c+d}$, $\rho_1 = \rho$ and $c \mapsto \rho_c$ is weakly continuous. (These conditions imply that $\lim_{c\to 0} \rho_c = \delta_1$ weakly.) We call such a family a *product convolution semigroup*. It is a (continuous) convolution semigroup in the abstract sense of [5] or [12]. A p'th root $\tau(p)$ is unique and one defines $\rho_{1/p} = \tau(p), \rho_{m/p} = (\tau(p))^{\diamond m}, m = 1, 2, \ldots$. Finally ρ_c is defined by continuity when $c > 0$ is irrational.

The family of image measures $(\log(\rho_c))$ under the log-function is a convolution semigroup of infinitely divisible measures in the ordinary sense on the real line considered as an additive group.

The following result generalizes Theorem 1.8 in [2], which treats the special case $\alpha = \beta = 1$. In addition we express the Mellin transform of the product convolution semigroup (ρ_c) in terms of the measure κ from (3).

Theorem 2.2 *Let $\alpha \geq 0, \beta > 0$ and let $f \in \mathcal{B}$ be such that $f(\alpha) > 0$. The uniquely determined probability measure ρ with moments*

$$s_n = f(\alpha) f(\alpha+\beta) \cdot \ldots \cdot f(\alpha + (n-1)\beta), \quad n \geq 1$$

is concentrated on $]0,\infty[$ and is infinitely divisible with respect to the product convolution. The unique product convolution semigroup $(\rho_c)_{c>0}$ with $\rho_1 = \rho$ has the moments

$$\int_0^\infty x^n \, d\rho_c(x) = \left(f(\alpha) f(\alpha+\beta) \cdot \ldots \cdot f(\alpha + (n-1)\beta)\right)^c, \quad c > 0, n = 1, 2, \ldots. \tag{5}$$

The Mellin transform of ρ_c is given by

$$\int_0^\infty t^z \, d\rho_c(t) = e^{-c\psi(z)}, \ \operatorname{Re} z \geq 0, \tag{6}$$

where

$$\psi(z) = -z\log f(\alpha) + \int_0^\infty \left((1-e^{-z\beta x}) - z(1-e^{-\beta x})\right) \frac{e^{-\alpha x}}{x(1-e^{-\beta x})}\, d\kappa(x), \quad (7)$$

and κ is given by (3).

The proof of the theorem is given in Section 3.

In connection with questions of determinacy the following result is useful.

Lemma 2.3 *Assume that a Stieltjes moment sequence (u_n) is the product $u_n = s_n t_n$ of two Stieltjes moment sequences $(s_n), (t_n)$. If $t_n > 0$ for all n and (s_n) is S-indeterminate, then also (u_n) is S-indeterminate.*

This is proved in Lemma 2.2 and Remark 2.3 in [4]. It follows that if $(s_n) \in \mathcal{I}$ and (s_n^c) is S-indeterminate for $c = c_0$, then it is S-indeterminate for any $c > c_0$. Therefore one of the following three cases occur

- (s_n^c) is S-determinate for all $c > 0$.
- There exists $c_0, 0 < c_0 < \infty$ such that (s_n^c) is S-determinate for $0 < c < c_0$ and S-indeterminate for $c > c_0$.
- (s_n^c) is S-indeterminate for all $c > 0$.

We have already mentioned examples of the first two cases, and the third case occurs in Remark 2.7. It follows also from the second case that the product of two S-determinate Stieltjes moment sequences can be S-indeterminate.

The question of characterizing the set of normalized Stieltjes moment sequences (s_n) with the property that (s_n^c) is a Stieltjes moment sequence for each $c > 0$ is essentially answered in the monograph [3]. (This was written without knowledge about [19].) In fact, δ_{0n} has clearly this property, so let us restrict the attention to the class of non-vanishing normalized Stieltjes moment sequences (s_n) for which we can apply the general theory of infinitely divisible positive definite kernels, see [3, Proposition 3.2.7]. Combining this result with Theorem 6.2.6 in the same monograph we can formulate the solution in the following way, where (iii) and (iv) are new:

Theorem 2.4 *For a sequence $s_n > 0$ the following conditions are equivalent:*

(i) *s_n^c is a normalized Stieltjes moment sequence for each $c > 0$, i.e. $(s_n) \in \mathcal{I}$.*
(ii) *There exist $a \in \mathbb{R}, b \geq 0$ and a positive Radon measure σ on $[0,\infty[\setminus\{1\}$ satisfying*

$$\int_0^\infty (1-x)^2\, d\sigma(x) < \infty, \quad \int_2^\infty x^n\, d\sigma(x) < \infty,\ n \geq 3$$

such that

$$\log s_n = an + bn^2 + \int_0^\infty (x^n - 1 - n(x-1))\, d\sigma(x),\ n = 0, 1, \ldots. \quad (8)$$

(iii) There exist $0 < \varepsilon \le 1$ *and an infinitely divisible probability* ω *on* $\mathbb{R}$ *such that*

$$s_n = (1-\varepsilon)\delta_{0n} + \varepsilon \int_{-\infty}^{\infty} e^{-ny}\, d\omega(y). \quad (9)$$

(iv) There exist $0 < \varepsilon \le 1$ *and a product convolution semigroup* $(\rho_c)_{c>0}$ *of probabilities on* $]0,\infty[$ *such that*

$$s_n^c = (1-\varepsilon^c)\delta_{0n} + \varepsilon^c \int_0^\infty x^n\, d\rho_c(x), n \ge 0,\ c > 0. \quad (10)$$

Assume $(s_n) \in \mathcal{I}$*. If* (s_n^c) *is S-determinate for some* $c = c_0 > 0$*, then the quantities* $a, b, \sigma, \varepsilon, \omega, (\rho_c)_{c>0}$ *from (ii)-(iv) are uniquely determined. Furthermore* $a = \log s_1, b = 0$ *and the finite measure* $(1-x)^2\, d\sigma(x)$ *is S-determinate.*

Remark 2.5 The measure σ in condition (ii) can have infinite mass close to 1. There is nothing special about the lower limit 2 of the second integral. It can be any number > 1. The conditions on σ can be formulated that $(1-x)^2\, d\sigma(x)$ has moments of any order.

Remark 2.6 Concerning condition (iv) notice that the measures

$$\tilde{\rho}_c = (1-\varepsilon^c)\delta_0 + \varepsilon^c \rho_c,\ c > 0 \quad (11)$$

satisfy the convolution equation

$$\tilde{\rho}_c \diamond \tilde{\rho}_d = \tilde{\rho}_{c+d} \quad (12)$$

and (10) can be written

$$s_n^c = \int_0^\infty x^n\, d\tilde{\rho}_c(x),\ c > 0. \quad (13)$$

On the other hand, if we start with a family $(\tilde{\rho}_c)_{c>0}$ of probabilities on $[0,\infty[$ satisfying (12), and if we define $h(c) = 1 - \tilde{\rho}_c(\{0\}) = \tilde{\rho}_c(]0,\infty[)$, then $h(c+d) = h(c)h(d)$ and therefore $h(c) = \varepsilon^c$ with $\varepsilon = h(1) \in [0,1]$. If $\varepsilon = 0$ then $\tilde{\rho}_c = \delta_0$ for all $c > 0$, and if $\varepsilon > 0$ then $\rho_c := \varepsilon^{-c}(\tilde{\rho}_c |]0,\infty[)$ is a probability on $]0,\infty[$ satisfying $\rho_c \diamond \rho_d = \rho_{c+d}$.

Remark 2.7 In [4] was introduced a transformation $\mathcal{T}$ from normalized non-vanishing Hausdorff moment sequences (a_n) to normalized Stieltjes moment sequences (s_n) by the formula

$$\mathcal{T}[(a_n)] = (s_n); \quad s_n = \frac{1}{a_1 \cdot \ldots \cdot a_n},\ n \ge 1. \quad (14)$$

We note the following result:

If (a_n) is a normalized Hausdorff moment sequence in $\mathcal{I}$, then $\mathcal{T}[(a_n)] \in \mathcal{I}$.

As an example consider the Hausdorff moment sequence $a_n = q^n$, where $0 < q < 1$ is fixed. Clearly $(q^n) \in \mathcal{I}$ and the corresponding product convolution semigroup is $(\delta_{q^c})_{c>0}$. The transformed sequence $(s_n) = \mathcal{T}[(q^n)]$ is given by

$$s_n = q^{-\binom{n+1}{2}},$$

which again belongs to $\mathcal{I}$. The sequence (s_n^c) is S-indeterminate for all $c > 0$ e.g. by [16]. The family of densities

$$v_c(x) = \frac{q^{c/8}}{\sqrt{2\pi \log(1/q^c)}} \frac{1}{\sqrt{x}} \exp\left[-\frac{(\log x)^2}{2\log(1/q^c)}\right],\ x > 0$$

form a product convolution semigroup because

$$\int_0^\infty x^z v_c(x)\, dx = q^{-cz(z+1)/2}, \quad z \in \mathbb{C}.$$

In particular

$$\int_0^\infty x^n v_c(x)\, dx = q^{-c\binom{n+1}{2}}.$$

Each of the measures $v_c(x)\, dx$ is infinitely divisible for the additive structure as well as for the multiplicative structure. The additive infinite divisibility is deeper than the multiplicative and was first proved by Thorin, cf. [18].

3 Proofs

We start by proving Theorem 2.4 and will deduce Theorem 2.1 and 2.2 from this result.

Proof of Theorem 2.4: The proof of "(i)⇒(ii)"is a modification of the proof of Theorem 6.2.6 in [3]: For each $c > 0$ we choose a probability measure $\tilde{\rho}_c$ on $[0, \infty[$ such that for $n \geq 0$

$$s_n^c = \int_0^\infty x^n\, d\tilde{\rho}_c(x),$$

hence

$$\int_0^\infty (x^n - 1 - n(x-1))\, d\tilde{\rho}_c(x) = s_n^c - 1 - n(s_1^c - 1).$$

(Because of the possibility of S-indeterminacy we cannot claim the convolution equation $\tilde{\rho}_c \diamond \tilde{\rho}_d = \tilde{\rho_{c+d}}$.) If μ denotes a vague accumulation point for $(1/c)(x-$

$1)^2\, d\tilde{\rho}_c(x)$ as $c \to 0$, we obtain the representation

$$\log s_n - n \log s_1 = \int_0^\infty \frac{x^n - 1 - n(x-1)}{(1-x)^2} d\mu(x),$$

which gives (8) by taking out the mass of μ at $x = 1$ and defining $\sigma = (x-1)^{-2} d\,\mu(x)$ on $[0,\infty[\setminus\{1\}$. For details see [3].

"(ii)⇒(iii)" Define $m = \sigma(\{0\}) \geq 0$ and let λ be the image measure on $\mathbb{R}\setminus\{0\}$ of $\sigma - m\delta_0$ under $-\log x$. We get

$$\int_{[-1,1]\setminus\{0\}} y^2\, d\lambda(y) = \int_{[1/e,e]\setminus\{1\}} (1-x)^2 \left(\frac{-\log x}{1-x}\right)^2 d\sigma(x) < \infty,$$

and for $n \geq 0$

$$\int_{\mathbb{R}\setminus]-1,1[} e^{-ny}\, d\lambda(y) = \int_{]0,\infty[\setminus]1/e,e[} x^n\, d\sigma(x) < \infty. \tag{15}$$

This shows that λ is a Lévy measure, which allows us to define a negative definite function

$$\psi(x) = i\tilde{a}x + bx^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{-ixy} - \frac{ixy}{1+y^2}\right) d\lambda(y),$$

where

$$\tilde{a} := \int_{\mathbb{R}\setminus\{0\}} \left(\frac{y}{1+y^2} + e^{-y} - 1\right) d\lambda(y) - a.$$

Let $(\tau_c)_{c>0}$ be the convolution semigroup on $\mathbb{R}$ with

$$\int_{-\infty}^\infty e^{-ixy}\, d\tau_c(y) = e^{-c\psi(x)}, x \in \mathbb{R}.$$

Because of (15) we see that ψ and then also $e^{-c\psi}$ has a holomorphic extension to the lower halfplane. By a classical result (going back to Landau for Dirichlet series), see [20, p.58], this implies

$$\int_{-\infty}^\infty e^{-ny}\, d\tau_c(y) < \infty, n = 0, 1, \ldots.$$

For $z = x + is, s \leq 0$ the holomorphic extension of ψ is given by

$$\psi(z) = i\tilde{a}z + bz^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{-izy} - \frac{izy}{1+y^2}\right) d\lambda(y),$$

and we have

$$\int_{-\infty}^\infty e^{-izy}\, d\tau_c(y) = e^{-c\psi(z)}.$$

In particular we get

$$
\begin{aligned}
-\psi(-in) &= -\tilde{a}n + bn^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{-ny} - 1 + \frac{ny}{1+y^2}\right) d\lambda(y) \\
&= -\tilde{a}n + bn^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{-ny} - 1 - n(e^{-y}-1)\right) d\lambda(y) \\
&\quad +n \int_{\mathbb{R}\setminus\{0\}} \left(\frac{y}{1+y^2} + e^{-y} - 1\right) d\lambda(y) \\
&= an + bn^2 + \int_{]0,\infty[\setminus\{1\}} \left(x^n - 1 - n(x-1)\right) d\sigma(x),
\end{aligned}
$$

and therefore

$$
\log s_n = (n-1)m - \psi(-in) \text{ for } n \geq 1, \tag{16}
$$

while $\log s_0 = \psi(0) = 0$.

The measure $\omega = \delta_{-m} * \tau_1$ is infinitely divisible on $\mathbb{R}$ and we find for $n \geq 1$

$$
s_n = e^{-m} e^{nm - \psi(-in)} = e^{-m} \int_{-\infty}^{\infty} e^{-ny} \, d\omega(y),
$$

so (9) holds with $\varepsilon = e^{-m}$.

"(iii)⇒(iv)" Suppose (9) holds and let $(\omega_c)_{c>0}$ be the unique convolution semigroup on $\mathbb{R}$ such that $\omega_1 = \omega$. Let $(\rho_c)_{c>0}$ be the product convolution semigroup on $]0,\infty[$ such that ρ_c is the image of ω_c under e^{-y}. Then (10) holds for $c = 1, n \geq 0$ and for $c > 0$ when $n = 0$. For $n \geq 1$ we shall prove that

$$
s_n^c = \varepsilon^c \int_0^{\infty} x^n \, d\rho_c(x), \quad c > 0,
$$

but this follows from (9) first for c rational and then for all $c > 0$ by continuity.

"(iv)⇒(i)" is clear since (s_n^c) is the Stieltjes moment sequence of $\tilde{\rho}_c$ given by (11).

Assume now $(s_n) \in \mathcal{I}$. We get $\log s_1 = a + b$. If $b > 0$ then (s_n^c) is S-indeterminate for all $c > 0$ by Lemma 2.3 because the moment sequence $(\exp(cn^2))$ is S-indeterminate for all $c > 0$ by Remark 2.7.

If $(1-x)^2 \, d\sigma(x)$ is S-indeterminate there exist infinitely many measures τ on $[0,\infty[$ with $\tau(\{1\}) = 0$ and such that

$$
\int_0^{\infty} x^n (1-x)^2 \, d\sigma(x) = \int_0^{\infty} x^n \, d\tau(x), \quad n \geq 0.
$$

For any of these measures τ we have

$$
\log s_n = an + bn^2 + \int_0^{\infty} \frac{x^n - 1 - n(x-1)}{(1-x)^2} \, d\tau(x),
$$

because the integrand is a polynomial. Therefore (s_n^c) has the S-indeterminate factor

$$\exp\left(c\int_0^\infty \frac{x^n-1-n(x-1)}{(1-x)^2}\,d\tau(x)\right)$$

and is itself S-indeterminate for all $c>0$.

We conclude that if (s_n^c) is S-determinate for $0<c<c_0$, then $b=0$ and $(1-x)^2\,d\sigma(x)$ is S-determinate. Then $a=\log s_1$ and σ is uniquely determined on $[0,\infty[\setminus\{1\}$. Furthermore, if $\varepsilon,(\rho_c)_{c>0}$ satisfy (10) then

$$s_n^c=\int_0^\infty x^n\,d\tilde{\rho}_c(x),\quad c>0$$

with the notation of Remark 2.6, and we get that $\tilde{\rho}_c$ is uniquely determined for $0<c<c_0$. This determines ε and ρ_c for $0<c<c_0$, but then ρ_c is unique for any $c>0$ by the convolution equation.

We see that ε,ω are uniquely determined by (9) since (iii) implies (iv). □

Proof of Theorem 2.1 and 2.2:

To verify directly that the sequence

$$s_n=f(\alpha)f(\alpha+\beta)\cdot\ldots\cdot f(\alpha+(n-1)\beta)$$

of the form considered in Theorem 2.1 satisfies (8), we integrate formula (3) from α to s and get

$$\log f(s)=\log f(\alpha)+\int_0^\infty(e^{-\alpha x}-e^{-sx})\frac{d\kappa(x)}{x}.$$

Applying this formula we find

$$\begin{aligned}\log s_n=&\sum_{k=0}^{n-1}\log f(\alpha+k\beta)\\=&\,n\log f(\alpha)+\int_0^\infty\left(n(1-e^{-\beta x})-(1-e^{-n\beta x})\right)\frac{e^{-\alpha x}d\,\kappa(x)}{x(1-e^{-\beta x})}\\=&\,n\log f(\alpha)+\int_0^1(x^n-1-n(x-1))\,d\,\sigma(x),\end{aligned}\tag{17}$$

where σ is the image measure of

$$\frac{e^{-\alpha x}d\,\kappa(x)}{x(1-e^{-\beta x})}$$

under $e^{-\beta x}$. Note that σ is concentrated on $]0,1[$. This shows that $(s_n)\in\mathcal{I}$. It follows from the proof of Theorem 2.4 that the constant ε of (iii) is $\varepsilon=1$, so

(10) reduces to (5). The sequence (s_n^c) is S-determinate for $c \le 2$ by Carleman's criterion stating that if

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt[2n]{s_n^c}} = \infty, \tag{18}$$

then (s_n^c) is S-determinate, cf. [1],[14]. To see that this condition is satisfied we note that $f(s) \le (f(\beta)/\beta)s$ for $s \ge \beta$, and hence

$$s_n = f(\alpha)f(\alpha+\beta)\cdot\ldots\cdot f(\alpha+(n-1)\beta)$$

$$\le f(\alpha)(\frac{f(\beta)}{\beta})^{n-1}\prod_{k=1}^{n-1}(\alpha+k\beta) = f(\alpha)f(\beta)^{n-1}(1+\frac{\alpha}{\beta})_{n-1}.$$

It follows from Stirling's formula that (18) holds for $c \le 2$.

We claim that

$$\int_1^\infty \frac{e^{-\alpha x}}{x}\,d\kappa(x) < \infty. \tag{19}$$

This is clear if $\alpha > 0$, but if $\alpha = 0$ we shall prove

$$\int_1^\infty \frac{d\kappa(x)}{x} < \infty.$$

For $\alpha = 0$ we assume that $f(0) = a > 0$ and therefore the potential kernel

$$p = \int_0^\infty \eta_t\,dt$$

has finite total mass $1/a$. Furthermore we have $\kappa = p * (b\delta_0 + x\,d\nu(x))$ since

$$f'(s) = b + \int_0^\infty e^{-sx}x\,d\nu(x),$$

so we can write $\kappa = \kappa_1 + \kappa_2$ with

$$\kappa_1 = p * (b\delta_0 + x1_{]0,1[}(x)\,d\nu(x)), \quad \kappa_2 = p * (x1_{[1,\infty[}(x)\,d\nu(x)),$$

and κ_1 is a finite measure. Finally

$$\int_1^\infty \frac{d\kappa_2(x)}{x} = \int_1^\infty \left(\int_0^\infty \frac{y}{x+y}\,dp(x)\right)d\nu(y) \le \frac{\nu([1,\infty[)}{a} < \infty.$$

The function ψ given by (7) is continuous in the closed half-plane $\operatorname{Re} z \ge 0$ and holomorphic in $\operatorname{Re} z > 0$ because of (19). Note that $\psi(n) = -\log s_n$ by (17). We also notice that $\psi(iy)$ is a continuous negative definite function on the additive group $(\mathbb{R}, +)$, cf. [5], because

$$1 - e^{-iyx} - iy(1 - e^{-x})$$

is a continuous negative definite function of y for each $x \geq 0$. Therefore there exists a unique product convolution semigroup $(\tau_c)_{c>0}$ of probabilities on $]0,\infty[$ such that

$$\int_0^\infty t^{iy}\, d\tau_c(t) = e^{-c\psi(iy)},\ c > 0, y \in \mathbb{R}. \tag{20}$$

By a classical result, see [20, p. 58]), the holomorphy of ψ in the right half-plane implies that t^z is τ_c-integrable for $\mathrm{Re}\, z \geq 0$ and

$$\int_0^\infty t^{z}\, d\tau_c(t) = e^{-c\psi(z)},\ c > 0, \mathrm{Re}\, z \geq 0. \tag{21}$$

In particular the n'th moment is given by

$$\int_0^\infty t^n\, d\tau_c(t) = e^{-c\psi(n)} = e^{c \log s_n} = s_n^c,$$

so by S-determinacy of (s_n^c) for $c \leq 2$ we get $\rho_c = \tau_c$ for $c \leq 2$.This is however enough to ensure that $\rho_c = \tau_c$ for all $c > 0$ since (ρ_c) and (τ_c) are product convolution semigroups. □

4 Tyan's thesis revisited

In [19] Tyan defines a normalized Hamburger moment sequence

$$s_n = \int_{-\infty}^\infty x^n \mu(x), \quad n \geq 0$$

to be *infinitely divisible* if

(i) $s_n \geq 0$ for all $n \geq 0$
(ii) (s_n^c) is a Hamburger moment sequence for all $c > 0$.

Since the set of Hamburger moment sequences is closed under limits and products, we can replace (ii) by the weaker

(ii') $\sqrt[k]{s_n}$ is a Hamburger moment sequence for all $k = 1, 2, \ldots$.

Lemma 4.1 *Let (s_n) be an infinitely divisible Hamburger moment sequence. Then one of the following cases hold:*

- $s_n > 0$ *for all* n.
- $s_{2n} > 0, s_{2n+1} = 0$ *for all* n.
- $s_n = 0$ *for* $n \geq 1$.

Proof: The sequence (u_n) defined by

$$u_n = \lim_{k\to\infty} \sqrt[k]{s_n} = \begin{cases} 1 \text{ if } s_n > 0 \\ 0 \text{ if } s_n = 0 \end{cases}$$

is a Hamburger moment sequence, and since it is bounded by 1 we have

$$u_n = \int_{-1}^{1} x^n \, d\mu(x)$$

for some probability μ on $[-1, 1]$.

Either $u_2 = 1$ and then $\mu = \alpha\delta_1 + (1-\alpha)\delta_{-1}$ for some $\alpha \in [0, 1]$, or $u_2 = 0$ and then $\mu = \delta_0$, which gives the third case of the Lemma.

In the case $u_2 = 1$ we have $u_1 = 2\alpha - 1$, which is either 1 or 0 corresponding to either $\alpha = 1$ or $\alpha = \frac{1}{2}$, which gives the two first cases of the Lemma. $\square$

The symmetric case $s_{2n} > 0, s_{2n+1} = 0$ is equivalent to studying infinitely divisible Stieltjes moment sequences, while the third case is trivial.

Theorem 4.2 of [19] can be formulated:

Theorem 4.2 *A Hamburger moment sequence (s_n) such that $s_n > 0$ for all n is infinitely divisible if and only if the following representation holds*

$$\log s_n = an + bn^2 + \int_{-\infty}^{\infty} (x^n - 1 - n(x-1))\, d\sigma(x), \quad n \geq 0,$$

where $a \in \mathbb{R}, b \geq 0$ and σ is a positive measure on $\mathbb{R} \setminus \{1\}$ such that $(1-x)^2\, d\sigma(x)$ is a measure with moments of any order. Furthermore (s_n) is a Stieltjes moment sequence if and only if σ can be chosen supported by $[0, \infty[$.

The proof is analogous to the proof of Theorem 2.4.

Tyan also discusses infinitely divisible multidimensional moment sequences and obtains analogous results.

5 An application to Hermite polynomials

It follows from equation (4) that

$$\sqrt{n!} = \int_0^{\infty} u^n \, d\sigma(u) \tag{22}$$

for the unique probability σ on the half-line satisfying $\sigma \diamond \sigma = \exp(-t)1_{]0,\infty[}(t)\,dt$. Even though σ is not explicitly known, it can be used to prove that a certain generating function for the Hermite polynomials is non-negative.

Let $H_n, n = 0, 1, \ldots$ denote the sequence of Hermite polynomials satisfying the orthogonality relation

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2}\, dx = 2^n n! \delta_{nm}.$$

The following generating function is well known:

$$\sum_{k=0}^{\infty} \frac{H_k(x)}{k!} z^k = e^{2xz - z^2}, \quad x, z \in \mathbb{C}. \tag{23}$$

The corresponding orthonormal polynomials are given by

$$h_n(x) = \frac{H_n(x)}{\sqrt{2^n n!}},$$

and they satisfy the following inequality of Szasz, cf. [17]

$$|h_n(x)| \le e^{x^2/2}, x \in \mathbb{R}, n = 0, 1, \ldots. \tag{24}$$

Let $\mathbb{D}$ denote the open unit disc in the complex plane.

Theorem 5.1 *The orthonormal generating function*

$$G(t, x) = \sum_{k=0}^{\infty} h_k(x) t^k \tag{25}$$

is continuous for $(t, x) \in \mathbb{D} \times \mathbb{R}$ *and satisfies* $G(t, x) > 0$ *for* $-1 < t < 1, x \in \mathbb{R}$.

Proof: The series for the generating function (25) converges uniformly on compact subsets of $\mathbb{D} \times \mathbb{R}$ by the inequality of Szasz (24), so it is continuous.

By (22) we find

$$\sum_{k=0}^{n} h_k(x) t^k = \int_0^{\infty} \left(\sum_{k=0}^{n} \frac{H_k(x)}{k!} (\frac{tu}{\sqrt{2}})^k \right) d\sigma(u),$$

which by (23) converges to

$$\int_0^{\infty} \exp(\sqrt{2}tux - t^2u^2/2)\, d\sigma(u) > 0 \text{ for } -1 < t < 1, x \in \mathbb{R},$$

provided we have dominated convergence. This follows however from (24) because

$$\int_0^\infty \left| \sum_{k=0}^n \frac{H_k(x)}{k!} (\frac{tu}{\sqrt{2}})^k \right| \, d\sigma(u) \leq e^{x^2/2} \int_0^\infty \left(\sum_{k=0}^\infty \frac{(|t|u)^k}{\sqrt{k!}} \right) \, d\sigma(u)$$

$$= e^{x^2/2}(1-|t|)^{-1} < \infty.$$

□

6 The moment sequences $(a)_n^c$ and $((a)_n/(b)_n)^c$

For each $a > 0$ the sequence $(a)_n := a(a+1) \cdot \ldots \cdot (a+n-1)$ is the Stieltjes moment sequence of the Γ-distribution γ_a:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \int x^n \, d\gamma_a(x) = \frac{1}{\Gamma(a)} \int_0^\infty x^{a+n-1} e^{-x} \, dx.$$

For $a = 1$ we get the moment sequence $n!$, so the following result generalizes Theorem 2.5 of [2].

Theorem 6.1 *The sequence $(a)_n$ belongs to $\mathcal{I}$ for each $a > 0$. There exists a unique product convolution semigroup $(\gamma_{a,c})_{c>0}$ such that $\gamma_{a,1} = \gamma_a$. The moments are given as*

$$\int_0^\infty x^n \, d\gamma_{a,c}(x) = (a)_n^c, \quad c > 0,$$

and

$$\int_0^\infty x^z \, d\gamma_{a,c}(x) = \left(\frac{\Gamma(a+z)}{\Gamma(a)} \right)^c, \quad \operatorname{Re} z > -a.$$

The moment sequence $((a)_n^c)$ is S-determinate for $c \leq 2$ and S-indeterminate for $c > 2$.

Proof: We apply Theorem 2.1 and 2.2 to the Bernstein function $f(s) = a + s$ and put $\alpha = 0, \beta = 1$. The formula for the Mellin transform follows from a classical formula about $\log \Gamma$, cf. [11, 8.3417].

We shall prove that $(a)_n^c$ is S-indeterminate for $c > 2$. In [2] it was proved that $(n!)^c$ is S-indeterminate for $c > 2$, and so are all the shifted sequences $((n+k-1)!)^c, k \in \mathbb{N}$. This implies that

$$(k)_n^c = \left(\frac{(n+k-1)!}{(k-1)!} \right)^c$$

is S-indeterminate for $k \in \mathbb{N}, c > 2$. To see that also $(a)_n^c$ is S-indeterminate for $a \notin \mathbb{N}$, we choose an integer $k > a$ and factorize

$$(a)_n^c = \left(\frac{(a)_n}{(k)_n}\right)^c (k)_n^c.$$

By the following theorem the first factor is a non-vanishing Stieltjes moment sequence, and by Lemma 2.3 the product is S-indeterminate. □

For $0 < a < b$ we have

$$\frac{(a)_n}{(b)_n} = \frac{1}{B(a,b-a)} \int_0^1 x^{n+a-1}(1-x)^{b-a-1}\,dx, \tag{26}$$

where B denotes the Beta-function.

Theorem 6.2 *Let $0 < a < b$. Then $((a)_n/(b)_n)$ belongs to $\mathcal{I}$ and all powers of the moment sequence are Hausdorff moment sequences. There exists a unique product convolution semigroup $(\beta(a,b)_c)_{c>0}$ on $]0,1]$ such that*

$$\int_0^1 x^z\, d\beta(a,b)_c(x) = \left(\frac{\Gamma(a+z)}{\Gamma(a)} / \frac{\Gamma(b+z)}{\Gamma(b)}\right)^c, \quad \operatorname{Re} z > -a.$$

Proof: We apply Theorem 2.1 and 2.2 to the Bernstein function $f(s) = (a+s)/(b+s)$ and put $\alpha = 0, \beta = 1$.

The Stieltjes moment sequences $(((a)_n/(b)_n)^c))$ are all bounded and hence Hausdorff moment sequences. The measures $\gamma_{b,c} \diamond \beta(a,b)_c$ and $\gamma_{a,c}$ have the same moments and are therefore equal for $c \leq 2$ and hence for any $c > 0$ by the convolution equations. The Mellin transform of $\beta(a,b)_c$ follows from Theorem 6.1.

□

7 The q-extension $((a;q)_n/(b;q)_n)^c$

In this section we fix $0 < q < 1$ and consider the q-shifted factorials

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k), z \in \mathbb{C}, n = 1, 2, \ldots, \infty$$

and $(z;q)_0 = 1$. We refer the reader to [10] for further details about q-extensions of various functions.

For $0 \le b < a < 1$ the sequence $s_n = (a;q)_n/(b;q)_n$ is a Hausdorff moment sequence for the measure

$$\mu(a,b;q) = \frac{(a;q)_\infty}{(b;q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a;q)_k}{(q;q)_k} a^k \delta_{q^k}, \tag{27}$$

which is a probability on $]0,1]$ by the q-binomial Theorem, cf. [10]. The calculation of the n'th moment follows also from this theorem since

$$s_n(\mu(a,b;q)) = \frac{(a;q)_\infty}{(b;q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a;q)_k}{(q;q)_k} a^k q^{kn} = \frac{(a;q)_\infty}{(b;q)_\infty} \frac{((b/a)aq^n;q)_\infty}{(aq^n;q)_\infty} = \frac{(a;q)_n}{(b;q)_n}.$$

Replacing a by q^a and b by q^b and letting $q \to 1$ we get the moment sequences $(a)_n/(b)_n$, so the present example can be thought of as a q-extension of the former. The distribution $\mu(q^a, q^b; q)$ is called the q-Beta law in Pakes [13] because of its relation to the q-Beta function.

Theorem 7.1 *For $0 \le b < a < 1$ the sequence $s_n = (a;q)_n/(b;q)_n$ belongs to $\mathcal{I}$. The measure $\mu(a,b;q)$ is infinitely divisible with respect to the product convolution and the corresponding product convolution semigroup $(\mu(a,b;q)_c)_{c>0}$ satisfies*

$$\int t^z \, d\mu(a,b;q)_c(t) = \left(\frac{(bq^z;q)_\infty}{(b;q)_\infty} / \frac{(aq^z;q)_\infty}{(a;q)_\infty} \right)^c, \quad \operatorname{Re} z > -\frac{\log a}{\log q}. \tag{28}$$

In particular

$$s_n^c = ((a;q)_n/(b;q)_n)^c \tag{29}$$

is the moment sequence of $\mu(a,b;q)_c$, which is concentrated on $\{q^k \mid k = 0,1,\ldots\}$ for each $c > 0$.

Proof: It is easy to prove that $(a;q)_n/(b;q)_n$ belongs to $\mathcal{I}$ using Theorem 2.1 and 2.2 applied to the Bernstein function

$$f(s) = \frac{1-aq^s}{1-bq^s} = 1 - (a-b) \sum_{k=0}^{\infty} b^k q^{(k+1)s},$$

but it will also be a consequence of the following considerations, which give information about the support of $\mu(a,b;q)_c$.

For a probability μ on $]0,1]$ let $\tau = -\log(\mu)$ be the image measure of μ under $-\log$. It is concentrated on $[0,\infty[$ and

$$\int_0^1 t^{ix} \, d\mu(t) = \int_0^\infty e^{-itx} \, d\tau(t).$$

This shows that μ is infinitely divisible with respect to the product convolution if and only if τ is infinitely divisible in the ordinary sense, and in the affirmative case the negative definite function ψ associated to μ is related to the Bernstein function f associated to τ by $\psi(x) = f(ix), x \in \mathbb{R}$, cf. [5, p.69].

We now prove that $\mu(a, b; q)$ is infinitely divisible for the product convolution. As noticed this is equivalent to proving that the measure

$$\tau(a, b; q) := \frac{(a; q)_\infty}{(b; q)_\infty} \sum_{k=0}^{\infty} \frac{(b/a; q)_k}{(q; q)_k} a^k \delta_{k \log(1/q)},$$

is infinitely divisible in the ordinary sense. To see this we calculate the Laplace transform of $\tau(a, b; q)$ and get by the q-binomial Theorem

$$\int_0^\infty e^{-st}\, d\tau(a, b; q)(t) = \frac{(bq^s; q)_\infty}{(b; q)_\infty} / \frac{(aq^s; q)_\infty}{(a; q)_\infty}, \quad s \geq 0. \tag{30}$$

Putting

$$f_a(s) = \log \frac{(aq^s; q)_\infty}{(a; q)_\infty},$$

we see that f_a is a bounded Bernstein function of the form

$$f_a(s) = -\log(a; q)_\infty - \varphi_a(s),$$

where

$$\varphi_a(s) = -\log(aq^s; q)_\infty = \sum_{k=1}^{\infty} \frac{a^k}{k(1 - q^k)} q^{ks}$$

is completely monotonic as Laplace transform of the finite measure

$$\nu_a = \sum_{k=1}^{\infty} \frac{a^k}{k(1 - q^k)} \delta_{k \log(1/q)}.$$

From (30) we get

$$\int_0^\infty e^{-st}\, d\tau(a, b; q)(t) = \frac{(a; q)_\infty}{(b; q)_\infty} e^{\varphi_a(s) - \varphi_b(s)},$$

and it follows that $\tau(a, b; q)$ is infinitely divisible and the corresponding convolution semigroup is given by the infinite series

$$\tau(a, b; q)_c = \left(\frac{(a; q)_\infty}{(b; q)_\infty} \right)^c \sum_{k=0}^{\infty} \frac{c^k (\nu_a - \nu_b)^{*k}}{k!}, \quad c > 0.$$

Note that each of these measures are concentrated on $\{k \log(1/q) \mid k = 0, 1, \ldots\}$. The associated Lévy measure is the finite measure $\nu_a - \nu_b$ concentrated on $\{k \log(1/q) \mid k = 1, 2, \ldots\}$. This shows that the image measures

$$\mu(a, b; q)_c = \exp(-\tau(a, b; q)_c), \quad c > 0$$

form a product convolution semigroup concentrated on $\{q^k \mid k = 0, 1, \ldots\}$.

The product convolution semigroup $(\mu(a,b;q)_c)_{c>0}$ has the negative definite function $f(ix)$, where $f(s) = f_a(s) - f_b(s)$ for $\operatorname{Re} s \geq 0$, hence

$$\int t^{ix}\, d\mu(a,b;q)_c(t) = \left(\frac{(bq^{ix};q)_\infty}{(b;q)_\infty} \Big/ \frac{(aq^{ix};q)_\infty}{(a;q)_\infty} \right)^c, \quad x \in \mathbb{R},$$

and the equation (28) follows by holomorphic continuation. Putting $z = n$ gives (29).

□

8 Complements

Example 8.1 Let $0 < a < b$ and consider the Hausdorff moment sequence $a_n = (a)_n/(b)_n \in \mathcal{I}$. By Remark 2.7 the moment sequence $(s_n) = \mathcal{T}[(a_n)]$ belongs to $\mathcal{I}$. We find

$$s_n = \prod_{k=1}^{n} \frac{(b)_k}{(a)_k} = \prod_{k=0}^{n-1} \left(\frac{b+k}{a+k} \right)^{n-k}.$$

Example 8.2 Applying $\mathcal{T}$ to the Hausdorff moment sequence $((a;q)_n/(b;q)_n)$ gives the Stieltjes moment sequence

$$s_n = \prod_{k=1}^{n} \frac{(b;q)_k}{(a;q)_k} = \prod_{k=0}^{n-1} \left(\frac{1-bq^k}{1-aq^k} \right)^{n-k}. \tag{31}$$

We shall now give the measure with moments (31).

For $0 \leq p < 1, 0 < q < 1$ we consider the function of z

$$h_p(z;q) = \prod_{k=0}^{\infty} \left(\frac{1-pzq^k}{1-zq^k} \right)^k,$$

which is holomorphic in the unit disk with a power series expansion

$$h_p(z;q) = \sum_{k=0}^{\infty} c_k z^k \tag{32}$$

having non-negative coefficients $c_k = c_k(p,q)$. To see this, notice that

$$\frac{1-pz}{1-z} = 1 + \sum_{k=1}^{\infty} (1-p) z^k.$$

For $0 \le b < a < 1$ and $\gamma > 0$ we consider the probability measure with support in $[0, \gamma]$

$$\sigma_{a,b,\gamma} = \frac{1}{h_{b/a}(a;q)} \sum_{k=0}^{\infty} c_k a^k \delta_{\gamma q^k},$$

where the numbers c_k are the (non-negative) coefficients of the power series for $h_{b/a}(z;q)$.

The n'th moment of $\sigma_{a,b,\gamma}$ is given by

$$s_n(\sigma_{a,b,\gamma}) = \gamma^n \frac{h_{b/a}(aq^n;q)}{h_{b/a}(a;q)}.$$

For $\gamma = (b;q)_\infty/(a;q)_\infty$ it is easy to see that

$$s_n(\sigma_{a,b,\gamma}) = \prod_{k=0}^{n-1} \left(\frac{1 - bq^k}{1 - aq^k} \right)^{n-k},$$

which are the moments (31).

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