Combinatorial aspects of pyramids of one-dimensional pieces of fixed integer length

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Outline

1. Background (3D)
2. Motivation (2D)
3. Results
4. Decoding
LEGO facts and figures

- It would take 40,000,000,000 LEGO bricks stacked on top of each other to reach from the Earth to the Moon.

- A LEGO set is sold across the counter somewhere in the world every 7 seconds.

- The eight robots in the LEGO Warehouse in Billund can move 660 crates of LEGO bricks an hour.

- Children all over the world spend 5 billion hours a year playing with LEGO bricks.

- There are 102,981,500 different ways of combining six eight-stud bricks of the same colour.

- On average each person on earth owns 52 LEGO bricks.
Selected LEGO statistics

- More than 400,000,000 children and adults will play with LEGO bricks this year.
- LEGO products are on sale in more than 130 countries.
- If you built a column of about 40,000,000,000 LEGO bricks, it would reach the moon.
- Approx. four LEGO sets are sold each second.
- There are 915,103,765 different ways of combining six eight-stud bricks of the same colour.
- On average every person on earth has 52 LEGO bricks.
- With a production of about 306 million tyres a year, the LEGO Group is the world's largest tyre manufacturer.
- If all the LEGO sets sold over the past 10 years were placed end to end, they would reach from London, England, to Perth, Australia.
Theorem (Abrahamsen/Durhuus-E)

The number of LEGO buildings constructable by $n$ blocks of size $b \times w$ grows asymptotically as $h_{b \times w}^n$ with

$$w^2 + b^2 + 6bw - 4b - 4w + 2 \leq h_{b \times w} \leq 24w^2 + 36bw - 48w$$

if $b \neq w$, and

$$4b^2 - 4b + 1 \leq h_{b \times b} \leq 18b^2$$

otherwise. We have

$$78 \leq h_{2 \times 4} \leq 192$$
Quadratic dependence (empirical evidence)
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Flat buildings
**Theorem (Abrahamsen-E)**

The number of flat LEGO buildings constructable by \( n \) blocks of size \( 1 \times w \) grows asymptotically as \( \hat{h}_w^n \) with

\[
2w - 1 \leq \hat{h}_w \leq 7w
\]

**Conjecture and wild guess**

\( \hat{h}_w \) grows linearly in \( w \)

\( \hat{h}_2 = 5 \)
Theorem [Bousquet-Mélou & Rechnitzer]

The number of pyramids constructable by \( m \) dimers equals

\[
\binom{2m - 1}{m - 1}
\]

and hence grows like

\[
\frac{1}{\sqrt{4\pi m}} 4^m
\]

The average width of such a pyramid is (caveat!) asymptotic to

\[
16\sqrt{\pi m}
\]
Bousquet-Mélou & Rechnitzer

\[ Q = \bigoplus \bigoplus \bigoplus \bigoplus \]

\[ P = Q = Q \bigoplus Q \]
Worth pondering

\[ \binom{2m-1}{m-1} \] is the number of strings of \(2m\) symbols drawn from \(\{0, 1\}\) with exactly \(m\) ones and starting with a one, like

10010011

visualizable as
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Theorem

The number of pyramids constructable by $m$ polymers/LEGOs of width $a$ equals

$$\binom{am - 1}{m - 1}$$

and hence grows like

$$\frac{1}{\sqrt{2\pi a(a - 1)m}} \left(\frac{a^a}{(a - 1)^{a-1}}\right)^m$$

The average width of such a pyramid is asymptotic to

$$\sqrt{\frac{\pi}{2} a(a - 1)m}$$
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Positive strings

**Definition**

A string

\[ x_1 \cdots x_n \]

with \( n \) symbols in \( \{0, 1\} \) is \textbf{a-positive} when

\[
\forall j \in \{1, \ldots, n\} : \sum_{i=1}^{j} (ax_i - 1) \geq 0
\]

We say that \( x_n \cdots x_1 \) is \textbf{a-negative} in this case.

**Examples**

110100 is 2-positive. 100011 is not.
P case ($a = 2$)
P case ($a = 2$)
P case ($a = 2$)
P case \((a = 2)\)
P case \((a = 2)\)
P case \((a = 2)\)
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P case \((a = 2)\)
PN case \((a = 2)\)
PN case \((a = 2)\)
Generalizing to $a > 2$

**Ambiguity ($a = 3$)**

- Pink blocks
- Yellow blocks

**Indecomposability ($a = 3$)**

- Pink blocks: 100010
- Yellow blocks: 100010
Lemma

Any \( \{0, 1\} \)-string of length \( am \) which starts with one and has exactly \( m \) ones may be uniquely decomposed into a sequence of strings \( P, N, T, U \) satisfying the constraints of
Lemma

Fix \( a \geq 2 \). The number \( A_n \) of one-sided pyramids coincides with the number of sequences in \( P \), or \( N \), by the coding procedure outlined earlier. Thus the number of pyramids is

\[
\sum_{r \geq 1} \sum_{m_1 + \cdots + m_r = m} (a - 1)^{r-1} A_{m_1} \cdots A_{m_r},
\]
Observation

The number of \( \{0,1\} \)-strings of length \( am \) which starts with one and has exactly \( m \) ones can be written in the form

\[
\sum_{r \geq 1} \sum_{m_1 + \ldots + m_r = m} a_r A_{m_1} \ldots A_{m_r},
\]

where \( r \) denotes the total number of substrings \( P \) or \( N \), with sizes \( m_1, \ldots, m_r \geq 1 \), in a composition and the factor \( a_r \) counts the number of admissible compositions subject to the boundary conditions specified by

\[
\begin{array}{c}
\bullet \rightarrow P \rightarrow T \rightarrow U \rightarrow N
\end{array}
\]
Theorem

\[ a_r = (a - 1)^{r-1} \]

Corollary

*The exponential rate of growth is*

\[ \frac{a^a}{(a - 1)^{a-1}} \sim e(a - 1) \]