Flow equivalence of shift spaces (and their C^*-algebras), VII

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Content

Outline

1. The extension theorem
2. Proof in 11 steps
3. The 2-sofic case

Theorem (Boyle-Carlsen-E)

Let X and Y be two irreducible sofic shift spaces with Fischer covers $\mathcal{A}$ and $\mathcal{B}$, respectively, and assume that $\widetilde{\mathsf{X}_{\mathcal{A}}}$ and $\widetilde{\mathsf{X}_{\mathcal{B}}}$ are both AFT. Then X and Y are flow equivalent exactly when the following conditions hold:

(1) $\mathsf{X}_{\mathcal{A}} \sim \mathsf{X}_{\mathcal{B}}$

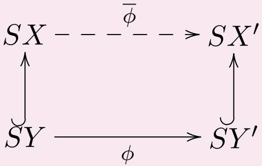
(2)
$$\begin{array}{ccc} S\widetilde{\mathsf{X}_{\mathcal{A}}} & \xrightarrow{\sim_+} & S\widetilde{\mathsf{X}_{\mathcal{B}}} \\ {\scriptstyle S\widetilde{\lambda_{\mathcal{A}}}}\downarrow & & \downarrow{\scriptstyle S\widetilde{\lambda_{\mathcal{B}}}} \\ S\widetilde{\mathsf{L}_{\mathcal{A}}} & \xrightarrow{\sim_+} & S\widetilde{\mathsf{L}_{\mathcal{B}}} \end{array}$$

Proof

The extension theorem!

Extension theorem

Let X and X' be flow equivalent irreducible SFTs with proper subshifts Y, Y' which are also flow equivalent. Then any given flow equivalence $\phi : SY \to SY'$ extends to $\overline{\phi} : SX \to SX'$:



Key results

Theorem (Cf. Krieger's embedding theorem)

Let X be a non-trivial irreducible SFT. For any n, $X \sim Y$ with $\mathsf{X}^{(n)} \hookrightarrow Y$

Theorem (Boyle-Krieger)

Let X be an irreducible SFT with Y, Y' disjoint subshifts such that $X \backslash (Y \sqcup Y')$ contains a copy of $\mathsf{X}^{(2)}$. Assume $\phi : Y \to Y'$ is a conjugacy.
Then ϕ extends to an automorphism $\overline{\phi} : X \to X$.

Theorem (Nasu)

Let Y and X be SFTs such that $Y \hookrightarrow X$. Then there exists a block matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

such that

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \simeq \downarrow & & \downarrow \simeq \\ \mathsf{X}_{A_{11}} & \hookrightarrow & \mathsf{X}_{A} \end{array}$$

Outline

Step 1

Set

$$\begin{aligned} Z' &= \{x \in X' \mid [(x,0)] \in \mathrm{Im}(\phi)\} \\ Z'' &= \{x \in X' \mid [(x,0)] \in \mathrm{Im}(\psi)\} \end{aligned}$$

and find subshift $W \subseteq X'$ such that $h(W) < h(X')$ and $Z \cup Z'' \subseteq W$

Step 2

Assume WLOG that W is an SFT.

Step 3

Find

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

such that $X' \simeq \mathsf{X}_A$, $W \hookrightarrow \mathsf{X}_{A_{11}}$, and A_{22} is $k \times k$ with $k > 1$.

Step 4

Find n such that $Z'' \hookrightarrow \mathsf{X}^{(n)}$.

Step 5

Assume WLOG that $\mathsf{X}^{(n)} \sqcup \mathsf{X}^{(2)} \hookrightarrow \mathsf{X}_{A_{22}}$

Step 6

Find $\chi \in \mathrm{Aut}(\mathsf{X}_A)$ with $\chi(Z'') \subseteq \mathsf{X}^{(n)} \hookrightarrow \mathsf{X}_{A_{22}}$.

Step 7

Assume WLOG that $\mathrm{Im}(\phi) \cap \mathrm{Im}(\psi) = \emptyset$.

Step 8

Consider $\phi_0 : Y \to SX'$ and $\psi_0 : Y \to SX'$ and assume WLOG that

$$\begin{aligned} \phi_0(Y) &= \bigcup_{j=1}^{J} C_j \times \{\epsilon_j\} \\ \psi_0(Y) &= \bigcup_{k=1}^{K} D_k \times \{\delta_k\} \end{aligned}$$

Step 9

Build a cross section $R \subset SX'$ such that

$$\phi_0(Y) \sqcup \psi_0(Y) \sqcup \mathsf{X}^{(2)} \subseteq R$$

Step 10

Find $\zeta \in \mathrm{Aut}(R)$ such that $\zeta \circ \psi_0 = \phi_0$

Step 11

Find $\overline{\zeta} \in \mathrm{Aut}(SR) = \mathrm{Aut}(SX')$ such that $\overline{\zeta} \circ \psi = \phi$ on SY.

Outline

Definition

Let $\pi : X \to Y$ be an SFT cover of an irreducible sofic shift Y. The **fiber product** $F[\pi]$ is given by

$$F[\pi] = \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}$$

Lemma

$F[\pi]$ is an SFT which is irreducible if and only if Y is SFT.

Lemma

$$\Delta = \{(x, x) \mid x \in X\}$$

is an irreducible component of $F[\pi]$ which is isolated when Y is AFT.

Lemma

$F[\pi]$ has a $\mathbb{Z}/2$-action

$$\chi((x,y)) = (y,x)$$

Definition

We say that the irreducible sofic shift Y is 2-sofic if both its left and right Fischer cover $\pi : X \to Y$ is $2-1$.

Theorem

When Y is 2-sofic and AFT we have that

$$F[\pi] \backslash \Delta \simeq M^{-1}(\pi)$$

and $(M^{-1}(\pi), \chi)$ is a free $\mathbb{Z}/2$-SFT.

Theorem

When Y and Y' are 2-sofic and AFT with Fischer covers

$$\pi : X \to Y \qquad \pi' : X' \to Y'$$

we have $Y \sim Y'$ precisely when

1. $X \sim X'$
2. $(M^{-1}(\pi), \chi) \sim (M^{-1}(\pi'), \chi)$