On the order of indeterminate moment problems

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Abstract

For an indeterminate moment problem we denote the orthonormal polynomials by P_n. We study the relation between the growth of the function $P(z) = (\sum_{n=0}^{\infty} |P_n(z)|^2)^{1/2}$ and summability properties of the sequence $(P_n(z))$. Under certain assumptions on the recurrence coefficients from the three term recurrence relation $zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1}P_{n-1}(z)$, we show that the function P is of order α with $0 < \alpha < 1$, if and only if the sequence $(P_n(z))$ is absolutely summable to any power greater than 2α. Furthermore, the order α is equal to the exponent of convergence of the sequence (b_n). Similar results are obtained for logarithmic order and for more general types of slow growth. To prove these results we introduce a concept of an order function and its dual.

We also relate the order of P with the order of certain entire functions defined in terms of the moments or the leading coefficient of P_n.

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1 Introduction and results

Stieltjes discovered the indeterminate moment problem in the memoir [25] from 1894, and one can follow his discoveries in the correspondence with Hermite, cf. [4]. Stieltjes only considered distribution functions on the half-line $[0, \infty)$ corresponding to what is now called the Stieltjes moment problem. It took about 25 years before Hamburger, Nevanlinna and Marcel Riesz laid the foundation of the Hamburger moment problem described by (1). Nevanlinna proved the Nevanlinna parametrization of the full set of solutions to the Hamburger moment problem. Using the four entire functions A, B, C, D, obtained from (3) by letting $n \to \infty$,

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any solution to the moment problem can be described via a universal parameter space, namely the one-point compactification of the space of Pick functions. Nevanlinna also pointed out what is now called the Nevanlinna extremal solutions corresponding to the degenerate Pick functions, which are a real constant or infinity. Since the same solutions appear in spectral theory for self-adjoint extensions of Jacobi-matrices, Simon [24] proposed to call them von-Neumann solutions. The classical monographs describing the Nevanlinna parametrization are [1],[23],[26]. None of these treatises contain a fully calculated example with concrete functions A, B, C, D. Although it was well known that the zeros of B, D interlace and similarly with A, C, nobody seem to have noticed that these functions have the same growth properties before it was done in [5]. In that paper it was proved that the four entire functions A, B, C, D as well as P, Q from Theorem 1.1 have the same order and type called the *order ρ and type τ of the indeterminate moment problem*. Long before, Marcel Riesz had proved the deep result that A, B, C, D are of minimal exponential type, i.e., that $0 \leq \rho \leq 1$ and if $\rho = 1$, then $\tau = 0$, cf. [1, p. 56].

I a series of papers in the beginning of the 1990'ies, Ismail-Masson [16], Chihara-Ismail [12], Berg-Valent [8] calculated a number of examples. One source of indeterminate moment problems is q-series, cf. [14], and formulas of Ramanujan, see [2]. The indeterminate moment problems within the q-Askey scheme were identified by Christiansen in [13]. All these moment problems have order zero, and in Ismail [15] it was conjectured that A, B, C, D should have the same growth properties on a more refined scala than ordinary order. This was proved in [6], by the introduction of a refined scale called logarithmic order and type, so we can speak about logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of a moment problem of order zero. In [21] it was proved that if (ρ, τ) or $(\rho^{[1]}, \tau^{[1]})$ are prescribed, then there exist indeterminate moment problems with these (logarithmic) orders and types. In Ramis [22] the notion of logarithmic order and type appears for entire solutions to q-difference equations.

The main achievement of the present paper is that we present some conditions on the coefficients $(a_n), (b_n)$ of the three term recurrence relation (2), such that when these hold, then summability properties of the sequence $(P_n^2(z))$ and order properties of the moment problem are equivalent. Furthermore, the order as well as the logarithmic order of the moment problem can be calculated from the growth properties of the sequence (b_n).

These conditions are of two different types. There is a *regularity condition* that (b_n) is either log-convex eventually or log-concave eventually, cf. (27) or (28), and a *growth condition* (29).

The last condition is also necessary in the symmetric case $a_n = 0$ because of Carleman's condition.

We shall now give a more detailed introduction to the content.

Consider a normalized Hamburger moment sequence (s_n) given as

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0, \tag{1}$$

where μ is a probability measure with infinite support and moments of any order.

Denote the corresponding orthonormal polynomials by $P_n(z)$ and those of the second kind by $Q_n(z)$, following the notation and terminology of [1]. These polynomials satisfy a three term recurrence relation of the form

$$zr_n(z) = b_n r_{n+1}(z) + a_n r_n(z) + b_{n-1} r_{n-1}(z), \quad n \geq 0, \tag{2}$$

where $a_n \in \mathbb{R}, b_n > 0$ for $n \geq 0$ and $b_{-1} = 1$, and with the initial conditions $P_0(z) = 1,\ P_{-1}(z) = 0$ and $Q_0(z) = 0,\ Q_{-1}(z) = -1$.

The following polynomials will be used, cf. [1, p.14]

$$\begin{array}{l} A_n(z) = z \sum_{k=0}^{n-1} Q_k(0) Q_k(z), \\ B_n(z) = -1 + z \sum_{k=0}^{n-1} Q_k(0) P_k(z), \\ C_n(z) = 1 + z \sum_{k=0}^{n-1} P_k(0) Q_k(z), \\ D_n(z) = z \sum_{k=0}^{n-1} P_k(0) P_k(z). \end{array} \tag{3}$$

We need the coefficients of the orthonormal polynomials

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k, \tag{4}$$

and by (2) we have

$$b_{n,n} = 1/(b_0 b_1 \cdots b_{n-1}) > 0. \tag{5}$$

The indeterminate case is characterized by the equivalent conditions in the following result, cf. [1, Section 1.3].

Theorem 1.1. *For (s_n) as in (1) the following conditions are equivalent:*

(i) $\sum_{n=0}^{\infty} (P_n^2(0) + Q_n^2(0)) < \infty$,

(ii) $P(z) = \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2} < \infty, \quad z \in \mathbb{C}$.

If (i) and (ii) hold (the indeterminate case), then $Q(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{1/2} < \infty$ for $z \in \mathbb{C}$, and P, Q are continuous functions.

Concerning order and type as well as logarithmic order and type of an (entire) function, we refer to Section 2, but we warn the reader that the logarithmic order treated in this paper differs from the logarithmic order of [6] by subtracting 1.

Our first main result extends Theorem 1.1. For $0 < \alpha$ we consider the complex linear sequence space

$$\ell^\alpha = \{(x_n) | \sum_{n=0}^{\infty} |x_n|^\alpha < \infty\}.$$

Theorem 1.2. *For a moment problem and $0 < \alpha \leq 1$ the following conditions are equivalent:*

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$,

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$. *Furthermore,* $(1/b_n) \in \ell^\alpha$ *and*

$$P(z) \leq C \exp(K|z|^\alpha), \tag{6}$$

where

$$C = \left(\sum_{n=0}^{\infty}(P_n^2(0) + Q_n^2(0))\right)^{1/2}, \quad K = \frac{1}{\alpha}\sum_{n=0}^{\infty}(|P_n(0)|^{2\alpha} + |Q_n(0)|^{2\alpha}). \tag{7}$$

In particular the moment problem has order $\rho \leq \alpha$, and if the order is α, then the type $\tau \leq K$.

Remark 1.3. The main point in Theorem 1.2 is that (i) or (ii) imply (6). The equivalence between (i) and (ii) is in principle known, since it can easily be deduced from formula [1.23a] in Akhiezer [1]. The theorem is proved in Section 4 as Theorem 4.7.

For an indeterminate moment problem the recurrence coefficients (b_n) satisfy $\sum 1/b_n < \infty$ by Carleman's Theorem. On the other hand the condition $\sum 1/b_n < \infty$ is not sufficient for indeterminacy, but if a condition of log-concavity is added, then indeterminacy holds by a result of Berezanskiĭ [3], see [1, p.26]. This result is extended in Section 4 to include log-convexity, leading to the following main result, which is an almost converse of Theorem 1.2 in the sense that (6) implies (i) and (ii) except for an ε, but under additional assumptions of the recurrence coefficients.

Theorem 1.4. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1 + |a_n|}{b_{n-1}} < \infty, \tag{8}$$

and that either (27) *or* (28) *holds. Assume in addition that P satisfies*

$$P(z) \leq C \exp(K|z|^\alpha)$$

for some α such that $0 < \alpha < 1$ and suitable constants $C, K > 0$.

Then

$$1/b_n, P_n^2(0), Q_n^2(0) = O(n^{-1/\alpha}), \tag{9}$$

so in particular $(1/b_n), (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$ *for any* $\varepsilon > 0$.

Theorem 1.4 is proved as Theorem 4.8, where we have replaced condition (8) by the slightly weaker condition (29). Under the same assumptions we prove in Theorem 4.11 that the order of the moment problem is equal to the convergence exponent of the sequence (b_n). In case of order zero it is also possible to characterize the logarithmic order of the moment problem as the convergence exponent of the sequence $(\log b_n)$, cf. Theorem 5.12.

In Section 5 the results of Theorem 1.2 and of Theorem 1.4 are extended to more general types of growth, based on a notion of an order function and its dual. See Theorem 5.8 and Theorem 5.9.

In Section 6 we focus on order functions of the form $\alpha(r) = (\log\log r)^\alpha$, which lead to the concept of double logarithmic order and type, giving a refined classification of entire functions and moment problems of logarithmic order 0. The six functions A, B, C, D, P, Q have the same double logarithmic order and type called the double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ of the moment problem.

We establish a number of formulas expressing the double logarithmic order and type of an entire function in terms of the coefficients in the power series expansion and the zero counting function. The proof of these results are given in the Appendix.

For an indeterminate moment problem the numbers

$$c_k = \left(\sum_{n=k}^{\infty} b_{k,n}^2\right)^{1/2}$$

were studied by the authors in [7], and c_k tends to zero so quickly that

$$\Phi(z) = \sum_{k=0}^{\infty} c_k z^k$$

determines an entire function of minimal exponential type. We study this function in Section 3 and prove that Φ has the same order and type as the moment problem, and if the common order is zero, then Φ has the same logarithmic order and type as the moment problem. This is extended to double logarithmic order and type in Section 6.

In Section 7 we revisit a paper [19] by Livšic, where it was proved that the function

$$F(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{s_{2n}}$$

has order less than or equal to the order of the entire function

$$B(z) = -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z).$$

We give a another proof of this result and extend it to logarithmic and double logarithmic order, using results about Φ. It seems to be unknown whether the order of F is always equal to the order of the moment problem. We prove in Theorem 7.5 that this the case, if the recurrence coefficients satisfy the conditions of Theorem 4.2, and at the same time it turns out that the entire function

$$H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n,$$

where $b_{n,n}$ is the leading coefficient of P_n, cf. (4), also has this common order.

2 Preliminaries

For a continuous function $f : \mathbb{C} \to \mathbb{C}$ we define the maximum modulus $M_f : [0, \infty[\to [0, \infty[$ by

$$M_f(r) = \max_{|z| \leq r} |f(z)|.$$

The order ρ_f of f is defined as the infimum of the numbers $\alpha > 0$ for which there exists a majorization of the form

$$\log M_f(r) \leq_{\text{as}} r^{\alpha},$$

where we use a notation inspired by [18], meaning that the above inequality holds for r sufficiently large. We will only discuss these concepts for unbounded functions f, so that $\log M_f(r)$ is positive for r sufficiently large.

It is easy to see that

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \,|\, \log M_f(r) \leq_{\text{as}} c r^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

The logarithmic order as defined in [6],[21] is a number in the interval $[1, \infty]$, and the functions studied in Ramis [22] are of logarithmic order 2. A detailed study of meromorphic functions of finite logarithmic order has been published in Chern [11].

We find it appropriate to renormalize this definition by subtracting 1, so the new logarithmic order of this paper belongs to the interval $[0, \infty]$. This will simplify certain formulas, which will correspond to formulas for the double logarithmic order developed in Section 6.

For an unbounded continuous function f we define the *logarithmic order* $\rho_f^{[1]}$ as

$$\rho_f^{[1]} = \inf\{\alpha > 0 \,|\, \log M_f(r) \leq_{\text{as}} (\log r)^{\alpha+1}\} = \inf\{\alpha > 0 \,|\, M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha}\},$$

where $\rho_f^{[1]} = \infty$, if there are no $\alpha > 0$ satisfying the asymptotic inequality. Of course $\rho_f^{[1]} < \infty$ is only possible for functions of order 0.

Note that an entire function f satisfying $\log M_f(r) \leq_{\text{as}} (\log r)^\alpha$ for some $\alpha < 1$ is constant by the Cauchy estimate

$$\frac{|f^{(n)}(0)|}{n!} \leq \frac{M_f(r)}{r^n}.$$

It is easy to obtain that

$$\rho_f^{[1]} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} - 1.$$

When $\rho_f^{[1]} < \infty$ we define the *logarithmic type* $\tau_f^{[1]}$ as

$$\begin{aligned} \tau_f^{[1]} &= \inf\{c > 0 \,|\, \log M_f(r) \leq_{\text{as}} c(\log r)^{\rho_f^{[1]}+1}\} \\ &= \inf\{c > 0 \,|\, M_f(r) \leq_{\text{as}} r^{c(\log r)^{\rho_f^{[1]}}}\}, \end{aligned}$$

and it is readily found that

$$\tau_f^{[1]} = \limsup_{r\to\infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}+1}}.$$

An entire function f satisfying $\rho_f^{[1]} = 0$ and $\tau_f^{[1]} < \infty$ is necessarily a polynomial of degree $\leq \tau_f^{[1]}$.

The shifted moment problem is associated with the cut off sequences (a_{n+1}) and (b_{n+1}) from (2). In terms of Jacobi matrices, the Jacobi matrix J_s of the shifted problem is obtained from the original Jacobi matrix J by deleting the first row and column. It is well-known that a moment problem and the shifted one are either both determinate or both indeterminate. If indeterminacy holds, Pedersen [20] studied the relationship between the A, B, C, D-functions of the two problems and deduced that the shifted moment problem has the same order and type as the original problem. We mention that the P-function of the shifted problem equals $b_0 Q(z)$. This equation shows that the two problems have the same logarithmic order and type in case the common order is zero.

By repetition, the N-times shifted problem is then indeterminate with the same growth properties as the original problem. This means that it is the large n behaviour of the recurrence coefficients which determine the order and type of an indeterminate moment problem. This is in contrast to the behaviour of the moments, where a modification of the zero'th moment can change an indeterminate moment problem to a determinate one, see e.g. [7, Section 5].

In the indeterminate case we can define an entire function of two complex variables

$$K(z,w)=\sum_{n=0}^{\infty}P_n(z)P_n(w)=\sum_{j,k=0}^{\infty}a_{j,k}z^jw^k, \tag{10}$$

called the *reproducing kernel* of the moment problem, and we collect the coefficients of the power series as the symmetric matrix $\mathcal{A}=(a_{j,k})$ given by

$$a_{j,k}=\sum_{n=\max(j,k)}^{\infty}b_{j,n}b_{k,n}. \tag{11}$$

It was proved in [7] that the series (11) is absolutely convergent and that the matrix $\mathcal{A}$ is of trace class with

$$\operatorname{tr}(\mathcal{A})=\rho_0,$$

where ρ_0 is given by

$$\rho_0=\frac{1}{2\pi}\int_0^{2\pi}K(e^{it},e^{-it})\,dt=\frac{1}{2\pi}\int_0^{2\pi}P^2(e^{it})\,dt<\infty. \tag{12}$$

Define

$$c_k=\sqrt{a_{k,k}}=\left(\sum_{n=k}^{\infty}b_{k,n}^2\right)^{1/2}. \tag{13}$$

From (4) we have

$$b_{k,n}=\frac{1}{2\pi i}\int_{|z|=r}P_n(z)z^{-(k+1)}\,dz=r^{-k}\frac{1}{2\pi}\int_0^{2\pi}P_n(re^{it})e^{-ikt}\,dt. \tag{14}$$

By (14) and by Parseval's identity we have for $r>0$

$$\sum_{k=0}^{\infty}r^{2k}\sum_{n=k}^{\infty}|b_{k,n}|^2=\sum_{n=0}^{\infty}\sum_{k=0}^{n}r^{2k}|b_{k,n}|^2=\sum_{n=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}|P_n(re^{it})|^2\,dt, \tag{15}$$

hence

$$\sum_{k=0}^{\infty}r^{2k}c_k^2=\frac{1}{2\pi}\int_0^{2\pi}P^2(re^{it})\,dt, \tag{16}$$

an identity already exploited in [7].

3 The order and type of Φ

The heading refers to the function

$$\Phi(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{17}$$

where c_k is defined in (13). By [7, Prop. 4.2] we know that $\lim_{k\to\infty} k\sqrt[k]{c_k} = 0$, which shows that Φ is an entire function of minimal exponential type.

Theorem 3.1. *The order and type of Φ are equal to the order ρ and type τ of the moment problem.*

Proof. By (4) and (11) we have

$$D(z) = z\sum_{k=0}^{\infty} P_k(0)P_k(z) = z\sum_{k=0}^{\infty} b_{0,k}\sum_{j=0}^{k} b_{j,k}z^j = z\sum_{j=0}^{\infty} a_{j,0}z^j. \tag{18}$$

Therefore,

$$|D(z)| \le |z|\sum_{j=0}^{\infty} |a_{j,0}||z|^j \le c_0|z|\sum_{j=0}^{\infty} c_j|z|^j, \tag{19}$$

where we used $|a_{j,k}| \le c_j c_k$. This leads to the following inequality for the maximum moduli

$$M_D(r) \le c_0 r M_\Phi(r), \tag{20}$$

from which we clearly get $\rho = \rho_D \le \rho_\Phi$.

Since $\rho_P = \rho$ (the order of the moment problem), we get for any $\varepsilon > 0$

$$P(re^{i\theta}) \le \exp(r^{\rho+\varepsilon}) \quad \text{for } \ r \ge R(\varepsilon).$$

Defining

$$\Psi(z) = \sum_{k=0}^{\infty} c_k^2 z^{2k}, \tag{21}$$

we get by (16)

$$M_\Psi(r) = \sum_{k=0}^{\infty} c_k^2 r^{2k} \le \exp(2r^{\rho+\varepsilon}) \le \exp(r^{\rho+2\varepsilon}) \quad \text{for } \ r \ge \max(R(\varepsilon), 2^{1/\varepsilon}),$$

hence $\rho_\Psi \le \rho + 2\varepsilon$ and finally $\rho_\Psi \le \rho$.

However, $\rho_\Psi = \rho_\Phi$ because for an entire function $f(z) = \sum_{n=0}^\infty a_n z^n$ it is known ([18]) that

$$\rho_f = \limsup_{n\to\infty} \frac{\log n}{\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)}. \tag{22}$$

This shows the assertion of the theorem concerning order.

Concerning type, let us assume that the common order of the moment problem and Φ is ρ, satisfying $0 < \rho < \infty$ in order to define type. For a function f as above with order ρ, the type τ_f can be determined as

$$\tau_f = \frac{1}{e\rho} \limsup_{n\to\infty} \left(n|a_n|^{\rho/n}\right), \tag{23}$$

cf. [18].

From (20) we get $\tau = \tau_D \le \tau_\Phi$, where τ is the type of the moment problem.

Since P has type τ, we know that $|P(re^{i\theta})| \le e^{(\tau+\varepsilon)r^\rho}$ for r sufficiently large depending on $\varepsilon > 0$, hence by (16)

$$M_\Psi(r) = \sum_{k=0}^\infty c_k^2 r^{2k} \le \exp(2(\tau+\varepsilon)r^\rho),$$

and we conclude that $\tau_\Psi \le 2\tau$. Fortunately $\tau_\Psi = 2\tau_\Phi$, as is easily seen from (23), so we get $\tau_\Phi \le \tau$, and the assertion about type has been proved. $\square$

Theorem 3.2. *Suppose the order of the moment problem is zero. Then Φ has the same logarithmic order $\rho^{[1]}$ and type $\tau^{[1]}$ as the moment problem.*

Proof. The logarithmic order $\rho_f^{[1]}$ of an entire function $f = \sum_0^\infty a_n z^n$ of order zero can be calculated as

$$\rho_f^{[1]} = \limsup_{n\to\infty} \frac{\log n}{\log\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)}, \tag{24}$$

cf. [6]. From (20) we want to see that $\rho^{[1]} = \rho_D^{[1]} \le \rho_\Phi^{[1]}$. This is clear if $\rho_\Phi^{[1]} = \infty$, so assume it to be finite. For any $\varepsilon > 0$ we have for r sufficiently large

$$M_D(r) \le c_0 r r^{(\log r)^{\rho_\Phi^{[1]}+\varepsilon}} \le r^{(\log r)^{\rho_\Phi^{[1]}+2\varepsilon}},$$

which gives the assertion.

We next use that for given $\varepsilon > 0$ we have for r sufficiently large

$$P(re^{i\theta}) \le r^{(\log r)^{\rho^{[1]}+\varepsilon}},$$

which by (16) yields

$$M_\Psi(r) \le r^{2(\log r)^{\rho^{[1]}+\varepsilon}} \le_{\rm as} r^{(\log r)^{\rho^{[1]}+2\varepsilon}},$$

hence $\rho_\Psi^{[1]} \le \rho^{[1]}$. From (24) we see that $\rho_\Phi^{[1]} = \rho_\Psi^{[1]}$, hence $\rho^{[1]} = \rho_\Phi^{[1]}$.

We next assume that the common value $\rho^{[1]}$ of the logarithmic order is a finite number > 0. (Transcendental function of logarithmic order 0 have necessarily logarithmic type ∞.) We shall show that $\tau^{[1]} = \tau_\Phi^{[1]}$ and recall that the logarithmic type $\tau_f^{[1]}$ of a function $f = \sum_0^\infty a_n z^n$ with logarithmic order $0 < \rho^{[1]} < \infty$ is given by the formula, cf. [6],

$$\tau_f^{[1]} = \frac{(\rho^{[1]})^{\rho^{[1]}}}{(\rho^{[1]}+1)^{\rho^{[1]}+1}} \limsup_{n\to\infty} \frac{n}{\left(\log \frac{1}{\sqrt[n]{|a_n|}}\right)^{\rho^{[1]}}}. \tag{25}$$

Again it is clear that $\tau_\Psi^{[1]} = 2\tau_\Phi^{[1]}$, and from (20) we get $\tau^{[1]} \le \tau_\Phi^{[1]}$, while (16) leads to $\tau_\Psi^{[1]} \le 2\tau^{[1]}$. This finally gives $\tau^{[1]} = \tau_\Phi^{[1]}$. □

4 Berezanskiĭ's method

We are going to use and extend a method due to Berezanskiĭ [3] giving a sufficient condition for indeterminacy. The method is explained in [1, p.26]. Berezanskiĭ treated the case below of log-concavity.

Lemma 4.1. *Let $b_n > 0, n \ge 0$ satisfy*

$$\sup_{n\ge 0} b_n = \infty \tag{26}$$

and either

$$\textit{log-convexity:} \qquad b_n^2 \le b_{n-1}b_{n+1}, \quad n \ge n_0, \tag{27}$$

or

$$\textit{log-concavity:} \qquad b_n^2 \ge b_{n-1}b_{n+1}, \quad n \ge n_0. \tag{28}$$

Then (b_n) is eventually strictly increasing to infinity.

Proof. Suppose first that (27) holds. For $n \ge n_0$, b_{n+1}/b_n is increasing, say to $\lambda \le \infty$. If $\lambda \le 1$, then b_n is decreasing for $n \ge n_0$ in contradiction to (26). Therefore $1 < \lambda \le \infty$ and for any $1 < \lambda_0 < \lambda$ we have $b_{n+1} \ge \lambda_0 b_n$ for n sufficiently large.

If (28) holds, then b_{n+1}/b_n is decreasing for $n \ge n_0$, say to $\lambda \ge 0$. If $\lambda < 1$ then $\sum b_n < \infty$ in contradiction to (26). Therefore $\lambda \ge 1$ and finally $b_{n+1} \ge b_n$ for $n \ge n_0$. If $b_n = b_{n-1}$ for some $n > n_0$, then (28) implies $b_n \ge b_{n+1}$, hence $b_n = b_{n+1}$, so (b_n) is eventually constant in contradiction to (26). □

Theorem 4.2 (Berezanskiĭ)**.** *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty, \tag{29}$$

and that either (27) *or* (28) *holds.* [1]

For any non-trivial solution (r_n) *of* (2) *there exists a constant* c*, depending on the* a_n, b_n *and the initial conditions* $(r_0, r_{-1}) \neq (0,0)$ *but independent of* z*, such that*

$$\sqrt{b_{n-1}}|r_n(z)| \le c\,\Pi(|z|), \quad \Pi(z) = \prod_{k=0}^{\infty}\left(1+\frac{z}{b_{k-1}}\right), \quad n \ge 0, \tag{30}$$

and there exists a constant $K_z > 0$ *for* $z \in \mathbb{C}$ *such that*

$$\max\{|r_n(z)|, |r_{n+1}(z)|\} \ge \frac{K_z}{\sqrt{b_{n+1}}}, \quad n \ge 0. \tag{31}$$

In particular,

$$P_n^2(0), Q_n^2(0) = O(1/b_{n-1}) \tag{32}$$

and

$$\frac{K}{b_{n+1}} \le |r_n(z)|^2 + |r_{n+1}(z)|^2 \le \frac{L}{b_{n-1}} \tag{33}$$

for suitable constants K, L *depending on* z*.*

The moment problem is indeterminate.

Proof. By Lemma 4.1 we have $b_{n-1} < b_n$ for $n \ge n_1 > n_0$.

By the recurrence relation we get

$$\frac{b_{n-1}}{b_n}|r_{n-1}(z)| - \frac{|z|+|a_n|}{b_n}|r_n(z)| \le |r_{n+1}(z)| \le \\ \frac{b_{n-1}}{b_n}|r_{n-1}(z)| + \frac{|z|+|a_n|}{b_n}|r_n(z)|. \tag{34}$$

Let

$$u_n = \sqrt{b_{n-1}}|r_n(z)|, \quad v_n = \max(u_n, u_{n-1}), \quad \varepsilon_n = \frac{|z|+|a_n|}{\sqrt{b_n b_{n-1}}}.$$

Since $(r_0, r_{-1}) \neq (0,0)$ we have $v_n > 0$ for $n \ge 1$, and by assumption $\varepsilon_n < 1$ for n sufficiently large depending on z, say for $n \ge n_z \ge n_1$.

[1] In [1] it is assumed that $|a_n| \le M$, $\sum 1/b_n < \infty$ and that (28) holds. The assertion (31) is not discussed.

From the second inequality in (34) we then get

$$u_{n+1} \leq \frac{b_{n-1}}{\sqrt{b_n b_{n-2}}} u_{n-1} + \varepsilon_n u_n \leq v_n(1+\varepsilon_n),$$

where the last inequality requires log-convexity, assumed for $n \geq n_0$. For $n \geq n_1$ we then get

$$v_{n+1} \leq (1+\varepsilon_n)v_n \leq \left(1 + \frac{|a_n|}{\sqrt{b_n b_{n-1}}}\right)\left(1 + \frac{|z|}{b_{n-1}}\right) v_n.$$

Therefore

$$v_{n_1+n}(z) \leq \prod_{k=n_1}^{\infty} \left(1 + \frac{|a_k|}{\sqrt{b_k b_{k-1}}}\right) \prod_{k=n_1}^{\infty} \left(1 + \frac{|z|}{b_{k-1}}\right) v_{n_1}(z), \quad n \geq 1,$$

and since

$$v_{n_1}(z) \prod_{k=0}^{n_1-1} (1 + |z|/b_{k-1})^{-1}$$

is bounded in the complex plane, we get (30) for $n > n_1$, hence for all n by modifying the constant. (Remember that $b_{-1} := 1$.)

From the first inequality in (34) we get for $n \geq n_z$ now using log-concavity

$$u_{n+1} \geq \frac{b_{n-1}}{\sqrt{b_n b_{n-2}}} u_{n-1} - \varepsilon_n u_n \geq u_{n-1} - \varepsilon_n u_n. \tag{35}$$

We claim that

$$v_{n+1} \geq (1-\varepsilon_n)v_n, \quad n \geq n_z.$$

This is clear if $v_n = u_n$, and if $v_n = u_{n-1}$, then $u_{n-1} \geq u_n$ so (35) gives $v_{n+1} \geq u_{n+1} \geq (1-\varepsilon_n)u_{n-1}$. For $n > n_z$ we then get

$$v_n \geq v_{n_z} \prod_{k=n_z}^{\infty} (1-\varepsilon_k) > 0,$$

hence $d := \inf_{n\geq 1} v_n > 0$. Therefore either $\sqrt{b_n}|r_{n+1}(z)| \geq d$ or $\sqrt{b_{n-1}}|r_n(z)| \geq d$, which shows (31) (even with the denominator $\sqrt{b_n}$).

We still have to prove the inequalities (30) and (31) when the assumptions of log-convexity and log-concavity are interchanged. To do so we change the definition of u_n to $u_n = \sqrt{b_n}|r_n(z)|$, and we get from the second inequality in (34)

$$u_{n+1} \leq \frac{\sqrt{b_{n-1}b_{n+1}}}{b_n} (u_{n-1} + \varepsilon_n u_n) \leq v_n(1+\varepsilon_n),$$

where the last inequality requires log-concavity, assumed for $n \geq n_0$. Therefore $v_{n+1} \leq (1+\varepsilon_n)v_n$, and (30) follows as above.

From the first inequality in (34) we similarly get

$$u_{n+1} \geq \frac{\sqrt{b_{n-1}b_{n+1}}}{b_n}\left(u_{n-1} - \varepsilon_n u_n\right).$$

We now claim that in the log-convex case

$$v_{n+1} \geq (1-\varepsilon_n)v_n, \quad n \geq n_z,$$

where $n \geq n_z$ implies $\varepsilon_n < 1$. This is clear if $v_n = u_n$, and if $v_n = u_{n-1}$ we have $u_{n-1} \geq u_n$, hence $u_{n-1} - \varepsilon_n u_n \geq (1-\varepsilon_n)u_{n-1} \geq 0$.

The proof is finished as in the first case.

From (30) we get for $z = 0$ with $r_n = P_n$ and $r_n = Q_n$ that (32) holds, and this implies indeterminacy by Theorem 1.1. Finally, (33) is obtained by combining (30) and (31). $\square$

Remark 4.3. The lower bound (31) for non-real z can be obtained differently based on the Christoffel-Darboux formula, cf. [1, p.9],

$$(\operatorname{Im} z)\sum_{k=0}^{n-1}|P_k(z)|^2 = b_{n-1}\operatorname{Im}\left[P_n(z)\overline{P_{n-1}(z)}\right].$$

Hence

$$\frac{|\operatorname{Im} z|}{b_{n-1}} \leq |P_{n-1}(z)|\,|P_n(z)|, \quad n \geq 1.$$

Similarly, we can get the same inequality with Q_n in place of P_n. So far we do not need any extra assumptions on the coefficients in the recurrence relation.

If we know that $r_n(z)$ is bounded above by $c\,\Pi(|z|)/\sqrt{b_{n-1}}$ for any solution of the recurrence relation, we immediately get

$$|P_n(z)| \geq \frac{|\operatorname{Im} z|}{c\,\Pi(|z|)\sqrt{b_n}}.$$

The same is true for Q_n in place of P_n.

Corollary 4.4. *Under the assumptions of Theorem 4.2 we have*

$$1/b_n, P_n^2(0), Q_n^2(0) = o(1/n).$$

Proof. Since (b_n) is eventually increasing by Lemma 4.1, we obtain from the convergence of $\sum 1/b_n$ that (n/b_n) tends to zero. Using (32) we see that also $(nP_n^2(0))$ and $(nQ_n^2(0))$ tend to zero. $\square$

Remark 4.5. Note that (29) is a weaker condition than (8) because (b_n) is eventually increasing.

By a theorem of Carleman, $\sum 1/b_n = \infty$ is a sufficient condition for determinacy, and it is well-known that there are determinate moment problems for which $\sum 1/b_n < \infty$. The converse of Carleman's Theorem holds under the additional conditions of Theorem 4.2.

We give next a family of examples of determinate symmetric moment problems for which $\sum 1/b_n < \infty$.

In the symmetric case $a_n = 0$ for all n, we have $P_{2n+1}(0) = Q_{2n}(0) = 0$, and it follows from (2) that

$$P_{2n}(0) = (-1)^n \frac{b_0 b_2 \cdots b_{2n-2}}{b_1 b_3 \cdots b_{2n-1}}, \quad Q_{2n+1}(0) = (-1)^n \frac{b_1 b_3 \cdots b_{2n-1}}{b_0 b_2 \cdots b_{2n}},$$

so the moment problem is determinate by Theorem 1.1 if and only if

$$\sum_{n=1}^{\infty} \left(\frac{b_0 b_2 \ldots b_{2n-2}}{b_1 b_3 \ldots b_{2n-1}} \right)^2 + \left(\frac{b_1 b_3 \ldots b_{2n-1}}{b_0 b_2 \ldots b_{2n}} \right)^2 = \infty. \tag{36}$$

If $\beta_n > 0$ is arbitrary such that $\sum 1/\beta_n < \infty$, then defining $b_{2n} = b_{2n+1} = \beta_n$ for $n \geq 0$, we get a symmetric moment problem which is determinate because of (36) since

$$\frac{b_0 b_2 \cdots b_{2n-2}}{b_1 b_3 \cdots b_{2n-1}} = 1.$$

Clearly $\sum 1/b_n < \infty$ and (b_n) does not satisfy the conditions (27) or (28).

Proposition 4.6. *Let $0 < \alpha \leq 1$, let $(u_n) \in \ell^\alpha$ be a sequence of positive numbers and define*

$$K := \sum_{n=1}^{\infty} u_n^\alpha.$$

Then

$$\prod_{n=1}^{\infty} (1 + r u_n) \leq \exp(\alpha^{-1} K r^\alpha).$$

Proof. The conclusion follows immediately from the inequalities below

$$1 + r u_n \leq (1 + r^\alpha u_n^\alpha)^{\frac{1}{\alpha}} \leq \exp(\alpha^{-1} r^\alpha u_n^\alpha).$$

□

We shall now prove Theorem 1.2, and in order to make the reading easier we repeat the result:

Theorem 4.7. *For a moment problem and $0 < \alpha \leq 1$ the following conditions are equivalent:*

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$*,*

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ *for all* $z \in \mathbb{C}$*.*

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$*. Furthermore,* $(1/b_n) \in \ell^\alpha$ *and*

$$P(z) \le C \exp(K|z|^\alpha), \tag{37}$$

where

$$C = \left(\sum_{n=0}^{\infty}(P_n^2(0) + Q_n^2(0))\right)^{1/2}, \quad K = \frac{1}{\alpha}\sum_{n=0}^{\infty}(|P_n(0)|^{2\alpha} + |Q_n(0)|^{2\alpha}). \tag{38}$$

In particular the moment problem has order $\rho \le \alpha$*, and if the order is* α*, then the type* $\tau \le K$*.*

Proof. Condition (ii) is clearly stronger than condition (i).

Assume next that (i) holds, and in particular the indeterminate case occurs because $\ell^\alpha \subseteq \ell^1$.

Following ideas of Simon [24], we can write (3) as

$$\begin{pmatrix} A_{n+1}(z) & B_{n+1}(z) \\ C_{n+1}(z) & D_{n+1}(z) \end{pmatrix} = \left[I + z\begin{pmatrix} -P_n(0)Q_n(0) & Q_n^2(0) \\ -P_n^2(0) & P_n(0)Q_n(0) \end{pmatrix}\right]\begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix}. \tag{39}$$

and evaluating the operator norm of the matrices gives

$$\begin{aligned} \left\| \begin{pmatrix} A_n(z) & B_n(z) \\ C_n(z) & D_n(z) \end{pmatrix} \right\| &\le \prod_{k=0}^{n-1}\left[1 + |z|(P_k^2(0) + Q_k^2(0))\right] \\ &\le \prod_{k=0}^{n-1}\left[1 + |z|P_k^2(0)\right] \prod_{k=0}^{n-1}\left[1 + |z|Q_k^2(0)\right]. \end{aligned}$$

In particular we have

$$\left.\begin{array}{l} \sqrt{|A_n(z)|^2 + |C_n(z)|^2} \\ \sqrt{|B_n(z)|^2 + |D_n(z)|^2} \end{array}\right\} \le \prod_{k=0}^{\infty}\left[1 + |z|P_k^2(0)\right] \prod_{k=0}^{\infty}\left[1 + |z|Q_k^2(0)\right]. \tag{40}$$

By Proposition 4.6 we obtain

$$\left.\begin{array}{l} \sqrt{|A_n(z)|^2 + |C_n(z)|^2} \\ \sqrt{|B_n(z)|^2 + |D_n(z)|^2} \end{array}\right\} \le \exp(\alpha^{-1}K(\alpha)|z|^\alpha), \tag{41}$$

where

$$K(\alpha)=\sum_{k=0}^{\infty}(|P_k(0)|^{2\alpha}+|Q_k(0)|^{2\alpha}). \tag{42}$$

We also have ([1, p.14])

$$P_n(z)=-P_n(0)B_n(z)+Q_n(0)D_n(z), \tag{43}$$

so by the Cauchy-Schwarz inequality

$$|P_n(z)|^2\le (P_n^2(0)+Q_n^2(0))(|B_n(z)|^2+|D_n(z)|^2). \tag{44}$$

Combined with (41) we get

$$|P_n(z)|^{2\alpha}\le (P_n^2(0)+Q_n^2(0))^{\alpha}\exp(2K(\alpha)|z|^{\alpha}), \tag{45}$$

which shows that $\sum_{n=0}^{\infty}|P_n(z)|^{2\alpha}$ converges uniformly on compact subsets of $\mathbb{C}$.

Similarly we have

$$Q_n(z)=-P_n(0)A_n(z)+Q_n(0)C_n(z),$$

leading to the estimate

$$|Q_n(z)|^{2\alpha}\le (P_n^2(0)+Q_n^2(0))^{\alpha}\exp(2K(\alpha)|z|^{\alpha}),$$

and the assertion $(Q_n^2(z))\in\ell^{\alpha}$. By (44) and (41) we also get

$$\begin{aligned}P^2(z)=\sum_{n=0}^{\infty}|P_n(z)|^2&\le\sum_{n=0}^{\infty}(P_n^2(0)+Q_n^2(0))(|B_n(z)|^2+|D_n(z)|^2)\\&\le\left(\sum_{n=0}^{\infty}(P_n^2(0)+Q_n^2(0))\right)\exp(2\alpha^{-1}K(\alpha)|z|^{\alpha}),\end{aligned} \tag{46}$$

showing (37), from which we clearly get that $\rho=\rho_P\le\alpha$, and if $\rho=\alpha$, then $\tau=\tau_P\le K$.

From the well-known formula

$$P_{n-1}(z)Q_n(z)-P_n(z)Q_{n-1}(z)=\frac{1}{b_{n-1}}, \tag{47}$$

cf. [1, p. 9], we get

$$\frac{2}{b_{n-1}}\le |P_{n-1}(z)|^2+|P_n(z)|^2+|Q_{n-1}(z)|^2+|Q_n(z)|^2, \tag{48}$$

hence

$$\frac{2^{\alpha}}{b_{n-1}^{\alpha}}\le |P_{n-1}(z)|^{2\alpha}+|P_n(z)|^{2\alpha}+|Q_{n-1}(z)|^{2\alpha}+|Q_n(z)|^{2\alpha},$$

which shows that $(1/b_n)\in\ell^{\alpha}$. □

We next give an almost converse theorem to Theorem 4.7, under the Berezanskiĭ assumptions. It is a slight sharpening of Theorem 1.4 because we have replaced (8) by (29).

Theorem 4.8. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that either (27) *or* (28) *holds. Assume in addition that P satisfies*

$$P(z) \le C \exp(K|z|^\alpha)$$

for some α such that $0 < \alpha < 1$ and suitable constants $C, K > 0$.

Then

$$1/b_n, P_n^2(0), Q_n^2(0) = O(n^{-1/\alpha}), \tag{49}$$

so in particular $(1/b_n), (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$ for any $\varepsilon > 0$.

Proof. Using that $b_{n-1} < b_n$ for $n \ge n_1$, we get $b := \min\{b_k\} > 0$. For $n \ge n_1$ we find

$$\frac{1}{b_{n-1}^{2n}} \le \frac{1}{b^{2n_1} b_{n-1}^{2(n-n_1)}} \le A b_{n,n}^2 \le A c_n^2, \tag{50}$$

where we have used (5), (13) and

$$A = \left(\frac{b_0 \cdots b_{n_1-1}}{b^{n_1}}\right)^2.$$

Next, (16) leads to

$$\sum_{n=n_1}^{\infty} \left(\frac{r}{b_{n-1}}\right)^{2n} \le A \sum_{n=0}^{\infty} c_n^2 r^{2n} = \frac{A}{2\pi} \int_0^{2\pi} P^2(re^{it})\, dt \le AC^2 \exp[2Kr^\alpha].$$

Therefore, for any $n \ge n_1, r > 0$

$$\frac{r}{b_{n-1}} \le (AC^2)^{1/2n} \exp[Kr^\alpha/n]. \tag{51}$$

For $r = n^{1/\alpha}$ we obtain

$$\frac{1}{b_{n-1}} = O(n^{-1/\alpha}),\ n \to \infty.$$

Now in view of (32) we get (49). □

Definition 4.9. For a sequence (z_n) of complex numbers for which $|z_n| \to \infty$, we introduce the *exponent of convergence*

$$\mathcal{E}(z_n) = \inf \left\{ \alpha > 0 \mid \sum_{n=n^*}^{\infty} \frac{1}{|z_n|^\alpha} < \infty \right\},$$

where $n^* \in \mathbb{N}$ is such that $|z_n| > 0$ for $n \geq n^*$.

The counting function of (z_n) is defined as

$$n(r) = \#\{n \mid |z_n| \leq r\}.$$

The following result is well-known, cf. [9],[18].

Lemma 4.10.

$$\mathcal{E}(z_n) = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.$$

Theorem 4.11. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that either (27) *or* (28) *holds.*

Then the order ρ *of the moment problem is given by* $\rho = \mathcal{E}(b_n)$.

Proof. We first show that $\mathcal{E}(b_n) \leq \rho_P$. This is clear if $\rho_P = 1$ because by assumption $\mathcal{E}(b_n) \leq 1$. If $\rho_P < 1$ then P satisfies

$$M_P(r) \leq_{as} \exp(r^\alpha)$$

for any $\alpha > \rho_P$. By (49) we then have $\sum 1/b_n^{\alpha+\varepsilon} < \infty$ for $\alpha > \rho_P$ and $\varepsilon > 0$, hence $\mathcal{E}(b_n) \leq \rho_P$.

By (30) we get for $r_n = P_n$

$$P(z) \leq c \left(\sum_{n=0}^{\infty} \frac{1}{b_{n-1}} \right)^{1/2} \Pi(|z|), \tag{52}$$

and the infinite product $\Pi(z)$ is an entire function of order equal to $\mathcal{E}(b_n)$ by Borel's Theorem, cf. [18], hence $\rho_P \leq \mathcal{E}(b_n)$. $\square$

Example 4.12. For $\alpha > 1$ let $b_n = (n+1)^\alpha, a_n = 0, n \geq 0$. The three-term recurrence relation (2) with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying (26) and (28). By Theorem 4.11 the order of the moment problem is $1/\alpha$.

Similarly, $b_n = (n+1)\log^\alpha(n+2), a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.

Theorem 4.7 and Theorem 4.8 can be generalized in order to capture much slower types of growth of the moment problem, as well as growth faster than any order. This is done in the following section.

5 Order functions

Definition 5.1. By an **order function**[2] we understand a continuous, positive and increasing function $\alpha : (r_0, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} \alpha(r) = \infty$ and such that the function $r/\alpha(r)$ is also increasing with $\lim_{r\to\infty} r/\alpha(r) = \infty$. Here $0 \le r_0 < \infty$.

If α is an order function , then so is $r/\alpha(r)$.

Definition 5.2. For an order function α as above, the function

$$\beta(r) = \frac{1}{\alpha(r^{-1})}, \quad 0 < r < r_0^{-1}$$

will be called the **dual function**. Since $\lim_{r\to 0} \beta(r) = 0$, we define $\beta(0) = 0$. Note that β as well as $r/\beta(r)$ are increasing.

Observe that the dual function satisfies

$$\beta(Kr) \le K\beta(r), \quad K > 1,\ 0 < Kr < r_0^{-1}, \tag{53}$$

$$\beta(r_1 + r_2) \le \beta(2\max(r_1, r_2)) \le 2\beta(\max(r_1, r_2)) \le 2\beta(r_1) + 2\beta(r_2), \tag{54}$$

for $2\max(r_1, r_2) < 1/r_0$.

Example 5.3. Order functions.

1. The function $\alpha(r) = r^\alpha$ with $0 < \alpha < 1$ satisfies the assumptions of an order function with $r_0 = 0$, and $\beta(r) = \alpha(r)$.

2. The function $\alpha(r) = \log^\alpha r$ with $\alpha > 0$ satisfies the assumptions of an order function with $r_0 = \exp(\alpha)$ and

$$\beta(r) = \frac{1}{(-\log r)^\alpha}.$$

3. The function $\alpha(r) = \log^\alpha \log r$ with $\alpha > 0$ is an order function with $r_0 > \mathrm{e}$ being the unique solution to $(\log r)\log\log r = \alpha$.

4. If α is an order function, the so are $c\alpha(r)$ and $\alpha(cr)$ for $c > 0$.

5. If α_1 and α_2 are order functions, then also $\alpha_1(\alpha_2(r))$ is an order function for r sufficiently large.

6. The function $\alpha(r) = (\log^\alpha r)\log^\beta \log r$ is an order function for any $\alpha, \beta > 0$, because

$$\frac{r}{\alpha(r)} = \left[\frac{r^{1/(\alpha+\beta)}}{(\alpha+\beta)\log r^{1/(\alpha+\beta)}}\right]^{\alpha+\beta} \left[\frac{\log r}{\log\log r}\right]^\beta$$

shows that $r/\alpha(r)$ is increasing for $r > r_0 := \exp(\max(e, \alpha + \beta))$.

[2] There is no direct relation between this concept and Valiron's concept of a proximate order studied in [18].

Definition 5.4. *Let α be an order function. A continuous unbounded function $f : \mathbb{C} \to \mathbb{C}$ is said to have order bounded by $\alpha(r)$ if*

$$M_f(r) \leq_{as} e^{K\alpha(r)\log r} = r^{K\alpha(r)},$$

for some constant K.

For f as above to have order bounded by $\alpha(r) = \log^\alpha r$ for some $\alpha > 0$, is the same as to have finite logarithmic order in the sense of Section 2.

Given an order function $\alpha : (r_0, \infty) \to \mathbb{R}$ and its dual β, we are in the following going to consider expressions $\beta(u_n)$, where $\{u_n\}$ is a sequence of non-negative numbers tending to zero. This means that $\beta(u_n)$ is only defined for n sufficiently large, so assertions like

$$\sum_n^\infty \beta(u_n) < \infty, \quad \beta(u_n) = O(1/n)$$

make sense. The first assertion means that

$$\sum_{n=N}^\infty \beta(u_n) < \infty$$

for one N (and then for all N) so large that $\beta(u_n)$ is defined for $n \geq N$.

We begin by proving two lemmas.

Lemma 5.5. *Let $\alpha : (r_0, \infty) \to (0, \infty)$ be an order function with dual function β and let $\{u_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $u_n \to 0$ and $u_n < 1/r_0$ for all $n \geq n_0$.*

For any number $r > 0$ let $A_r = \{n \mid u_n \geq r^{-1}\}$ and $N_r = \#A_r$.

(a) Assume $\sum_n^\infty \beta(u_n) < \infty$. Then $N_r = O(\alpha(r))$.

(b) Assume $N_r = O(\alpha(r))$. Then for any $\varepsilon > 0$

$$\sum_n^\infty \beta^{1+\varepsilon}(u_n) < \infty.$$

Proof. Let v_n be the decreasing rearrangement of the sequence u_n. Then

$$N_r = \#\{n \mid v_n \geq r^{-1}\},$$

and since $\beta(r)$ is increasing, we find for $r > r_0$

$$N_r \leq n_0 - 1 + \#\{n \geq n_0 \mid \beta(v_n) \geq \beta(r^{-1})\}.$$

(a) We have $\sum_n^\infty \beta(v_n) < \infty$, hence $n\beta(v_n) \to 0$ and thus $n\beta(v_n) \le K$ for $n \ge n_0$ and a suitable constant K. Furthermore,

$$\begin{aligned} N_r &\le n_0 - 1 + \#\left\{ n \ge n_0 \mid \frac{K}{n} \ge \beta(r^{-1}) \right\} \\ &= n_0 - 1 + \#\left\{ n \ge n_0 \mid n \le K\alpha(r) \right\}, \end{aligned}$$

showing that $N_r = O(\alpha(r))$.
(b) Assume $N_r = O(\alpha(r))$. Observing that $N_{v_n^{-1}} \ge n$ we get $n \le K\alpha(v_n^{-1})$, for n sufficiently large and suitable K, i.e., $\beta(v_n) = O(1/n)$, which implies the conclusion. □

Lemma 5.6. *Assume the conditions of Lemma 5.5(a). For $r > r_0$ we then have*

$$\log \prod_{n=1}^{\infty} (1 + ru_n) \le N_r[\log r + C] + \alpha(r) \sum_{n \notin A_{r_0}} \beta(u_n),$$

where $C = \max\{\log(2u_n)\}$.

Proof. For $n \in A_r$ we have $ru_n \ge 1$, hence

$$\log(1 + ru_n) \le \log 2ru_n = \log r + \log(2u_n) \le \log r + C.$$

Furthermore, for $r > r_0, n \notin A_r$ we have $u_n < r^{-1}$, and using that $s/\beta(s)$ is increasing leads to

$$ru_n = \frac{u_n}{r^{-1}} \le \frac{\beta(u_n)}{\beta(r^{-1})} = \alpha(r)\beta(u_n).$$

Thus, for $r > r_0$

$$\begin{aligned} \log \prod_{n=1}^{\infty} (1 + ru_n) &= \sum_{n \in A_r} \log(1 + ru_n) + \sum_{n \notin A_r} \log(1 + ru_n) \\ &\le N_r[\log r + C] + \sum_{n \notin A_r} \alpha(r)\beta(u_n) \le N_r[\log r + C] + \alpha(r) \sum_{n \notin A_{r_0}} \beta(u_n). \end{aligned}$$

□

Combining Lemma 5.5(a) and Lemma 5.6 gives immediately the following.

Proposition 5.7. *Let $\alpha : (r_0, \infty) \to (0, \infty)$ be an order function with dual function β, and let $\{u_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $u_n \to 0$ and $u_n < 1/r_0$ for all $n \ge n_0$. Under the assumption $\sum_n^\infty \beta(u_n) < \infty$*

$$\log \prod_{n=1}^{\infty} (1 + ru_n) = O(\alpha(r) \log r),$$

and in particular the entire function

$$f(z) = \prod_{n=1}^{\infty}(1 + zu_n)$$

has order bounded by α.

Theorem 4.7 and 4.8 can be considered as results about the order function $\alpha(r) = r^\alpha, 0 < \alpha < 1$.

Theorem 5.8 and 5.9 below are similar results for arbitrary order functions. The price for the generality is an extra log-factor, so the generalization is mainly of interest for orders of slower growth than $\alpha(r) = r^\alpha$. For the order $\alpha(r) = r^\alpha$ it is better to refer directly to the results of Section 4.

Theorem 5.8. *For an order function α with dual function β the following conditions are equivalent for a given indeterminate moment problem:*

(i) $\beta(P_n^2(0)), \beta(Q_n^2(0)) \in \ell^1$,

(ii) $\beta(|P_n(z)|^2), \beta(|Q_n(z)|^2) \in \ell^1$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, then the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$.

Furthermore, $\beta(1/b_n) \in \ell^1$ and P has order bounded by α.

Proof. Condition (ii) is clearly stronger than condition (i).

Assume next that (i) holds. By (45) for $\alpha = 1$

$$|P_n(z)|^2 \leq (P_n^2(0) + Q_n^2(0)) \exp(2K(1)|z|), \tag{55}$$

so by (53) and (54) we get for n sufficiently large

$$\beta(|P_n(z)|^2) \leq 2 \exp(2K(1)|z|) \left(\beta(P_n^2(0)) + \beta(Q_n^2(0))\right). \tag{56}$$

This shows that $\sum \beta(|P_n(z)|^2)$ converges uniformly on compact subsets of $\mathbb{C}$.

The assertion $\beta(|Q_n(z)|^2) \in \ell^1$ is proved similarly.

By (40) and Proposition 5.7 we obtain

$$\sqrt{|B_n(z)|^2 + |D_n(z)|^2} \leq \exp(L\alpha(|z|) \log |z|), \tag{57}$$

for some constant L and $|z|$ sufficiently large. Using (44) and (42) (with $\alpha = 1$) we then get for large $|z|$

$$P^2(z) = \sum_{n=0}^{\infty} |P_n(z)|^2 \leq K(1) \exp(2L\alpha(|z|) \log |z|),$$

which shows that P has order bounded by α.

From the inequality (48) we immediately get that $\beta(1/b_n) \in \ell^1$. $\square$

Theorem 5.9. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that either (27) *or* (28) *holds. Assume in addition that the function* $P(z)$ *has order bounded by some given order function* α.

(i) If there is $0 < \alpha < 1$ *so that* $r^\alpha \leq_{as} \alpha(r)$, *then*

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) = O\left(\frac{\log n}{n}\right).$$

(ii) If $\alpha(r^2) = O(\alpha(r))$, *then*

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) = O(1/n).$$

In both cases

$$\beta(1/b_n), \beta(P_n^2(0)), \beta(Q_n^2(0)) \in \ell^{1+\varepsilon}$$

for any $\varepsilon > 0$.

Proof. Inserting the estimate

$$M_P(r) \leq_{\text{as}} \exp(K\alpha(r)\log r)$$

in (16), we get

$$\sum_{k=0}^{\infty} r^{2k} c_k^2 \leq_{\text{as}} \exp(2K\alpha(r)\log r),$$

hence by (50)

$$\sum_{n=n_1}^{\infty} \left(\frac{r}{b_{n-1}}\right)^{2n} \leq_{\text{as}} A\exp(2K\alpha(r)\log r). \tag{58}$$

Choose $r_1 > \max(1, r_0)$ so large that the inequality in (58) holds for $r \geq r_1$. In particular we have

$$\frac{r}{b_{n-1}} \leq A^{1/2n}\exp((K/n)\alpha(r)\log r), \quad n \geq n_1, r \geq r_1. \tag{59}$$

Consider (i). For any $n > K\alpha(r_1)\log r_1$ it is possible by continuity of α to choose $r = r_n > r_1$ such that

$$K\alpha(r_n)\log r_n = n. \tag{60}$$

For sufficiently large n we then have

$$\frac{1}{b_{n-1}} \leq \frac{A^{1/(2n)}e}{r_n} < \frac{3}{r_n}.$$

Since β is increasing, we get for sufficiently large n by (53) and (60)

$$\beta(1/b_{n-1}) \leq \beta(3/r_n) \leq 3\beta(1/r_n) = \frac{3}{\alpha(r_n)} = \frac{3K\log r_n}{n}. \tag{61}$$

But (60) and the assumption $r^\alpha \leq_{\text{as}} \alpha(r)$ imply that $Kr_n^\alpha \log r_1 \leq n$, for large n. Thus $\log r_n = O(\log n)$, and by (61) we get

$$\beta(1/b_{n-1}) = O(\frac{\log n}{n}).$$

In view of (32) we get that $\beta(P_n^2(0)), \beta(Q_n^2(0)) = O(\log n/n)$.

We turn now to the case (ii), where $\alpha(r^2) = O(\alpha(r))$. For any $n > 2K\alpha(r_1)$ we now choose r_n such that

$$K\alpha(r_n) = \frac{n}{2}. \tag{62}$$

Then (59) yields

$$\frac{1}{b_{n-1}} \leq \frac{A^{1/2n}}{\sqrt{r_n}} < \frac{2}{\sqrt{r_n}}$$

for n sufficiently large. Thus

$$\beta(1/b_{n-1}) \leq \beta(2/\sqrt{r_n}) \leq 2\beta(1/\sqrt{r_n}) = \frac{2}{\alpha(\sqrt{r_n})}.$$

By assumption there exists $d > 0$ such that $\alpha(\sqrt{r_n}) \geq d\alpha(r_n)$ for n large enough. Thus in view of (62) we find

$$\beta(1/b_{n-1}) \leq \frac{2}{d\,\alpha(r_n)} = \frac{4K}{d\,n}.$$

As above, the conclusion follows from (32). $\square$

Remark 5.10. The following order functions satisfy the assumption (i) of Theorem 5.9:

$$\alpha(r) = r^\alpha,\ 0 < \alpha < 1, \qquad \alpha(r) = \frac{r}{\log^\alpha r},\ \alpha > 0.$$

On the other hand the functions

$$\alpha(r) = \log^\alpha r, \quad \alpha(r) = \log^\alpha \log r, \quad \alpha(r) = (\log^\alpha r)\log^\beta \log r,\ \alpha, \beta > 0$$

satisfy (ii).

Although $\alpha(r) = r/\log^\alpha r$ is an order function for any $\alpha > 0$, then an entire function f of order bounded by $\alpha(r)$ is only of minimal exponential type under the assumption $\alpha > 1$.

Example 5.11. Consider a moment problem of logarithmic order $\rho^{[1]}$ satisfying $0 < \rho^{[1]} < \infty$ and of finite logarithmic type $\tau^{[1]}$. Assume that a_n, b_n satisfy the conditions of Theorem 5.9. Then P has order bounded by the order $\alpha(r) = (\log r)^{\rho^{[1]}}$. Since the case (ii) occurs, and since $\beta(r) = \log^{-\rho^{[1]}}(1/r)$, we have

$$\log^{-\rho^{[1]}}(b_n), \log^{-\rho^{[1]}}(P_n^{-2}(0)), \log^{-\rho^{[1]}}(Q_n^{-2}(0)) = O(1/n).$$

Therefore

$$1/b_n, P_n^2(0), Q_n^2(0) = O(e^{-Cn^{1/\rho^{[1]}}})$$

for a suitable constant $C > 0$. From (55) we also get

$$|P_n^2(z)| = O(e^{-Cn^{1/\rho^{[1]}}}),$$

uniformly on compact subsets of $\mathbb{C}$. These results can be applied to Discrete q-Hermite II polynomials, where $a_n = 0,\ b_n = q^{-n-1/2}(1-q^{n+1})^{1/2}$, cf. [17], and to q^{-1}-Hermite polynomials, where $a_n = 0,\ b_n = (1/2)q^{-(n+1)/2}(1-q^{n+1})^{1/2}$, cf. [16]. In both cases $0 < q < 1$ and (b_n) is log-concave, $\rho^{[1]} = 1$.

In analogy with Theorem 4.11 the logarithmic order of an indeterminate moment problem of order zero can be determined by the growth of (b_n), provided the Berezanskiĭ conditions hold.

Theorem 5.12. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty$$

and that either (27) *or* (28) *holds. Assume further that the moment problem has order 0.*

Then the logarithmic order $\rho^{[1]}$ of the moment problem is given as $\rho^{[1]} = \mathcal{E}(\log b_n)$.

Proof. We first establish that $\rho^{[1]} \geq \mathcal{E}(\log b_n)$, which is clear if $\rho^{[1]} = \infty$. If $\rho^{[1]} < \infty$ we know that for every $\varepsilon > 0$

$$M_P(r) \leq_{\text{as}} r^{(\log r)^{\rho^{[1]}+\varepsilon}}.$$

In other words P has order bounded by $\alpha(r) = (\log r)^{\rho^{[1]}+\varepsilon}$, so by Theorem 5.9(ii) we know that

$$\beta(1/b_n) = \frac{1}{(\log b_n)^{\rho^{[1]}+\varepsilon}} \in \ell^{1+\varepsilon},$$

hence $\mathcal{E}(\log b_n) \leq (\rho^{[1]}+\varepsilon)(1+\varepsilon)$ for any $\varepsilon > 0$, thus $\mathcal{E}(\log b_n) \leq \rho^{[1]}$.

From (52) we get $\rho_P^{[1]} \leq \rho_\Pi^{[1]}$. However, $\rho_\Pi^{[1]} = \mathcal{E}(\log b_n)$ by Proposition 5.4 in [6]. $\square$

Example 5.13. For $a > 1, \alpha > 0$ let $b_n = a^{n^{1/\alpha}}$, and let $|a_n| \leq a^{cn^{1/\alpha}}$ for some $0 < c < 1$. The three-term recurrence relation (2) with these coefficients determine orthogonal polynomials of an indeterminate moment problem satisfying (26) and (27) or (28) according to

$$b_n^2 \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} b_{n-1}b_{n+1} \;\Leftrightarrow\; \left\{ \begin{array}{l} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{array} \right. .$$

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so by Theorem 4.11 and Theorem 5.12 the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$.

Example 5.14. For $a > 1$ and $\alpha > 0$ consider the product

$$f(r) = \prod_{n=1}^{\infty} \left(1 + \frac{r}{a^{n^{1/\alpha}}}\right)$$

appearing in Lemma 5.6 with $u_n = a^{-n^{1/\alpha}}$. Let

$$\alpha(r) = (\log^{\alpha} r)\,(\log\log r)^2$$

be an order function of the type considered in Example 5.3 (6). We can use $r_0 = \exp(\max(e, 2+\alpha))$ and $u_n < 1/r_0$ for $n > n_0$ with

$$n_0 = \left(\frac{\max(e, 2+\alpha)}{\log a}\right)^{\alpha}.$$

For $N_r = \#\{n \,|\, a^{n^{1/\alpha}} \leq r\}$ we have

$$\left(\frac{\log r}{\log a}\right)^{\alpha} - 1 < N_r \leq \left(\frac{\log r}{\log a}\right)^{\alpha}. \tag{63}$$

Moreover,

$$\beta(u_n) = \frac{1}{\alpha(u_n^{-1})} = \frac{1}{(\log a)^{\alpha}}\ \frac{1}{n\,[(1/\alpha)\log n + \log\log a]^2}$$

satisfies

$$C := \sum_{n > (1/\log a)^{\alpha}}^{\infty} \beta(u_n) < \infty.$$

The proof of Lemma 5.6 gives

$$\log f(r) \leq \sum_{n=1}^{N_r} \log\left(2\frac{r}{a^{n^{1/\alpha}}}\right) + C\alpha(r) = \sum_{n=1}^{N_r} \log\left(\frac{r}{a^{n^{1/\alpha}}}\right) + N_r \log 2 + C\alpha(r).$$

On the other hand

$$\log f(r) \geq \sum_{n=1}^{N_r} \log\left(1 + \frac{r}{a^{n^{1/\alpha}}}\right) \geq \sum_{n=1}^{N_r} \log\left(\frac{r}{a^{n^{1/\alpha}}}\right).$$

We have

$$\sum_{n=1}^{N_r} \log\left(\frac{r}{a^{n^{1/\alpha}}}\right) = N_r \log r - \log a \sum_{n=1}^{N_r} n^{1/\alpha}$$

and

$$\frac{1}{1+1/\alpha} N_r^{1+1/\alpha} \leq \sum_{n=1}^{N_r} n^{1/\alpha} \leq \frac{1}{1+1/\alpha} (N_r + 1)^{1+1/\alpha}.$$

Therefore, in view of (63) we get

$$\log f(r) = \frac{1}{(\alpha+1)(\log a)^\alpha} (\log r)^{1+\alpha} [1 + o(1)],$$

showing that the logarithmic order is α (as we already know from Example 5.13), and the logarithmic type is

$$\frac{1}{(\alpha+1)(\log a)^\alpha}.$$

Example 5.15. For $a, b > 1$ let $b_n = a^{b^n}$ and $|a_n| \leq a^{cb^n}$ with $bc < 1$. In this case (b_n) is logarithmic convex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0.

This motivates a study of functions bounded by the order function $\alpha(r) = (\log\log r)^\alpha$, considered in the next section.

6 Double logarithmic order

For an unbounded continuous function f we define the *double logarithmic order* $\rho_f^{[2]}$ as

$$\rho_f^{[2]} = \inf\{\alpha > 0 \,|\, M_f(r) \leq_{\text{as}} r^{(\log\log r)^\alpha}\},$$

where $\rho_f^{[2]} = \infty$, if there are no $\alpha > 0$ satisfying the asymptotic inequality. Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$.

In case $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$ we define the *double logarithmic type* as

$$\tau_f^{[2]} = \inf\{c > 0 \,|\, M_f(r) \leq_{\text{as}} r^{c(\log\log r)^{\rho^{[2]}}}\}.$$

Theorem 6.1. *For an indeterminate moment problem of logarithmic order zero the functions A, B, C, D, P, Q have the same double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ called the double logarithmic order and type of the moment problem.*

The proof of this result can be done exactly in the same way as the corresponding proof for logarithmic order and type in [6], so we leave the details to the reader.

For an entire transcendental function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of logarithmic order 0 the double logarithmic order and type can be expressed in terms of the coefficients a_n by the following formulas.

Theorem 6.2.

$$\rho_f^{[2]} = \limsup_{n\to\infty} \frac{\log n}{\log\log\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)}, \tag{64}$$

and if $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$

$$\tau_f^{[2]} = \limsup_{n\to\infty} \frac{n}{\left(\log\log\frac{1}{\sqrt[n]{|a_n|}}\right)^{\rho^{[2]}}}. \tag{65}$$

The proof is given in the Appendix.

The results of Section 3 about Φ can also be generalized:

Theorem 6.3. *Suppose the logarithmic order of the moment problem is zero. Then Φ has the same double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ as the moment problem.*

Proof. From the inequality $M_D(r) \le c_0 r M_\Phi(r)$, cf. (20), we get $\rho^{[2]} = \rho_D^{[2]} \le \rho_\Phi^{[2]}$. For any $\varepsilon > 0$ we have

$$P(re^{i\theta}) \le r^{(\log\log r)^{\rho^{[2]}+\varepsilon}}$$

for r sufficiently large, which by (16) leads to $\rho_\Psi^{[2]} \le \rho^{[2]}$, where Ψ is given by (21). From Theorem 6.2 we see that $\rho_\Phi^{[2]} = \rho_\Psi^{[2]}$ and hence $\rho^{[2]} = \rho_\Phi^{[2]}$. The proof concerning type follows using similar ideas. $\square$

Theorem 6.4. *Assume that the coefficients of* (2) *satisfy*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty$$

and that either (27) *or* (28) *holds.*

Then the double logarithmic order $\rho^{[2]}$ of the moment problem is given as $\rho^{[2]} = \mathcal{E}(\log\log b_n)$.

Proof. We first establish that $\rho^{[2]} \geq \mathcal{E}(\log\log b_n)$, which is clear if $\rho^{[2]} = \infty$. If $\rho^{[2]} < \infty$ we know that for every $\varepsilon > 0$

$$M_P(r) \leq_{\mathrm{as}} r^{(\log\log r)^{\rho^{[2]}+\varepsilon}}.$$

In other words P has order bounded by $\alpha(r) = (\log\log r)^{\rho^{[2]}+\varepsilon}$, so by Theorem 5.9(ii) we know that

$$\beta(1/b_n) = \frac{1}{(\log\log b_n)^{\rho^{[2]}+\varepsilon}} \in \ell^{1+\varepsilon},$$

hence $\mathcal{E}(\log\log b_n) \leq (\rho^{[2]}+\varepsilon)(1+\varepsilon)$ for any $\varepsilon > 0$, thus $\mathcal{E}(\log\log b_n) \leq \rho^{[2]}$.

From (52) we get $\rho_P^{[2]} \leq \rho_\Pi^{[2]}$, hence $\rho^{[2]} = \rho_P^{[2]} = \mathcal{E}(\log\log b_n)$, if we prove that $\rho_\Pi^{[2]} \leq \mathcal{E}(\log\log b_n)$. This is a consequence of Theorem 8.3, but follows directly in the following way: It is clear if $\mathcal{E}(\log\log b_n) = \infty$. If $\rho = \mathcal{E}(\log\log b_n) < \infty$ we use Proposition 5.7 for the order function $\alpha(r) = (\log\log r)^{\rho+\varepsilon}$ and $u_n = 1/b_n$, and since

$$\sum_n \beta(u_n) = \sum_n \frac{1}{(\log\log b_n)^{\rho+\varepsilon}} < \infty$$

we conclude that $\log M_\Pi(r) = O(\alpha(r)\log r)$, hence $\rho_\Pi^{[2]} \leq \rho$, because $\varepsilon > 0$ can be chosen arbitrarily small.

$\square$

Example 6.5. Consider

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{\exp(e^{n^{1/\alpha}})}\right),$$

where $0 < \alpha < \infty$. We prove that $\rho_f^{[2]} = \alpha$, $\tau_f^{[2]} = 1$. Note that $b_n = \exp(e^{n^{1/\alpha}})$ is eventually log-convex because $\exp(x^{1/\alpha})$ is convex for $x > (\alpha-1)^\alpha$ when $\alpha > 1$ and convex for $x > 0$ when $0 < \alpha \leq 1$. This means that the indeterminate moment problem with recurrence coefficients $a_n = 0$ and b_n as above has double logarithmic order equal to $\mathcal{E}(\log\log b_n) = \alpha$.

Define

$$\alpha(r) = (\log\log r)^{2\alpha},$$

which is an order function with $r_0 = \exp(\max(e, 2\alpha))$.

For $N_r = \#\{n \mid \exp(e^{n^{1/\alpha}}) \leq r\}$ we have

$$(\log\log r)^\alpha - 1 < N_r \leq (\log\log r)^\alpha. \tag{66}$$

Moreover, for $u_n = 1/b_n$ we have $\beta(u_n) = 1/\alpha(b_n) = 1/n^2$. Observe that $\max\{\log(2u_n)\} \leq 0$. Hence Lemma 5.6 gives

$$\log f(r) \leq N_r \log r + C\alpha(r),$$

where

$$C = \sum_{n \notin A_{r_0}}^{\infty} \beta(u_n) \le \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus

$$\log f(r) \le (\log\log r)^{\alpha} \log r + C(\log\log r)^{2\alpha}. \tag{67}$$

To minorize $\log f(r)$ we need

$$\begin{aligned}\sum_{n=1}^{N} e^{n^{1/\alpha}} &\le e^{N^{1/\alpha}} + \int_1^N e^{x^{1/\alpha}}\,dx = e^{N^{1/\alpha}} + \alpha \int_e^{e^{N^{1/\alpha}}} (\log t)^{\alpha-1} dt \\ &\le \begin{cases} e^{N^{1/\alpha}}(1+\alpha) & \text{for } 0<\alpha\le 1 \\ e^{N^{1/\alpha}}(1+\alpha N^{1-1/\alpha}) & \text{for } 1<\alpha. \end{cases}\end{aligned}$$

This gives

$$\begin{aligned}\log f(r) &\ge \sum_{n=1}^{N_r} \log\left(1+\frac{r}{\exp(e^{n^{1/\alpha}})}\right) \ge N_r \log r - \sum_{n=1}^{N_r} e^{n^{1/\alpha}} \\ &\ge \begin{cases} \log r\,((\log\log r)^{\alpha} - 2 - \alpha) & \text{for } 0<\alpha\le 1 \\ \log r\,((\log\log r)^{\alpha} - 2 - \alpha(\log\log r)^{\alpha-1}) & \text{for } 1<\alpha. \end{cases}\end{aligned}$$

These inequalities together with (67) leads to

$$\lim_{r\to\infty} \frac{\log f(r)}{(\log\log r)^{\alpha}\log r} = 1,$$

showing the assertion about double logarithmic order and type of f.

7 Livšic's function

For an indeterminate moment sequence (s_n) Livšic [19] considered the function

$$F(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{s_{2n}}. \tag{68}$$

It is entire of minimal exponential type because $\lim n/\sqrt[2n]{s_{2n}} = 0$, which holds by Carleman's criterion giving that

$$\sum_{n=0}^{\infty} 1/\sqrt[2n]{s_{2n}} < \infty.$$

Moreover, $\sqrt[2n]{s_{2n}}$ is increasing for $n \ge 1$.

Livšic proved that $\rho_F \le \rho$, where ρ is the order of the moment problem. It is interesting to know whether the equality sign holds. In fact, we do not know any example with $\rho_F < \rho$. We will rather consider a modification of Livšic's function given by

$$L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}. \tag{69}$$

It is easy to see that $\rho_L = \rho_F$ and that $\tau_F = 2\tau_L$ by the formulas (22) and (23).

We shall give a new proof of the inequality $\rho_F \le \rho$ using the function Φ from Section 2. We shall also consider the entire function

$$H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n, \tag{70}$$

where $b_{n,n}$ is the leading coefficient of P_n, cf. (4).

Proposition 7.1. *For an indeterminate moment problem of order ρ we have*

(i) $1 \le s_{2n} b_{n,n}^2 \le c_n^2 s_{2n}$.

(ii) $M_L(r) \le M_H(r) \le M_\Phi(r), \quad r \ge 0$.

(iii) $\rho_L \le \rho_H \le \rho_\Phi = \rho$.

(iv) $\rho_L^{[1]} \le \rho_H^{[1]} \le \rho_\Phi^{[1]} = \rho^{[1]}$, *provided* $\rho = 0$.

(v) $\rho_L^{[2]} \le \rho_H^{[2]} \le \rho_\Phi^{[2]} = \rho^{[2]}$, *provided* $\rho^{[1]} = 0$.

Proof. By orthogonality we have

$$1 = \int P_n^2(x)\, d\mu(x) = b_{n,n} \int x^n P_n(x)\, d\mu(x),$$

so by the Cauchy-Schwarz inequality

$$\frac{1}{b_{n,n}} \le \left(\int x^{2n}\, d\mu(x) \right)^{1/2} \left(\int P_n^2(x)\, d\mu(x) \right)^{1/2} = \sqrt{s_{2n}},$$

which gives the first inequality of (i). The second follows from (13).

The maximum modulus M_f for an entire function $f(z) = \sum a_n z^n$ with $a_n \ge 0$ is given by $M_f(r) = f(r)$, $r \ge 0$, and therefore (ii) follows from (i). Finally (iii), (iv) and (v) follow from (ii). □

The following result gives a sufficient condition for equality in Proposition 7.1.

Proposition 7.2. *If*

$$\log \sqrt[2n]{c_n^2 s_{2n}} = o(\log n)$$

and in particular if

$$c_n^2 s_{2n} = O(K^n)$$

for some $K > 1$, then $\rho = \rho_L$.

If $\rho = 0$ then $\rho^{[1]} = \rho_L^{[1]}$, and if $\rho^{[1]} = 0$ then $\rho^{[2]} = \rho_L^{[2]}$.

Proof. Given $\varepsilon > 0$ we have for n sufficiently large

$$\log \sqrt[2n]{s_{2n}} \le \varepsilon \log n + \log \frac{1}{\sqrt[n]{c_n}}. \tag{71}$$

Dividing by $\log n$ leads to

$$\liminf_{n\to\infty} \frac{\log \sqrt[2n]{s_{2n}}}{\log n} \le \varepsilon + \liminf_{n\to\infty} \frac{\log \frac{1}{\sqrt[n]{c_n}}}{\log n},$$

so by (22)

$$\frac{1}{\rho_L} \le \varepsilon + \frac{1}{\rho},$$

but this gives $\rho \le \rho_L$.

From (71) we get

$$\log\log \sqrt[2n]{s_{2n}} \le \log\log \frac{1}{\sqrt[n]{c_n}} + \log\left(1 + \frac{\varepsilon \log n}{\log \frac{1}{\sqrt[n]{c_n}}}\right).$$

If $\rho = 0$ the last term tends to 0, and dividing by $\log n$ we get as above $\rho^{[1]} \le \rho_L^{[1]}$. Similarly, if $\rho^{[1]} = 0$ we find $\rho^{[2]} \le \rho_L^{[2]}$. $\square$

In the next results we shall use the function

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_n^n}, \tag{72}$$

which is entire if $b_n \to \infty$.

Lemma 7.3. *Suppose that the recurrence coefficients of* (2) *satisfy*

(i) $a_n = O(b_n)$,

(ii) (b_n) *is eventually increasing,*

(iii) $b_n \to \infty$.

Then there exist constants $A, C \geq 1$ such that

$$\sqrt{s_{2n}} \leq A(3C)^n b_0 b_1 \cdots b_{n-1}, n \geq 0. \tag{73}$$

Proof. Because of the assumption (i) there exists a constant $C \geq 1$ such that $|a_n| \leq C b_n$ for all $n \geq 0$. By (ii) there exists $n_0 \geq 1$ such that $b_{n-1} \leq b_n$ for $n \geq n_0$ and by (iii) there exists $n_1 \geq n_0$ such that $b_{n_1} \geq \max(1, b_0, \ldots, b_{n_0-1})$, hence

$$B := \max(1, b_0, \ldots, b_{n_1-1}) \leq b_{n_1}. \tag{74}$$

The three term recurrence relation (2) for P_n applied successively leads to

$$\begin{aligned} x &= a_0 P_0 + b_0 P_1, \\ x^2 &= x(a_0 P_0 + b_0 P_1) = a_0(a_0 P_0 + b_0 P_1) + b_0(b_0 P_0 + a_1 P_1 + b_1 P_2), \end{aligned}$$

and in general there exist an index set I_n with $|I_n| \leq 3^n$, a mapping J_n from I_n to $\{0, 1, \ldots, n\}$ and real coefficients $d_{n,k}, k \in I_n$ such that

$$x^n = \sum_{k \in I_n} d_{n,k} P_{J_n(k)}. \tag{75}$$

In the next step we get

$$x^{n+1} = \sum_{k \in I_n} d_{n,k} \left(b_{J_n(k)-1} P_{J_n(k)-1} + a_{J_n(k)} P_{J_n(k)} + b_{J_n(k)} P_{J_n(k)+1} \right),$$

which shows how each element $k \in I_n$ gives rise to two or three elements in I_{n+1} depending on $J_n(k) = 0$ or $J_n(k) > 0$.

Each $d_{n,k}$ is a product of n terms from $\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}\}$, hence

$$|d_{n,k}| \leq C^n (\max(b_0, \ldots, b_{n-1}))^n.$$

For $n \leq n_1$ we have in particular $|d_{n,k}| \leq (BC)^n \leq B^{n_1} C^n$.

We claim that in general

$$|d_{n,k}| \leq B^{n_1} C^n b_{n_1} \cdots b_{n-1}, \quad k \in I_n,\ n \geq 1, \tag{76}$$

which is already established for $n \leq n_1$, where the empty product $b_{n_1} \cdots b_{n-1}$ is to be understood as 1. Assume now that (76) holds for some $n \geq n_1$. If $J_n(k) \geq n_1$ we have

$$\begin{aligned} |d_{n,k}| b_{J_n(k)-1} &\leq |d_{n,k}| b_{J_n(k)} \leq |d_{n,k}| b_n \leq B^{n_1} C^n b_{n_1} \cdots b_{n-1} b_n \\ |d_{n,k}||a_{J_n(k)}| &\leq C|d_{n,k}| b_{J_n(k)} \leq B^{n_1} C^{n+1} b_{n_1} \cdots b_{n-1} b_n, \end{aligned}$$

while if $J_n(k) \leq n_1 - 1$

$$\begin{aligned} |d_{n,k}| b_{J_n(k)-1}, |d_{n,k}| b_{J_n(k)} &\leq |d_{n,k}| B \leq B^{n_1} C^n b_{n_1} \cdots b_{n-1} b_n \\ |d_{n,k}||a_{J_n(k)}| &\leq C|d_{n,k}| b_{J_n(k)} \leq B^{n_1} C^{n+1} b_{n_1} \cdots b_{n-1} b_n, \end{aligned}$$

where we have used that $B \le b_{n_1} \le b_n$. This finishes the induction proof of (76), which may be written

$$|d_{n,k}| \le AC^n b_0 b_1 \cdots b_{n-1}, \quad k \in I_n \; n \ge 1,$$

where $A = B^{n_1}/(b_0 b_1 \cdots b_{n_1-1})$.

Now (73) follows because

$$\begin{aligned} s_{2n} &= \int x^{2n}\,d\mu(x) = \sum_{k\in I_n}\sum_{l\in I_n} d_{n,k}d_{n,l}\int P_{J_n(k)}P_{J_n(l)}\,d\mu(x) \\ &\le \sum_{k\in I_n}\sum_{l\in I_n} |d_{n,k}|\,|d_{n,l}| = \left(\sum_{k\in I_n}|d_{n,k}|\right)^2 \le \left(3^n AC^n b_0 b_1\cdots b_{n-1}\right)^2. \end{aligned}$$

□

Proposition 7.4. *Let (s_n) denote an indeterminate moment sequence for which the recurrence coefficients (2) satisfy the conditions of Lemma 7.3. Then*

(i) $\rho_G \le \rho_L = \rho_H$.

(ii) $\rho_G^{[1]} \le \rho_L^{[1]} = \rho_H^{[1]}$, *provided* $\rho_H = 0$.

(iii) $\rho_G^{[2]} \le \rho_L^{[2]} = \rho_H^{[2]}$, *provided* $\rho_H^{[1]} = 0$.

Proof. From (73), (5) and $b_{n-1} \le b_n$ for $n \ge n_1$, it follows for such n that

$$\sqrt{s_{2n}} \le \frac{A(3C)^n}{b_{n,n}} \le B^{n_1}(3C)^n b_n^{n-n_1},$$

where B is given by (74), hence

$$\sqrt{s_{2n}} \le \frac{\alpha(3C)^n}{b_{n,n}} \le \gamma(3C)^n b_n^{n-n_1}, \quad n \ge 0,$$

for suitable constants $\alpha, \gamma > 0$. Introducing

$$G^*(z) = \sum_{n=0}^{\infty} \frac{1}{b_n^{n-n_1}} z^n,$$

this gives

$$M_L(r) \ge (1/\alpha) M_H(r/3C) \ge (1/\gamma) M_{G^*}(r/3C), \quad r > 0,$$

showing that $\rho_L \ge \rho_H \ge \rho_G^*$ and similar inequalities for the logarithmic and double logarithmic orders. If this is combined with Proposition 7.1, we get the equality sign between the orders of L and H. Furthermore, by (22)

$$\rho_{G^*} = \limsup \frac{\log n}{(1 - n_1/n)\log b_n} = \limsup \frac{\log n}{\log b_n} = \rho_G,$$

and similarly $\rho_G^{[1]} = \rho_{G^*}^{[1]}$ and $\rho_G^{[2]} = \rho_{G^*}^{[2]}$. □

Theorem 7.5. *Given an (indeterminate) moment problem where*

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and where either (27) *or* (28) *holds.*

The following holds

(i) $\rho = \rho_F = \rho_G = \rho_H = \rho_L = \mathcal{E}(b_n)$.

If $\rho = 0$ *then*

(ii) $\rho^{[1]} = \rho_F^{[1]} = \rho_G^{[1]} = \rho_H^{[1]} = \rho_L^{[1]} = \mathcal{E}(\log b_n)$.

If $\rho^{[1]} = 0$ *then*

(iii) $\rho^{[2]} = \rho_F^{[2]} = \rho_G^{[2]} = \rho_H^{[2]} = \rho_L^{[2]} = \mathcal{E}(\log\log b_n)$.

Proof. By Lemma 4.1 we know that $b_{n-1} \leq b_n$ for $n \geq n_1$, so the conditions of Proposition 7.4 are fulfilled. By (50) we have

$$\frac{1}{b_n^{2n}} \leq \frac{1}{b_{n-1}^{2n}} \leq A b_{n,n}^2, \quad n \geq n_1$$

for a certain constant A, and by replacing A by a larger constant if necessary, we see that there exists a constant a such that $1/b_n^n \leq a b_{n,n}$ for all n. This gives $M_G(r) \leq a M_H(r)$, hence $\rho_G \leq \rho_H$. By (22) we have

$$\rho_G = \limsup_{n\to\infty} \frac{\log n}{\log b_n},$$

so for any $\varepsilon > 0$ we get $n \leq b_n^{\rho_G+\varepsilon}$ for n sufficiently large. This gives

$$\sum_{n=0}^{\infty} \frac{1}{b_n^{(\rho_G+\varepsilon)(1+\varepsilon)}} < \infty,$$

hence $\mathcal{E}(b_n) \leq \rho_G$. Finally, by Theorem 4.11, Proposition 7.1 and Proposition 7.4 we get $\rho = \mathcal{E}(b_n) \leq \rho_G \leq \rho_H = \rho_L \leq \rho$.

If the common order $\rho = 0$, we get as above $\rho_G^{[1]} \leq \rho_H^{[1]}$, and by (24) we know that

$$\rho_G^{[1]} = \limsup_{n\to\infty} \frac{\log n}{\log\log b_n}.$$

For given $\varepsilon > 0$ we get for n sufficiently large that

$$n \leq (\log b_n)^{\rho_G^{[1]}+\varepsilon},$$

showing that $\mathcal{E}(\log b_n) \leq \rho_G^{[1]}$. We finally use Theorem 5.12 combined with Proposition 7.1 and Proposition 7.4 to get (ii), and proceed similarly concerning the double logarithmic order. □

Example 7.6. In [10] symmetric polynomials with the recurrence coefficients $b_{n-1} = 2n\sqrt{4n^2-1}, n \geq 1$, are considered. The sequence is log-concave and the order of the moment problem is 1/2 by Theorem 4.11.

The case of $b_{n-1} = q^{-n}$ for $0 < q < 1$ is also considered, and Chen and Ismail find explicit representations of P_n and the entire functions A, B, C, D. Clearly $b_n^2 = b_{n-1}b_{n+1}$ and we find that the order is 0 and the logarithmic order is 1 in accordance with the estimates of the paper.

8 Appendix

Proof of Theorem 6.2. To establish (64), we first show that if

$$M_f(r) \leq r^{(\log\log r)^\alpha}, \quad \alpha > 0, r \geq r_0,$$

then

$$\limsup_{n\to\infty} \frac{\log n}{\log\log\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)} \leq \alpha. \tag{77}$$

This will yield $\geq$ in (64).

By the Cauchy estimates

$$|a_n| \leq \frac{M_f(r)}{r^n} \leq r^{(\log\log r)^\alpha - n}, \quad r \geq r_0.$$

In this inequality we will choose an r approximately minimizing

$$\varphi(r) = ((\log\log r)^\alpha - n)\log r.$$

Note that $\varphi'(r) = 0$ if $x = \log\log r$ satisfies

$$x^\alpha + \alpha x^{\alpha-1} - n = 0. \tag{78}$$

Motivated by Lemma 8.1 below we choose r such that $\log\log r = n^{1/\alpha} - 1$. This is certainly larger than r_0 if n is large enough. Inserting this value for r, we get

$$\log|a_n| \leq ((n^{1/\alpha}-1)^\alpha - n)\exp(n^{1/\alpha}-1) = -n\left(1-(1-n^{-1/\alpha})^\alpha\right)\exp(n^{1/\alpha}-1),$$

hence

$$\log\log\frac{1}{\sqrt[n]{|a_n|}} \geq n^{1/\alpha} - 1 + \log\left(1-(1-n^{-1/\alpha})^\alpha\right) = n^{1/\alpha}(1+o(1)),$$

showing (77).

We next show that the double logarithmic order of f satisfies

$$\rho_f^{[2]} \le \limsup_{n\to\infty} \frac{\log n}{\log\log\log\left(\frac{1}{\sqrt[n]{|a_n|}}\right)}. \tag{79}$$

This is clear if the right-hand side is infinity. Let μ be an arbitrary number larger than the right-hand side, now assumed finite. Then there exists n_0 such that

$$\log n \le \mu \log\log\log \frac{1}{\sqrt[n]{|a_n|}}, \quad n \ge n_0,$$

or

$$|a_n| \le \exp\left(-n\exp(n^{1/\mu})\right), \quad n \ge n_0.$$

Fix $r > e$ so large that $\log r > \exp(n_0^{1/\mu}) - 1$. We next determine $n_1 > n_0$ so that

$$\exp((n_1-1)^{1/\mu}) - 1 < \log r \le \exp(n_1^{1/\mu}) - 1.$$

For this r we find with $C_1 = \sum_{n=0}^{n_0-1} |a_n|$

$$\begin{aligned} M_f(r) &\le \sum_{n=0}^{n_0-1} |a_n| r^n + \sum_{n=n_0}^{\infty} |a_n| r^n \\ &\le C_1 r^{n_0} + \sum_{n=n_0}^{\infty} \exp\left(-n\exp(n^{1/\mu}) + n\log r\right) \\ &\le C_1 r^{n_0} + \sum_{n=n_0}^{n_1-1} \exp\left(-n\exp(n^{1/\mu}) + (\log(1+\log r))^{\mu}\log r\right) \\ &+ \sum_{n=n_1}^{\infty} \exp\left(-n\exp(n^{1/\mu}) + n\exp(n^{1/\mu}) - n\right), \end{aligned}$$

where we have used in the second sum that for $n_0 \le n < n_1$: $\exp(n^{1/\mu}) - 1 < \log r$, hence $n < (\log(1+\log r))^{\mu}$, and in the last sum that for $n \ge n_1$

$$\log r \le \exp(n_1^{1/\mu}) - 1 \le \exp(n^{1/\mu}) - 1.$$

We then get

$$\begin{aligned} M_f(r) &\le C_1 r^{n_0} + r^{(\log(1+\log r))^{\mu}} \sum_{n=n_0}^{n_1-1} \exp\left(-n\exp(n^{1/\mu})\right) + \sum_{n=n_1}^{\infty} \exp(-n) \\ &< C_1 r^{n_0} + r^{(\log(1+\log r))^{\mu}} + 1, \end{aligned}$$

where we have majorized the two sums by $\sum_1^{\infty} \exp(-n) = 1/(e-1) < 1$. For any given $\varepsilon > 0$ we have

$$(\log(1+\log r))^{\mu} \le_{\text{as}} (\log\log r)^{\mu+\varepsilon},$$

hence

$$M_f(r) \leq_{\text{as}} 2r^{(\log\log r)^{\mu+\varepsilon}} \leq_{\text{as}} r^{(\log\log r)^{\mu+2\varepsilon}}.$$

This establishes $\rho_f^{[2]} \leq \mu + 2\varepsilon$, which shows $\leq$ in (64).

We next prove (65). For simplicity of notation we put $\alpha = \rho_f^{[2]}$ and assume that $0 < \alpha < \infty$. We show first that if

$$M_f(r) \leq r^{K(\log\log r)^\alpha}, \quad K > 0,\ r \geq r_0,$$

then

$$\limsup_{n\to\infty} \frac{n}{\left(\log\log \frac{1}{\sqrt[n]{|a_n|}}\right)^\alpha} \leq K, \tag{80}$$

which establishes $\geq$ in (65).

By the Cauchy estimates

$$|a_n| \leq \frac{M_f(r)}{r^n} \leq r^{K(\log\log r)^\alpha - n}, \quad r \geq r_0,$$

hence

$$\log|a_n| \leq (K(\log\log r)^\alpha - n)\log r, \quad r \geq r_0.$$

In this inequality we will choose $\log\log r = (n/K)^{1/\alpha} - 1$ by inspiration from the proof in the first part. This gives

$$\log|a_n| \leq -n\left(1 - [1-(n/K)^{-1/\alpha}]^\alpha\right)\exp((n/K)^{1/\alpha} - 1),$$

hence

$$\log\log\frac{1}{\sqrt[n]{|a_n|}} \geq (n/K)^{1/\alpha} - 1 + \log\left(1 - [1-(n/K)^{-1/\alpha}]^\alpha\right) = (n/K)^{1/\alpha}(1+o(1)),$$

showing (80).

We next show that the double logarithmic type of f satisfies

$$\tau_f^{[2]} \leq \limsup_{n\to\infty} \frac{n}{\left(\log\log \frac{1}{\sqrt[n]{|a_n|}}\right)^\alpha}. \tag{81}$$

This is clear if the right-hand side is infinity. Let μ be an arbitrary number larger than the right-hand side, now assumed finite. Then there exists n_0 such that

$$n \leq \mu\left(\log\log\frac{1}{\sqrt[n]{|a_n|}}\right)^\alpha, \quad n \geq n_0,$$

or

$$|a_n| \leq \exp\left(-n\exp((n/\mu)^{1/\alpha})\right), \quad n \geq n_0.$$

Fix $r > e$ so large that $\log r > \exp((n_0/\mu)^{1/\alpha}) - 1$. We next determine $n_1 > n_0$ so that

$$\exp\left(\left(\frac{n_1 - 1}{\mu}\right)^{1/\alpha}\right) - 1 < \log r \le \exp((n_1/\mu)^{1/\alpha}) - 1.$$

For this r we find with $C_1 = \sum_{n=0}^{n_0-1} |a_n|$

$$\begin{aligned}
M_f(r) &\le C_1 r^{n_0} + \sum_{n=n_0}^{\infty} \exp\left(-n\exp((n/\mu)^{1/\alpha}) + n\log r\right) \\
&\le C_1 r^{n_0} + \sum_{n=n_0}^{n_1-1} \exp\left(-n\exp((n/\mu)^{1/\alpha}) + \mu(\log(1+\log r))^{\alpha}\log r\right) \\
&+ \sum_{n=n_1}^{\infty} \exp(-n),
\end{aligned}$$

where we have used that $n < \mu\,(\log(1+\log r))^{\alpha}$ when $n_0 \le n \le n_1 - 1$, and that $\log r \le \exp\left((n/\mu)^{1/\alpha}\right) - 1$ when $n \ge n_1$.

We then get

$$\begin{aligned}
M_f(r) &\le C_1 r^{n_0} + r^{\mu(\log(1+\log r))^{\alpha}} \sum_{n=n_0}^{n_1-1} \exp\left(-n\exp((n/\mu)^{1/\alpha})\right) + \sum_{n=n_1}^{\infty} \exp(-n) \\
&< C_1 r^{n_0} + r^{\mu(\log(1+\log r))^{\alpha}} + 1.
\end{aligned}$$

For any given $\varepsilon > 0$ we have

$$\mu(\log(1+\log r))^{\alpha} \le_{\text{as}} (\mu+\varepsilon)(\log\log r)^{\alpha},$$

hence

$$M_f(r)) \le_{\text{as}} 2r^{(\mu+\varepsilon)(\log\log r)^{\alpha}} \le_{\text{as}} r^{(\mu+2\varepsilon)(\log\log r)^{\alpha}}.$$

This establishes $\tau_f^{[2]} \le \mu + 2\varepsilon$, which shows $\le$ in (65).

$\square$

Lemma 8.1. *Let $n \in \mathbb{N}, n \ge 4$ and $\alpha > 0$. Then the function in* (78)

$$h(x) = x^{\alpha} + \alpha x^{\alpha-1} - n$$

has a zero in

$$\begin{cases} [n^{1/\alpha} - 1, n^{1/\alpha}] & \text{if } \alpha > 1, \\ n - 1 & \text{if } \alpha = 1, \\ [n^{1/\alpha} - 2, n^{1/\alpha} - 1] & \text{if } 0 < \alpha < 1. \end{cases}$$

Proof. We find $h(n^{1/\alpha}) = \alpha n^{1-1/\alpha} > 0$ for all $\alpha > 0$. Putting $y = n^{1/\alpha} - 1$ we find for some $\xi \in (0,1)$

$$(y+1)^\alpha - y^\alpha = \alpha(y+\xi)^{\alpha-1} \begin{cases} > \alpha y^{\alpha-1} & \text{if} \quad \alpha > 1, \\ < \alpha y^{\alpha-1} & \text{if} \quad 0 < \alpha < 1. \end{cases}$$

This shows that $h(n^{1/\alpha} - 1) < 0$ (resp. > 0) for $\alpha > 1$ (resp. $0 < \alpha < 1$).

Finally, for $0 < \alpha < 1$ we put $y = n^{1/\alpha} - 2$ and get for some $0 < \eta < 2$

$$(y+2)^\alpha - y^\alpha = 2\alpha(y+\eta)^{\alpha-1} > \alpha y^{\alpha-1}$$

if $y \geq 2$. This shows that $h(n^{1/\alpha} - 2) < 0$. Note that $y = n^{1/\alpha} - 2 \geq 2$ for $n \geq 4$. □

Propositions 5.3 and 5.4 from [6] can be extended to double logarithmic order.

These results deal with transcendental entire functions f of ordinary order strictly less than 1. They have infinitely many zeros, which we label $\{z_n\}$ and number according to increasing order of magnitude. We repeat each zero according to its multiplicity. Supposing $f(0) = 1$ we get from Hadamard's factorization theorem

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \tag{82}$$

The growth of f is thus determined by the distribution of the zeros. We shall use the following quantities to describe this distribution.

The usual zero counting function $n(r)$ is

$$n(r) = \#\{n \mid |z_n| \leq r\},$$

and we define

$$N(r) = \int_0^r \frac{n(t)}{t}\, dt,$$

and

$$Q(r) = r \int_r^\infty \frac{n(t)}{t^2}\, dt.$$

These quantities are related to $M_f(r)$ in the following way

$$N(r) \leq \log M_f(r) \leq N(r) + Q(r) \tag{83}$$

for $r > 0$. (This is relation (3.5.4) in Boas [9]).

By a theorem of Borel it is known that $\rho_f = \mathcal{E}(z_n)$, and if the order is 0, then $\rho_f^{[1]} = \mathcal{E}(\log|z_n|)$ by Proposition 5.4 in [6]. Furthermore, by Proposition 5.3 in [6] we have

$$\mathcal{E}(\log|z_n|) = \limsup_{n\to\infty} \frac{\log n(r)}{\log\log r}.$$

The following proposition expresses the double logarithmic convergence exponent $\mathcal{E}(\log\log|z_n|)$ in terms of the zero counting function of f.

Proposition 8.2. *We have*

$$\mathcal{E}(\log\log|z_n|) = \limsup_{r\to\infty} \frac{\log n(r)}{\log\log\log r}. \tag{84}$$

Proof. We have

$$n(e^{e^r}) = \#\left\{n \mid |z_n| \le e^{e^r}\right\} = \#\{n \mid \log\log|z_n| \le r\},$$

hence by Lemma 4.10

$$\mathcal{E}(\log\log|z_n|) = \limsup_{r\to\infty} \frac{\log n(e^{e^r})}{\log r} = \limsup_{s\to\infty} \frac{\log n(s)}{\log\log\log s}.$$

□

Theorem 8.3. *The double logarithmic order of the canonical product* (82) *is equal to the double logarithmic convergence exponent of the zeros, i.e.,* $\rho_f^{[2]} = \mathcal{E}(\log\log|z_n|)$.

Proof. We shall prove that $L = \rho_f^{[2]}$, where L is given by the right-hand side of (84). Let $\alpha > 0$ be such that

$$M_f(r) \le r^{(\log\log r)^\alpha}, \quad r \ge r_0.$$

For $r \ge r_0$ we then get by the left-hand side of (83)

$$n(r)\log r \le \int_r^{r^2} \frac{n(t)}{t}\,dt \le N(r^2) \le \log M_f(r^2) \le 2(\log\log r^2)^\alpha \log r,$$

hence for any $\varepsilon > 0$

$$n(r) \le 2(\log 2 + \log\log r)^\alpha \le_{\text{as}} (\log\log r)^{\alpha+\varepsilon},$$

which shows that $L \le \alpha + \varepsilon$, leading to $L \le \rho_f^{[2]}$.

To prove the converse inequality we let $\varepsilon > 0$ be given. There exists $r_0 > 1$ such that

$$n(r) \le (\log\log r)^{L+\varepsilon}, \quad r \ge r_0.$$

For $r > r_0$ we then get

$$N(r) \le \int_0^{r_0} \frac{n(t)}{t}\,dt + \int_{r_0}^r (\log\log t)^{L+\varepsilon}\frac{dt}{t} < \int_0^{r_0} \frac{n(t)}{t}\,dt + (\log\log r)^{L+\varepsilon}\log r.$$

We also get

$$Q(r) \le r\int_r^\infty \frac{(\log\log t)^{L+\varepsilon}}{t^{1/2}}\,\frac{dt}{t^{3/2}}.$$

We next use that
$$\frac{t^{1/2}}{(\log\log t)^{L+\varepsilon}} = \left[\frac{t}{(\log\log t)^{2(L+\varepsilon)}}\right]^{1/2}$$
is increasing for t sufficiently large, because $(\log\log r)^\alpha$ is an order function for any $\alpha > 0$. We can therefore write
$$Q(r) \le r\frac{(\log\log r)^{L+\varepsilon}}{r^{1/2}}\int_r^\infty \frac{dt}{t^{3/2}} = 2(\log\log r)^{L+\varepsilon},$$
so by the right-hand side of (83) we find
$$\limsup_{r\to\infty}\frac{\log M_f(r)}{(\log\log r)^{L+\varepsilon}\log r} \le 1,$$
and it follows that $\rho_f^{[2]} \le L$. $\square$

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