LOGARITHMIC ORDER AND TYPE OF INDETERMINATE MOMENT PROBLEMS

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WITH AN APPENDIX BY WALTER HAYMAN

We investigate a refined growth scale, logarithmic growth, for indeterminate moment problems of order zero. We show that the four entire functions appearing in the Nevanlinna parametrization have the same logarithmic order and type. In the appendix it is shown that the logarithmic indicator is constant.

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1. Introduction and results

This paper deals with the indeterminate moment problem on the real line. We are given a positive measure μ on $\mathbb{R}$ having moments of all orders and we assume that μ is not determined by its moments. For details about the indeterminate moment problem see the monographs by Akhiezer[1], by Shohat and Tamarkin[26] or the survey paper by Berg[3]. Our notation follows that of Akhiezer[1].

In this indeterminate situation the solutions ν to the moment problem form an infinite convex set V, which is compact in the vague topology. Nevanlinna has obtained a parametrization of V in terms of the so-called Pick functions. We recall that a holomorphic function φ defined in the upper half plane is called a Pick function if $\Im\varphi(z) \geq 0$ for $\Im z > 0$. The class of Pick functions is denoted by $\mathcal{P}$.

The Nevanlinna parametrization is the one-to-one correspondence $\nu_\varphi \leftrightarrow \varphi$ between V and $\mathcal{P} \cup \{\infty\}$ given by

$$\int_{-\infty}^{\infty} \frac{d\nu_\varphi(t)}{t-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1}$$

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Here A, B, C and D are certain entire functions defined in terms of the orthonormal polynomials $\{P_k\}$ and the polynomials of the second kind $\{Q_k\}$ in the following way:

$$\begin{aligned} A(z) &= z\sum_{k=0}^{\infty} Q_k(0)Q_k(z), \\ B(z) &= -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z), \\ C(z) &= 1 + z\sum_{k=0}^{\infty} P_k(0)Q_k(z) \quad \text{and} \\ D(z) &= z\sum_{k=0}^{\infty} P_k(0)P_k(z). \end{aligned} \tag{2}$$

These functions are closely related due to the relation

$$A(z)D(z) - B(z)C(z) \equiv 1. \tag{3}$$

Two other functions play a role, namely

$$p(z) = \left(\sum_{k=0}^{\infty} |P_k(z)|^2\right)^{1/2} \quad \text{and} \quad q(z) = \left(\sum_{k=0}^{\infty} |Q_k(z)|^2\right)^{1/2}.$$

We recall that $1/p(x)^2$ is the maximal point mass of any solution to the moment problem at the point $x \in \mathbb{R}$. The function q has a similar property when one considers the so-called shifted moment problem, cf. Pedersen[23].

In Berg and Pedersen[4] the entire functions A, B, C and D were shown to have the same order, type and indicator function. It was also shown that the logarithmically subharmonic functions p and q had that order, type and indicator. A result of M. Riesz states that each of the entire functions is of minimal exponential type and therefore the common order is a number between 0 and 1.

The point of this paper is to investigate moment problems of order 0. The question arises if the growth of the four entire functions and p and q is also the same when one considers a refined growth scale for functions of order 0. We shall use a logarithmic scale, which has been used by other authors in connection with q-special functions.

Several examples of indeterminate moment problems of order 0 have been investigated. The indeterminate moment problems within the so-called q-Askey scheme have been identified by Christiansen[11]. As examples of moment problems of order zero we mention in particular the moment problems associated with the q-Meixner, q-Charlier, Al-Salam–Carlitz II, q-Laguerre and Stieltjes–Wigert polynomials. Also the discrete q-Hermite II, q^{-1}-Meixner–Pollaczek, symmetric Al-Salam–Chihara II and continuous q^{-1}-Hermite polynomials lead to moment problems of order zero. See Section 4.

For an entire function f the quantity $M(f,r)$ denotes the maximum modulus of f on the closed disk centered at the origin and of radius r.

We recall that an entire function f is of order 0 if for any $\varepsilon > 0$ there is $r_0 > 0$ such that

$$\log M(f,r) \le r^{\varepsilon}, \quad \text{for} \quad r \ge r_0.$$

The inequality $\log M(f,r) \le r^{\varepsilon}$ is thus true for r sufficiently large, and this we write as

$$\log M(f,r) \le_{\text{as}} r^{\varepsilon},$$

adopting a notation from Levin[21].

For an entire function f of order zero we define the *logarithmic order* $\rho = \rho_f$ as

$$\rho = \inf\{\alpha > 0 \,|\, \log M(f,r) \le_{\text{as}} (\log r)^{\alpha}\,\}.$$

For non-constant f we must have $\rho \ge 1$, by the usual proof of Liouville's theorem. It is easy to obtain that

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log\log r}.$$

When $\rho < \infty$ we define the *logarithmic type* $\tau = \tau_f$ as

$$\tau = \inf\{\beta > 0 \,|\, \log M(f,r) \le_{\text{as}} \beta(\log r)^{\rho}\,\},$$

and it is readily found that

$$\tau = \limsup_{r\to\infty} \frac{\log M(f,r)}{(\log r)^{\rho}}.$$

It is easily seen that if $f(z)$ has logarithmic order ρ and logarithmic type τ then so has the function $f(az+b)$ (for $a \neq 0$). Furthermore, the function $f(z)^n$ is again of logarithmic order ρ but of logarithmic type $n\tau$, while $f(z^n)$ has logarithmic order ρ and logarithmic type τn^{ρ}. It is also clear that if a transcendental entire function has logarithmic order equal to 1, then the logarithmic type must be infinite. For a polynomial of degree $k \ge 1$ the logarithmic order is 1 and the type is k.

The indicator function for an entire function of finite logarithmic order ρ and finite logarithmic type is defined in the natural way as

$$h(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{(\log r)^\rho}, \qquad 0 \le \theta \le 2\pi.$$

However it turns out that the indicator of any entire function of finite logarithmic order and type is actually constant equal to the type. This fact can be deduced (at least when $\rho \geq 2$) from results in a paper by Barry[2] (see p. 469 in Barrys paper). M. Sodin has kindly informed us that the result can also be deduced from a result of Grishin[13]. In the Appendix we present a self-contained proof of this result by Walter Hayman.

With these definitions we have the following result proving the conjecture 24.4.4 p. 651 in Ismail[14]. Because of the applications to the q-Askey scheme, Ismail called q-order, q-type and q-Phragmén-Lindelöf-indicator what we have called logarithmic order, type and indicator, see Ismail[14] p. 532.

Theorem 1.1. *The functions A, B, C, D, p and q appearing in an indeterminate moment problem of order 0 have the same logarithmic order $\rho \geq 1$. If $\rho < \infty$ then they have the same logarithmic type.*

Any combination $A(z)t - C(z)$ and $B(z)t - D(z)$, where $t \in \mathbb{R} \cup \{\infty\}$, has also the same logarithmic order and type.

The common logarithmic order and type of the functions of Theorem 1.1 are called the logarithmic order and type of the indeterminate Hamburger moment problem.

The four entire functions occuring in the indeterminate Hamburger moment problem can be regarded as the entries of a certain 2×2 matrix of entire functions. This leads to the concept of a *Nevanlinna matrix*, which was introduced by Krein[19], see also Akhiezer[1]. In Berg and Pedersen[5] the common growth of the entries was investigated, and it was shown that all four entries have the same ordinary order and type.

Definition 1.1. An entire function $N : \mathbb{C} \to \mathrm{SL}_2(\mathbb{C})$ of the form

$$N(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is called a (real) Nevanlinna matrix if the entries are real transcendental entire functions and

$$\Im\left\{\frac{A(z)t + B(z)}{C(z)t + D(z)}\right\} > 0, \quad \text{for} \quad t \in \mathbb{R} \cup \{\infty\}, \quad \Im z > 0.$$

If we consider the entire functions A, B, C and D from an indeterminate Hamburger moment problem, then the matrix

$$\begin{pmatrix} -A(z) & C(z) \\ B(z) & -D(z) \end{pmatrix}$$

defines a real Nevanlinna matrix, taking into account the relations (1) and (3).

Part of Theorem 1.1 can be generalized to real Nevanlinna matrices. We have

Theorem 1.2. *For any real Nevanlinna matrix of order zero*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the entries A, B, C and D have the same logarithmic order and logarithmic type.

This common order and type is also the logarithmic order and type of any of the functions $At + B$, $Ct + D$, where $t \in \mathbb{R}$.

Theorem 1.1 and 1.2 are proved in Section 2.

It was shown in Bergweiler, Ishizaki and Yanagihara[7] and in Ramis[25] that entire transcendental solutions of certain q-difference equations are of logarithmic order 2 and finite logarithmic type. Refined results about the zeros of such solutions are given in Bergweiler and Hayman[6].

Remark 1.1. There is a notion of proximate or refined order for entire functions, originally introduced by Valiron[28], see also Levin's book[22]. In this general setup it is still true that the four entire functions in a Nevanlinna matrix have the same growth, due to the quite accurate estimates between the functions that we shall use in the proof of Theorem 1.2.

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2. Proof of the main results

In this section we prove Theorem 1.1 and 1.2. The key to this is the following lemma.

Lemma 2.1. *Let f and g be two transcendental entire functions such that f/g is a Pick function. Then f and g have the same logarithmic order and type.*

Proof. We use the fact that any Pick function p admits an integral representation of the form

$$p(z) = az + b + \int_{-\infty}^{\infty} \frac{tz+1}{t-z}\, d\tau(t),$$

where $a \geq 0$, $b \in \mathbb{R}$ and τ is a finite positive measure on the real line. The function f/g is a *meromorphic* Pick function, so from the integral representation we easily obtain

$$\frac{f(z)}{g(z)} = az + b - \frac{b_0}{z} - z \sum_{n=1}^{\infty} \frac{b_n}{(z-a_n)a_n},$$

where $a, b_0 \geq 0$, $b \in \mathbb{R}$, $\{a_n\}$ is the set of nonzero poles, $b_n > 0, n \geq 1$ and $\sum_{n=1}^{\infty} b_n/a_n^2 < \infty$. From this series representation we see that

$$\left|\frac{f(z)}{g(z)}\right| \leq K \frac{|z|^2+1}{|y|},$$

for some constant K and all $z \in \mathbb{C} \setminus \mathbb{R}$.

For $|y| \geq 1$ this estimate gives us (with $r = |z|$)

$$|f(z)| \leq K(r^2+1)M(g,r).$$

For $|y| < 1$, we get, since $\log|f|$ is subharmonic,

$$\begin{aligned}\log|f(z)| &\leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(z+e^{it})|\, dt \\ &\leq \log K((r+1)^2+1) - \frac{1}{2\pi}\int_0^{2\pi} \log|y+\sin t|\, dt + \log M(g, r+1).\end{aligned}$$

If we combine this with the estimate for $|y| \geq 1$, then we get

$$\log M(f,r) \leq K_1 + K_2 \log(r+1) + \log M(g, r+1), \tag{4}$$

for suitable positive constants K_1 and K_2. From this relation it follows that the logarithmic order ρ_f of f is less than or equal to the logarithmic order ρ_g of g: this is clear if $\rho_g = \infty$, so we may suppose that $\rho_g < \infty$. Let $\varepsilon > 0$ be given. Then

$$M(g,r) \leq_{\text{as}} e^{(\log r)^{\rho_g+\varepsilon}},$$

and hence,

$$M(f,r) \leq_{\text{as}} e^{K_1+K_2\log(r+1)+(\log(r+1))^{\rho_g+\varepsilon}} \leq_{\text{as}} e^{(\log r)^{\rho_g+2\varepsilon}},$$

since $\rho_g \geq 1$. In this way we see that $\rho_f \leq \rho_g$.

Clearly, the function $-g/f$ is also a Pick function, and so we get $\rho_g \leq \rho_f$. Therefore the two logarithmic orders must be identical. This common order we denote by ρ. Assume now $1 < \rho < \infty$. From the relation (4) it also follows that

$$\frac{\log M(f,r)}{(\log r)^\rho} \leq \frac{K_1 + K_2 \log(r+1)}{(\log r)^\rho} + \frac{\log M(g, r+1)}{(\log r)^\rho}.$$

This implies that the logarithmic type of f is less than or equal to the logarithmic type of g. Again, by considering $-g/f$, we find that the two logarithmic types are equal. When $\rho = 1$ both f, g have logarithmic type ∞. □

Proof of Theorem 1.2. For any $t \in \mathbb{R} \cup \{\infty\}$, the meromorphic function

$$\frac{A(z)t + B(z)}{C(z)t + D(z)}$$

is a Pick function. Hence, by Lemma 2.1, the logarithmic order and the logarithmic type of the two functions $At + B$ and $Ct + D$ (for fixed t) are identical. In particular ($t = \infty$) the logarithmic order and type of A is the same as the logarithmic order and type of C, and similarly for B and D ($t = 0$).

In a real Nevanlinna matrix, the function D/C is also a Pick function, see e.g. Berg and Pedersen[5], so the logarithmic order and type of D and C are also identical.

For fixed $t \in \mathbb{R}$, the function

$$\frac{C(z)t + D(z)}{C(z)} = t + \frac{D(z)}{C(z)}$$

is thus also a meromorphic Pick function and therefore the logarithmic growth of $Ct + D$ is the same as the logarithmic growth of C.

It is easy to see that also the matrix

$$\begin{pmatrix} -C(z) & -D(z) \\ A(z) & B(z) \end{pmatrix}$$

is a real Nevanlinna matrix. Therefore also B/A and hence $(At + B)/A$ is a Pick function. Consequently, the logarithmic growth of $At + B$ is equal to the logarithmic growth of A. □

Proof of Theorem 1.1. The assertions about the functions A, B, C and D follow from Theorem 1.2. We turn to the functions p and q. We claim that we also have $\rho_p = \rho_q = \rho$, where ρ is the common logarithmic order of the four entire functions. Indeed, it is enough to prove that $\rho_p = \rho_D$, as mentioned in Berg and Pedersen[4]. From the definition of D, (2), and the

Cauchy-Schwarz inequality we see $M(D,r) \leq p(0)rM(p,r)$ so that $\rho_D \leq \rho_p$. Furthermore, formula (22) in Berg and Pedersen[4], stating $M(p,r)^2 \leq M(B,r+1)M(D,r+1)$, yields $\rho_p \leq \rho_D$.

We also obtain $\tau_D \leq \tau_p$ and $2\tau_p \leq \tau_B + \tau_D$ so that $\tau_p = \tau$, the common logarithmic type. □

3. Stieltjes moment problems

A Stieltjes moment problem may be determinate on the half-line, but indeterminate on the whole real line. One defines the quantity α as

$$\alpha = \lim_{n\to\infty} \frac{P_n(0)}{Q_n(0)}.$$

It is a fact that $\alpha \leq 0$ and that the problem is indeterminate on the half-line (or in the sense of Stieltjes) if and only if $\alpha < 0$, cf. Chihara[10] or Berg[3].

The set $V_+ = \{\sigma \in V \,|\, \mathrm{supp}(\sigma) \subseteq [0,\infty)\}$ of solutions to a Stieltjes moment problem, which is indeterminate in the sense of Stieltjes, can be parameterized via the one-to-one correspondence $\nu_\sigma \leftrightarrow \sigma$ between V_+ and $\mathcal{S} \cup \{\infty\}$ given by

$$\int_0^\infty \frac{d\nu_\sigma(t)}{t+w} = \frac{P(w)+\sigma(w)R(w)}{Q(w)+\sigma(w)S(w)}, \quad w \in \mathbb{C} \setminus [0,\infty),$$

where the functions P, Q, R and S can be defined as limits of convergents of the Stieltjes continued fraction. The parameter space $\mathcal{S}$ is the set of Stieltjes transforms, i.e. functions of the form

$$\sigma(w) = a + \int_0^\infty \frac{d\tau(x)}{x+w}, \quad w \in \mathbb{C} \setminus (-\infty, 0],$$

where $a \geq 0$ and τ is a positive measure on $[0,\infty)$ such that the integral makes sense. This is the Krein parametrization of the solutions to an indeterminate Stieltjes moment problem, see Krein and Nudelman[20] or Berg[3]. The functions P, Q, R and S are related to A, B, C and D as follows.

$$P(z) = A(-z) - \frac{1}{\alpha}C(-z), \quad R(z) = C(-z),$$
$$Q(z) = -\left(B(-z) - \frac{1}{\alpha}D(-z)\right), \quad S(z) = -D(-z).$$

Concerning these functions we have

Proposition 3.1. *The entire functions P, Q, R and S all have the same logarithmic order and type as the indeterminate Hamburger moment problem.*

Proof. This follows directly from the definitions of P, Q, R and S in terms of linear combinations of A, B, C and D and Theorem 1.1. □

For a Stieltjes moment sequence $\{t_n\}_{n\geq 0}$ one considers the corresponding symmetric Hamburger moment sequence given by $\{t_0, 0, t_1, 0, t_2, 0, \ldots\}$. There is a close connection between these two moment problems and relations between the entire functions in the two Nevanlinna parametrizations can be found in e.g. Chihara[10] or Pedersen[24]. Let us just mention that the D-functions D_S and D_H for the Stieltjes and Hamburger problems are connected by $zD_H(z) = D_S(z^2)$. From this relation we easily obtain

Proposition 3.2. *Let ρ_S and τ_S denote the logarithmic order and type of an indeterminate Stieltjes moment problem and let ρ_H and τ_H denote the logarithmic order and type of the corresponding symmetric Hamburger moment problem. Then we have*

$$\rho_H = \rho_S, \quad \tau_H = \tau_S 2^{\rho_S}.$$

4. Examples

In this section we determine the logarithmic order and type of some indeterminate moment problems from the q-Askey scheme, which is discussed in Koekoek and Swarttouw[18]. To do so we apply results from Section 5 below. The moment problem associated with the q-Meixner polynomials, which we denote as $\{M_n(z+1; b, c; q)\}_n$ is indeterminate in the sense of Stieltjes. The four entire functions in the Krein parametrization have been computed in Theorem 1.3 in Christiansen[11]. In particular the function Q is shown to be given by

$$Q(z) = {}_1\phi_1\left(\begin{matrix} 1-z \\ bq \end{matrix}\middle| q; -q/c\right),$$

where $b < 1/q$ and $c > 0$. For the definition of basic hypergeometric series see Gasper and Rahman[12] or Koekoek and Swarttouw[18]. We denote the zeros of Q by $\{x_n\}$, where $0 > x_1 > x_2 > \ldots$. By a result of Bergweiler and Hayman[6] there is a constant $A > 0$ such that

$$x_n \sim -Aq^{-2n} \quad \text{as} \quad n \to \infty, \tag{5}$$

see Proposition 1.5 in Christiansen[11] for details.

Proposition 4.1. *The indeterminate Stieltjes moment problem associated with the q-Meixner polynomials have logarithmic order equal to 2 and logarithmic type equal to $-1/(4\log q)$.*

Proof. We see from (5) and Proposition 5.6 that Q has logarithmic order equal to 2 and logarithmic type equal to $-1/(4\log q)$. Then the result follows from Proposition 3.1. □

By specialization or taking limits of the parameters in the q-Meixner case we obtain:

Corollary 4.1. *The indeterminate Stieltjes moment problems associated with the q-Charlier, Al-Salam–Carlitz II, q-Laguerre and Stieltjes–Wigert polynomials are all of logarithmic order* 2 *and logarithmic type* $-1/(4\log q)$.

The Discrete q-Hermite II moment problem is symmetric and the corresponding Stieltjes moment problem is the q^2-Laguerre moment problem with $\alpha = \frac{1}{2}$. Applying Proposition 3.2 we get the following.

Corollary 4.2. *The indeterminate Hamburger moment problem associated with the Discrete q-Hermite II polynomials is of logarithmic order* 2 *and logarithmic type* $-1/(2\log q)$.

The q^{-1}-Hermite moment problem was treated in detail by Ismail and Masson[15]. The zeros of the function D are given explicitly as

$$x_n = \frac{1}{2}\left(q^n - q^{-n}\right), \quad n \in \mathbb{Z}.$$

Therefore the counting function satisfies

$$n(r) \sim \frac{2\log r}{-\log q}, \quad r \to \infty.$$

¿From Proposition 5.6 we obtain:

Proposition 4.2. *The indeterminate Hamburger moment problem associated with the Continuous* q^{-1}*-Hermite polynomials has logarithmic order* 2 *and logarithmic type equal to* $-1/\log q$.

5. The logarithmic growth scale

In this section we collect some facts about entire functions of finite logarithmic growth. Most of these facts can be found in the literature, but for the readers convenience we have included the proofs.

For an entire function f with Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

the (ordinary) order is 0 if and only if

$$\lim_{n\to\infty} \frac{\log n}{\log\left(\frac{1}{\sqrt[n]{|c_n|}}\right)} = 0.$$

One can also express the logarithmic order and type in terms of the Taylor coefficients.

Proposition 5.1. *For an entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ of order 0 its logarithmic order ρ satisfies*

$$\rho = 1 + \limsup_{n\to\infty} \frac{\log n}{\log\log\left(\frac{1}{\sqrt[n]{|c_n|}}\right)}.$$

Proof. We put

$$\mu = \liminf_{n\to\infty} \frac{\log|\log|c_n||}{\log n}$$

and we first show that

$$\mu = \frac{\rho}{\rho - 1}. \tag{6}$$

Suppose that ρ is finite and let $\lambda > \rho$. Then there exists $r_0 > 1$ such that

$$\log M(f,r) \leq (\log r)^\lambda, \quad \text{for} \quad r \geq r_0.$$

By applying the Cauchy estimates we find, for any $n \geq 0$ and $r \geq r_0$,

$$\log|c_n| \leq (\log r)^\lambda - n \log r.$$

The function

$$\varphi(r) = (\log r)^\lambda - n \log r$$

(defined for $r \geq 1$) attains its minimum for

$$r = \exp\left(\left(\frac{n}{\lambda}\right)^{1/(\lambda-1)}\right)$$

which is bigger than r_0 for $n \geq \lambda(\log r_0)^{\lambda-1}$. For such n the minimum value over $[r_0, \infty[$ is

$$\left(\frac{n}{\lambda}\right)^{1/(\lambda-1)} n(1/\lambda - 1),$$

which is a negative quantity. It follows that for all sufficiently large n

$$\log|c_n| \leq \left(\frac{n}{\lambda}\right)^{1/(\lambda-1)} n(1/\lambda - 1),$$

so that

$$\log|\log|c_n|| \geq \frac{\lambda}{\lambda-1}\log n + \log\left(\frac{1-1/\lambda}{\lambda^{1/(\lambda-1)}}\right),$$

or

$$\frac{\log|\log|c_n||}{\log n} \geq \frac{\lambda}{\lambda-1} + \frac{\log\left(\frac{1-1/\lambda}{\lambda^{1/(\lambda-1)}}\right)}{\log n},$$

for all sufficiently large n. Hence

$$\mu = \liminf_{n\to\infty}\frac{\log|\log|c_n||}{\log n} \geq \frac{\lambda}{\lambda-1}.$$

Since this holds for any $\lambda > \rho$ we must have $\mu \geq \rho/(\rho-1)$. Notice that $\mu = \infty$ if $\rho = 1$ and also that $\rho = \infty$ if $\mu = 1$.

Conversely, if $\mu > 1$ we choose $\nu \in (1, \mu)$ and next n_0 such that

$$\frac{\log|\log|c_n||}{\log n} > \nu,$$

for all $n \geq n_0$. This implies $|c_n| < e^{-n^\nu}$ for $n \geq n_0$. Then we have, for $|z| = r \geq 1$,

$$\begin{aligned}|f(z)| &\leq \sum_{n=0}^{n_0-1}|c_n|r^n + \sum_{n=n_0}^{\infty}|c_n|r^n \\ &\leq \mathrm{Const}\, r^{n_0} + \sum_{n=n_0}^{\infty} e^{-n^\nu} r^n.\end{aligned}$$

For given r so large that $n_0^{\nu-1} - 1 < \log r$ we choose n_1, depending on r, such that $n_1 \geq n_0 + 1$ and

$$(n_1-1)^{\nu-1} - 1 < \log r \leq n_1^{\nu-1} - 1.$$

(This is possible since $\nu > 1$.) For $n \geq n_1$ we thus have $\log r \leq n^{\nu-1} - 1$ so that $\log r^n \leq n^\nu - n$, or $r^n \leq e^{n^\nu}e^{-n}$. This yields

$$\sum_{n=n_1}^{\infty} e^{-n^\nu} r^n \leq \sum_{n=n_1}^{\infty} e^{-n} < 1.$$

For $n \in \{n_0, \ldots, n_1 - 1\}$ we have $n^{\nu-1} - 1 < \log r$ so that

$$n < (\log r + 1)^{1/(\nu-1)}.$$

Therefore

$$\begin{aligned}\sum_{n=n_0}^{n_1-1} e^{-n^\nu} r^n &= \sum_{n=n_0}^{n_1-1} e^{n\log r} e^{-n^\nu} \\ &< \sum_{n=n_0}^{n_1-1} e^{(\log r+1)^{1/(\nu-1)}\log r} e^{-n^\nu} \\ &= e^{(\log r+1)^{1/(\nu-1)}\log r} \sum_{n=n_0}^{n_1-1} e^{-n^\nu} \\ &< e^{(\log r+1)^{1/(\nu-1)}\log r} \sum_{n=1}^{\infty} e^{-n}.\end{aligned}$$

This gives, for r sufficiently large,

$$\begin{aligned}\log M(f,r) &\le \log\left(\mathrm{Const}\, r^{n_0} + 1 + e^{(\log r+1)^{1/(\nu-1)}\log r}\right) \\ &\le \mathrm{Const}\,(\log r)^{1+1/(\nu-1)} \;=\; \mathrm{Const}\,(\log r)^{\nu/(\nu-1)}.\end{aligned}$$

Since ν was an arbitrary number between 1 and μ we conclude that f has logarithmic order $\le \mu/(\mu-1)$. We have therefore verified the relation (6) and from it we get

$$\begin{aligned}\rho &= 1 + \frac{1}{\mu-1} \\ &= 1 + \limsup_{n\to\infty} \frac{1}{\frac{\log|\log|c_n||}{\log n} - 1} \\ &= 1 + \limsup_{n\to\infty} \frac{\log n}{\log|\log|c_n|| - \log n} \\ &= 1 + \limsup_{n\to\infty} \frac{\log n}{\log\log\left(\frac{1}{\sqrt[n]{|c_n|}}\right)}.\end{aligned}$$

□

Remark 5.1. The logarithmic order is $\rho'(2)$ in the notation of Shah and Ishaq[27] and the $\rho(2,2)$ order of Juneja, Kapoor and Bajpai[16].

Proposition 5.2. *For an entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ of logarithmic order $\rho \in (1,\infty)$ its logarithmic type τ satisfies*

$$\tau = \frac{(\rho-1)^{\rho-1}}{\rho^\rho} \limsup_{n\to\infty} \frac{n}{\left(\log \frac{1}{\sqrt[n]{|c_n|}}\right)^{\rho-1}}.$$

Proof. Suppose that $M(f,r) \le e^{K(\log r)^\rho}$ for $r \ge r_0$. From the Cauchy estimates we see that

$$\log|c_n| \le K(\log r)^\rho - n\log r, \quad r \ge r_0.$$

The function $r \mapsto K(\log r)^\rho - n\log r$ attains its minimum for $r \ge 1$ when $K\rho(\log r)^{\rho-1} = n$, i.e. when $\log r = (n/(K\rho))^{\frac{1}{\rho-1}}$. For all sufficiently large n we must therefore have

$$\log|c_n| \le (\log r)\left(K(\log r)^{\rho-1} - n\right) = \left(\frac{n}{K\rho}\right)^{\frac{1}{\rho-1}} n\left(\frac{1}{\rho} - 1\right),$$

so that

$$|\log|c_n||^{\rho-1} \ge \frac{n^\rho}{K\rho}\left(1 - \frac{1}{\rho}\right)^{\rho-1},$$

or

$$K \ge \frac{1}{\rho}\left(1 - \frac{1}{\rho}\right)^{\rho-1} \frac{n^\rho}{|\log|c_n||^{\rho-1}} = \frac{(\rho-1)^{\rho-1}}{\rho^\rho} \frac{n}{\left(\log\frac{1}{\sqrt[n]{|c_n|}}\right)^{\rho-1}}.$$

Since this holds for all sufficiently large n and K is an arbitrary number greater than the logarithmic type τ we must have

$$\tau \ge \frac{(\rho-1)^{\rho-1}}{\rho^\rho} \limsup_{n\to\infty} \frac{n}{\left(\log\frac{1}{\sqrt[n]{|c_n|}}\right)^{\rho-1}}.$$

For the converse we argue as follows and put

$$\sigma = \limsup_{n\to\infty} \frac{n}{\left(\log\frac{1}{\sqrt[n]{|c_n|}}\right)^{\rho-1}} = \limsup_{n\to\infty} \frac{n^\rho}{|\log|c_n||^{\rho-1}}.$$

Let $\varepsilon > 0$ be given. We choose n_0 such that for all $n \ge n_0$ we have

$$\frac{n^\rho}{|\log|c_n||^{\rho-1}} \le \sigma + \varepsilon,$$

which means that $\log|c_n| \le -(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} n^{\frac{\rho}{\rho-1}}$.

Hence, for $|z| = r \ge 1$,

$$\begin{aligned} |f(z)| &\le \sum_{n=0}^{n_0-1} |c_n| r^n + \sum_{n=n_0}^{\infty} |c_n| r^n \\ &\le \mathrm{Const}\, r^{n_0} + \sum_{n=n_0}^{\infty} e^{-(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} n^{\frac{\rho}{\rho-1}} + n\log r}. \end{aligned}$$

When r is so large that

$$-(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} n_0^{\frac{\rho}{\rho-1}} + n_0 \log r > -n_0,$$

we choose the smallest integer $n_1 = n_1(r) > n_0$ such that for $n \geq n_1$

$$-(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} n^{\frac{\rho}{\rho-1}} + n \log r \leq -n.$$

This implies first of all that $n_1 - 1 < (\sigma+\varepsilon)(\log r + 1)^{\rho-1}$, but to treat the n's between n_0 and $n_1 - 1$ we look at the concave function of s

$$\varphi(s) = s \log r - (\sigma+\varepsilon)^{-\frac{1}{\rho-1}} s^{\frac{\rho}{\rho-1}}.$$

We find

$$\varphi'(s) = \log r - \frac{\rho}{\rho-1}(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} s^{\frac{1}{\rho-1}} = 0$$

for

$$s = s_0(r) = \left(\frac{\rho-1}{\rho}\right)^{\rho-1} (\sigma+\varepsilon)(\log r)^{\rho-1}.$$

Furthermore, $\varphi(s)$ attains its maximum at $s = s_0(r)$ and

$$\varphi(s_0(r)) = \frac{(\rho-1)^{\rho-1}}{\rho^\rho}(\sigma+\varepsilon)(\log r)^\rho.$$

We thus get

$$\sum_{n=n_0}^{n_1-1} e^{-(\sigma+\varepsilon)^{-\frac{1}{\rho-1}} n^{\frac{\rho}{\rho-1}} + n \log r} \leq (\sigma+\varepsilon)(1+\log r)^{\rho-1} \cdot e^{\varphi(s_0(r))}.$$

Therefore, for $|z| = r$ sufficiently large,

$$\begin{aligned} M(f,r) &\leq \mathrm{Const}\, r^{n_0} + 1 + (\sigma+\varepsilon)(\log r + 1)^{\rho-1} e^{\varphi(s_0(r))} \\ &\leq \mathrm{Const}\, r^{n_0} + 1 + (\sigma+\varepsilon)(\log r + 1)^{\rho-1} \cdot \\ &\quad \exp\left(\frac{(\rho-1)^{\rho-1}}{\rho^\rho}(\sigma+\varepsilon)(\log r)^\rho\right), \end{aligned}$$

and hence

$$M(f,r) \leq_{\text{as}} \exp\left(\frac{(\rho-1)^{\rho-1}}{\rho^\rho}(\sigma+2\varepsilon)(\log r)^\rho\right).$$

Therefore the logarithmic type τ satisfies

$$\tau \leq \frac{(\rho-1)^{\rho-1}}{\rho^\rho}(\sigma+2\varepsilon),$$

and letting $\varepsilon \to 0$, we see that

$$\tau \leq \frac{(\rho-1)^{\rho-1}}{\rho^\rho}\sigma.$$

$\square$

Remark 5.2. The logarithmic type is the $T(2,2)$ type of Juneja, Kapoor and Bajpai[17].

Example 5.1. We let $q \in \mathbb{C}$ and suppose that $0 < |q| < 1$. The basic hypergeometric series

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle| q; z\right),$$

defines an entire function of z when $r \leq s$ and the parameters are such that none of the denominators become zero. We assume also that ${}_r\phi_s$ is not a polynomial.

The logarithmic order of ${}_r\phi_s$ is equal to 2, as can be seen from Proposition 5.1. The logarithmic type is equal to

$$\frac{1}{2(1+s-r)\log 1/|q|},$$

which follows from Proposition 5.2.

In particular,

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1-zq^k) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n}\, z^n = {}_0\phi_0\left(\begin{matrix} - \\ - \end{matrix}\middle| q; z\right)$$

is of logarithmic order 2 and logarithmic type $1/(2\log 1/|q|)$.

Example 5.2. We let $q \in \mathbb{C}$ and suppose that $0 < |q| < 1$. Then

(1)

$$f(z) = \sum_{n=0}^{\infty} q^{n^{\frac{\alpha}{\alpha-1}}}\, z^n$$

is (for $\alpha > 1$) of logarithmic order α and logarithmic type equal to

$$\tau = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{1}{(\log 1/|q|)^{\alpha-1}}.$$

(2)

$$f(z) = \sum_{n=0}^{\infty} q^{e^n} z^n$$

is of logarithmic order 1 and infinite logarithmic type.

(3)

$$f(z) = \sum_{n=0}^{\infty} q^{n(\log n)^2} z^n$$

is of order zero, but its logarithmic order is ∞.

A transcendental entire function f of ordinary order less than 1 must have infinitely many zeros, which we label $\{a_n\}$ and number according to increasing order of magnitude and repeating each zero according to its multiplicity. We suppose that $f(0) = 1$ and from Hadamard's factorization theorem, we get that

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

The growth of an entire function of ordinary order less than 1 is thus in principle determined by the zero distribution. We shall use the following quantities to describe this distribution.

We define the usual zero counting function $n(r)$ as

$$n(r) = \#\{n \geq 1 \,|\, |a_n| \leq r\}.$$

We define the following quantities in terms of the zero counting function

$$N(r) = \int_0^r \frac{n(t)}{t}\, dt,$$

and

$$Q(r) = r \int_r^{\infty} \frac{n(t)}{t^2}\, dt.$$

These quantities are related to $M(r) = M(f, r)$ in the following way

$$N(r) \leq \log M(r) \leq N(r) + Q(r) \tag{7}$$

for $r > 0$. (See e.g. the relation (3.5.4) in Boas[9]).

If f is of (ordinary) order 0 we get from Hadamard's first theorem that the convergence exponent of the zeros is also equal to 0. It means that we have

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\varepsilon}} < \infty$$

for all $\varepsilon > 0$. In this situation we define the logarithmic convergence exponent ρ_l as

$$\rho_l = \inf \left\{ \alpha > 0 \,\middle|\, \sum_{n=1}^{\infty} \frac{1}{(|\log|a_n|| + 1)^{\alpha}} < \infty \right\}.$$

The following proposition expresses the logarithmic convergence exponent in terms of the logarithmic order of the zero counting function.

Proposition 5.3. *We have*

$$\rho_l = \limsup_{r\to\infty} \frac{\log n(r)}{\log\log r}.$$

Proof. First of all, we see by integration by parts that for $a > 0$ and $r > r_0 > 1$,

$$\int_{r_0}^{r} \frac{dn(t)}{(\log t)^a} = \frac{n(r)}{(\log r)^a} - \frac{n(r_0)}{(\log r_0)^a} + a\int_{r_0}^{r} \frac{n(t)}{(\log t)^{a+1}}\frac{dt}{t}. \tag{8}$$

To ease notation we let

$$L = \limsup_{r\to\infty} \frac{\log n(r)}{\log\log r}.$$

If $\alpha > L$ we choose $a \in (L, \alpha)$ and notice that $\frac{n(r)}{(\log r)^a}$ is bounded and hence that $\lim_{r\to\infty} n(r)/(\log r)^{\alpha} = 0$. Furthermore, since

$$\int_{r_0}^{r} \frac{n(t)}{(\log t)^{\alpha+1}}\frac{dt}{t} = \int_{r_0}^{r} \frac{n(t)}{(\log t)^{a}} \frac{dt}{t(\log t)^{\alpha-a+1}},$$

the limit of the integral on the right-hand side of (8) (with a replaced by α) as $r \to \infty$ is finite. Therefore

$$\sum_{n\geq n_0} \frac{1}{(\log|a_n|)^{\alpha}} = \int_{|a_{n_0}|}^{\infty} \frac{dn(t)}{(\log t)^{\alpha}} < \infty$$

and consequently $\rho_l \leq \alpha$ and thus we obtain that $\rho_l \leq L$.

Conversely, if $a > \rho_l$ we have

$$\int_{r_0}^{\infty} \frac{dn(t)}{(\log t)^a} < \infty.$$

If we look again at (8) it means that

$$\frac{n(r)}{(\log r)^a}$$

remains bounded as $r \to \infty$, hence $L \leq a$. We conclude that $\rho_l \geq L$. $\square$

It is also possible to relate the logarithmic order and logarithmic convergence exponent. The proposition below is mentioned in the assumptions of Theorem 3.6.1 in Boas[9] in the special case where $\rho = 2$.

Proposition 5.4. *For an entire function of order 0 we have*

$$\rho = \rho_l + 1.$$

To prove this proposition we give two lemmas.

Lemma 5.1. *Suppose that $n(r) \leq const\,(\log r)^\alpha$ for $r > 1$ and some $\alpha > 0$. Then, for $r > 1$,*

$$N(r) \leq const\,(\log r)^{\alpha+1}$$

and (where $[\cdot]$ denotes the integer part)

$$Q(r) \leq const\left(\sum_{l=0}^{[\alpha]} \alpha \cdot \ldots \cdot (\alpha - l + 1)\,(\log r)^{\alpha-l} + \delta(r)\right),$$

where $0 \leq \delta(r) \leq const\,(\log r)^{\alpha-[\alpha]-1}$.

Proof. By definition we have

$$N(r) \leq \text{const } + \int_1^r \frac{n(t)}{t}\,dt \leq \text{const } (\log r)^\alpha \int_1^r \frac{1}{t}\,dt = \text{const } (\log r)^{\alpha+1}.$$

Concerning $Q(r)$ we have

$$Q(r) = r\int_r^\infty \frac{n(t)}{t^2}\,dt \leq \text{const}\,r \int_r^\infty \frac{(\log t)^\alpha}{t^2}\,dt.$$

Here, by repeated integrations by parts,

$$\begin{aligned} r\int_r^\infty \frac{(\log t)^\alpha}{t^2}\,dt &= r\left\{\left[-\frac{(\log t)^\alpha}{t}\right]_r^\infty + \int_r^\infty \frac{\alpha(\log t)^{\alpha-1}}{t^2}\,dt\right\} \\ &= (\log r)^\alpha + \alpha r\int_r^\infty \frac{(\log t)^{\alpha-1}}{t^2}\,dt \\ &\;\vdots \\ &= \sum_{l=0}^{[\alpha]} \alpha \cdot \ldots \cdot (\alpha - l + 1)\,(\log r)^{\alpha-l} + \delta(r), \end{aligned}$$

where

$$\delta(r) = \alpha \cdot \ldots \cdot (\alpha - [\alpha]) \cdot r\int_r^\infty \frac{(\log t)^{\alpha-[\alpha]-1}}{t^2}\,dt.$$

Since the exponent $\alpha - [\alpha] - 1$ is negative we find

$$\delta(r) \leq \text{const}\,(\log r)^{\alpha - [\alpha] - 1}.$$

$\square$

Lemma 5.2. *If, for some $\alpha > 0$, $\log M(r) \leq$ const$(\log r)^{\alpha+1}$ then $n(r) \leq$ const$(\log r)^{\alpha}$.*

Proof. Since $N(r) \leq \log M(r)$ we have

$$n(r)\log r = n(r)\int_r^{r^2}\frac{dt}{t} \leq N(r^2) \leq \text{const}\,\left(\log r^2\right)^{\alpha+1} \leq \text{const}\,(\log r)^{\alpha+1}.$$

Hence $n(r) \leq \text{const}\,(\log r)^{\alpha}$. $\square$

Proof of Proposition 5.4. Suppose that $\alpha > \rho_l$. From Proposition 5.3 we have $n(r) \leq (\log r)^{\alpha}$ for all sufficiently large r. From (7) and Lemma 5.1 we thus have $\log M(r) \leq \text{const}\,(\log r)^{\alpha+1}$, and therefore we see that $\rho \leq \alpha + 1$. From this we conclude that $\rho \leq \rho_l + 1$.

On the other hand, if $\beta > \rho$ then $\log M(r) \leq (\log r)^{\beta}$ for all sufficiently large r. By Lemma 5.2 we therefore have $n(r) \leq \text{const}\,(\log r)^{\beta-1}$, so that $\rho_l \leq \beta - 1$. We have shown that $\rho_l \leq \rho - 1$. $\square$

It is also possible to relate the logarithmic type to the growth of the zero counting function. For an entire function of finite logarithmic order $\rho > 1$ we put

$$\kappa = \kappa(f) = \limsup_{r\to\infty}\frac{n(r)}{(\log r)^{\rho-1}}.$$

Proposition 5.5. *For an entire function of finite logarithmic order $\rho > 1$ we have the following relation between the quantity κ and the logarithmic type τ:*

$$\tau \leq \kappa/\rho \leq e\tau.$$

Proof. For any given $\varepsilon > 0$ we choose $r_0 > 1$ such that

$$n(r) \leq (\kappa + \varepsilon)(\log r)^{\rho-1},$$

for $r \geq r_0$. Then we get

$$\begin{aligned} N(r) &= \int_0^{r_0}\frac{n(t)}{t}\,dt + \int_{r_0}^{r}\frac{n(t)}{t}\,dt \\ &\leq \text{Const} + (\kappa+\varepsilon)\int_{r_0}^{r}\frac{(\log t)^{\rho-1}}{t}\,dt \\ &\leq \text{Const} + \frac{\kappa+\varepsilon}{\rho}(\log r)^{\rho}. \end{aligned}$$

Since we have $Q(r) \leq \text{Const}\,(\log r)^{\rho-1}$ we see that

$$\log M(f,r) \leq N(r) + Q(r) \leq \text{Const} + \frac{\kappa+\varepsilon}{\rho}(\log r)^{\rho} + \text{Const}\,(\log r)^{\rho-1}.$$

Therefore $\tau \leq \kappa/\rho$.

For $\varepsilon > 0$ we have $\log M(f,r) \leq (\tau+\varepsilon)(\log r)^{\rho}$ for $r \geq r_0$ and hence, for any $s > 1$,

$$n(r)(s-1)\log r \leq \int_r^{r^s} \frac{n(t)}{t}\,dt \leq N(r^s) \leq \log M(r^s) \leq (\tau+\varepsilon)s^{\rho}(\log r)^{\rho}.$$

This gives

$$n(r) \leq (\tau+\varepsilon)\frac{s^{\rho}}{s-1}(\log r)^{\rho-1},$$

so that

$$\kappa \leq \tau\frac{s^{\rho}}{s-1}.$$

It is easily found that the function $\varphi(s) = s^{\rho}/(s-1)$, $s > 1$ attains its minimum for $s = \rho/(\rho-1)$ and that the minimum is

$$\varphi\left(\frac{\rho}{\rho-1}\right) = \frac{\rho^{\rho}}{(\rho-1)^{\rho-1}}.$$

Hence

$$\kappa \leq \tau\frac{\rho^{\rho}}{(\rho-1)^{\rho-1}}.$$

Since $(\rho/(\rho-1))^{\rho-1} \leq e$ we finally see that $\kappa \leq \tau\rho e$. $\square$

We shall now see that $\tau = \kappa/\rho$ provided that the zero distribution has some regularity.

Proposition 5.6. *Let f be an entire function of finite logarithmic order $\rho > 1$. Then the following are equivalent for $r \to \infty$.*

(i) $n(r) \sim \lambda(\log r)^{\rho-1}$.
(ii) $\log M(r) \sim \frac{\lambda}{\rho}(\log r)^{\rho}$.

Proof. Since the function $Q(r)$ in (7) is $O((\log r)^{\rho-1})$ under each of the conditions *(i)*,*(ii)*, we have for $\lambda > 0$

$$\log M(r) \sim \frac{\lambda}{\rho}(\log r)^{\rho} \quad \Leftrightarrow \quad N(r) \sim \frac{\lambda}{\rho}(\log r)^{\rho}.$$

It is therefore enough to show that

$$n(r) \sim \lambda(\log r)^{\rho-1} \quad \Leftrightarrow \quad N(r) \sim \frac{\lambda}{\rho}(\log r)^{\rho}.$$

We have

$$N(r) = \int_0^{r_0} \frac{n(t)}{t}\,dt + \int_{r_0}^{r} \frac{(\log t)^{\rho-1}}{t}\left(\frac{n(t)}{(\log t)^{\rho-1}} - \lambda\right) dt + \lambda \int_{r_0}^{r} \frac{(\log t)^{\rho-1}}{t}\,dt.$$

If $n(r) \sim \lambda(\log r)^{\rho-1}$ then we choose r_0 such that

$$\left|\frac{n(r)}{(\log r)^{\rho-1}} - \lambda\right| \le \varepsilon,$$

for $r \ge r_0$, and this gives

$$\left|N(r) - \lambda \int_{r_0}^{r} \frac{(\log t)^{\rho-1}}{t}\,dt\right| \le \mathrm{Const} + \frac{\varepsilon}{\rho}(\log r)^{\rho}.$$

Therefore

$$\lim_{r\to\infty} \frac{N(r)}{(\log r)^{\rho}} = \frac{\lambda}{\rho}.$$

For $s > 1$ we have

$$(s-1)n(r)\log r \le \int_r^{r^s} \frac{n(t)}{t}\,dt = N(r^s) - N(r).$$

Therefore, if $N(r) \sim \frac{\lambda}{\rho}(\log r)^{\rho}$ we find

$$\begin{aligned}
\frac{(s-1)n(r)\log r}{(\log r)^{\rho}} &\le \frac{N(r^s)}{(\log r)^{\rho}} - \frac{N(r)}{(\log r)^{\rho}} \\
&= s^{\rho}\frac{N(r^s)}{(\log r^s)^{\rho}} - \frac{N(r)}{(\log r)^{\rho}} \\
&= (s^{\rho}-1)\left(\frac{\lambda}{\rho} + o(1)\right),
\end{aligned}$$

and conclude that

$$\limsup_{r\to\infty} \frac{n(r)}{(\log r)^{\rho-1}} \le \frac{\lambda}{\rho}\frac{s^{\rho}-1}{s-1}.$$

If we let s tend to 1 we find $\kappa \le \lambda$. We next use the relation

$$(s-1)n(r^s)\log r \ge \int_r^{r^s} \frac{n(t)}{t}\,dt = N(r^s) - N(r)$$

to find

$$\liminf_{r\to\infty} \frac{n(r)}{(\log r)^{\rho-1}} \ge \frac{\lambda}{\rho}\frac{s^{\rho}-1}{(s-1)s^{\rho-1}},$$

and therefore

$$\kappa = \lim_{r\to\infty} \frac{n(r)}{(\log r)^{\rho-1}} = \lambda.$$

□

6. Appendix: The Phragmén-Lindelöf indicator of some functions of order zero

This appendix was written by Walter Hayman during the "International Conference on Difference Equations, Special Functions and Applications" held in Munich, Germany in the period July 25 - July 30, 2005. We appreciate that Haymans result could be included in this appendix.

Introduction and statement of results

Suppose that $f(z)$ is a transcendental entire function. We write

$$m(r) = \inf_{|z|=r} |f(z)|, \qquad M(r) = \sup_{|z|=r} |f(z)|, \tag{9}$$

for the minimum and maximum modulus of f respectively.

Next we define a function $\Psi(r)$ to be of slow growth (s.g.) if $\Psi(r)$ is positive nondecreasing in $[0,\infty)$ and

$$\Psi(2r) \sim \Psi(r), \quad \text{as} \quad r\to\infty. \tag{10}$$

It follows immediately from (10) that

$$\Psi(Kr) \sim \Psi(r), \quad \text{as} \quad r\to\infty, \tag{11}$$

whenever $K > 1$. For we may take $K = 2^p$, for $p = 1, 2, \ldots$, and prove the result by induction on p, using (10). Since Ψ is increasing, (11) then follows also for $2^p < K < 2^{p+1}$.

We can now state our results.

Theorem 6.1. *Suppose that with the above hypotheses,*

$$\limsup_{r\to\infty} \frac{\log M(r)}{\Psi(r)} = \alpha < \infty. \tag{12}$$

Then

$$\int_r^{2r} \log\left(\frac{M(t)}{m(t)}\right)\frac{dt}{t} = o(\Psi(r)), \tag{13}$$

as $r\to\infty$.

Corollary 6.1.

$$\limsup_{r\to\infty} \frac{\log m(r)}{\Psi(r)} = \alpha. \tag{14}$$

Corollary 6.2.

$$h_\Psi(\theta) = \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{\Psi(r)} = \alpha, \tag{15}$$

for $0 \le \theta \le 2\pi$.

Some preliminary results

We assume from now on that $f(0) = 1$. Otherwise we apply our conclusions to $f(z)/(cz^p)$, where p is a nonnegative integer and c a non-zero constant. This does not affect the relations (12), (13), (14) and (15).

Next it follows from (11) that

$$\Psi(r) = o(r^\rho), \quad \text{as} \quad r \to \infty \tag{16}$$

whenever ρ is positive, cf. Theorem 1.3.1 in Bingham, Goldie and Teugels[8]. Now (12) shows that f has zero order. Thus by Hadamard's theorem

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu}\right), \tag{17}$$

where z_ν are the zeros of f. We deduce that, for $|z| = r$,

$$\prod_{\nu=1}^{\infty} \left| \left(1 - \frac{r}{|z_\nu|}\right) \right| \le |f(z)| \le \prod_{\nu=1}^{\infty} \left(1 + \frac{r}{|z_\nu|}\right).$$

Hence

$$\log \frac{M(r)}{m(r)} \le \sum_{\nu=1}^{\infty} \log \frac{r + |z_\nu|}{|r - |z_\nu||}, \quad 0 < r < \infty. \tag{18}$$

We now have

Lemma 6.1. *If* $n(r)$ *denotes the number of zeros of* f *in* $|z| \le r$, *then*

$$n(r) = o(\Psi(r)), \quad \textit{as} \quad r \to \infty.$$

In fact we have by Jensen's formula, cf. (7), if $K > 1$ and $\varepsilon > 0$,

$$\int_0^{Kr} \frac{n(t)dt}{t} \le \log M(Kr) \le (\alpha + \varepsilon)\Psi(r), \quad r > r_0(K, \varepsilon),$$

using (11) and (12). Hence

$$n(r)\log K \le \int_r^{Kr} \frac{n(t)dt}{t} \le (\alpha+\varepsilon)\Psi(r), \quad r > r_0.$$

This yields Lemma 6.1, since K can be chosen as large as we please.

Lemma 6.2. *For $s > 0$*

$$\int_0^\infty \log\left|\frac{1+\frac{s}{t}}{1-\frac{s}{t}}\right| \frac{dt}{t} = \frac{\pi^2}{2}.$$

We first put $s/t = x$ so that our integral becomes

$$\int_0^\infty \log\left|\frac{1+x}{1-x}\right| \frac{dx}{x} = \int_0^1 \log\left(\frac{1+x}{1-x}\right)\frac{dx}{x} + \int_0^1 \log\left(\frac{1+y}{1-y}\right)\frac{dy}{y},$$

where we have put $x = 1/y$, when $x > 1$. Also

$$\int_0^1 \log\left(\frac{1+x}{1-x}\right)\frac{dx}{x} = 2\sum_{m=1}^\infty \int_0^1 \frac{x^{2m-2}}{2m-1}dx = 2\sum_{m=1}^\infty \left(\frac{1}{2m-1}\right)^2 = \frac{\pi^2}{4}.$$

This proves Lemma 6.2.

Lemma 6.3. *We have*

$$\sum_{|z_\nu|>2r} \log\frac{1+\frac{r}{|z_\nu|}}{1-\frac{r}{|z_\nu|}} = o(\Psi(r)), \quad \text{as} \quad r\to\infty.$$

We can write the sum as

$$\begin{aligned}\int_{2r}^\infty \log\left(\frac{1+\frac{r}{t}}{1-\frac{r}{t}}\right)dn(t) &= \left[n(t)\log\left(\frac{1+\frac{r}{t}}{1-\frac{r}{t}}\right)\right]_{2r}^\infty + 2r\int_{2r}^\infty \frac{n(t)dt}{t^2-r^2}\\ &= -n(2r)\log 3 + o\left\{r\int_{2r}^\infty \frac{\Psi(t)dt}{t^2}\right\} \qquad (19)\end{aligned}$$

by Lemma 6.1. Also it follows from (11) that, for $p = 1, 2, \ldots$ and large r,

$$\begin{aligned}\int_{r2^p}^{r2^{p+1}} \frac{\Psi(t)dt}{t^2} &< \int_{r2^p}^{r2^{p+1}} \frac{\Psi(2r)2^{p/2}dt}{t^2}\\ &= \frac{\Psi(2r)2^{p/2}}{r2^{p+1}}\\ &= \frac{\Psi(2r)}{2r}2^{-p/2},\end{aligned}$$

since for large r (and $p \geq 1$), $\Psi(2r2^p) < 2^{1/2}\Psi(2r2^{p-1}) < \ldots < 2^{p/2}\Psi(2r)$ by (10). We get

$$r \int_{2r}^{\infty} \frac{\Psi(t)dt}{t^2} < \frac{\Psi(2r)}{2} \sum_{p=1}^{\infty} 2^{-p/2} = O\left(\Psi(r)\right).$$

Thus (19) and Lemma 6.1 yields Lemma 6.3.

Proof of the Theorem and its Corollaries

We deduce from Lemma 6.3 that, for $r < t < 2r$, we have

$$\sum_{|z_\nu|>4r} \log \frac{|z_\nu| + t}{|z_\nu| - t} = o\left\{\Psi(r)\right\}.$$

Thus

$$\int_r^{2r} \left(\sum_{|z_\nu|>4r} \log \frac{|z_\nu| + t}{|z_\nu| - t} \right) \frac{dt}{t} = o(\Psi(r)) \int_r^{2r} \frac{dt}{t} = o(\Psi(r)). \tag{20}$$

Again by Lemma 6.2

$$\int_r^{2r} \sum_{|z_\nu|\leq 4r} \log \left| \frac{|z_\nu| + t}{|z_\nu| - t} \right| \frac{dt}{t} \leq n(4r) \int_0^{\infty} \log \frac{1+t}{|1-t|} \frac{dt}{t}$$

$$\leq \frac{\pi^2}{2} n(4r) \;=\; o(\Psi(r)). \tag{21}$$

Putting together (18), (20) and (21) we deduce (13) and the theorem is proved.

To prove Corollary 6.1, we suppose given a positive ε and then choose a large r, such that

$$\log M(r) > (\alpha - \varepsilon)\Psi(r).$$

In view of (11) and the fact that $\log M(r)$ increases with r, we deduce that if r is sufficiently large

$$\log M(t) > (\alpha - 2\varepsilon)\Psi(t), \quad r \leq t \leq 2r. \tag{22}$$

Next it follows from (13) that we can choose t, such that $r \leq t \leq 2r$ and

$$\log m(t) > \log M(t) - \varepsilon\Psi(r) \geq \log M(t) - \varepsilon\Psi(t),$$

if r is sufficiently large.

On combining this with (22) we obtain

$$\log m(t) > (\alpha - 3\varepsilon)\Psi(t).$$

Since ε is arbitrarily small we obtain

$$\limsup_{t\to\infty} \frac{\log m(t)}{\Psi(t)} \geq \alpha.$$

Since $m(r) \leq M(r)$ we have from (12)

$$\limsup_{t\to\infty} \frac{\log m(t)}{\Psi(t)} \leq \alpha.$$

This proves Corollary 6.1.

Clearly for every θ

$$m(r) \leq |f(re^{i\theta})| \leq M(r).$$

Thus (15) follows from (12) and (14) and Corollary 6.2 is proven.

We remark that for $\Psi(r)$ we may take not only $(\log r)^\alpha$, but

$$(\log r)^\alpha \exp\{(\log r)^\beta (\log\log r)^\gamma\}$$

etc. provided that $\beta < 1$.

The conclusion (14) is clearly false if $\alpha = \infty$. We may take $f(z) = e^z$, $\Psi(r) = (\log r)^2$. Then $\alpha = \infty$ in (12) and

$$h_\Psi(\pi) = -\infty$$

and

$$\frac{\log m(r)}{\Psi(r)} \to -\infty \quad \text{as} \quad r \to \infty.$$

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