Flow equivalence of shift spaces (and their $C^*$-algebras), I

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2. Conjugacy
3. Classification
4. Flow equivalence
5. Flow classification
Outline

1. Definitions
2. Conjugacy
3. Classification
4. Flow equivalence
5. Flow classification
Baker’s map

\[ b(x, y) = \left( 2x - \text{floor}(2x), \frac{y + \text{floor}(2x)}{2} \right) \]

\[
\text{seq}(\text{floor}(2 \cdot b(n)(0.243453, 0.7232)[1]), n=0..20); \\
0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 0, 1, 0, 0, 1
\]

\[
\alpha := 1.74565; r(n) = (\log(2 + n/10) \cos(\alpha n), \log(2 + n/10) \sin(\alpha n))
\]

\[
101110100101001111100 \ldots
\]
Irrational rotation

\[ r := \frac{n}{\log 2 C_1} \]

\[ 00010001001000100100010001000100010001000100010001000100010001000 \cdots \]
Let $\alpha$ be a finite set and equip $\alpha^\mathbb{Z}$ with the product topology based on the discrete topology on $\alpha$.

**Definition**

A **shift space** is a subset $X$ of $\alpha^\mathbb{Z}$ which is closed and closed under the **shift map**

$$\sigma : \alpha^\mathbb{Z} \rightarrow \alpha^\mathbb{Z}, \quad \sigma((x_i)) = (x_{i+1})$$

**Definition**

A shift space is **irreducible** if some forward orbit $\{\sigma^n(x) \mid n \in \mathbb{N}\}$ is dense.
### 3 constructions

<table>
<thead>
<tr>
<th>Name</th>
<th>Input</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X(W)$</td>
<td>List of words $W$</td>
<td>Sequences not containing words from $W$</td>
<td>$W = {11}$</td>
</tr>
<tr>
<td>$X_G$</td>
<td>Graph $G$</td>
<td>Infinite paths on $G$</td>
<td>$e_1 \circlearrowleft \bullet \quad \quad e_2 \quad e_3 \circlearrowright \bullet$</td>
</tr>
<tr>
<td>$L_A$</td>
<td>Labelled graph $A$</td>
<td>Infinite paths on $A$</td>
<td>$0 \circlearrowleft \bullet \quad \quad 1 \quad \quad 0 \circlearrowright \bullet$</td>
</tr>
</tbody>
</table>
Forbidden word shifts

Let $W$ be a set of finite words on $\alpha$.

Definition

$X(W)$ is the shift space $\{x \in \alpha^\mathbb{Z} \mid \forall i < j : x_i \cdots x_j \not\in W\}$

Example

With $\alpha = \{0, 1\}$ and $W = \{11\}$ the shift space $X(W)$ contains elements such as

\[\cdots 01000010001000100001010101001001000100010 \cdots\]

Lemma

For any shift space $X$, $X = X(W)$ where $W$ is chosen as the complement of the language

$$\mathcal{L}(X) = \{x_i \cdots x_j \mid x \in X, i < j\}$$
Edge shifts

Let a graph $G = (V, E, r, s)$ be given with

- Vertices $V$
- Edges $E$ enumerated $\{e_1, \ldots, e_n\}$
- Range and source maps $r, s : E \to V$.

**Definition**

$X_G$ is the shift space $X^{(W)}$ with alphabet $E$ and

$$W = \{e_i e_j \mid r(e_i) \neq s(e_j)\}$$

**Example**

With $G = e_1 \xrightarrow{e_2} \bullet \xrightarrow{e_3} \bullet$, $X_G$ contains elements such as

$$\cdots e_1 e_1 e_2 e_3 e_2 e_3 e_2 e_3 e_1 e_2 e_3 e_2 e_3 e_1 e_1 e_1 e_1 e_1 e_1 e_2 \cdots$$
Labelled edge shifts

Convention

A labelled graph $\mathcal{A} = (V, E, r, s, a, \lambda)$ is given by an underlying graph $(V, E, r, s)$ and a labelling map $\Lambda : E \to a$

Definition

We denote by $X_{\mathcal{A}}$ the edge shift associated to the underlying graph of $\mathcal{A}$ and by

$$\lambda : X_{\mathcal{A}} \to a^\mathbb{Z}$$

the labelling map induced by $\Lambda$. The shift defined by $\mathcal{A}$ is $L_{\mathcal{A}} = \lambda(X_{\mathcal{A}})$. 
Labelled edge shifts

**Example**

With $\mathcal{A} = \begin{array}{c} 0 \\ \circ \\ 0 \end{array}$, the shift space $X_{\mathcal{A}}$ contains elements such as

\[ \cdots 01000010001000100001010101010010010001000100010 \cdots \]
Definition

Let $X \subseteq \mathbb{a}^\mathbb{Z}$ and $Y \subseteq \mathbb{b}^\mathbb{Z}$. $\phi : X \rightarrow Y$ is the $(m, n)$ sliding block code given by a map

$$\Phi : \mathbb{a}^{n+1+m} \rightarrow \mathbb{b}$$

when

$$\phi(x)_i = \Phi(x_{i-m} \cdots x_{i+n})$$

Lemma

The following are equivalent:

- $\phi$ is continuous and shift-commuting
- $\phi$ is a sliding block code

Definition

$X$ and $Y$ are conjugate when there is a bijective sliding block code $\phi : X \rightarrow Y$
With $\mathcal{A}$ as above,

$$
\begin{align*}
\lambda : & \begin{array}{c}
\bullet \\
\longrightarrow
\end{array}
\begin{array}{c}
e_1 \\
e_2 \\
e_3 \\
e_1
\end{array}
\begin{array}{c}
\bullet \\
\longrightarrow
\end{array} \\
\begin{array}{c}
\bullet \\
\longrightarrow
\end{array}
\begin{array}{c}
0 \\
1 \\
0
\end{array}
\begin{array}{c}
\bullet \\
\longrightarrow
\end{array}
\begin{array}{c}
0 \\
1 \\
0
\end{array}
\begin{array}{c}
\bullet
\end{array}
\end{align*}
$$

becomes a conjugacy. Indeed, the labelling map is always a $(0, 0)$ sliding block code induced by $\Lambda$. And in this case it has a $(1, 0)$ block inverse $\mu$ given by

$$
\begin{align*}
00 & \mapsto e_1 \\
01 & \mapsto e_2 \\
10 & \mapsto e_3
\end{align*}
$$

For instance,

$$
\begin{align*}
\mu \circ \lambda(\cdots e_1 e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \cdots) &= \\
\mu(\cdots 010000100 \cdots) &= \\
\cdots e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \cdots
\end{align*}
$$
A shift space is a *shift of finite type (SFT)* if it has the form $X^{(W)}$ with $W$ finite.

**Lemma**

*The following are equivalent:*

- $X$ is an SFT
- $X \simeq X_G$ for some graph $G$
Sofic shifts

Definition
A shift space is sofic if there is a surjective sliding block code \( \phi : Y \to X \) with \( Y \) an SFT.

Lemma
The following are equivalent:
- \( X \) is sofic
- \( X \cong L_A \) for some labelled graph \( A \)

Theorem
When \( X \) is irreducible and sofic, there is a unique labelled graph \( A \) with fewest possible vertices and each pair of edges emanating from the same vertex distinctly labelled, such that \( X \cong L_A \). \( A \) is called the Fischer cover of \( X \).
Outline

1. Definitions
2. Conjugacy
3. **Classification**
4. Flow equivalence
5. Flow classification
Classification

The classification problem
Let $X$ and $Y$ be shift spaces finitely presented by objects $A$ and $B$, respectively. Determine in terms of $A$ and $B$ when $X$ and $Y$ are conjugate.

The SFT classification problem
Let $X$ and $Y$ be irreducible shifts of finite type given by graphs $G$ and $H$, respectively. Determine in terms of $G$ and $H$ when $X$ and $Y$ are conjugate.
State splitting

\[
\begin{bmatrix}
2 & 1 \\
2 & 0
\end{bmatrix}
= 
\begin{bmatrix}
2 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 0 & 1 \\
2 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
Theorem (Williams)

Let $X_G$ and $X_H$ be two irreducible SFTs given by graphs with adjacency matrices $A$ and $B$, respectively. The following conditions are equivalent.

(i) $X_G$ and $X_H$ are conjugate.

(ii) There exist nonnegative integral matrices $D_i$ and $E_i$ with

$$A = D_0E_0, E_0D_0 = D_1E_1, \ldots, E_nD_n = B$$

Arsenal of invariants

Real numbers (entropy), power series (zeta function), ordered abelian groups (Dimension group), finitely generated abelian groups (Bowen-Franks groups), $C^*$-algebras (Cuntz-Krieger algebra),...
### 4 examples

<table>
<thead>
<tr>
<th>$A$</th>
<th>$G$</th>
<th>$h(X_G)$</th>
<th>$BF(X_G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 2 &amp; 2 \ 2 &amp; 2 \end{bmatrix}$</td>
<td><img src="Diagram1.png" alt="Diagram" /></td>
<td>4</td>
<td>$(\mathbb{Z}_3, -)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 3 &amp; 1 \ 3 &amp; 1 \end{bmatrix}$</td>
<td><img src="Diagram2.png" alt="Diagram" /></td>
<td>4</td>
<td>$(\mathbb{Z}_3, -)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 1 &amp; 1 \ 3 &amp; 2 \end{bmatrix}$</td>
<td><img src="Diagram3.png" alt="Diagram" /></td>
<td>$\frac{3+\sqrt{13}}{2}$</td>
<td>$(\mathbb{Z}_3, -)$</td>
</tr>
<tr>
<td>$\begin{bmatrix} 2 &amp; 2 \ 1 &amp; 3 \end{bmatrix}$</td>
<td><img src="Diagram4.png" alt="Diagram" /></td>
<td>4</td>
<td>$(\mathbb{Z}, 0)$</td>
</tr>
</tbody>
</table>
### 4 examples

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</thead>
<tbody>
<tr>
<td>[ \begin{bmatrix} 2 &amp; 2 \ 2 &amp; 2 \end{bmatrix} ]</td>
<td>[ \begin{array}{c} \bigcirc \bullet \bigcirc \bullet \bigcirc \end{array} ]</td>
<td>4</td>
<td>$(\mathbb{Z}_3, -)$</td>
</tr>
<tr>
<td>[ \begin{bmatrix} 3 &amp; 1 \ 3 &amp; 1 \end{bmatrix} ]</td>
<td>[ \begin{array}{c} \bigcirc \bullet \bigcirc \bullet \bigcirc \end{array} ]</td>
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<tr>
<td>[ \begin{bmatrix} 1 &amp; 1 \ 3 &amp; 2 \end{bmatrix} ]</td>
<td>[ \begin{array}{c} \bigcirc \bullet \bigcirc \bullet \bigcirc \end{array} ]</td>
<td>$\frac{3+\sqrt{13}}{2}$</td>
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<td>[ \begin{bmatrix} 2 &amp; 2 \ 1 &amp; 3 \end{bmatrix} ]</td>
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<td>4</td>
<td>$(\mathbb{Z}, 0)$</td>
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Outline

1. Definitions
2. Conjugacy
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4. Flow equivalence
5. Flow classification
Fix $a \in \mathfrak{a}$ and $\star \not\in \mathfrak{a}$ and define $\eta : \mathfrak{a}^\mathbb{Z} \rightarrow (\mathfrak{a} \cup \{\star\})^\mathbb{Z}$ as the map inserting a $\star$ after each $a$:  

$$
\cdots babbbaba \cdots \quad \mapsto \quad \cdots ba \star bbba \star ba \star \cdots
$$

**Definition**

The “$a \mapsto a\star$” symbol expansion of a shift space $X$ is the shift space $X_{a \mapsto a\star} = \eta(X)$.  

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Definitions | Conjugacy | Classification | Flow equivalence | Flow classification
Flow equivalence

Associated to any shift space there is a **suspension flow** given by product topology on

\[ SX = \frac{X \times \mathbb{R}}{(x, t) \sim (\sigma(x), t + 1)} \]

**Definition**

\( X \) and \( Y \) are **flow equivalent** (written \( X \simeq_{fe} Y \)) when \( SX \) and \( SY \) are homeomorphic in a way preserving direction in \( \mathbb{R} \).

**Theorem (Parry-Sullivan)**

*Flow equivalence is the coarsest equivalence relation containing conjugacy and* \( X \sim X_{a\rightarrow a^*} \)
Flow classification

Lemma

If $X \simeq_{fe} Y$ and $X$ is SFT, sofic or irreducible, then so is $Y$.

The flow classification problem

Let $X$ and $Y$ be shifts finitely presented by objects $A$ and $B$, respectively. Determine in terms of $A$ and $B$ when $X$ and $Y$ are flow equivalent.

The SFT flow classification problem

Let $X$ and $Y$ be irreducible shifts of finite type given by graphs $G$ and $H$, respectively. Determine in terms of $G$ and $H$ when $X$ and $Y$ are flow equivalent.
Outline

1. Definitions
2. Conjugacy
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5. Flow classification
Flow classification of SFTs

**Theorem (Franks)**

Let $X_G$ and $X_H$ be two irreducible SFTs given by graphs with adjacency matrices $A$ and $B$, respectively. The following conditions are equivalent.

(i) $X_G \simeq_{fe} X_H$

(ii) $\mathbb{Z}^m / (1 - A)\mathbb{Z}^m \simeq \mathbb{Z}^n / (1 - B)\mathbb{Z}^n$

and

\[ \text{sgn det}(1 - A) = \text{sgn det}(1 - B) \]
4 examples

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<td>$\begin{pmatrix} 2 &amp; 2 \ 2 &amp; 2 \end{pmatrix}$</td>
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<th>$A$</th>
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</thead>
</table>
| \[
\begin{pmatrix}
2 & 2 \\
2 & 2 \\
3 & 1 \\
3 & 1 \\
1 & 1 \\
3 & 2 \\
\end{pmatrix}
\] | \[
\begin{array}{c}
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\end{array}
\] | \[
(Z_3, -) \\
(Z_3, -) \\
(Z_3, -) \\
\] |
| \[
\begin{pmatrix}
2 & 2 \\
1 & 3 \\
\end{pmatrix}
\] | \[
\begin{array}{c}
\circlearrowleft \bullet \\
\circlearrowleft \bullet \\
\end{array}
\] | \[
(Z, 0) \\
\] |
Flow classification of sofics

**Theorem**

Let $X$ and $Y$ be two irreducible sofic shifts and let $A, B$ be their Fischer covers. The following conditions are equivalent.

(i) $X \simeq_{fe} Y$

(ii) $S\lambda_A \sim^+ \rightarrow S\lambda_B$

\[ SL_A \sim^+ \rightarrow SL_B \]
**Definition**

With a given map $\lambda : \mathcal{X}_A \to \mathcal{L}_A$ we set

$$
\tilde{\mathcal{L}}_A = \{ x \in \mathcal{L}_A \mid |\lambda^{-1}\{\{x\}\}| > 1 \} \\
\tilde{\mathcal{X}}_A = \lambda^{-1}(\tilde{\mathcal{L}}_A)
$$

and restrict $\lambda$ to

$$\tilde{\lambda} : \tilde{\mathcal{X}}_A \to \tilde{\mathcal{L}}_A$$

**Example**

With $\mathcal{A} = \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}$ and $\mathcal{B} = \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array}$ we get $\tilde{\mathcal{L}}_A = \emptyset$ and $\tilde{\mathcal{L}}_B = \{0^\infty\}$. 
Theorem (Boyle-Carlsen-E)

Let $X$ and $Y$ be two irreducible sofic shift spaces with Fischer covers $A$ and $B$, respectively, and assume that $\tilde{X_A}$ and $\tilde{X_B}$ are both closed. Then $X$ and $Y$ are flow equivalent exactly when the following conditions hold:

1. $X_A \simeq_{fe} X_B$
2. $\tilde{SX_A} \sim_+ \tilde{S X_B}$

$$
\begin{align*}
S\tilde{\lambda}_A & \sim_+ S\tilde{\lambda}_B \\
S\tilde{L}_A & \sim_+ S\tilde{L}_B
\end{align*}
$$
\[ \lambda : X_A \rightarrow L_A \]
### Definitions

#### Conjugacy

#### Classification

#### Flow equivalence

#### Flow classification

<table>
<thead>
<tr>
<th>λ : X_A → L_A</th>
<th>(\tilde{\lambda} : \tilde{X}_A \rightarrow \tilde{L}_A)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
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<td><img src="image8" alt="Diagram" /></td>
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