Classification of $C^*$-algebras associated to minimal dynamics

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A *shift space* is a subset of $\mathcal{A}^\mathbb{Z}$ which is closed in the product topology and under the shift map

$$\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.$$ 

The *alphabet* $\mathcal{A}$ is a finite set.

A shift space is *of finite type* when it can be specified by a finite list of forbidden words.

**Example** $X_{\{11,21\}} \ni \cdots 00100122200010201012 \cdots$
Example  With primitive substitutions such as

\[ \tau(0) = 01 \quad \tau(1) = 0 \]

one gets a shift space

\[ X_\tau = \{ \sigma^m(u) \mid m \in \mathbb{Z} \} \]

where \( u \) is any periodic point of \( \tau \), i.e. \( \tau^n(u) = u \).

In this case we could take \( n = 2 \) and

\[ u = \cdots 01001001.01001010010010100101001 \cdots \]
Some substitutions

$\tau_1(\aleph) = \aleph \aleph \aleph$ \hspace{1cm} $\tau_1(\beth) = \beth \beth \beth$

$\tau_2(\alpha) = \alpha \beta$ \hspace{1cm} $\tau_2(\beta) = \alpha \beta \gamma \delta \epsilon$ \hspace{1cm} $\tau_2(\gamma) = \alpha \beta$
$\tau_2(\delta) = \gamma \delta \epsilon$ \hspace{1cm} $\tau_2(\epsilon) = \alpha \beta \gamma \delta \epsilon$

$\tau_3(1) = 1212345$
$\tau_3(2) = 12123451234512345$
$\tau_3(3) = 1212345$ \hspace{1cm} $\tau_3(4) = 1234512345$
$\tau_3(5) = 12123451234512345$

$\tau_4(a) = ababacb$ \hspace{1cm} $\tau_4(b) = ababacbabacbabacb$
$\tau_4(c) = abacbabacb$
Let $X^+$ denote the projection of $X$ down on $a^\mathbb{N}$. We set for finite words $u, v$ written with letters from $a$

$$C(u|v) = \{ux \in X^+ \mid vx \in X^+\}$$

and define a $C^*$-algebra $\mathcal{O}_X$ by generators $(S_a)_{a \in a}$ and relations

$$a \in a : \quad S_a S_a^* S_a = S_a$$

$$C(u|v) = \bigcap_{i=1}^n C(u_i|v_i) : \quad S_u S_v S_v^* S_u^* = \prod_{i=1}^n S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^*$$

$$C(u|v) = \bigsqcup_{i=1}^n C(u_i|v_i) : \quad S_u S_v S_v S_u^* = \sum_{i=1}^n S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^*$$

with “$\bigsqcup$” denoting disjoint unions. [Matsumoto]
Key features:

• Matsumoto algebras associated to shifts of finite type are Cuntz-Krieger algebras.

• [Matsumoto, Carlsen]

\[ X \sim_{\text{flow}} Y \implies \mathcal{O}_X \otimes K \sim \mathcal{O}_Y \otimes K \]

Complication:

• Not always simple, even for minimal shift spaces like \( X_\tau \).
When are these maps injective?

This question has been successfully worked out in the case of irreducible shifts of finite type. [Cuntz, Krieger, Bowen, Franks, Rørdam]

The case for

\[ \mathcal{O}_\tau := \mathcal{O}_{X_T} \]

is structurally very different but shares the property of the shift space being finitely presented. Work by Carlsen-E shows that not both maps are injective.
Good and bad news
\( \mathcal{O}_\tau \) is never simple.

\[ \text{\textcolor{red}{:-(} Carlsen} \quad 0 \rightarrow \mathbb{K}^{\mathbb{N}_\tau} \rightarrow \mathcal{O}_\tau \rightarrow C(\mathbb{X}_\tau) \rtimes_\sigma \mathbb{Z} \rightarrow 0 \]

\[ \text{\textcolor{green}{:-)} Carlsen-E} \quad \text{Computable } K\text{-theory:} \]

\[
K_1(\mathcal{O}_\tau) \rightarrow K_1(C(\mathbb{X}_\tau) \rtimes_\sigma \mathbb{Z}) \rightarrow K_0(\mathbb{K}^{\mathbb{N}_\tau}) \rightarrow K_0(\mathcal{O}_\tau) \rightarrow K_0(C(\mathbb{X}_\tau) \rtimes_\sigma \mathbb{Z}) \]

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^{n_\tau} \rightarrow \text{DG}(\tilde{\mathbb{A}}_\tau) \rightarrow \text{DG}(\mathbb{A}_\tau) \]

\[ \text{\textcolor{green}{:-)} Putnam} \quad C(\mathbb{X}_\tau) \rtimes_\sigma \mathbb{Z} \text{ is simple, } AT \text{ and of real rank zero.} \]

\( \mathcal{O}_\tau \) has real rank zero.

\[ \text{\textcolor{red}{:-(} Putnam} \quad \mathcal{O}_\tau \text{ is not stably finite, nor purely infinite.} \]
Bootstrap classification

**Theorem** [Rørdam 1994]

Let $A$ and $B$ be $C^*$-algebras each having one essential ideal $I$ and $J$ such that $I, J, A/I$ and $B/J$ are stable Kirchberg algebras satisfying the UCT. Then $A \simeq B$ precisely when

\[
\begin{align*}
K_0(I) & \to K_0(A) \to K_0(A/I) \to K_1(I) \to K_1(A) \to K_1(A/I) \\
\simeq & \downarrow \simeq \downarrow \simeq \downarrow \simeq \\
K_0(J) & \to K_0(B) \to K_0(B/J) \to K_1(J) \to K_1(B) \to K_1(B/J) \\
\end{align*}
\]

Unfortunately, this has proven very difficult to generalize to more ideals. Restorff is able to do the next case $0 \triangleleft I_0 \triangleleft I_1 \triangleleft A$, but by completely different methods.
Another bootstrap classification

**Theorem** [E-Restorff-Ruiz]

Let $E_1$ and $E_2$ be $C^*$-algebras each having an ideal $B_i$ such that $B_i$ are stable AF algebras and $A_i = E_i/B_i$ are simple AT algebras of real rank zero. Then $E_1 \otimes \mathbb{K} \simeq E_2 \otimes \mathbb{K}$ precisely when

$$
egin{align*}
K_1(E_1) &\to K_1(A_1) \to K_0(B_1) \to K_0(E_1) \to K_0(A_1) \\
&\simeq \downarrow \quad \simeq \downarrow \quad \simeq \downarrow \quad \simeq \downarrow \quad \simeq \\
K_1(E_2) &\to K_1(A_2) \to K_0(B_2) \to K_0(E_2) \to K_0(A_2)
\end{align*}
$$
Combine with results by Carlsen-E to get

**Corollary** \( \mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_\nu \otimes \mathbb{K} \) precisely when

\[
\begin{align*}
\mathbb{Z} \rightarrow & \mathbb{Z}^{n_\tau} \rightarrow K_0(\mathcal{O}_\tau) \rightarrow K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}) \\
\mathbb{Z} \rightarrow & \mathbb{Z}^{n_\nu} \rightarrow K_0(\mathcal{O}_\nu) \rightarrow K_0(C(\underline{X}_\nu) \rtimes_\sigma \mathbb{Z})
\end{align*}
\]

**Corollary** \( \mathcal{O}_{\underline{X}_\tau} \otimes \mathbb{K} \simeq \mathcal{O}_{\underline{X}_\nu} \otimes \mathbb{K} \nRightarrow \underline{X}_\tau \simeq_{flow} \underline{X}_\nu \)

**Corollary** \( K_0(\mathcal{O}_{\underline{X}_\tau}) \simeq K_0(\mathcal{O}_{\underline{X}_\nu}) \nRightarrow \mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_\nu \otimes \mathbb{K} \)
Assume

\[
\begin{array}{c}
K_*(B_1) \to K_*(E_1) \to K_*(A_1) & \tau_1 : A_1 \to M(B_1)/B_1 \\
\beta_* & \eta_* & \alpha_* \\
K_*(B_2) \to K_*(E_2) \to K_*(A_2) & \tau_2 : A_2 \to M(B_2)/B_2
\end{array}
\]

① WLOG we may assume $A_1 = A_2 = A$, $B_1 = B_2 = B$, $\alpha_* = id$, $\beta_* = id$. [Strong classification/Elliott^2]
2 Then with \( x_i \in KK^1(A, B) \) representing the extensions we have
\[
x_1b = ax_2
\]
for some \( a \in KK(A, A)^{-1}, \ b \in KK(B, B)^{-1}. \) [Generalization of method by Rørdam]

3 \( b = KK(id_B) \). And WLOG we may assume that also \( a = KK(id_A) \) [Kishimoto-Kumjian]

4 Now \( x_1 = x_2 \). Thus \( \tau_1 \oplus \tau_0 \sim \tau_2 \oplus \tau_0 \). WLOG we may assume \( \tau_1 \sim \tau_2 \) [Kucerovsky-Ng]

5 Done.
Good and bad news
A weak classification result

Range of the invariant currently unknown.

Similar methods apply to

\[ 0 \to B \otimes K \to E \to A \to 0 \]

if \( A \) and \( B \) both are simple unital \(\text{AT}\) algebras of real rank zero.

Similar methods apply to

\[ 0 \to B \to E \to A \to 0 \]

if \( A \) and \( B \) both are simple unital nuclear algebras of tracial rank zero with finitely generated \( K\)-theory [Dadarlat].