Flow equivalence of shift spaces (and their $C^*$-algebras), IV

Søren Eilers
eilers@math.ku.dk
Department of Mathematical Sciences
University of Copenhagen
01.03.11
1. Franks’ theorem
2. Generalizations
Outline

1. Franks’ theorem
2. Generalizations
Franks’ theorem

Generalizations

Theorem

When an \( n \times n \)-matrix \( A \) and an \( n' \times n' \)-matrix \( A' \) define irreducible and infinite SFTs the following are equivalent

1. \( X_A \sim X_{A'} \)
2. \( \mathbb{Z}^n / (I - A) \mathbb{Z}^n \cong \mathbb{Z}^{n'} / (I - A') \mathbb{Z}^{n'} \) and
   \[ \text{sgn det}(I - A) = \text{sgn det}(I - A') \]
Basic operation

**Lemma**

When $A \geq 0$ with $a_{ij} > 0$ we have that $X_A \sim X_{A^{(ij)}}$ where

$$A^{(ij)} = \begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i1} + a_{j1} & \cdots & a_{ij} + a_{jj} - 1 & \cdots & a_{in} + a_{jn} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{bmatrix}.$$
Franks’ theorem

Generalizations

Step 1
Outsplit to go

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix} \rightarrow \begin{bmatrix}
    0 & 0 & 1 \\
    a_{11} & a_{11} & a_{12} - 1 \\
    a_{21} & a_{21} & a_{22}
\end{bmatrix}
\]

Step 2
Inspli to go

\[
\begin{bmatrix}
    0 & 0 & 1 \\
    a_{11} & a_{11} & a_{12} - 1 \\
    a_{21} & a_{21} & a_{22}
\end{bmatrix} \rightarrow \begin{bmatrix}
    0 & 0 & 1 & 0 \\
    a_{11} & a_{11} & 0 & a_{12} - 1 \\
    a_{21} & a_{21} & 0 & a_{22}
\end{bmatrix}
\]
Step 3
Symbol reduce to go

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
\alpha_{11} & \alpha_{11} & 0 & \alpha_{12} - 1 \\
\alpha_{21} & \alpha_{21} & 0 & \alpha_{22} \\
\alpha_{21} & \alpha_{21} & 0 & \alpha_{22}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\alpha_{11} & \alpha_{11} & \alpha_{12} - 1 \\
\alpha_{21} & \alpha_{21} & \alpha_{22} \\
\alpha_{21} & \alpha_{21} & \alpha_{22}
\end{bmatrix}
\]

Step 4
Out-amalgamate to go

\[
\begin{bmatrix}
\alpha_{11} & \alpha_{11} & \alpha_{12} - 1 \\
\alpha_{21} & \alpha_{21} & \alpha_{22} \\
\alpha_{21} & \alpha_{21} & \alpha_{22}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\alpha_{11} + \alpha_{21} & \alpha_{12} + \alpha_{22} - 1 \\
\alpha_{21} & \alpha_{22}
\end{bmatrix}
\]
Proposition
For any \( A \geq 0 \) there is a \( B \geq 0 \) such that
\[
X_A \sim X_{I+B}
\]

Proof
If all \( a_{ij} > 0 \) we are done. If not, employ that
\[
A^{(ij)} = \begin{bmatrix}
    a_{11} & \ldots & a_{1j} & \ldots & a_{1n} \\
    \vdots & & \vdots & & \vdots \\
    a_{i1} + a_{j1} & \ldots & a_{ij} - 1 & \ldots & a_{in} + a_{jn} \\
    \vdots & & \vdots & & \vdots \\
    a_{n1} & \ldots & a_{nj} & \ldots & a_{nn}
\end{bmatrix}
\]
to create a zero column, which may then be deleted.
Proposition

If a row or column addition takes an irreducible matrix $B \geq 0$ to $B' \geq 0$, we have

$$X_{I+B} \sim X_{I+B'}$$

Proof

Suppose row 2 of $B$ is added to row 1 to create $B'$. The first row of $I + B'$ is

$$\begin{bmatrix}
1 + b_{11} + b_{21} & b_{12} + b_{22} & b_{13} + b_{23} & \ldots \\
\end{bmatrix}$$

and the first two rows of $I + B$ are

$$\begin{bmatrix}
1 + b_{11} & b_{12} & b_{13} & \ldots \\
 b_{21} & 1 + b_{22} & b_{23} & \ldots \\
\end{bmatrix}$$

Note how this coincides with “basic move” when $b_{12} > 0$. In general, use irreducibility.
Proposition
Let an irreducible matrix $B \geq 0$ be of size $n \times n$ with $n > 1$. Then

$$X_{I+B} \sim X_{I+C}$$

where we may assume that $C > 0$ of any size $m \geq n$.

Proof
We may keep adding rows until all entries are $\geq N$ for any $N > 0$. New rows may be added as required by state splitting as soon as the entries are sufficiently large.
Proposition
When $C > 0$ we have $X_{I+C} \sim X_{I+D}$ where the first column of $D$ is identically $d$, with

$$d = \gcd\{c_{ij}\} = \gcd\{d_{ij}\}$$

Proof
Subsequent “column prepared row subtractions” and “row prepared column subtractions”. See example.
Franks’ theorem

Generalizations

**Standard form 1**

When \( C > 0 \) is a given \( n \times n \)-matrix with \( \mathbb{Z}^n / C \mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z} / d_i \mathbb{Z} \)

where

\[
d_1 \mid d_2 \mid \cdots \mid d_n
\]

and \( \det(-C') = (-1)^n \det(C) < 0 \) we have that \( X_{I+C} \sim X_{I+D} \)

where

\[
D = \begin{bmatrix}
0 & \cdots & 0 & d_n \\
d_1 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 \\
& \ddots & \ddots & \ddots \\
0 & \cdots & d_{n-1} & 0 \\
\end{bmatrix}
\]
Standard form 2

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C\mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\det(-C') = (-1)^n \det(C) > 0$ we have that $X_{I+C} \sim X_{I+D}$ where

$$D = \begin{bmatrix}
0 & \cdots & d_{n-1} & d_{n-1} \\
d_1 & 0 & 0 & 0 \\
0 & d_2 & 0 & 0 \\
& \ddots & \ddots & \ddots \\
0 & \cdots & d_{n-1} & d_{n-1} + d_n
\end{bmatrix}$$
Standard form 3

When $C > 0$ is a given $n \times n$-matrix with $\mathbb{Z}^n / C \mathbb{Z}^n = \sum_{i=1}^{n} \mathbb{Z} / d_i \mathbb{Z}$ where

$$d_1 \mid d_2 \mid \cdots \mid d_n$$

and $\text{rank}(C) = k < n$ we have that $X_{I+C} \sim X_{I+D}$ where

$$D = \begin{bmatrix} 0 & \cdots & 0 & d_k & \cdots & d_k \\ d_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{k-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_k & \cdots & d_k \end{bmatrix}$$
Outline

1 Franks’ theorem

2 Generalizations
Theorem

Suppose $A$ is in $\mathcal{M}_P,+(\mathbb{Z})$ and $A'$ is in $\mathcal{M}_{P'},+(\mathbb{Z})$ The following are equivalent.

1. The SFTs $X_A$ and $X_{A'}$ are flow equivalent.
2. For some permutation matrix implementing an isomorphism from $P$ to $P'$, there exists a positive $SL_P(\mathbb{Z})$ equivalence from $I - A$ to $I - P^{-1}A'P$.
3. For some permutation matrix implementing an isomorphism from $P$ to $P'$, and sending cycle components to cycle components, there exists an $SL_P(\mathbb{Z})$ equivalence from $I - A$ to $I - P^{-1}A'P$ which is positive on cycle components.
Equivariant case

Theorem

Let $G$ be a finite group, and let $A$ and $B$ be square matrices over $\mathbb{Z}^+G$. Then $X_A$ and $X_B$ are $G$-flow equivalent precisely when $I - A$ and $I - B$ are $G$-positively equivalent.

Theorem

Let $G$ be a finite group, and let $A$ and $B$ be essentially irreducible nontrivial matrices over $\mathbb{Z}^+G$. For $X_A$ and $X_B$ to be $G$-flow equivalent, it is necessary that $W(A) = W(B)$.

Suppose $W(A) = G$. Then the following are equivalent:

1. $X_A$ and $X_B$ are $G$-flow equivalent.

2. There exists $\gamma \in G$ and an $E(\mathbb{Z}G)$ equivalence from $(I - A)$ to $I - \gamma B \gamma^{-1}$. 