Correction to a paper by A. G. Pakes

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Abstract

Starting from a probability σ on the half-line with moments of any order A.G. Pakes has defined probabilities σ_r by length biasing of order r and g_r by the stationary-excess operation of order $r, r = 1, 2, \ldots$. Examples are given to show that σ can be determined in the Stieltjes sense while σ_1 and g_1 are indeterminate in the Stieltjes sense. This shows that a statement in a recent paper by Pakes does not hold.

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1 Introduction

In a recent paper [11] Pakes is considering the criteria of Carleman and Krein together with some converse results. We shall use the notation of [11]. For a measure σ on the half-line $\mathbb{R}_+$ with moments of any order and distribution function F, Pakes introduces the measure σ_r with distribution function F_r given by

$$F_r(x) = \mu_r^{-1} \int_0^x v^r \, dF(v),$$

where $\{\mu_n\}$ is the moment sequence of F. The moment sequence of F_r is $\mu_n(r) = \mu_{r+n}/\mu_r$. The construction is called length biasing of order r, and r can be any non-negative integer.

In [11, page 92] Pakes remarks: 'Obviously $\{\mu_n\}$ is S-determining if and only if $\{\mu_n(r)\}$ is.'

This is not true. While it is clear indeed that S-indeterminacy of σ implies S-indeterminacy of σ_r, the converse is false.

In fact, in our paper with Thill [6] we completely characterized the probabilities σ on the half-line which are S-determinate but for which σ_1 is not S-determinate. This characterization was the starting point for the solution of the Challifour problem solved in [6].

This lead us in [6] to introduce an index of determinacy:

For a measure σ on the half-line with moments of any order and which is S-determinate (det(S) in short) the *index (of determinacy)* of σ is

$$\mathrm{ind}(\sigma) = \sup\{r \in \mathbb{N}_0 | \sigma_r \text{ is det(S)}\}\,.$$

Theorems 5.5 and 5.6 of [6] contain a complete characterization of the measures with $\mathrm{ind}(\sigma) = k$. In a continuation [7] we considered the relation between the index and the denseness of the polynomials in L^2-spaces.

I later papers with Duran [4, 5] we extended this to the Hamburger case, that is, for measures on the real line with moments of any order. For a survey of the these results see [2]. It should be added that the remark of Pakes is true if σ is a non-discrete measure, because such a measure is either S-indeterminate or S-determinate with $\mathrm{ind}(\sigma) = \infty$. Our observation has also the consequence that σ can be S-determinate although the *stationary-excess operation of order* 1 defined in [11] leads to an S-indeterminate probability density

$$g_1(x) = \overline{F}(x)/\mu_1, \quad \overline{F}(x) = 1 - F(x).$$

In particular, the first part of Theorem 5 in [11] is not true:

Theorem 1.1 *There exists S-determinate measures σ for which*

$$\int_{x'}^{\infty} x^{-3/2}(-\log \overline{F}(x))\,dx < \infty, \quad x' > 0, \tag{1}$$

and the density $g_1(x)$ is S-indeterminate.

We shall explain why the result fails and also give a concrete counterexample in the next section.

2 Counterexamples

For the general theory of the moment problem see [1]. Let us first recall that if σ is S-indeterminate, there are infinitely many solutions to the corresponding Stieltjes moment problem. Among those are the N(evanlinna)-extremal solutions ν_t supported by $[0,\infty[$. Here the parameter t can be any real number in a well-defined interval $[\alpha, 0]$ where $\alpha < 0$, see [9, page 179] for details. The particular value $t = 0$ gives a measure of the form

$$\nu_0 = \beta_0\varepsilon_0 + \sum_{n=1}^{\infty} \beta_n \varepsilon_{x_n}, \tag{2}$$

where the masses $\beta_n > 0$ sum to 1 and $0 < x_1 < x_2 < \cdots$ tend to infinity. If the mass at zero is removed from ν_0, and we rescale to a probability σ, that is

$$\sigma = (\nu_0 - \beta_0\varepsilon_0)/(1-\beta_0), \tag{3}$$

then σ is S-determinate and determinate even for the corresponding Hamburger moment problem. For different proofs of this see [1, page 115] and [3]. Let as before $\{\mu_n\}$ be the moment sequence of σ.

The probability measure of length biasing of order 1

$$\sigma_1 = \frac{t}{\mu_1} d\sigma(t)$$

is indet(S) because σ_1 is proportional to $td\nu_0(t)$, which is clearly indet(S) because ν_0 is so.

Let F be the distribution function of σ and define $\overline{F}(x) = 1 - F(x)$, $g_1(x) = \overline{F}(x)/\mu_1$.

Then g_1 is a probability density with moments of any order and moment sequence

$$\overline{\mu}_n(1) = \frac{1}{1+n}\frac{\mu_{n+1}}{\mu_1}.$$

We claim that g_1 is indet(S), because it is the product of the S-indeterminate sequence μ_{n+1}/μ_1 with the moment sequence of Lebesgue measure on $[0,1]$, see Lemma 2.1 below.

As a preparation for Lemma 2.1 we shall recall the Mellin transformation.

The (open) positive half-line is a locally compact abelian group under multiplication, and the Mellin transformation is the Fourier transformation in the sense of harmonic analysis on such groups.

The corresponding convolution of measures is denoted $\diamond$, so $\tau \diamond \chi$ is the image measure under $(x,y) \mapsto xy$ of the product measure $\tau \otimes \chi$. The Mellin transformation $\mathcal{M}$ is defined for finite (complex) measures by

$$\mathcal{M}(\tau)(x) = \int_0^\infty t^{ix}\, d\tau(t), \quad x \in \mathbb{R}.$$

The Mellin transform of the convolution product is the ordinary product of the Mellin transforms. Furthermore, for the n'th moments we have $\mu_n(\tau \diamond \chi) = \mu_n(\tau)\mu_n(\chi)$.

The Mellin transform of the Lebesgue measure m on the unit interval $[0,1]$ is

$$\mathcal{M}(m)(x) = \frac{1}{1+ix},$$

hence non-vanishing. The Mellin transformation is one-to-one which implies the first statement of Lemma 2.1.

Lemma 2.1 *The mapping $\tau \mapsto \tau \diamond m$ is one-to-one. If τ is indet(S), then so is $\tau \diamond m$.*

The second statement follows from the first, because if τ and χ are different positive measures with the same moments, then $\tau \diamond m$ and $\chi \diamond m$ are different, and they also have identical moments. □

Remark 2.2 There exists a measure τ which is det(S) and yet $\tau \diamond m$ is indet(S).

The measure ν_0 from (2) can be written $\nu_0 = \beta_0 \varepsilon_0 + \rho$ and $\nu_0 \diamond m = \beta_0 \varepsilon_0 + \rho \diamond m$ is indet(S) by Lemma 2.1. Since $\rho \diamond m$ is absolutely continuous we can conclude that $\rho \diamond m$ is indeterminate. In fact, if $\rho \diamond m$ was determinate, then the polynomials are dense in $L^2(\rho \diamond m)$ and hence in $L^2(\nu_0 \diamond m)$ by [3, Lemma 2]. Therefore the indeterminate measure $\nu_0 \diamond m$ is N-extremal, but this contradicts the fact that it is non-discrete.

The probability $\tau = \rho/(1-\beta_0)$ ($= \sigma$ from (3)) satisfies the claim of the remark.

The author does not know if the phenomenon of Remark 2.2 can hold if τ is non-discrete or absolutely continuous. □

Remark 2.3 The Krein condition (1) cannot distinguish between the measures ν_0 and σ given by (2) and (3).

If we let F and G denote the corresponding distribution functions, condition (1) for $\overline{F}$ takes the form

$$2\sum_{n=N}^{\infty} -\log(1-\beta_0-\cdots-\beta_n)\left(\frac{1}{\sqrt{x_n}}-\frac{1}{\sqrt{x_{n+1}}}\right) < \infty, \tag{4}$$

while for $\overline{G}$ it has the form

$$2\sum_{n=N}^{\infty} -\log(\frac{1-\beta_0-\cdots-\beta_n}{1-\beta_0})\left(\frac{1}{\sqrt{x_n}}-\frac{1}{\sqrt{x_{n+1}}}\right) < \infty. \tag{5}$$

Since

$$\sum_{n=N}^{\infty}\left(\frac{1}{\sqrt{x_n}}-\frac{1}{\sqrt{x_{n+1}}}\right) = \frac{1}{\sqrt{x_N}},$$

the two series in (4), (5) converge simultaneously, and we know that ν_0 is indet(S), but σ is det(S). □

We shall now give a concrete example of a probability of the form (2), which leads to a probability σ which is det(S) and for which the Krein condition (1) nevertheless holds by direct verification. This gives a concrete example showing that the first part of Theorem 5 in [11] is not correct.

The example comes from a birth and death process with quartic rates studied by Berg and Valent, see [8, 9].

A birth and death process is defined by the sequences $(\lambda_n)_{n\geq 0}$ of birth rates and $(\mu_n)_{n\geq 0}$ of death rates, restricted by $\lambda_n > 0, \mu_{n+1} > 0$ for $n \geq 0$ and $\mu_0 \geq 0$, see for example [10].

In order to solve the so-called Kolmogorov equation, one studies the polynomials $F_n(x)$ defined by the recurrence

$$(\lambda_n + \mu_n - x)F_n(x) = \mu_{n+1}F_{n+1}(x) + \lambda_{n-1}F_{n-1}(x) \quad , \; n \geq 0$$

with the initial conditions

$$F_{-1}(x) = 0, \quad F_0(x) = 1.$$

Defining

$$\pi_0 = 1, \quad \pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \; , \; n \geq 1$$

and

$$a_n = \lambda_n + \mu_n \; , \; b_n = \sqrt{\lambda_n \mu_{n+1}} \; , \; n \geq 0 \; ,$$

it is wellknown that the polynomials

$$P_n(x) = (-1)^n \frac{1}{\sqrt{\pi_n}} F_n(x)$$

satisfy the three term recurrence relation

$$xP_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1}P_{n-1}(x) \; , \; n \geq 1$$

together with the initial conditions

$$P_0(x) = 1 \; , \; P_1(x) = \frac{1}{b_0}(x - a_0).$$

By Favard's Theorem the polynomials $\{P_n\}$ form an orthonormal system with respect to some probability measure on the half-line and the corresponding moment sequence is a Stieltjes moment sequence.

We shall consider the following quartic rates

$$\lambda_n = (4n+1)(4n+2)^2(4n+3) \; , \; \mu_n = (4n-1)(4n)^2(4n+1) \; , \; n \geq 0$$

initially considered in [12, 13, 14]. Note that $\mu_0 = 0$ and

$$\pi_n = \frac{1}{4n+1}\left(\frac{(1/2)_n}{n!}\right)^2 \sim \frac{1}{4\pi}\frac{1}{n^2} \; , \; \lambda_{n-1}\pi_{n-1} = \mu_n \pi_n \sim \frac{64}{\pi} n^2,$$

and it follows from known criteria that the corresponding moment problem is indet(S), see for example [8].

The N-extremal measure ν_0 is given by

$$\nu_0 = \frac{\pi}{K_0^2}\varepsilon_{x_0} + \frac{4\pi}{K_0^2}\sum_{n=1}^{\infty}\frac{2n\pi}{\sinh(2n\pi)}\varepsilon_{x_n},\ x_n = \left(\frac{2n\pi}{K_0}\right)^4,$$

and the constant K_0 is given by en elliptic integral, see [8].

From the general theory mentioned above

$$\sigma = c\sum_{n=1}^{\infty}\frac{2n\pi}{\sinh(2n\pi)}\varepsilon_{x_n}$$

is determinate. The normalization constant c (expressible by K_0) is chosen so that σ is a probability. The function $\overline{F}$ is piecewise constant and to establish (1), we have to prove that

$$\sum_{n=1}^{\infty} -\log(y_n)\left(\frac{1}{\sqrt{x_n}} - \frac{1}{\sqrt{x_{n+1}}}\right) < \infty, \tag{6}$$

where x_n is as above and

$$y_n = c\sum_{k=n+1}^{\infty}\frac{2k\pi}{\sinh(2k\pi)}.$$

Using

$$y_n \geq c\int_{n+1}^{\infty}\frac{2x\pi}{\sinh(2x\pi)}\,dx \geq 4\pi c\int_{n+1}^{\infty} xe^{-2\pi x}\,dx \geq 2c(n+1)e^{-2\pi(n+1)},$$

we see that (6) holds.

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