Augmenting dimension group invariants for substitutional dynamical systems

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Substitutions

Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite non-empty words with letters from $\mathfrak{a}$. A *substitution* is a map

$$\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp$$

We write "$w \dashv v$" when $w, v \in \mathfrak{a}^\sharp$ and w is a subword of v.

Define

$$\underline{\mathsf{X}}_\tau = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \exists n, a : x_{[i,j]} \dashv \tau^n(a)\}.$$

and equip with

$$\sigma(x_n) = (x_{n+1})$$

Focus Invariants of conjugacy and flow equivalence.

Recurrent examples

We shall consider the substitutions

$$\tau(a) = accdadbb, \tau(b) = acdcbadb,$$
$$\tau(c) = aacdcdbb, \tau(d) = accbdadb,$$

and

$$\upsilon(a) = accbbadd, \upsilon(b) = accdbabd,$$
$$\upsilon(c) = aacbbcdd, \upsilon(d) = acbcdabd,$$

throughout.

Indispensible properties: Primitive and aperiodic.
Dispensible properties: Proper, injective, of constant length.

The dimension group

To a substitution τ one associates the abelianization

$$\mathbf{A}_\tau : \mathbb{Z}^{|\mathfrak{a}|} \longrightarrow \mathbb{Z}^{|\mathfrak{a}|}$$

given by the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix

$$(\mathbf{A}_\tau)_{a,b} = \# \text{ of occurrences of } b \text{ in } \tau(a)$$

Theorem [Giordano/Putnam/Skau^2/Durand/Host]
When τ is aperiodic, primitive and proper*,

$$G_\tau = \varinjlim\left(\mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{\mathbf{A}_\tau} \mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{\mathbf{A}_\tau} \cdots \right)$$

defines an ordered group which is an invariant of flow equivalence.

Origin $K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z})$

*No loss of generality

Recurrent example, I

$$\mathbf{A}_\tau = \mathbf{A}_\upsilon = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

The augmented dimension group

Theorem [Carlsen/Eilers]
Let τ be a primitive, aperiodic, proper* and injective† substitution of constant length‡.

There exists a finitely generated abelian group H_τ and a map

$$\mathbf{E}_\tau : \mathbb{Z}^{|\mathfrak{a}|} \longrightarrow H_\tau$$

such that

$$\widetilde{\mathbf{A}}_\tau = \begin{bmatrix} \mathbf{A}_\tau & 0 \\ \mathbf{E}_\tau & \mathbf{Id} \end{bmatrix}$$

defines an ordered group

$$\widetilde{G}_\tau = \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau \xrightarrow{\widetilde{\mathbf{A}}_\tau} \mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau \xrightarrow{\widetilde{\mathbf{A}}_\tau} \cdots).$$

which is an invariant of flow equivalence.

Corollary $\underline{\mathsf{X}}_\tau$ and $\underline{\mathsf{X}}_\upsilon$ are not flow equivalent.

*No loss of generality

†No loss of generality

‡Dispensible

Interjection concerning special elements

Definition $x \in \underline{\mathsf{X}}_\tau$ is right special if

$$\exists n : x_{[n,\infty[} = y_{[n,\infty[} \wedge x \neq y$$

Definition $x \in \underline{\mathsf{X}}_\tau$ is left special if

$$\exists n : x_{]-\infty,n]} = y_{]-\infty,n]} \wedge x \neq y$$

Theorem [Queffélec]
If τ is a primitive and aperiodic substitution on $\mathfrak{a}$, the number of orbit classes of special words is nonzero, but finite.

Configuration data

Define a bipartite graph with vertices

$$\begin{aligned} V_l &= \{[x]_{\text{l.orbit}} \mid x \in \mathsf{X}_\tau^-\} \\ V_r &= \{[x]_{\text{r.orbit}} \mid x \in \mathsf{X}_\tau^+\} \end{aligned}$$

and where each orbit class $[x]_{\text{orbit}}$ yields an edge

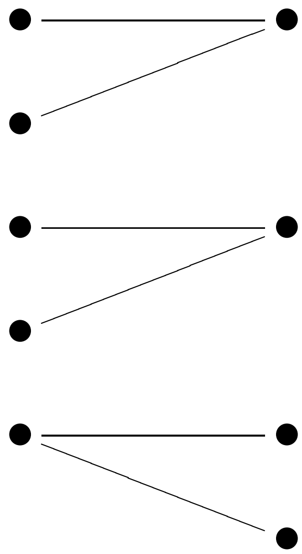
$$[x_{]-\infty,-1]}]_{\text{l.orbit}} \qquad \bullet\!\!\longrightarrow\!\!\bullet \qquad [x_{[0,\infty[}]_{\text{r.orbit}}$$

but delete an edge if both its vertices have valency one.

Observation The bipartite graph thus defined is finite, and an invariant of flow equivalence.

Recurrent example, II

The configuration data graph of both τ and υ is



What is H_τ?

Definition

$$\mathsf{n}_\tau = \#\{[x]_{\mathsf{orbit}} \mid x \text{ is right special}\} \in \mathbb{N}$$

Enumerate one-sided representatives of the right special orbits for τ as

$$x_1, \ldots, x_{\mathsf{n}_\tau}.$$

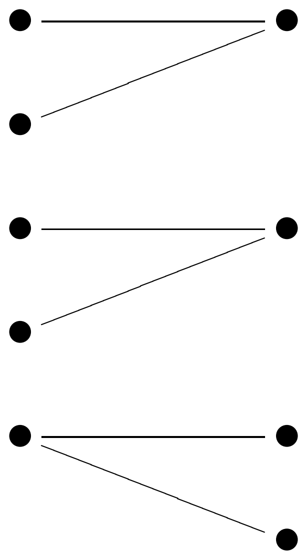
Definition

$$(\mathsf{p}_\tau)_i = \#\{y \in \underline{\mathsf{X}}_\tau \mid y_{[0,\infty[} = x_i\} - 1 \in \mathbb{N}^{\mathsf{n}_\tau}$$

Answer

$$H_\tau = \mathbb{Z}^{\mathsf{n}_\tau} / \mathsf{p}_\tau \mathbb{Z}$$

Recurrent example, III



Thus

$$\mathsf{n}_\tau = \mathsf{n}_\upsilon = 2$$
$$\mathsf{p}_\tau = \mathsf{p}_\upsilon = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$H_\tau = H_\upsilon = \mathbb{Z}$$

What is E_τ*?*

When τ is of constant length ℓ, the right special elements are all of the form

$$\cdots \tau^{3k}(w)\tau^{2k}(w)\tau^{k}(w)a\tau^{k}(v)\tau^{2k}(v)\tau^{3k}(v)\cdots$$

where w and v are unique if $|w|, |v| < \ell$. Fix v_i representing the right orbit class x_i and note that if we enumerate the corresponding w as

$$w_1, \ldots, w_{m_i},$$

then $(\mathsf{p}_\tau)_i = m_i - 1$.

Answer With this setup,

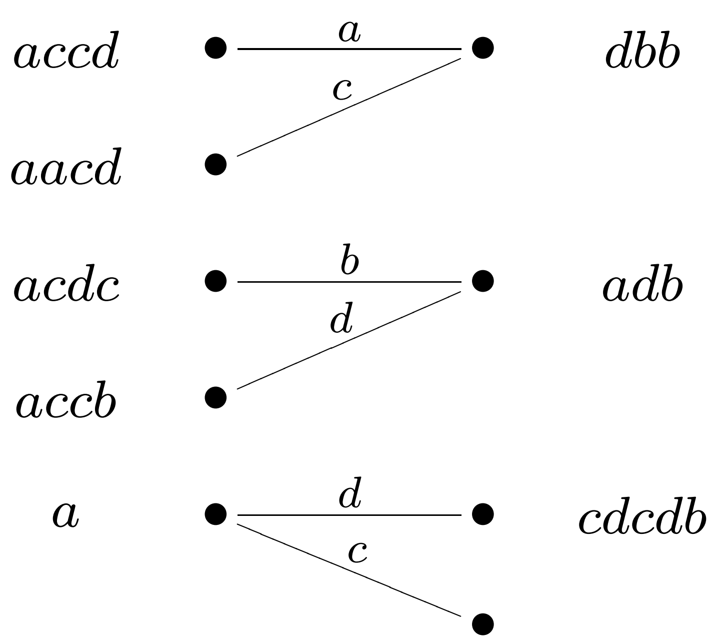
$$(\mathbf{E})_{i,b} = \sum_{i=1}^{m_i-1} [\#\text{ of occurrences of } b \text{ in } w_i]$$

Recurrent example IV

With

$$\tau(a) = accdadbb, \tau(b) = acdcbadb$$
$$\tau(c) = aacdcdbb, \tau(d) = accbdadb,$$

one gets



and thus

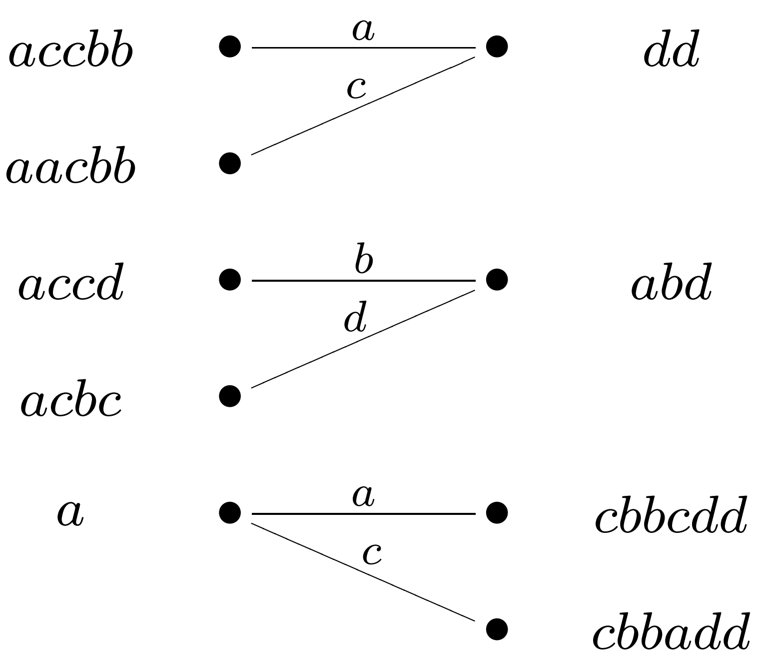
$$\mathbf{E}_\tau = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}$$

Recurrent example IV'

With

$$\upsilon(a) = accbbadd, \upsilon(b) = accdbabd$$
$$\upsilon(c) = aacbbcdd, \upsilon(d) = acbcdabd,$$

one gets



and thus

$$\mathbf{E}_\mathcal{T} = \begin{bmatrix} 1 & 0 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}$$

Computatibility

[Previous work by Barge/Diamond, related unpublished work by Barge/Diamond/Holton]

There are efficient algorithms for computing $n_\mathcal{T}, p_\mathcal{T}$ and $\mathbf{E}_\mathcal{T}$:

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CONCERNING tau GIVEN BY:
[a->bcada, b->bdbca, c->bccda, d->bddca]

COMPUTING ND_tau:
(a,c) <-- da,+ -- (a,c)
(b,d) <-- ca,+ -- (b,d)
(c,d) <-- a,- -- (c,d)

COMPUTING CONFIGURATION DATA [Pass to (tau)^2]
[0--0, 0--0, 2--2, 2--2, 4--4, 5--4, 6--6, 6--7]

COMPUTING p-VECTOR AND E-MATRIX [Pass to (tau)^2]
Enumerating: [dabddcabcada, cabccdabcada, abcada]
p_tau: [1, 1, 1]
E_tau: [[2, 4, 4, 2], [2, 4, 2, 4], [3, 5, 5, 5]]
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This extends to all primitive and aperiodic substitutions as regards $n_\mathcal{T}$ and $p_\mathcal{T}$.