Flow equivalence of shift spaces (and their C^*-algebras), II

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Shift spaces

Let $\mathfrak{a}$ be a finite set and equip $\mathfrak{a}^{\mathbb{Z}}$ with the product topology based on the discrete topology on $\mathfrak{a}$.

Definition

A *shift space* is a subset X of $\mathfrak{a}^{\mathbb{Z}}$ which is closed and invariant (ie. $\sigma(X) = X$) under the *shift map*

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}} \qquad \sigma((x_i)) = (x_{i+1}).$$

Definition

Two shift spaces X and Y are *conjugate* when there is a homeomorphism $\phi : X \to Y$ such that $\phi \circ \sigma_X = \sigma_Y \circ \phi$.

Flow equivalence

Associated to any shift space there is a *suspension flow* given by product topology on

$$SX = \frac{X \times \mathbb{R}}{(x,t) \sim (\sigma(x), t-1)}.$$

Definition

X and Y are *flow equivalent* (written $X \simeq_{\mathrm{fe}} Y$) when SX and SY are homeomorphic in a way preserving direction in $\mathbb{R}$.

Symbol expansion

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \to (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a $\star$ after each a:

$$\cdots babbbaba \cdots \qquad \mapsto \qquad \cdots ba \star bbba \star ba \star \cdots$$

Definition

The "$a \mapsto a\star$" symbol expansion of a shift space X is the shift space $X_{a \mapsto a\star} = \eta(X)$.

Theorem (Parry and Sullivan)

Flow equivalence is the coarsest equivalence relation containing conjugacy and $X \sim X_{a \mapsto a\star}$.

Flows

By a *flow* we will mean a continuous action α of $\mathbb{R}$ on a compact metrizable space Y.
For $y \in Y$ we let

$$\begin{aligned} \mathrm{Orb}(y) &= \{\alpha_s(y) \mid s \in \mathbb{R}\} \\ \mathrm{Orb}_+(y) &= \{\alpha_s(y) \mid s > 0\} \\ \mathrm{Orb}_-(y) &= \{\alpha_s(y) \mid s < 0\}. \end{aligned}$$

- Two flows are *conjugate* if there is a homeomorphism between their domains intertwining the $\mathbb{R}$-actions.
- Two flows are *equivalent* if there is a homeomorphism between their domains taking orbits to orbits and preserving orientation.

Suspensions

Let $T : X \to X$ be a homeomorphism of a compact metrizable zero-dimensional space X, and $f : X \to \mathbb{R}$ a continuous strictly positive function. The *suspension* of T by f is the quotient space

$$S_f T = \frac{X \times \mathbb{R}}{(x,t) \sim (Tx, t - f(x))},$$

and the *suspension flow* is the action $\alpha^{T,f}$ of $\mathbb{R}$ on $S_f T$ given by

$$\alpha_s^{T,f}([x,t]) = [x, s+t].$$

The suspension $S_1 T$ is called the *standard suspension* of T.

Cross sections

A *cross section* to a flow α on Y is a closed set $C \subseteq Y$ such that $\alpha : C \times \mathbb{R} \to Y$ is a surjective local homeomorphism.

Proposition

If C is a cross section then

1. $\mathrm{Orb}_+(y) \cap C \neq \emptyset$ *and* $\mathrm{Orb}_-(y) \cap C \neq \emptyset$ *for every* $y \in Y$,
2. *the map*

$$x \mapsto r_C(x) := \inf\{t > 0 \mid \alpha_t(x) \in C\}$$

   is a continuous strictly positiv function from C *to* $\mathbb{R}$,
3. *the map* $x \mapsto R_C(x) := \alpha_{r_C(x)}(x)$ *is a homeomorphism from* C *to* C.

Sections

Definition

We say that a homeomorphism T is a *section* to a flow if it is conjugate to R_C for some cross section C of the flow.

- If $S_f X$ is a suspension of a homeomorphism $T : X \to X$ of a compact metrizable zero-dimensional space X, then $C := \{[x, 0] \mid x \in X\}$ is a cross section to $\alpha^{T,f}$ and R_C is conjugate to T.
- If C is a cross section to a flow α, then α^{R_C, r_C} is conjugate to α.

Flow equivalence

Proposition

If T and S are two homeomorphisms on compact metrizable zero-dimensional spaces then the following are equivalent:

1. *T is a section to some suspension of S.*
2. *T is a section to the standard suspension of S.*
3. *T is a section to any flow for which S is a section.*
4. *T and S are sections to a common flow.*
5. *The standard suspension of T is equivalent to the standard suspension of S.*
6. *Any suspension of T is equivalent to any suspension of S.*

Definition

We say that T and S are *flow equivalent* if the above conditions are satisfied.

Discrete suspensions

Let T be a homeomorphism on of a compact metrizable zero-dimensional space X and $p : X \to \mathbb{N}$ a continuous map. For each $k \in \mathbb{N}$ let $A_k = \{x \in X \mid p(x) = k\}$, and let

$$X_p = \bigcup_{0 \leq i < k} A_k \times \{i\}.$$

Then X_p is a compact metrizable zero-dimensional subspace of $X \times \mathbb{N}_0$.
Define $T_p : X_p \to X_p$ by

$$T_p((x,i)) = \begin{cases} (x, i+1) & \text{if } i+1 < p(x) \\ (T(x), 0) & \text{if } i = p(x) - 1. \end{cases}$$

Then T_p is a homeomorphism.

Discrete suspensions

Definition

The homeomorphism T_p is called the *discrete suspension* of T by p and $X \times \{0\}$ is called the *base* of the suspension.

The Parry-Sullivan theorem recasted

Theorem

Suppose C_1 and C_2 are cross sections to some suspension flow of a homeomorphism on a zero-dimensional compact metrizable space. Let T_1 and T_2 denote their respective return maps. Then there exists a third cross section C_3 such that T_1 and T_2 are conjugate to discrete suspensions of the return map of C_3.

Back to shift spaces

Lemma

If X is the discrete suspension of a shift space X_0, then there exists a finite sequence $X_0, X_1, \ldots, X_n$ of shift spaces such that X_i is a symbolic expansion of X_{i-1} for each $i = 1, 2, \ldots, n$, and X_n is conjugate to X.

Corollary

Flow equivalence is the coarsest equivalence relation containing conjugacy and symbolic expansion.