Approximate homogeneity is not a local property

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- The **AF** algebras are those of the form:

$$\varinjlim F_i$$

where F_i is a finite sum of matrix algebras over $\mathbb{C}$.

- The **AT** algebras are those of the form:

$$\varinjlim T_i$$

where T_i is a finite sum of matrix algebras over $C(S^1)$.

- The **AD** algebras are those of the form:

$$\varinjlim D_i$$

where D_i is a finite sum of matrix algebras over $C(S^1)$ or $\mathbb{I}_n^{\sim}$.

Proposition An inductive limit of **AF** algebras is again an **AF** algebra.

Proof [Bratteli] Such an inductive limit A is *locally finite-dimensional* (**LF**) in the sense that

- $\forall\ \varepsilon > 0$
  $\forall$ finite sets $\mathcal{F} \subseteq A$

- $\exists\ C^*$-algebra $B = \bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(\mathbb{C})$ which is a finite sum of matrix algebras over $\mathbb{C}$
  $\exists\ *$-homomorphism $\varphi : B \to A$.

- Such that $\mathcal{F} \subseteq^{\varepsilon} \varphi(B)$

This *local criterion* implies that A is **AF**.

Observation

$$\mathbf{AF} = \mathbf{LF}$$
$$\mathbf{AT} = \mathbf{LT}$$
$$\mathbf{AD} = \mathbf{LD}$$

$\mathbb{C}$, $C(S^1)$ and $\mathbb{I}_n^{\sim}$ are all (weakly) semiprojective C^*-algebras.

Question

$$\mathbf{AH} \stackrel{?}{=} \mathbf{LH}$$

$C(X)$ is not in general weakly semiprojective!

- The **AH** algebras are those of the form:

$$\varinjlim H_i$$

where H_i is of the form

$$\bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i$$

where X_i is connected and p_i is a projection in $\mathbf{M}_{n_i}(C(X_i))$.

- A C^*-algebra A is **LH** when
  - $\forall\ \varepsilon > 0$
    $\forall$ finite sets $\mathcal{F} \subseteq A$
  - $\exists$ C^*-algebra $B = \bigoplus_{i=1}^{k} p_i \mathbf{M}_{n_i}(C(X_i))p_i$
    $\exists$ $*$-homomorphism $\varphi : B \to A$
  - Such that $\mathcal{F} \subseteq^{\varepsilon} \varphi(B)$

$\boxed{\textbf{?}}$ How much can be said about **LH** algebras?

$\boxed{\textbf{!}}$ An **LH** algebra is nuclear.

$\boxed{\textbf{!}}$ An **LH** algebra which is unital, separable and simple, and has real rank zero, stable rank one and weak unperforation is also **AH**. [H Lin]

K-theory with coefficients!

Consider $K_0(-;\mathbf{Z}/m)$ and note that

$$K_0(A;\mathbf{Z}/m) \cong K_0(A) \otimes \mathbf{Z}/m \oplus \mathrm{Tor}(K_1(A), \mathbf{Z}/m)$$

because the *Künneth sequence*

$$0 \longrightarrow K_0(A) \otimes \mathbf{Z}/m \xrightarrow{\tilde{\rho}_m} K_0(A;\mathbf{Z}/m) \xrightarrow{\tilde{\beta}_m} \mathrm{Tor}(K_1(A), \mathbf{Z}/m) \longrightarrow 0$$

splits.

However, it splits *unnaturally*. In particular, a square

$$\begin{tikzcd}
K_0(I) \otimes \mathbf{Z}/m \arrow[d] & K_0(I;\mathbf{Z}/m) \arrow[l, dotted] \arrow[d] \\
K_0(A) \otimes \mathbf{Z}/m & K_0(A;\mathbf{Z}/m) \arrow[l, dotted]
\end{tikzcd}$$

may fail to commute for an ideal I in A.

Hence a splitting map σ need not satisfy

$$\sigma(K_0(I;\mathbf{Z}/m)) \subseteq K_0(I) \otimes \mathbf{Z}/m.$$

$\boxed{\textbf{Definition}}$ An IMI_n*-splitting* is a group homomorphism α_m in

$$\begin{array}{ccc} K_0(A) \otimes \mathbf{Z}/m & \xrightarrow{\tilde{\rho}^0_m} & K_0(A;\mathbf{Z}/n) \\ {\scriptstyle q}\downarrow & \swarrow {\scriptstyle \alpha_m} & \\ K_0^{\mathrm{Inf}}(A) \otimes \mathbf{Z}/m & & \end{array}$$

which preserves all ideals I of A which are generated by their projections. Here,

$$K_0^{\mathrm{Inf}}(A) = K_0(A)/\,\mathrm{Inf}(K_0(A)).$$

$\boxed{\textbf{Theorem}}$ Every **AH** algebra has an IMI_m-splitting for any m.

$\boxed{\textbf{Theorem}}$ There exists an **LH** algebra which is not an **AH** algebra. It can be chosen with the following additional properties:

- separable
- unital
- real rank zero
- stable rank one
- **AD**

$\boxed{\textbf{Corollary}}$ There exists a C^*-algebra which is an inductive limit of **AH** algebras but not itself **AH**.

$\boxed{\textbf{Corollary}}$ There exists a *quasidiagonal* and *split* extension of C^*-algebras

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$$

in which both A and B are **AH** but E is not **AH**.

$\mathbf{AH}_n$	$\mathbf{LH}_n$	$\bigoplus_{i=1}^k p_i \mathbf{M}_{n_i}(C(X_i))p_i$ dim $X \leq n$
$\mathbf{AH}^-$	$\mathbf{LH}^-$	$\bigoplus_{i=1}^k \mathbf{M}_{n_i}(C(X_i))$
$\mathbf{AH}^+$	$\mathbf{LH}^+$	$\bigoplus_{i=1}^k A_i$ A_i is homogeneous

$\boxed{!}$

$$\mathbf{AH}_1 = \mathbf{LH}_1$$

$\boxed{?}$

$$\mathbf{AH}_2 \stackrel{?}{=} \mathbf{LH}_2$$

$\boxed{!}$

$$\mathbf{AH}_3 \neq \mathbf{LH}_3$$

$\boxed{!}$

$$\mathbf{AH}^- \neq \mathbf{LH}^-$$

$\boxed{?}$

$$\mathbf{AH}^+ \stackrel{?}{=} \mathbf{LH}^+$$