Ergod. Th. & Dynam. Sys. (200Y), **XX**, 1–35
Printed in the United Kingdom

Augmenting dimension group invariants for substitution dynamics

TOKE M. CARLSEN and SØREN EILERS

Department of Mathematics
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark
(e-mail: toke@math.ku.dk, eilers@math.ku.dk)

(*Received 2003*)

Abstract. We present new invariants for substitutional dynamical systems. Our main contribution is a flow invariant which is strictly finer than, but related and akin to, the dimension groups of Herman, Putnam and Skau. We present this group as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data based on the class of *special words* of the dynamical system.

1. *Introduction*

The topics of *topological dynamics* and *operator algebras* are tied together in a way allowing fruitful bidirectional (although asymmetrical) transport of ideas from one area of research to another. The main benefit for operator algebras from this transport of ideas seems to be the definition of several important classes of C^*-algebras associated to dynamical systems. The main benefit for topological dynamics seems to be the discovery of conjugacy invariants, especially ordered groups arising from K-theory for operator algebras.

The contribution in the present paper is of the latter kind, based on a contribution by Matsumoto of the former. Indeed, computing the K-groups of C^*-algebras associated to certain shift spaces, we shall arrive at a flow (and hence conjugacy) invariant for these. This invariant is closely related to, but finer than, the dimension groups for substitutional shift spaces defined by Herman, Putnam and Skau in [**18**], and studied in this particular setting by Durand, Host and Skau in [**14**].

The ground-breaking work of Cuntz and Krieger [**13**] showed how to associate, in a natural and conjugacy invariant way, a C^*-algebra to a shift space of finite type.

There has been a large amount of attention in the operator algebra community to endeavors to generalize this construction, as the Cuntz-Krieger class holds a pivotal position in the theory of *purely infinite* C^*-algebras. Some work has taken the graph picture of such a shift space as a starting point of generalization to a non-finite setting. Other work, notably that of K. Matsumoto, has looked towards the full class of shift spaces on a finite alphabet. Indeed, in a series of papers [**24**]–[**28**] Matsumoto has managed to associate a certain C^*-algebra to *any* such shift space, and to gather much information about the algebras.

At the core of the interplay between operator algebras and dynamics lies an idea originating with Krieger to study the K-groups of the operator algebras in question, employing the fact that these will be invariants of the shift spaces when the C^*-algebras are. This idea allowed the realization of the dimension groups originating in Elliott's work [**15**] on AF algebras as the conjugacy invariant now well known.

Such a strategy has been successfully pursued in work of Matsumoto ([**24**]-[**29**]) (and of Krieger and Matsumoto ([**21**])) leading to a complete description of these K-groups which does not involve C^*-algebras, and to new insight in several important classes of shift spaces. Taking the vastness of the class covered by Matsumoto's work into account, it is no surprise that the best general description of such algebras — given in terms of "past equivalence" — is not readily computable. However, for the class of *substitutional dynamical spaces* which is the focus of the present paper, a very concrete description of this group can be given taking into account the ordered group arising as the K-theory of a completely different C^*-algebraic construction.

Indeed, such shift spaces will give rise to *minimal* topological dynamical systems, and as shown in work by Putnam [**33**] and Giordano, Putnam and Skau [**17**], the canonical *crossed product* associated hereto falls in a well studied class of C^*-algebras. This work was the starting point for work by Durand, Host and Skau [**14**] and by Forrest [**16**] leading to new and readily computable conjugacy invariants for such systems.

In the present paper, we shall compute the Matsumoto K-groups for any primitive and aperiodic substitution shift space in terms of an integer matrix giving rise to a dimension group through a standard inductive limit construction. The starting point of our work is the intermediate presentation of the K-groups given in [**10**] which then, in the present paper, leads to a complete description of the Matsumoto K-group as an inductive limit of a stationary system just like the K-groups considered in [**14**]. Indeed, this part of our computation is a (not completely trivial) adaptation of methods from that paper.

1.1. *A recurring example* Throughout the paper we shall use the substitutions

$$\tau(a) = accdadbb \qquad \tau(b) = acdcbadb$$
$$\tau(c) = aacdcdbb \qquad \tau(d) = accbdadb$$

and

$$\upsilon(a) = accbbadd \qquad \upsilon(b) = accdbabd$$
$$\upsilon(c) = aacbbcdd \qquad \upsilon(d) = acbcdabd$$

to illustrate the nature of our invariant, and to demonstrate how it is computed. The corresponding shift spaces are strong orbit equivalent and hence indistinguishable by the invariant of Durand, Host and Skau. This pair of examples is also resistant to the method of comparing the configuration of the special elements or asymptotic orbits (cf. [**1**]), suggested to us by Charles Holton. Indeed, the "configuration data" of all right or left tail equivalence classes of special elements (see [**10**]) are identical. From [**1**, 3.10] τ and υ are flow equivalent precisely when the derived substitutions τ^* and υ^* defined there are weakly equivalent. However, since computer experiments indicate that both τ^* and υ^* allow squares ww but no triples www in their respective languages, the method given in [**1**] for establishing flow inequivalence does not seem to work here.

Nevertheless, we can use our invariant to prove that the shift spaces associated to these two substitutions are not flow equivalent. We will return to this example in 2.9, 3.4, 3.6, 3.8 and 5.17 below.

1.2. *Acknowledgments* This work is the result of a long process starting when we were both visiting The Mathematical Sciences Research Institute, Berkeley, CA, in the fall of 2000. We wish to thank the Danish Science Research Counsil and Herborgs Fond for making this visit possible. We are also grateful to Klaus Thomsen for directing our attention to the class of shift spaces considered in the paper, and to Ian Putnam for hospitality and suggestions during a visit by the first author to University of Victoria. We are also grateful to Charles Holton for a productive email exchange during the process.

2. *Preliminaries and notation*

Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite, nonempty words with letters from $\mathfrak{a}$. Thus with ϵ denoting the empty word, $\epsilon \notin \mathfrak{a}^\sharp$.

2.1. *Substitutions and shifts* We refer to [**14**] and [**34**] for an introduction to this subject. A *substitution* is simply a map

$$\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp.$$

We can extend τ to $\mathfrak{a}^\sharp$ in the obvious way, and thereby define powers of τ recursively by

$$\tau^n(a) = \tau(\tau^{n-1}(a)).$$

We find the following notation convenient:

Definition 2.1. Let $v, w \in \mathfrak{a}^\sharp$. We say that v *occurs in* w and write

$$v \dashv w$$

when $w = w'vw''$ for suitable $w', w'' \in \mathfrak{a}^\sharp \cup \{\epsilon\}$.

For $w \in \mathfrak{a}^\sharp$, we call the number of letters in w the *length* and denote it $|w|$.

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $-\mathbb{N} = \mathbb{Z} \backslash \mathbb{N}_0$, and equip $\mathfrak{a}^{\mathbb{Z}}$ and $\mathfrak{a}^{\mathbb{N}_0}$ with the product topology from the discrete topology on $\mathfrak{a}$, and define $\sigma : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{Z}}$ and $\sigma_+ : \mathfrak{a}^{\mathbb{N}_0} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$ by

$$(\sigma(x))_n = x_{n+1}, \qquad (\sigma_+(x))_n = x_{n+1}.$$

Such maps we will refer to as *shift maps*. A *two-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped onto itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$" with possible subscripts; note that $\sigma(\underline{\mathsf{X}}) = \underline{\mathsf{X}}$. A *one-sided shift space* is a closed subset of $\mathfrak{a}^{\mathbb{N}_0}$ which is mapped into itself by σ_+. We refer to such spaces by X^+, and remark that $\sigma_+(\mathsf{X}^+) \neq \mathsf{X}^+$ is possible. There is a rich theory of shift spaces; we refer to [**23**] and [**20**] and shall not give details here, but just establish notation. However, the method for describing such spaces by way of *languages* and *forbidden words* deserves explicit mentioning here.

We can further extend τ to $\mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}}$ and $\mathfrak{a}^{\mathbb{Z}}$ in the obvious way. It is necessary in the last case, however, to specify that the word resulting from the substitution of the letter at index 0 of a doubly infinite sequence x will be placed starting at index 0 in $\tau(x)$. Using a dot to indicate the position separating $-\mathbb{N}$ and $\mathbb{N}_0$, as we will do in the following, we thus have

$$\tau(y.x) = \tau(y).\tau(x)$$

The *language* of a shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$ given by

$$\mathcal{L}(\underline{\mathsf{X}}) = \{w \in \mathfrak{a}^\sharp \cup \{\epsilon\} \mid \exists x \in \underline{\mathsf{X}} : w \dashv x\}$$

(extending the notation "$\dashv$" in the obvious way). With the obvious restriction maps

$$\pi_+ : \underline{\mathsf{X}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0},$$

we get

$$\sigma_+ \circ \pi_+ = \pi_+ \circ \sigma$$

and immediately note that $\pi_+(\underline{\mathsf{X}})$ is a one-sided shift space. Sometimes it is more suggestive to write

$$x_{[n,\infty[} = \pi_+(\sigma^n(x))$$

for $n \in \mathbb{Z}$.

Whenever $\mathcal{F} \subseteq \mathfrak{a}^\sharp$ is given, we define a two-sided shift space by

$$\underline{\mathsf{X}}_{\mathcal{F}} = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \in \mathbb{Z} : x_i \cdots x_j \notin \mathcal{F}\}.$$

One can prove that every two-sided shift space has such a description.

We say that shift spaces are *conjugate*, denoted by "$\simeq$", when they are homeomorphic via a map which intertwines the relevant shift maps. A *conjugacy invariant* is a mapping associating to a class of shift spaces another mathematical object, called the *invariant*, in such a way that conjugate shift spaces give isomorphic invariants.

The weaker notion of *flow equivalence* among two-sided shift spaces is also of importance here. This notion is defined using the *suspension flow space* of $(\underline{\mathsf{X}}, \sigma)$ defined as $S\underline{\mathsf{X}} = (\underline{\mathsf{X}} \times \mathbb{R})/\sim$ where the equivalence relation $\sim$ is generated by requiring that $(x, t+1) \sim (\sigma(x), t)$. Equipped with the quotient topology, we get a compact space with a *continuous flow* consisting of a family of maps (ϕ_t) defined by $\phi_t([x, s]) = [x, s+t]$. We say that two shift spaces $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}'$ are *flow equivalent* and write $\underline{\mathsf{X}} \cong_f \underline{\mathsf{X}}'$ if a homeomorphism $F : S\underline{\mathsf{X}} \longrightarrow S\underline{\mathsf{X}}'$ exists with the property that for every $x \in S\underline{\mathsf{X}}$ there is a monotonically increasing map $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$F(\phi_t(x)) = \phi'_{f_x(t)}(F(x)).$$

In words, F takes flow orbits to flow orbits in an orientation-preserving way. It is not hard to see that conjugacy implies flow equivalence.

We derive shift spaces from substitutions as follows:

Definition 2.2. With τ a substitution, we set

$$\mathcal{F}_\tau = \mathfrak{a}^\sharp \backslash \{ w \in \mathfrak{a}^\sharp \mid \exists n \in \mathbb{N}, a \in \mathfrak{a} : w \dashv \tau^n(a) \}.$$

We abbreviate $\underline{\mathsf{X}}_{\mathcal{F}_\tau} = \underline{\mathsf{X}}_\tau$.

Clearly the maps derived from τ above sends $\underline{\mathsf{X}}_\tau$ back in itself.

2.2. *Classes of substitutions* In this section, we single out and discuss some important properties of substitutions:

Definition 2.3. A substitution τ is *primitive* if $|\mathfrak{a}| > 1$ and

$$\exists n \in \mathbb{N} \forall a, b \in \mathfrak{a} : b \dashv \tau^n(a).$$

Note that $\mathcal{F}_\tau = \mathcal{F}_{\tau^n}$ and (hence $\underline{\mathsf{X}}_\tau = \underline{\mathsf{X}}_{\tau^n}$) irrespective of $n \in \mathbb{N}$, when τ is primitive. Furthermore, in the primitive case, $\underline{\mathsf{X}}_\tau$ is *minimal* in the sense that every orbit $\{\sigma^m(x) \mid m \in \mathbb{Z}\}$ is dense, see [**34**, p. 90].

We are not interested in the case where $\underline{\mathsf{X}}_\tau$ is finite, and hence consider only the following class.

Definition 2.4. A substitution τ is *aperiodic* if $\underline{\mathsf{X}}_\tau$ is infinite.

The following concepts are useful in determining whether or not a substitution is aperiodic.

Definition 2.5. A substitution τ on the alphabet $\mathfrak{a}$ is *intertwined* with a substitution υ on the alphabet $\mathfrak{b}$ if $\tau = g \circ f$ and $\upsilon = f \circ g$ for some maps

$$f : \mathfrak{a} \longrightarrow \mathfrak{b}^\sharp \qquad g : \mathfrak{b} \longrightarrow \mathfrak{a}^\sharp.$$

We say that υ is a *simplification* of τ if $|\mathfrak{b}| < |\mathfrak{a}|$. In case τ has no simplification, we call it *elementary.*

It is computable whether a substitution is elementary or not, and there is an algorithmic way to produce a sequence of simplifications ending with an elementary substitution in the latter case, cf. [**35**, p. 17]. Since simplification preserves aperiodicity, this reduces the problem of deciding aperiodicity to the elementary case, in which it is readily decidable, cf. [**32**].

The final property we shall consider is perhaps less natural than the others:

Definition 2.6. A substitution τ is *proper* if for some $\tau' : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp \cup \{\epsilon\}$,

$$\exists n \in \mathbb{N} \exists l, r \in \mathfrak{a} \forall a \in \mathfrak{a} : \tau^n(a) = l\tau'(a)r.$$

In [**14**, Proposition 20, Lemma 21] an algorithmic way is given for passing from a primitive and aperiodic substitution τ' to a primitive, aperiodic and proper substitution τ such that $\underline{\mathsf{X}}_{\tau'} \simeq \underline{\mathsf{X}}_\tau$. There is hence no restriction, when the goal is to discuss conjugacy or flow equivalence of aperiodic and primitive substitution shift spaces, in working with the proper ones among them.

Furthermore, when a proper, primitive and aperiodic substitution τ' is simplified to an elementary substitution τ'', the resulting substitution will also be proper, primitive and aperiodic. That properness is preserved after incrementing the power n in Definition 2.6 is obvious, and the other two claims are proved in [**8**, Lemma

Proposition 2.7. *If τ and υ are intertwined primitive substitutions, then $\underline{\mathsf{X}}_\tau \cong_f \underline{\mathsf{X}}_\upsilon$.*

Proof: Assume that $g \circ f = \tau$ and $f \circ g = \upsilon$ with notation as in Definition 2.5. We prove the claim by defining

$$F : S\underline{\mathsf{X}}_\tau \longrightarrow S\underline{\mathsf{X}}_\upsilon$$

by $F([x, s]) = [f(x), s|f(x_0)|]$ when $s \in [0, 1[$ and $x \in \underline{\mathsf{X}}_\tau$. Checking that F is defined and continuous is straightforward; we shall give details for injectivity and surjectivity of F.

Suppose first that $[f(x), s|f(x_0)|] = [f(y), t|f(y_0)|]$ for some $s, t \in [0, 1[$ and $x, y \in \underline{\mathsf{X}}_\tau$. By definition, there is an $n \in \mathbb{Z}$ with the property

$$\sigma^n(f(x)) = f(y) \qquad s|f(x_0)| = t|f(y_0)| + n.$$

Reversing the roles of x and y if necessary, we may assume that $n \geq 0$. Choose $m \in \mathbb{N}_0$ maximal with the property that

$$|\tau(x_{[0,m[})| \leq |g(f(x)_{[0,n[})|$$

and set $i = |g(f(x)_{[0,n[})| - |\tau(x_{[0,m[})|$. Since $0 \leq i < |\tau(x_m)|$ and

$$\begin{aligned} \sigma^i(\tau(\sigma^m(x))) &= \sigma^{|g(f(x)_{[0,n[})|}(\tau(x)) \\ &= \sigma^{|g(f(x)_{[0,n[})|}(g(f(x))) \\ &= g(\sigma^n(f(x))) \\ &= g(f(y)) \\ &= \tau(y), \end{aligned}$$

we get by Mossé recognizability ([**30**], cf. [**14**, Corollary 12]) that $i = 0$, $\sigma^m(x) = y$ and $|\tau(x_{[0,m[})| = |g(f(x)_{[0,n[})|$. Hence $|f(x_{[0,m[})| = n$. We conclude that $m = n = 0$ because if $m > 0$, then

$$s|f(x_0)| < |f(x_0)| \leq |f(x_{[0,m[})| = n,$$

which would make it impossible for $s|f(x_0)|$ to equal $t|f(y_0)| + n$. Hence $x = y$, and $s|f(x_0)| = t|f(y_0)|$ so that $s = t$ as desired.

To see that F is surjective, let $x \in \underline{\mathsf{X}}_\upsilon$ and $s \in [0,1[$ be given. Choose $y \in \underline{\mathsf{X}}_\upsilon$ and $k \in [0, |\upsilon(y_0)|[$ such that $x = \sigma^k(\upsilon(y))$. Let

$$n = \max\{n' \in \mathbb{N}_0 \mid |f(g(y)_{[0,n'[})| \leq s + k\}$$

and

$$r = \frac{s + k - |f(g(y)_{[0,n[})|}{|f(g(y)_n)|}.$$

Then $r \in [0,1[$, and

$$\begin{aligned} F([\sigma^n(g(y)), r]) &= [f(\sigma^n(g(y))), r|f(g(y)_n)|] \\ &= [\sigma^{|f(g(y)_{[0,n[}))|}(f(g(y))), r|f(g(y)_n)|] \\ &= [\sigma^{|f(g(y)_{[0,n[}))|}(\upsilon(y)), r|f(g(y)_n)|] \\ &= [\sigma^k(\upsilon(y)), s] = [x, s] \end{aligned}$$

because $k + s = |f(g(y)_{[0,n[}))| + r|f(g(y)_n)|$.

We have established that F is a homeomorphism. Since it obviously maps orbits to orbits in an orientation-preserving way, $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_\upsilon$ are flow equivalent. □

COROLLARY 2.8. *If τ is simplified to τ', then $\underline{\mathsf{X}}_\tau \cong_f \underline{\mathsf{X}}_{\tau'}$.*

EXAMPLE 2.9. *The substitutions τ and υ are aperiodic, elementary, primitive, and proper on $\{a, b, c, d\}$.*

3. *Components of the invariant*

3.1. *Basic quantities* Fix a primitive and aperiodic substitution τ. In this section, we shall associate a collection of combinatoric data to τ which we shall employ in our theoretical work, as well as in our invariants, below.

As we are navigating mathematical waters close to known undecidable quantities, the reader might worry about computability of these data. Fortunately, we have been able to find efficient algorithms for computing all the data described below. The algorithms are suffiently simple that we have found ourselves capable of implementing them in a Java applet ([**9**]), and although we have not studied the complexity of these algorithms, we have found that they can compute invariants for substitutions such as τ and υ in a matter of seconds. A presentation of our algorithmic results, proved by methods partially related to [**2**], will appear elsewhere, in [**8**].

We say (cf. [**19**]) that $y \in \underline{\mathsf{X}}_\tau$ is *left special* if there exists $y' \in \underline{\mathsf{X}}_\tau$ such that

$$y_{-1} \neq y'_{-1} \qquad \pi_+(y) = \pi_+(y').$$

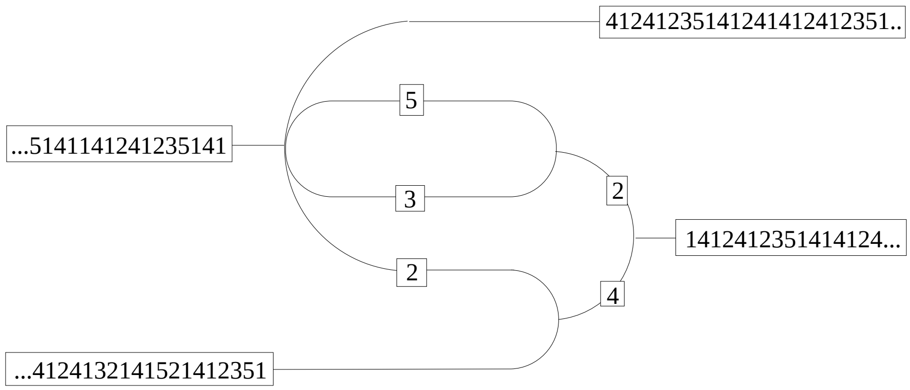

FIGURE 1. Special words for ω

By [**34**, p. 107] and [**3**, Theorem 3.9], there is a finite, but nonzero, number of left special words.

We say that the left special word y is *adjusted* if $\sigma^{-n}(y)$ is not left special for any $n \in \mathbb{N}$, and that y is *cofinal* if $\sigma^n(y)$ is not left special for any $n \in \mathbb{N}$. Thinking of left special words as those which are not deterministic from the right at index -1, the adjusted and cofinal left special words are those where this is the *leftmost* and *rightmost* occurrence of nondeterminacy, respectively.

Let $x, y \in \underline{\mathsf{X}}_\tau$. If there exist an n and an M such that $x_m = y_{n+m}$ for all $m > M$ then we say that x and y are *right shift tail equivalent* and write $x \sim_r y$. One defines right special elements using

$$y_0 \neq y'_0 \qquad \pi_-(y) = \pi_-(y'),$$

and left shift tail equivalence and $\sim_l$ in the obvious way. When a left special word y is cofinal, every word in its right shift tail equivalence class will end in $\pi_+(y)$.

REMARK 3.1. Quite often, all the special words of a substitution are simultaneously adjusted and cofinal. There are exceptions, though, as illustrated by Figure 1 which indicates the relations among all the special words of the aperiodic and primitive substitution ω on $\{1, 2, 3, 4, 5\}$ given by $\omega(1) = 123514, \omega(2) = 124, \omega(3) = 13214, \omega(4) = 14124, \omega(5) = 15214$. The element

$$...514114124123514152.1412412351414124...$$

is an example of a cofinal left special element which is not adjusted left special. Shifting it to

$$...51411412412351415.21412412351414124...$$

one achieves an element which is adjusted left special, but not cofinal. Shifting once more, one gets an element which is simultaneously adjusted and cofinal *right* special.

Definition 3.2. When τ is an aperiodic and primitive substitution, we denote the number of right shift tail equivalence classes of left special elements of $\underline{\mathsf{X}}_\tau$ by n_τ.

It is not hard to see directly that this number is a flow invariant for substitutional systems, but in fact it will follow from our main result as noted in Theorem 6.2.

As described in Section 2.2 there is an algorithmic way of passing from a given aperiodic and primitive substitution τ' to an aperiodic, elementary, primitive and proper substitution τ in the same flow equivalence class. Now as concluded in [**8**, Remark

Definition 3.3. We say that a substitution τ is *basic* if it has the form $(\tau')^n$ for some aperodic, elementary, primitive and proper substitution τ', and if all its left special words have the form

$$\cdots\tau^3(v)\tau^2(v)\tau(v)vu.w\tau(w)\tau^2(w)\tau^3(w)\dots \tag{3.1}$$

for suitable $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)\backslash\{\epsilon\}$ such that

$$\tau(u) = vuw. \tag{3.2}$$

As outlined above, there is an algorithm yielding for every aperiodic and primitive substitution υ a basic substitution $\tau = (\tau')^n$ with $\underline{\mathsf{X}}_\upsilon \cong_f \underline{\mathsf{X}}_\tau$. We may hence work only with basic substituions as long as we are interested in invariants of flow equivalence.

Our paper [**8**] provides algorithms for computing and representing each left special element in $\underline{\mathsf{X}}_\tau$ as in (3.1) and (3.2), to determine which of these elements are adjusted or cofinal, and which among them are right shift tail equivalent. Thus we may, in what follows, use the convenient notation

$$[w]^- = \cdots\tau^{n+1}(w)\tau^n(w)\tau^{n-1}(w)\cdots\tau(w)w \in \mathfrak{a}^{-\mathbb{N}}$$
$$[w]^+ = w\tau(w)\cdots\tau^{n-1}(w)\tau^n(w)\tau^{n+1}(w)\cdots \in \mathfrak{a}^{\mathbb{N}_0}$$

to describe all the (adjusted, cofinal) left special words.

Example 3.4. *Both τ and υ are basic substitutions. The left special elements of $\underline{\mathsf{X}}_\tau$ are*

$$[accd]^-a.[dbb]^+, [aacd]^-c.[dbb]^+, [acdc]^-b.[adb]^+, [accb]^-d.[adb]^+.$$

which are all simultaneously adjusted and cofinal. The left special elements of $\underline{\mathsf{X}}_\upsilon$ are

$$[accbb]^-a.[dd]^+, [aacbb]^-c.[dd]^+, [acbc]^-d.[abd]^+, [accd]^-b.[abd]^+,$$

also all adjusted and cofinal. Since $[dbb]^+ \not\sim_r [adb]^+$ and $[dbb]^+ \not\sim_r [adb]^+$, $\mathsf{n}_\tau = \mathsf{n}_\upsilon = 2$.

Definition 3.5. When τ is a basic substitution, equipped with some ordering of the right shift tail equivalence classes containing left special elements, we define $\mathsf{p}_\tau \in \mathbb{N}^{\mathsf{n}_\tau}$ by

$$\mathsf{p}_\tau = (p_1, \dots, p_{\mathsf{n}_\tau}),$$

where $p_i + 1$ is the number of adjusted left special words in each such class.

Note that by the definition of right special words, $p_i \geq 1$ for all i.

Enumerating the output of our algorithm we organize all the adjusted left special words as

$$\begin{array}{c} \mathsf{y}^1_1, \mathsf{y}^1_2, \ldots, \mathsf{y}^1_{p_1+1} \\ \mathsf{y}^2_1, \mathsf{y}^2_2, \ldots, \mathsf{y}^2_{p_2+1} \\ \vdots \\ \mathsf{y}^{\mathsf{n}_\tau}_1, \mathsf{y}^{\mathsf{n}_\tau}_2, \ldots, \mathsf{y}^{\mathsf{n}_\tau}_{p_{\mathsf{n}_\tau}+1} \end{array}$$

where

$$\mathsf{y}^j_k = [\mathsf{v}^j_k]^- \mathsf{u}^j_k . [\mathsf{w}^j_k]^+, \qquad \tau(\mathsf{u}^j_k) = \mathsf{v}^j_k \mathsf{u}^j_k \mathsf{w}^j_k.$$

Finally, we denote the last letter of each word u^j_k by a^j_k.

We further choose *one* cofinal left special element in each right tail equivalence class, and denote it $\widetilde{\mathsf{y}}^j$. As above, we write

$$\widetilde{\mathsf{y}}^j = [\widetilde{\mathsf{v}}^j]^- \widetilde{\mathsf{u}}^j . [\widetilde{\mathsf{w}}^j]^+, \qquad \tau(\widetilde{\mathsf{u}}^j) = \widetilde{\mathsf{v}}^j \widetilde{\mathsf{u}}^j \widetilde{\mathsf{w}}^j.$$

and denote by $\widetilde{\mathsf{a}}^j$ the last letter of $\widetilde{\mathsf{u}}^j$.

3.2. *Matrices* The number $\#[a, w]$ counts the number of occurrences of the letter a in the word w. As usual (cf. [**14**]) one associates to any substitution τ the *abelianization matrix* which is the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix $\mathbf{A}_\tau$ given by

$$(\mathbf{A}_\tau)_{a,b} = \#[b, \tau(a)].$$

Example 3.6.

$$\mathbf{A}_\tau = \mathbf{A}_\upsilon = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$

We shall define a rectangular matrix based on different data of the same nature. The reader may share our initial surprise that this definition will eventually lead to an invariant of conjugacy and flow equivalence.

Definition 3.7. To a basic substitution τ one associates the $\mathsf{n}_\tau \times |\mathfrak{a}|$-matrix $\mathbf{E}_\tau$ given by

$$(\mathbf{E}_\tau)_{j,b} = \left(\sum_{k=1}^{p_j+1} e_{\tau, \mathsf{a}^j_k, \mathsf{w}^j_k}(b) \right) - e_{\tau, \widetilde{\mathsf{a}}^j, \widetilde{\mathsf{w}}^j}(b)$$

with

$$e_{\tau,a,w}(b) = \max(0, \#[b, \tau(a)] - \#[b, aw])$$

and with $\mathsf{a}^j_k, \widetilde{\mathsf{a}}^j$ and $\mathsf{w}^j_k, \widetilde{\mathsf{w}}^j$ given as in Section 3.1.

In all the applications of $e_{\tau,a,w}$ we either have that $\tau(a)$ is a proper subword of aw, in which case the term vanishes, or that $\tau(a)$ ends in aw, in which case the contribution of the term is a count of the remaining letters in $\tau(a)$

EXAMPLE 3.8. *Enumerating the elements given Example 3.6 in the order* $\mathsf{y}_1^1, \mathsf{y}_2^1 = \widetilde{\mathsf{y}}^1, \mathsf{y}_1^2, \mathsf{y}_2^2 = \widetilde{\mathsf{y}}^2$, *we would have*

$$\mathsf{E}_\tau = \begin{bmatrix} \#[\bullet, accd] + \#[\bullet, aacd] - \#[\bullet, aacd] \\ \#[\bullet, acdc] + \#[\bullet, accb] - \#[\bullet, accb] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

and similarly

$$\mathsf{E}_\upsilon = \begin{bmatrix} \#[\bullet, accbb] \\ \#[\bullet, acbc] \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}.$$

DEFINITION 3.9. To a basic substitution τ one associates the $(|\mathfrak{a}| + \mathsf{n}_\tau) \times (|\mathfrak{a}| + \mathsf{n}_\tau)$-matrix

$$\widetilde{\mathsf{A}}_\tau = \begin{bmatrix} \mathsf{A}_\tau & 0 \\ \mathsf{E}_\tau & \mathsf{Id} \end{bmatrix}.$$

4. *Matsumoto K-groups*

The *Matsumoto K-groups* with which we are concerned in the present paper can be efficiently defined directly, using the concept of past equivalence. They were, however, discovered as the (ordered) K-groups associated to certain classes of C^*-algebras. The results in the present paper do not depend directly or indirectly on an analysis of C^*-algebras, so we shall employ the most fundamental definition and repeat it for the benefit of the reader in section 4.1 below.

However, since our results were developed in this category and subsequently translated to a more basic setting, and since we do have further results (see Section 6) which we do not know how to get without this machinery, we find that a brief outline of how our work is positioned in an operator algebraic setting may be in order. We do this in section 4.3 below, which may be skipped by any reader not operator algebraically inclined.

4.1. *Past equivalence*

Let $\underline{\mathsf{X}}$ be a two-sided shift space. For every $x \in \pi_+(\underline{\mathsf{X}})$ and every $k \in \mathbb{N}$ we set

$$\mathcal{P}_k(x) = \{\mu \in \mathcal{L}(\underline{\mathsf{X}}) \mid \mu x \in \pi_+(\underline{\mathsf{X}}), |\mu| = k\},$$

and define for every $l \in \mathbb{N}$ an equivalence relation $\sim_l$ on $\pi_+(\underline{\mathsf{X}})$ by

$$x \sim_l y \Leftrightarrow \mathcal{P}_l(x) = \mathcal{P}_l(y).$$

Following Matsumoto ([**25**], [**27**]), we denote by $[x]_l$ the equivalence class of x and refer to the relation as *l-past equivalence*.

Obviously the set of equivalence classes of the l-past equivalence relation $\sim_l$ is finite. We will denote the number of such classes $m(l)$ and enumerate them $\mathcal{E}_s^l$ with $s \in \{1, \ldots, m(l)\}$. For each $l \in \mathbb{N}$, we define an $m(l+1) \times m(l)$-matrix I^l by

$$(\mathsf{I}^l)_{rs} = \begin{cases} 1 & \text{if } \mathcal{E}_r^{l+1} \subseteq \mathcal{E}_s^l \\ 0 & \text{otherwise,} \end{cases}$$

and note that I^l induces a group homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. We denote by $\mathbb{Z}_{\underline{\mathsf{X}}}$ the group given by the inductive limit

$$\varinjlim(\mathbb{Z}^{m(l)}, \mathsf{I}^l).$$

For a subset $\mathcal{E}$ of $\pi_+(\underline{\mathsf{X}})$ and a finite word μ we let $\mu\mathcal{E} = \{\mu x \in \pi_+(\underline{\mathsf{X}}) \mid x \in \mathcal{E}\}$.

For each $l \in \mathbb{N}$ and $a \in \mathfrak{a}$ we define an $m(l+1) \times m(l)$-matrix

$$(\mathsf{L}_a^l)_{rs} = \begin{cases} 1 & \text{if } \emptyset \neq a\mathcal{E}_r^{l+1} \subseteq \mathcal{E}_s^l \\ 0 & \text{otherwise,} \end{cases}$$

and letting $\mathsf{L}^l = \sum_{a\in\mathfrak{a}} \mathsf{L}_a^l$ we get a matrix inducing a group homeomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. Since one can prove that $\mathsf{L}^{l+1} \circ \mathsf{I}^l = \mathsf{I}^{l+1} \circ \mathsf{L}^l$, a group endomorphism λ on $\mathbb{Z}_{\underline{\mathsf{X}}}$ is induced.

Definition 4.1. [Cf. [**25**, Theorem 4.9], [**27**], [**29**, Theorem], [**5**, pp. 67-68]] Let $\underline{\mathsf{X}}$ be a two-sided shift space. The group

$$K_0(\underline{\mathsf{X}}) = \mathbb{Z}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \lambda)\mathbb{Z}_{\underline{\mathsf{X}}},$$

is a conjugacy invariant of $\underline{\mathsf{X}}$ and $\pi_+(\underline{\mathsf{X}})$, and a flow invariant of $\underline{\mathsf{X}}$.

4.2. *An intermediate description* The dimension group $DG(X, \sigma)$ of a Cantor minimal system (X, σ) — a dynamical system where X is a Cantor set in which every σ-orbit is dense — is the cokernel of the map

$$\mathrm{Id} - (\sigma^{-1})^\sharp : C(X, \mathbb{Z}) \longrightarrow C(X, \mathbb{Z}),$$

equipped with the quotient order induced from $C(X, \mathbb{N}_0)$.

When τ is an aperiodic and primitive substitution, $(\underline{\mathsf{X}}_\tau, \sigma)$ is a Cantor minimal system. The technical basis of our results is a similar description of the Matsumoto K_0-group of a basic substitution as the cokernel of a certain map, based on a set of choices made as described in Section 3.1. Such a description can, for the special kind of shift spaces considered here, be inferred from a theoretically straightforward, but rather technical, analysis of l-past equivalence relation (or the lambda-graph) of the substitutional dynamics, noting that it may be correlated with the structure of the left special words. It is not the same cokernal description as the – much more general – basic tool in [**27**]. We defer the proofs of this to our paper [**10**], and shall here just present the results, laying out notation along the way.

Fix a basic substitution τ. We shall work extensively with elements of $\mathbb{Z}^{\mathsf{n}_\tau}$, and fix here notation for such. We shall prefer the index $j \in \{1, \ldots, \mathsf{n}_\tau\}$ and write

$$\underline{x} = \left(x^j\right)_{j=1}^{\mathsf{n}_\tau}, \underline{y} = \left(y^j\right)_{j=1}^{\mathsf{n}_\tau}, \underline{x}_i = \left(x_i^j\right)_{j=1}^{\mathsf{n}_\tau},$$

etc., for such vectors. The vector $\underline{\delta}_{j_0}$ has zero entries except at index j_0, where the entry is 1.

We now define a group

$$G_\tau = C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \oplus \sum_{i=0}^{\infty} \mathbb{Z}^{\mathsf{n}_\tau}$$

and, based on a set of choices made as described in Section 3.1, a map $A_\tau : G_\tau \longrightarrow G_\tau$ given hereon by

$$A_\tau(f, [\underline{x}_0, \underline{x}_1, \dots]) = \left(f \circ \sigma^{-1}, \left[\left(\left(\sum_{k=1}^{p_j+1} f(\sigma^{-1}(\mathsf{y}_k^j)) \right) - f(\sigma^{-1}(\widetilde{\mathsf{y}}^j)) \right)_{j=1}^{\mathsf{n}_\tau}, \underline{x}_0, \underline{x}_1, \dots \right] \right),$$

with y_k^j and $\widetilde{\mathsf{y}}^j$ defined as in Section 3.1 above. The following result of [**10**] forms the basis of our alternative characterization of $K_0(\underline{\mathsf{X}}_\tau)$ involving G_τ and A_τ.

PROPOSITION 4.2. *When τ is a basic substitution, then $K_0(\underline{\mathsf{X}}_\tau)$ is isomorphic to the cokernel of the map A_τ.*

4.3. *Related C^*-algebras* There is a universal construction associating to most dynamical systems a C^*-algebra called the *crossed product*. In the seminal case of a $\mathbb{Z}$-action given by a homeomorphism ϕ of a compact Hausdorff space X, one first passes to the C^*-dynamical system of the transpose $\phi^\sharp$ acting on $C(X)$ by composition, and constructs therefrom a C^*-algebra denoted $C(X) \rtimes_{\phi^\sharp} \mathbb{Z}$ which captures the dynamics in non-commutative structure. A crossed product algebra $C(\underline{\mathsf{X}}) \rtimes_{\sigma^\sharp} \mathbb{Z}$ can hence be associated to each two-sided shift space, and the universality of the construction proves that such an associated C^*-algebra is a conjugacy invariant. But in this special case, another invariant C^*-algebra is available to us via the one-sided shift $\pi_+(\underline{\mathsf{X}})$.

The C^*-algebras first considered by Matsumoto can be constructed from such a one-sided shift space in several equivalent ways – by a universal construction based on generators and relations, or by invoking standard constructions in C^*-algebras based on either groupoids ([**7**]) or Hilbert C^*-bimodules ([**5**]). The original approach in [**24**] based on a Fock space construction may in some cases lead to a different algebra, see [**12**]. Each of these approaches have independent virtues and add to the accumulated value of this concept. When a one-sided shift space X^+ is given, we denote this C^*-algebra by $\mathcal{O}_{\mathsf{X}^+}$. Such C^*-algebras can be used to provide conjugacy invariants up to either one-sided or two-sided conjugacy as follows. Here and below, $\mathbb{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert spaces.

THEOREM 4.3. *[[**5**, Theorem 4.1.4], [**6**]] Let X^+ and Y^+ be one-sided shift spaces. We have*

$$\mathsf{X}^+ \simeq \mathsf{Y}^+ \Longrightarrow \mathcal{O}_{\mathsf{X}^+} \simeq \mathcal{O}_{\mathsf{Y}^+}.$$

Furthermore, when $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ are two-sided shift spaces, we have

$$\underline{\mathsf{X}} \simeq \underline{\mathsf{Y}} \Longrightarrow \underline{\mathsf{X}} \cong_f \underline{\mathsf{Y}} \Longrightarrow \mathcal{O}_{\pi_+(\underline{\mathsf{X}})} \otimes \mathbb{K} \simeq \mathcal{O}_{\pi_+(\underline{\mathsf{Y}})} \otimes \mathbb{K}.$$

This observation goes back to Matsumoto for a large family of shift spaces with the so-called *property (I)*, see [**24**, Proposition 5.8], [**28**, Corollary 6.2] and [**26**, Lemma 4.5]. However, the one-sided shift spaces associated to the two-sided shifts under investigation, those of the form $\pi_+(\underline{\mathsf{X}}_\tau)$, rarely have this property.

Definition 4.4. $\mathcal{O}_\tau = \mathcal{O}_{\pi_+(\underline{\mathsf{X}}_\tau)}$.

These C^*-algebras bear relevance for the groups considered in the present paper through a K-functor. In fact,

$$\begin{aligned} DG(\underline{\mathsf{X}}_\tau) &= K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z}) \\ K_0(\underline{\mathsf{X}}_\tau) &= K_0(\mathcal{O}_\tau). \end{aligned}$$

In general, the C^*-algebras $C(\underline{\mathsf{X}}) \rtimes_{\sigma^\sharp} \mathbb{Z}$ and $\mathcal{O}_{\pi_+(\underline{\mathsf{X}})}$ will be very different. For instance, when $\underline{\mathsf{X}} = \underline{\mathsf{X}}_{\mathcal{F}}$ where $\mathcal{F}$ is a finite set (a so-called shift of finite type), the crossed product will always have a very rich ideal structure, whereas the algebra considered by Matsumoto becomes the *Cuntz-Krieger* algebra associated to $\mathcal{F}$, which is a simple C^*-algebra under modest assumptions. When $\mathcal{F} = \mathcal{F}_\tau$, as proved in [**7**], there is an extension of C^*-algebras

$$0 \longrightarrow \mathbb{K}^{\mathsf{n}_\tau} \longrightarrow \mathcal{O}_\tau \longrightarrow C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z} \longrightarrow 0 \tag{4.3}$$

showing that $\mathcal{O}_\tau$ is non-simple, with the crossed product as a quotient.

Since $C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z}$ is simple because the underlying dynamical system is minimal, this gives a complete description of the ideal structure of $\mathcal{O}_\tau$. However, a reader unfamiliar with the extension theory of C^*-algebras should probably be explicitly warned that such a description offers very little concrete information about the algebra in general. In many cases, the theorem of Brown-Douglas-Fillmore in conjunction with the Universal coefficient theorem in Kasparov's theory proved by Rosenberg and Schochet shows that there are uncountably many nonisomorphic algebras having such a decomposition.

5. *Inductive limit descriptions*

A main accomplishment in [**14**] is the description of $DG(\underline{\mathsf{X}}_\tau)$ (see Section 4.2) as a stationary inducetive limit with matrices for the connecting maps read off directly from the substitution. A main result is the following.

Theorem 5.1. *[[**14**], Theorem 22(i)] There is an order isomorphism*

$$DG(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|}$ *is ordered by*

$$(x_a) \geq 0 \Longleftrightarrow \forall a \in \mathfrak{a} : x_a \geq 0.$$

We have found analogous results for the ordered group $K_0(\underline{\mathsf{X}}_\tau)$, but will in the present paper restrain ourselves to give, in Theorem 5.8 below, an inductive limit description of $K_0(\underline{\mathsf{X}}_\tau)$ as a group.

Computing the order structure requires a deeper analysis of the interrelations among certain C^*-algebras, employing the fact that $DG(\underline{\mathsf{X}}_\tau) = K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_{\sigma^\sharp} \mathbb{Z})$ and $K_0(\underline{\mathsf{X}}_\tau) = K_0(\mathcal{O}_\tau)$, cf. Section 4.3. We defer this to [**11**], but the interested reader is referred to Section 6 for a brief overview of our results.

5.1. *Kakutani-Rohlin partitions* Theorem 5.1 is achieved from the cokernal description of the dimension group (see Section 4.2 above) using a sequence of Kakutani-Rohlin partitions of $\underline{\mathsf{X}}_\tau$ and direct computations of the actions hereupon by τ. We are going to follow the lead of [**14**], adapting crucial techniques to our somewhat more complicated setting. As in that paper, we abbreviate

$$[a] = \{x \in \underline{\mathsf{X}}_\tau \mid x_0 = a\},$$

and note that by [**14**, Corollary 13] — a consequence of the work by Mossé [**31**], [**30**] — the family of sets

$$\sigma^{-i}\tau^m[a], \qquad a \in \mathfrak{a} \quad, i \in \{0, \dots, |\tau(a)^m| - 1\}, \tag{5.4}$$

forms a (clopen) disjoint partition of $\underline{\mathsf{X}}_\tau$ for each $m \in \mathbb{N}$, when τ is any aperiodic substitution.

To set up notation and motivate our adaptation, we will sketch how the Kakutani-Rohlin partitions are used in [**14**] to prove Theorem 5.1 in the case of proper substitutions. We do this to allow references to parts of this proof in our proof of Theorem 5.8 below.

For any fixed $m \in \mathbb{N}$, we use the notation $\Xi = (\xi_{i,a})$ to denote a collection of integers where $a \in \mathfrak{a}, i \in \{0, \dots, |\tau^m(a)| - 1\}$. For each such collection, we define a function on $\underline{\mathsf{X}}_\tau$ by

$$f_\Xi = \sum_{a \in \mathfrak{a}} \sum_{i=0}^{|\tau^m(a)|-1} \xi_{i,a} 1_{\sigma^{-i}\tau^m[a]}.$$

Definition 5.2. Fix $m \in \mathbb{N}$. We define $\mathcal{CE}_\tau[m]$ as the set all integer collections defined above, and let

$$\mathrm{rk}_\tau[m] = \{f_\Xi \in C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \mid \Xi \in \mathcal{CE}_\tau[m]\}.$$

The subset of $\mathcal{CE}_\tau[m]$ with the further property that

$$\xi_{0,a} = \xi_{0,b} \qquad \forall a, b \in \mathfrak{a}$$

we denote by $\mathcal{CE}^c_\tau[m]$, and let $\mathrm{rk}^c_\tau[m]$ be the corresponding subspace of $\mathrm{rk}_\tau[m]$.

Our properness assumption enters our proof as follows, cf. [**14**, Proposition 14(iv)]:

Proposition 5.3. *If τ is a proper, primitive and aperiodic substitution, then*

$$\bigcup_{m=1}^{\infty} \mathrm{rk}_\tau[m] = \bigcup_{m=1}^{\infty} \mathrm{rk}^c_\tau[m] = C(\underline{\mathsf{X}}_\tau, \mathbb{Z}).$$

As the family generates $C(\underline{\mathsf{X}}_\tau, \mathbb{Z})$, the proof of Theorem 5.1 may be reduced to check that $\psi_{m+1} = \mathbf{A}_\tau \circ \psi_m$ where

$$\psi_m : \mathrm{rk}_\tau[m] \longrightarrow \mathbb{Z}^{|\mathfrak{a}|}, \qquad \psi_m\left(\sum_{a \in \mathfrak{a}} \sum_{i=0}^{|\tau^m(a)|-1} \alpha_{i,a} 1_{\sigma^{-i}\tau^m[a]}\right) = \left(\sum_{i=0}^{|\tau^m(a)|-1} \alpha_{i,\bullet}\right)_{\bullet \in \mathfrak{a}}$$

so that a map

$$\psi_\infty : C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) \longrightarrow \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

is induced, and to check that this map is surjective and has the property that $\ker(\psi_\infty) = \operatorname{Im}(\operatorname{Id} - (\sigma^{-1})^\sharp)$. An isomorphism

$$\overline{\psi}_\infty : C(\underline{\mathsf{X}}_\tau, \mathbb{Z}) / \operatorname{Im}(\operatorname{Id} - (\sigma^{-1})^\sharp) \longrightarrow \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

is then induced.

We need to consider the interrelations between sets of the form $\sigma^{-n}\tau^{m+1}[a]$ and $\sigma^{-n'}\tau^m[a']$. Doing so is eased by the following perhaps somewhat counterintuitive notation, which we shall use for the remainder of Section 5.

Notation 5.4. *Let $w \in \mathfrak{a}^\sharp$. By $w^{[h]}$ we denote the letter at position h in w* from right to left, *starting with index* 0 *at the rightmost letter. By $w^{]h,0]}$ we denote the subword of w consisting of the h rightmost letters.*

It is straightforward (but tedious) to check that

$$\sigma^{-(|\tau^m(\tau(a)^{]h,0]})|+k)}\tau^{m+1}[a] \subseteq \sigma^{-k}\tau^m[\tau(a)^{[h]}] \tag{5.5}$$

for any $a \in \mathfrak{a}$, $m \in \mathbb{N}_0$, $h \in \{0, \ldots, |\tau(a)| - 1\}$ and $k \in \mathbb{N}_0$. Letting $k \in \{0, \ldots, |\tau^m(\tau(a)^{[h]})|\}$ one covers the sets in the $(m+1)$st level of the Rohlin-Kakutani partition exactly once. Consequently, $\operatorname{rk}_\tau[m] \subseteq \operatorname{rk}_\tau[m+1]$ and $\operatorname{rk}^c_\tau[m] \subseteq \operatorname{rk}^c_\tau[m+1]$.

We end this section by defining a numerical quantity associated to the kind of words used to describe right special elements and observing two basic properties of it:

Definition 5.5. For $w \in \mathcal{L}(\underline{\mathsf{X}}_\tau)$, and $m \in \mathbb{N}$ we set $\ell(m, w) = \sum_{i=0}^{m-1} |\tau^i(w)|$. We also let $\ell(0, w) = 0$.

Observation 5.6. When $[v]^- u.[w]^+ \in \underline{\mathsf{X}}$ for $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau) \backslash \{\epsilon\}$ with $\tau(u) = vuw$, and u ends in $a \in \mathfrak{a}$,

$$[v]^- u.[w]^+ \in \sigma^{-\ell(m,w)}\tau^m[a]$$

for every $m \in \mathbb{N}_0$.

Proof: An inductive argument based on

$$\tau([v]^- u.[w]^+) = [\tau(v)]^- vuw.[\tau(w)]^+ = \sigma^{|w|}([v]^- u.[w]^+).$$

□

Observation 5.7. Let $u, v, w \in \mathcal{L}(\underline{\mathsf{X}}_\tau) \backslash \{\epsilon\}$ with $\tau(u) = vuw$, and assume that u ends in $a \in \mathfrak{a}$. For any $m \in \mathbb{N}_0$ and $h \leq |\tau(a)|$, we have

$$|\tau^m(\tau(a)^{]h,0]})| \geq \ell(m+1, w) + 1$$

if and only if $h \geq |w| + 1$.

Proof: Suppose first that $h \leq |w|$. Then $\tau(a)^{]h,0]} \dashv w$ so that

$$|\tau^m(\tau(a)^{]h,0]})| \leq |\tau^m(w)| \leq \ell(m+1,w).$$

Induction after m is required to prove the other implication, so assume that $h \geq |w|+1$ and note that this assumption is equivalent with the case $m=0$. For $m>0$ we further note that $aw \dashv \tau(a)^{]h,0]}$ because of the way that u, a and w are interrelated. Thus

$$\begin{aligned} |\tau^m(\tau(a)^{]h,0]})| &\geq |\tau^m(a)| + |\tau^m(w)| \\ &\geq |\tau^{m-1}(\tau(a)^{]h,0]})| + |\tau^m(w)| \\ &\geq \ell(m,w) + 1 + |\tau^m(w)| \\ &= \ell(m+1,w)+1 \end{aligned}$$

using the induction hypothesis at the third inequality sign. □

5.2. *A stationary inductive system* The main result of our paper is the following:

THEOREM 5.8. *Let τ be a basic substitution. There is a group isomorphism*

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathbf{A}}_\tau).$$

We recall that there is an algorithmic way of passing from any aperiodic and primitive substitution to one which is basic, staying in the same flow equivalence class. Since K_0 is an invariant of flow equivalence, the result above can be used to compute the Matsumoto K_0-group of any aperiodic and primitive substitution.

We note right away that the group $K_0(\underline{\mathsf{X}}_\tau)$ has the group $DG(\underline{\mathsf{X}}_\tau)$ computed in [**14**], as a quotient. The corresponding kernel is simply $\mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$. But as we shall see, this extension is not split in general, making room for storage of additional information in the non-vanishing cross-term $\mathbf{E}_\tau$.

COROLLARY 5.9. *Let τ be a basic substitution. The short exact sequence*

$$0 \longrightarrow \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{P} \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{R} \mathbb{Z}^{|\mathfrak{a}|} \longrightarrow 0$$

induces a short exact sequence

$$0 \longrightarrow \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \xrightarrow{P_\infty} K_0(\underline{\mathsf{X}}_\tau) \xrightarrow{R_\infty} DG(\underline{\mathsf{X}}_\tau) \longrightarrow 0\ .$$

Proof: Observe that $P \circ \mathrm{Id}_{\mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}} = \widetilde{\mathbf{A}}_\tau \circ P$ and $R \circ \widetilde{\mathbf{A}}_\tau = \mathbf{A}_\tau \circ R$. □

DEFINITION 5.10. When $\Xi \in \mathcal{CE}_\tau[m]$, we define $\widetilde{\Xi}, \widehat{\Xi} \in \mathcal{CE}_\tau[m]$ by

$$\begin{aligned} \widetilde{\xi}_{i,a} &= \begin{cases} \xi_{i+1,a} & 0 \leq i < |\tau^m(a)|-1 \\ \xi_{0,a} & i = |\tau^m(a)|-1, \end{cases} \\ \widehat{\xi}_{i,a} &= \sum_{k=i}^{|\tau^m(a)|-1} \xi_{k,a}. \end{aligned}$$

LEMMA 5.11. *If* $\Xi \in \mathcal{CE}_\tau^c[m]$ *then* $f_{\widetilde{\Xi}} = f_\Xi \circ \sigma^{-1}$.

Proof: Let c denote the mutual value at the lower level of Ξ. First note that if $x \in \sigma^{-i}\tau^m[a]$ is given with $i < |\tau^m(a)| - 1$, $\sigma^{-1}(x) \in \sigma^{-(i+1)}\tau^m[a]$. Further, if $x \in \sigma^{-(|\tau^m(a)|-1)}\tau^m[a]$, say with $x = \sigma^{-(|\tau^m(a)|-1)}(y)$ where $y \in \tau^m([a])$, we can write

$$\sigma^{-1}(x) = \sigma^{-|\tau^m(a)|}\tau^m(y) = \tau^m(\sigma^{-1}(y))$$

such that $\sigma^{-1}(x) \in \sigma^{-0}\tau^m[b]$ for b chosen as the second letter of y. Thus for any $x \in \underline{\mathsf{X}}_\tau$, we have

$$f_{\widetilde{\Xi}}(x) = \begin{cases} \xi_{i+1,a} & x \in \sigma^{-i}\tau^m[a], i < |\tau^m(a)| - 1 \\ c & x \in \sigma^{-i}\tau^m[a], i = |\tau^m(a)| - 1 \end{cases} = f_\Xi(\sigma^{-1}(x)).$$

□

LEMMA 5.12. *If* $\Xi \in \mathcal{CE}_\tau[m]$ *and satisfies*

$$\sum_{i=0}^{|\tau^m(a)|-1} \xi_{i,a} = 0 \qquad \forall a \in \mathfrak{a}, \tag{5.6}$$

then $\widehat{\Xi} \in \mathcal{CE}_\tau^c[m]$, *and*

$$(\mathrm{Id} - A_\tau)(f_{\widehat{\Xi}}, [\underline{0}, \underline{0}, \dots]) = \left(f_\Xi, \left[\left(\sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} - \sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} \right)_{j=1}^{\mathsf{n}_\tau}, \underline{0}, \underline{0}, \dots \right] \right).$$

Proof: By (5.6), $\widehat{\Xi} \in \mathcal{CE}_\tau^c[m]$. So we get by Lemma 5.11 that $f_{\widehat{\Xi}} \circ \sigma^{-1} = f_{\widetilde{\widehat{\Xi}}}$. By (5.6) again,

$$\widehat{\Xi} - \widetilde{\widehat{\Xi}} = \Xi.$$

Finally according to Observation 5.6,

$$f_{\widehat{\Xi}}(\sigma^{-1}(\mathsf{y}_k^j)) = \widehat{\xi}_{\ell(m,\mathsf{w}_k^j)+1,\mathsf{a}_k^j} = \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j}$$

and similarly for $\widetilde{\mathsf{y}}^j$. □

We are now ready to define the family of maps which shall give the desired identification between $G_\tau / \mathrm{Im}(\mathrm{Id} - A_\tau)$ and a stationary inductive system.

DEFINITION 5.13. The maps

$$\Psi_m : \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

are given by

$$\Psi_m(f_\Xi, [\underline{x}_0, \underline{x}_1, \dots, \underline{x}_m]) = \left(\sum_{i=0}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,a}, \left(\sum_{i=0}^{m} x_i^j + \sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} \right)_{j=1}^{\mathsf{n}_\tau} + \mathsf{p}_\tau\mathbb{Z} \right).$$

Note that Ψ_m is well-defined because $\sigma^{-i}\tau^m[a] \neq \emptyset$.

We have seen in Lemma 5.11 that $(\sigma^{-1})^\sharp$ maps $\mathrm{rk}^c_\tau[m]$ to $\mathrm{rk}_\tau[m]$. Therefore, A_τ restricts to a map

$$\mathrm{rk}^c_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau}$$

which we shall also denote by A_τ.

Proposition 5.14. *The sequence*

$$\mathrm{rk}^c_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{\mathrm{Id}-A_\tau} \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{\Psi_m} \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

is exact.

Proof: Direct computations, using among other things that there are p_j+1 positive ξ-terms and one negative ξ-term in the j entry of the second coordinate of the image of Ψ_m, show that $\Psi_m \circ (\mathrm{Id}-A_\tau) = 0$. And if

$$(f_\Xi, [\underline{x}_0, \dots, \underline{x}_m]) \in \ker \Psi_m,$$

then the conditions of Lemma 5.12 are met for Ξ, and $\widehat{\widehat{\Xi}} \in \mathcal{CE}^c_\tau[m]$. Note also that for suitable $c \in \mathbb{Z}$,

$$\sum_{i=0}^{m} x_i^j + \left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} \right) - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} = p_j c$$

for each $j \in \{1, \dots, \mathsf{n}_\tau\}$. With $C \in \mathcal{CE}^c_\tau[m]$ a constant scheme which each entry set to c, we have that $\widehat{\widehat{\Xi}} + C$ induces a function $g \in \mathrm{rk}^c_\tau[m]$ for which

$$\begin{aligned}
&(\mathrm{Id}-A_\tau)\left(g, \left[-\sum_{i=1}^{m} \underline{x}_i, -\sum_{i=2}^{m} \underline{x}_i, \dots, -\sum_{i=m-1}^{m} \underline{x}_i, -\sum_{i=m}^{m} \underline{x}_i\right]\right) \\
&= \left(f_\Xi, \left[-\sum_{i=1}^{m} \underline{x}_i - \left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m,\mathsf{w}_k^j)+1}^{|\tau^m(\mathsf{a}_k^j)|-1} \xi_{i,\mathsf{a}_k^j} - \right.\right.\right. \\
&\qquad \left.\left.\left. \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} + c\mathsf{p}_\tau \right), \underline{x}_1, \dots, \underline{x}_m \right]\right) \\
&= (f_\Xi, [\underline{x}_0, \underline{x}_1, \dots, \underline{x}_m]).
\end{aligned}$$

□

We shall work with the following basic elements of $\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau}$. For each $\bullet \in \mathfrak{a}$, we let

$$e^m_\bullet = (1_{\tau^m\lceil\bullet\rceil}, [\underline{0}, \dots, \underline{0}])$$
$$f^m_i = (0, [\underline{\delta}_i, \underline{0}, \dots, \underline{0}])$$

with $\underline{\delta}_i$ referring to Kronecker delta. Further, we define a vector $\Delta_\bullet \in \mathbb{Z}^{|\mathfrak{a}|}$ using Kronecker delta again.

LEMMA 5.15. *For each* m,

$$\Psi_m(e^m_\bullet) = (\Delta_\bullet, 0) \qquad \Psi_m(f^m_j) = (0, \delta_j + \mathsf{p}_\tau\mathbb{Z}),$$

and under the imbedding $\mathrm{rk}_\tau[m] \hookrightarrow \mathrm{rk}_\tau[m+1]$

$$\Psi_{m+1}(e^m_\bullet) = (\mathbf{A}\Delta_\bullet, \mathbf{E}\Delta_\bullet + \mathsf{p}_\tau\mathbb{Z}) \qquad \Psi_{m+1}(f^m_j) = (0, \delta_j + \mathsf{p}_\tau\mathbb{Z}).$$

Proof: The set of claims concerning f^m_j are straightforward; the second coordinate of $\Psi_m(e^m_\bullet)$ vanishes as described because evaluation begins at a nonzero index. To compute $\Psi_{m+1}(e^m_\bullet)$, we note that as a consequence of (5.5)

$$\tau^m\lceil\bullet\rceil = \bigcup_{\substack{a\in\mathfrak{a}\\ j\in\{1,\dots,|\tau(a)|\}\\ \tau(a)^{[j]}=\bullet}} \sigma^{-|\tau^m(\tau(a)^{]j,0]})|}\tau^{m+1}[a].$$

This means that the element in $\mathcal{CE}_\tau[m+1]$ inducing the function of $e^m_\bullet$ is given by

$$\xi_{i,a} = \begin{cases} 1 & \exists h : \tau(a)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})| \\ 0 & \text{otherwise.} \end{cases}$$

Now

$$\begin{aligned} \sum_{i=0}^{|\tau^{m+1}(a)|-1} \xi_{i,a} &= \sum_{i=0}^{|\tau^{m+1}(a)|-1} \#\{h \mid \tau(a)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})|\} \\ &= \#\{h \mid \tau(a)^{[h]} = \bullet\} \\ &= \#[\bullet, \tau(a)] = (\mathbf{A})_{a,\bullet} \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{i=\ell(m+1,\mathsf{w}^j_k)+1}^{|\tau^{m+1}(\mathsf{a}^j_k)|-1} \xi_{i,\mathsf{a}^j_k} &= \sum_{i=\ell(m+1,\mathsf{w}^j_k)+1}^{|\tau^{m+1}(\mathsf{a}^j_k)|-1} \#\{h \mid \tau(\mathsf{a}^j_k)^{[h]} = \bullet, i = |\tau^m(\tau(a)^{]h,0]})|\} \\ &= \#\{h \mid \tau(\mathsf{a}^j_k)^{[h]} = \bullet, |\tau^m(\tau(\mathsf{a}^j_k)^{]h,0]})| \geq \ell(m+1,\mathsf{w}^j_k)+1\} \\ &= \#\{h \mid \tau(\mathsf{a}^j_k)^{[h]} = \bullet, h \geq |\mathsf{w}^j_k|+1\} \end{aligned}$$

according to Lemma 5.7. If $|\tau(\mathsf{a}^j_k)| \leq |\mathsf{w}^j_k| + 1$ this sum evaluates to 0, otherwise we get a count of the letter $\bullet$ in what is to the left of $\mathsf{a}^j_k\mathsf{w}^j_k$ in $\tau(\mathsf{a}^j_k)$, corresponding to our Definition 3.7. The same argument applies to $\widetilde{\mathsf{a}}^j$ and $\widetilde{\mathsf{w}}^j$. Thus

$$\left(\sum_{k=1}^{p_j+1} \sum_{i=\ell(m+1,\mathsf{w}^j_k)+1}^{|\tau^{m+1}(\mathsf{a}^j_k)|-1} \xi_{i,\mathsf{a}^j_k}\right) - \sum_{i=\ell(m,\widetilde{\mathsf{w}}^j)+1}^{|\tau^m(\widetilde{\mathsf{a}}^j)|-1} \xi_{i,\widetilde{\mathsf{a}}^j} = (\mathbf{E}_\tau)_{j,\bullet}$$

as desired. □

PROPOSITION 5.16. *The diagram*

$$\begin{array}{ccc}
\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\Psi_m} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \\
\big\downarrow & & \big\downarrow \widetilde{\mathsf{A}} \\
\mathrm{rk}_\tau[m+1] \oplus \sum_{i=0}^{m+1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow[\Psi_{m+1}]{} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}
\end{array}$$

commutes.

Proof: By Lemma 5.15 and the definition of $\widetilde{\mathsf{A}}_\tau$, the diagram commutes on $e_\bullet^m$ and f_j^m, and on the subgroup that they generate. For a general $(f, [\underline{x}_0, \dots, \underline{x}_m]) \in \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau}$ we note that Proposition 5.14 proves the claim as the images of $e_\bullet^m$ and f_j^m generate $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$. □

Proof of 5.8: By Proposition 5.14 and Proposition 5.16, the diagram

$$\begin{array}{ccccc}
\mathrm{rk}^c_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\mathrm{Id}-A_\tau} & \mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\Psi_m} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \widetilde{\mathsf{A}} \\
\mathrm{rk}^c_\tau[m+1] \oplus \sum_{i=0}^{m} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow{\mathrm{Id}-A_\tau} & \mathrm{rk}_\tau[m+1] \oplus \sum_{i=0}^{m+1} \mathbb{Z}^{\mathsf{n}_\tau} & \xrightarrow[\Psi_{m+1}]{} & \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}
\end{array}$$

is commutative and exact for each m. Furthermore, since Lemma 5.15 shows that Ψ_m is surjective for each m, the rightmost horizontal maps in the diagram are surjections. Since taking inductive limits is an exact functor, we get that

$$G_\tau \xrightarrow{\mathrm{Id}-A_\tau} G_\tau \xrightarrow{\Psi_\infty} \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathsf{A}}_\tau) \longrightarrow 0$$

is exact, where we have used Lemma 5.3 to identify

$$\bigcup_{m\in\mathbb{N}} \left(\mathrm{rk}^c_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \right) = \bigcup_{m\in\mathbb{N}} \left(\mathrm{rk}_\tau[m] \oplus \sum_{i=0}^{m-1} \mathbb{Z}^{\mathsf{n}_\tau} \right) = G_\tau.$$

□

EXAMPLE 5.17. *The matrices*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

induce maps $\chi : \mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z} \longrightarrow \mathbb{Z}^2$ *and* $\eta : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z}$ *with the property that* $\chi \circ \eta = \left[\begin{smallmatrix} 8 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ *and* $\eta \circ \chi = \widetilde{\mathsf{A}}_\tau$*; the latter since*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

as a map from $\mathbb{Z}^4 \oplus \mathbb{Z}^2/(1,1)\mathbb{Z}$ *to* $\mathbb{Z}^2/(1,1)\mathbb{Z}$*. Similarly, we may reduce our description of* $K_0(\underline{\mathsf{X}}_\upsilon)$ *to a stationary system with* $\left[\begin{smallmatrix} 8 & 0 \\ 2 & 1 \end{smallmatrix}\right]$*.*

One now easily finds that

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}$$

and, by $\left[\begin{smallmatrix} 8^{-k} & 0 \\ -2 & 7 \end{smallmatrix}\right] \left[\begin{smallmatrix} 8 & 0 \\ 2 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8^{1-k} & 0 \\ -2 & 7 \end{smallmatrix}\right]$*, that*

$$K_0(\underline{\mathsf{X}}_\upsilon) = \{(8^{-k}x, 7y - 2x) \in \mathbb{Q}^2 \mid k \in \mathbb{N}, x, y \in \mathbb{Z}\}.$$

One sees that $K_0(\underline{\mathsf{X}}_\tau) \not\simeq K_0(\underline{\mathsf{X}}_\upsilon)$ *– and hence that* $\underline{\mathsf{X}}_\tau \not\simeq_f \underline{\mathsf{X}}_\upsilon$ *– by proving that any element in* $K_0(\underline{\mathsf{X}}_\upsilon)$ *which is divisibe by any power of two is also divisible by seven. This is not the case for* $K_0(\underline{\mathsf{X}}_\tau)$*.*

6. *Finer invariants*

6.1. *Symmetrized invariants* Our focus on left special elements makes our invariant non-symmetric. It is easy to find examples of pairs of substitutions τ, υ which cannot be distinguished by our invariant, but such that their opposites τ^{-1}, υ^{-1} – the same substitutions, but read from right to left – can.

Thus a strictly finer flow invariant may be achieved by considering $K_0(\underline{\mathsf{X}}_\tau) \oplus K_0(\underline{\mathsf{X}}_{\tau^{-1}})$.

6.2. *Pointed groups* The K_0-group associated to a unital C^*-algebra posseses a distinguished element [1] corresponding to the unit of the C^*-algebra. This element is an invariant of isomorphism of such algebras, so according to Theorem 4.3 we have that $(K_0(\underline{\mathsf{X}}_\tau), [1])$ is an invariant of one-sided conjugacy of $\pi_+(\underline{\mathsf{X}}_\tau)$.

When τ is basic, [1] is the image of $(1, \dots, 1) \oplus 0$ from the first copy of $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ in our description of $K_0(\underline{\mathsf{X}}_\tau)$. Since this distinguished element is not an invariant of flow equivalence we do not at present know how to compute it when τ is simplifiable.

6.3. *Ordered groups* The K_0-group associated to any C^*-algebra posseses a canonical order structure stemming from the fact that it is given as a Grothendieck group of a semigroup of equivalence classes of self-adjoint projections. The order structure may be degenerate in the sense that elements can be simultaneously positive and negative, but often holds important and natural information on the algebras in question.

In the case of crossed products associated to Cantor minimal systems, for instance, the order on the K_0-group is part of the complete invariant for (strong) orbit equivalence given in [**17**]. Similarly, since it can be given as the K_0-groups of a C^*-algebra, $K_0(\underline{\mathsf{X}}_\tau)$ has an order structure which is a flow invariant for the underlying substitutional dynamics.

In our paper [**11**] we give examples showing that this ordered group carries more information than the group itself, by proving that $K_0(\underline{\mathsf{X}}_\tau)$ may fail to be order isomorphic to $K_0(\underline{\mathsf{X}}_{\tau^{-1}})$, even though the K-groups are isomorphic as groups. We also give the following complete description of this ordered group:

THEOREM 6.1. *Let τ be a basic substitution. There is an order isomorphism*

$$K_0(\underline{\mathsf{X}}_\tau) \simeq \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}, \widetilde{\mathsf{A}}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ is ordered by

$$[(x_a), (y_i)] \geq 0 \Longleftrightarrow \forall a \in \mathfrak{a} : x_a \geq 0.$$

This result shows in particular that the order on $K_0(\underline{\mathsf{X}}_\tau)$ is the quotient order induced by the order on $DG(\underline{\mathsf{X}}_\tau)$ via the map R_∞ considered in the proof of Theorem 5.9. As will be explained in [**11**], this phenomenon extends beyond substitutional shift spaces.

Let us quote another result from [**11**], stating the potentially finest invariant conceivable to us from our work above. Such an invariant can be extracted from the six term exacty sequence associated to the extension (4.3), which becomes

$$\begin{array}{ccccc} \mathbb{Z}^{\mathsf{n}_\tau} & \longrightarrow & K_0(\underline{\mathsf{X}}_\tau) & \longrightarrow & DG(\underline{\mathsf{X}}_\tau) \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

To describe the maps, apart from p_τ and R_∞, we use $Q : \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow \mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ defined by

$$Q(\underline{x}) = (0, \underline{x} + \mathsf{p}_\tau\mathbb{Z}),$$

and its composition Q_1 with the canonical mapping from the first instance of $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$ in the inductive system to the inductive limit in our description of $K_0(\underline{\mathsf{X}}_\tau)$:

COROLLARY 6.2. *Let τ be a basic substitution. The exact complex*

$$\mathbf{K}_\tau : \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{\mathsf{p}_\tau} \mathbb{Z}^{\mathsf{n}_\tau} \xrightarrow{Q_1} K_0(\underline{\mathsf{X}}_\tau) \xrightarrow{R_\infty} DG(\underline{\mathsf{X}}_\tau) \longrightarrow 0,$$

where $\mathbb{Z}, \mathbb{Z}^{\mathsf{n}_\tau}, K_0(\underline{\mathsf{X}}_\tau)$ and $DG(\underline{\mathsf{X}}_\tau)$ should be considered as ordered groups and $\mathsf{p}_\tau, Q_1, R_\infty$ as positive homomorphisms, is a flow invariant of $\underline{\mathsf{X}}_\tau$.

6.4. *Open questions* It would be most interesting to know exactly which relation on the substitution shift spaces $\underline{\mathsf{X}}_\tau$ is induced by isomorphism of the stabilized algebra $\mathcal{O}_\tau \otimes \mathbb{K}$, or by isomorphism of the invariants mentioned above. Our examples above show that this relation is stronger than strong orbit equivalence, cf. [**17**]. There are classification results, notably those of Lin and Su ([**22**]), which could apply to the class of C^*-algebras in question, but we have not yet attempted to pursue this question.

As mentioned above, and documented in [**8**], the constituents of our invariants are effectively computable. However, this does not in itself lead to the conclusion that isomorphism of our invariants is decidable. Related work by Bratteli *et al* ([**4**]) proves decidability of the invariant which is complete for strong orbit equivalence – it would seem reasonable to expect that the result can be extended.

References

[1] M. Barge and B. Diamond, *A complete invariant for the topology of one-dimensional substitution tiling spaces*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1333–1358.

[2] M. Barge, B. Diamond, and C. Holton, *Asymptotic orbits of primitive substitutions*, Theoret. Comput. Sci. **301**, (2003), no. 1-3, 439–450.

[3] M. Boyle and D. Lind, *Exponential subdynamics*, Trans. Amer. Math. Soc. **349** (1997), 55–102.

[4] O. Bratteli, P. Jørgensen, K.H. Kim, and F. Roush, *Decidability of the isomorphism problem for stationary AF-algebras and the associated ordered simple dimension groups*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1625–1655.

[5] T.M. Carlsen, *C^*-algebras associated to general shift spaces*, Master's thesis, Copenhagen University, 2001, available on `www.math.ku.dk/~toke`.

[6] ———, *Cuntz-Pimsner C^*-algebras associated with subshifts*, in preparation, will appear on `www.math.ku.dk/~toke`.

[7] ———, *Shift spaces, groupoids and C^*-algebras*, in preparation, will appear on `www.math.ku.dk/~toke`.

[8] T.M. Carlsen and S. Eilers, *A graph approach to computing nondeterminacy in substitutional dynamical systems*, preprint, `www.math.ku.dk/~eilers/papers/cei`, 2002.

[9] ———, *Java applet*, `www.math.ku.dk/~eilers/papers/cei`, 2002.

[10] ———, *Matsumoto K-groups associated to certain shift spaces*, in preparation, will appear on `www.math.ku.dk/~eilers/papers/ceiii`.

[11] ———, *Ordered K-groups associated to substitutional dynamics*, in preparation, will appear on `www.math.ku.dk/~eilers/papers/ceiv`.

[12] T.M. Carlsen and K. Matsumoto, *Some remarks on the C^*-algebras associated with subshifts*, to appear in *Math. Scand*.

[13] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological markov chains*, Invent. Math. **56** (1980), 251–268.

[14] F. Durand, B. Host, and C. Skau, *Substitutional dynamical systems, Bratteli diagrams and dimension groups*, Ergodic Theory Dynam. Systems **19** (1999), no. 4, 953–993.

[15] G.A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.

[16] A. H. Forrest, *K-groups associated with substitution minimal systems*, Israel J. Math. **98** (1997), 101–139.

[17] T. Giordano, I.F. Putnam, and C.F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.

[18] R.H. Herman, I.F. Putnam, and C.F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), no. 6, 827–864.

[19] C. Holton and L. Q. Zamboni, *Directed graphs and substitutions*, Theory Comput. Syst. **34** (2001), no. 6, 545–564.

[20] B.P. Kitchens, *Symbolic dynamics*, Springer-Verlag, Berlin, 1998, One-sided, two-sided and countable state Markov shifts.

[21] W. Krieger and K. Matsumoto, *Shannon graphs, subshifts and lambda-graph systems*, J. Japan Math. Soc. **54** (2002), 877–899.

[22] H. Lin and H. Su, *Classification of direct limits of generalized Toeplitz algebras*, Pacific J. Math. **181** (1997), no. 1, 89–140.

[23] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[24] K. Matsumoto, *On C^*-algebras associated with subshifts*, Internat. J. Math. **8** (1997), no. 3, 357–374.

[25] ———, *K-theory for C^*-algebras associated with subshifts*, Math. Scand. **82** (1998), no. 2, 237–255.

[26] ———, *Dimension groups for subshifts and simplicity of the associated C^*-algebras*, J. Math. Soc. Japan **51** (1999), no. 3, 679–698.

[27] ———, *Presentations of subshifts and their topological conjugacy invariants*, Doc. Math. **4** (1999), 285–340 (electronic).

[28] ———, *Stabilized C^*-algebras constructed from symbolic dynamical systems*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 821–841.

[29] ———, *Bowen-Franks groups as an invariant for flow equivalence of subshifts*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1831–1842.

[30] B. Mossé, *Puissances de mots et reconnaissabilité des points fixes d'une substitution*, Theoret. Comput. Sci. **99** (1992), no. 2, 327–334.

[31] ———, *Reconnaissabilité des substitutions et complexité des suites automatiques*, Bull. Soc. Math. France **124** (1996), no. 2, 329–346.

[32] J.-J. Pansiot, *Decidability of periodicity for infinite words*, RAIRO Inform. Théor. Appl. **20** (1986), no. 1, 43–46.

[33] I.F. Putnam, *On the topological stable rank of certain transformation group C^*-algebras*, Ergodic Theory Dynam. Systems **10** (1990), 197–207.

[34] M. Queffélec, *Substitution dynamical systems—spectral analysis*, Springer-Verlag, Berlin, 1987.

[35] G. Rozenberg and A. Salomaa, *The mathematical theory of L systems*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980.