Classifying naturally occurring graph C^*-algebras

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Content

Observation

Most classification results by K-theoretic invariants apply to classes of C^*-algebras $\mathfrak{A}$ enjoying at least one of the following properties:

- $\mathfrak{A}$ is simple
- $\mathfrak{A}$ is stably finite with real rank zero
- $\mathfrak{A}$ is purely infinite

Leitmotif

The classification theory for graph C^*-algebras applies in some cases satisfying none of these properties.

C^*-algebras of orthogonal or commuting isometries

Example

- $\mathcal{T} = C^*\langle S \mid S^*S = 1\rangle$
- $\mathcal{T} \otimes \mathcal{T} = C^*\left\langle S_1, S_2 \,\middle|\, \begin{array}{c} S_i^*S_i = 1 \\ S_1S_2 = S_2S_1 \\ S_1S_2^* = S_2^*S_1 \end{array} \right\rangle$
- $\mathcal{E}_2 = C^*\langle S_1, S_2 \mid S_i^*S_i = 1, S_1^*S_2 = 0\rangle$
- $C^*\left\langle S_1, S_2, S_3 \,\middle|\, \begin{array}{c} S_i^*S_i = 1, S_1^*S_2 = 0, S_1^*S_3 = 0 \\ S_2S_3 = S_2S_3, S_3S_2^* = S_2^*S_3 \end{array} \right\rangle$

Encoding by graphs

We think of finite, simple, undirected graphs with no self-loops $\Gamma = (\Gamma^0, \Gamma^1)$ as irreflexive and symmetric relations in Γ^0.

Right-angled Artin-Tits monoid

$$A_\Gamma^+ = \left\langle \{\sigma_v\}_{v \in V} \mid \sigma_v \sigma_w = \sigma_w \sigma_v \text{ if } (v,w) \in \Gamma^1 \right\rangle^+$$

Definition (Crisp–Laca 2002)

The C^*-algebra associated to the Artin-Tits monoid of Γ is

$$C^*(A_\Gamma^+) = C^* \left\langle \{s_v\}_{v \in V} \;\middle|\; \begin{array}{ll} s_v s_w = s_w s_v & (v,w) \in \Gamma^1 \\ s_v s_w^* = s_w^* s_v & (v,w) \in \Gamma^1 \\ s_v^* s_w = \delta_{v,w} \cdot 1 & (v,w) \notin \Gamma^1 \end{array} \right\rangle.$$

Example

- $\mathcal{T} = C^*(A^+_{\Gamma_1})$ with $\Gamma_1 = \bullet$
- $\mathcal{T} \otimes \mathcal{T} = C^*(A^+_{\Gamma_2})$ with $\Gamma_2 = \bullet - \bullet$
- $\mathcal{E}_2 = C^*(A^+_{\Gamma_3})$ with $\Gamma_3 = \bullet \quad \bullet$
- $C^*(A^+_{\Gamma_4})$ with $\Gamma_4 = \bullet \quad \bullet - \bullet$

Definition

For $\Gamma = (\Gamma^0, \Gamma^1)$ we let

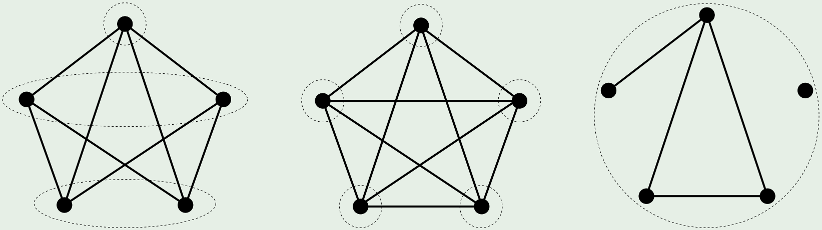
$$\Gamma^{\mathsf{op}} = (\Gamma^0, (\Gamma^0 \times \Gamma^0) \setminus (\Gamma^1 \cup \{(v,v) \mid v \in \Gamma^0\}).$$

We call Γ co-irreducible when Γ^{op} is irreducible., and for non-co-irreducible graphs consider co-irreducible components:

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$$

Graphs (cont'd)

Examples



Definition (Euler characteristic)

$$\chi(\Gamma) = \sum_{K \ \Gamma\text{-simplex}} (-1)^{|K|}$$

χ is multiplicative over co-irreducible components.

Structure results [Crisp-Laca, Ivanov, Cuntz-Echterhoff-Li]

- $C^*(A_\Gamma^+) = C^*(A_{\Gamma_1}^+) \otimes C^*(A_{\Gamma_2}^+) \otimes \cdots \otimes C^*(A_{\Gamma_n}^+)$ when

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$$

- $\mathbb{K} \triangleleft C^*(A_\Gamma^+)$ with

$$\begin{array}{ccc} K_0(\mathbb{K}) & \longrightarrow & K_0(C^*(A_\Gamma^+)) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow[\chi(\Gamma)]{} & \mathbb{Z} \end{array}$$

- When Γ is co-irreducible we have

$$A_\Gamma^+/\mathbb{K} \simeq \begin{cases} C(S^1) \text{ when } |\Gamma_0| = 1 \\ \text{a Kirchberg algebra when } |\Gamma_0| > 1 \end{cases}$$

Obstructions for isomorphism

Suppose $C^*(A_\Gamma^+) \simeq C^*(A_{\Gamma'}^+)$ with Γ co-irreducible. Then also Γ' is co-irreducible, and

$$\chi(\Gamma) = \chi(\Gamma').$$

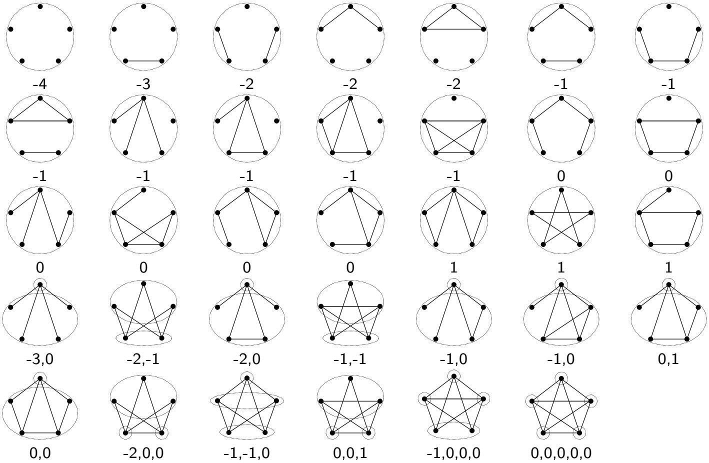
Further,

$$|\Gamma_0| = 1 \Longleftrightarrow |\Gamma_0'| = 1$$

Question

Are these the only obstructions? What happens in the non-co-irreducible case?

$n = 5$


-4
-3
-2
-2
-2
-1
-1
-1
-1
-1
-1
-1
0
0
0
0
0
0
1
1
1
-3,0
-2,-1
-2,0
-1,-1
-1,0
-1,0
0,1
0,0
-2,0,0
-1,-1,0
0,0,1
-1,0,0,0
0,0,0,0,0

Definition

The *Vaksman-Soibelman odd quantum sphere* $C(S_q^{2n-1})$ is the universal C^*-algebra for generators $z_1, \ldots, z_n$ subject to

$$\begin{aligned} z_j z_i &= q z_i z_j \qquad i < j \\ z_j^* z_i &= q z_i z_j^* \qquad i \neq j \\ z_i^* z_i &= z_i z_i^* + (1 - q^2) \sum_{j>i} z_j z_j^* \\ 1 &= \sum_{i=1}^{n} z_i z_i^* \end{aligned}$$

Let n and r be given, set $\theta = e^{2\pi i/r}$ and note that

$$\Lambda_{\underline{m}}(z_i) = \theta^{m_i} z_i$$

defines $\Lambda_{\underline{m}} \in \mathrm{Aut}\, C(S_q^{2n-1})$ when $(m_i, r) = 1$ for all i.

Definition [Hong-Szymanski 2002]

Given r, n, and $\underline{m} \in \mathbb{N}^n$. The quantum lens space $C(L_q(r; \underline{m}))$ is the fixed point space

$$C(S_q^{2n-1})^{\Lambda_{\underline{m}}}$$

Structure [Hong-Szymanski]

- $C(L_q(r;\underline{m}))$ has a decomposition series
$$0 = \mathfrak{I}_0 \lhd \mathfrak{I}_1 \lhd \mathfrak{I}_2 \lhd \cdots \mathfrak{I}_n = C(L_q(r;\underline{m}))$$
with $\mathfrak{I}_i/\mathfrak{I}_{i-1} = C(S^1)\otimes\mathbb{K}$ for $i < n$ and $\mathfrak{I}_n/\mathfrak{I}_{n-1} \simeq C(S^1)$.
- $C(L_q(r;\underline{m}))$ is stably finite (in fact type I) of real rank 1.
- $\mathrm{Prim}(C(L_q(r;\underline{m}))) = [1;n]\times S^1$ with the order topology on $[1;n] = \{1,2,\ldots,n\}$.
- $K_0(C(L_q(r;\underline{m}))) = \mathbb{Z}\oplus G$ with $|G| = r^{n-1}$

Obstructions for isomorphism

Suppose r, r', n, n' and $\underline{m} \in \mathbb{N}^n, \underline{m}' \in \mathbb{N}^{n'}$ are given with $C(L_q(r;\underline{m})) \simeq C(L_q(r';\underline{m}'))$. Then $r = r'$ and $n = n'$.

Question

Are these the only obstructions? Is it possible that $C(L_q(r;\underline{m})) \simeq C(L_q(r;\underline{1}))$ irrespective of $\underline{m}$?

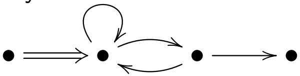
Definition

A graph is a tuple (E^0, E^1, r, s) with

$$r, s : E^1 \to E^0$$

and E^0 and E^1 countable sets.

We think of $e \in E^1$ as an edge from $s(e)$ to $r(e)$ and often represent graphs visually



or by an adjacency matrix

$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Singular and regular vertices

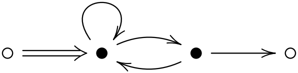
Definitions

Let E be a graph and $v \in E^0$.

- v is a *sink* if $|s^{-1}(\{v\})| = 0$
- v is an *infinite emitter* if $|s^{-1}(\{v\})| = \infty$

Definition

v is *singular* if v is a sink or an infinite emitter. v is *regular* if it is not singular.



Graph algebras

Definition

The *graph C^*-algebra* $C^*(E)$ is given as the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges subject to the Cuntz-Krieger relations

1. $s_e^* s_e = p_{r(e)}$
2. $s_e s_e^* \leq p_{s(e)}$
3. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every regular v

$C^*(E)$ is unital precisely when E has finitely many vertices.

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Theorem

Gauge invariant ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies$ *[$v \in V$ or v is singular]*

and when there are no **breaking vertices**, *all such ideals arise this way.*

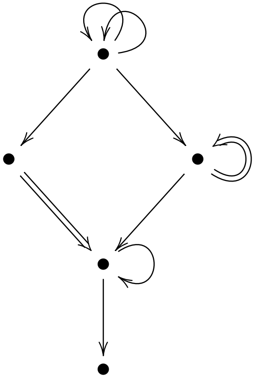
The gauge simple case

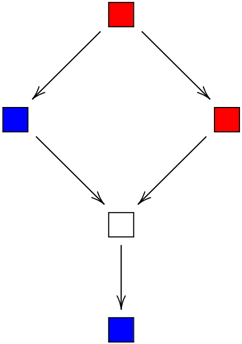
Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

- *a simple AF algebra;*
- *a Kirchberg algebra; or*
- *$C(\mathbb{T}) \otimes \mathbb{K}(H)$ for some Hilbert space H.*

It is easy to tell from the graph which case occurs: The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.





Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra with only finitely many gauge invariant ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with gauge invariant $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $FK^{\gamma}(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $FK^{\gamma,+}(\mathfrak{A})$.

$FK^{\gamma,+}(C^*(E))$ is readily computable when $|E^0| < \infty$.

Working conjecture [E-Restorff-Ruiz 2010]

$FK^{\gamma,+}(-)$ is a complete invariant, up to stable isomorphism, for graph C^*-algebras of real rank zero (*i.e.*, with no $\square$ subquotients) and finitely many ideals.

No counterexamples are known, not even allowing for $\square$ subquotients.

General graph C^*-algebras

Status of working conjecture:

	■	■	□	■/■	■/□	■/□	■/■/□
1	El76	KiPh00					
2	El76	Rø97		ET10			
3	El76	BKö12	ERS				
4	El76	ARR14	ERS				
n	El76	(BMe14)	ERS				

Xx: Elliott, Kirchberg, Köhler, Meyer, Phillips, Rørdam.
Y: Arklint, Bentmann, Restorff, Ruiz, Sørensen, Tomforde.

Unital graph C^*-algebras

Status of working conjecture:

				/	/	/	/ /
1		KiPh00					
2		Rø97		ET10			
3		BKö12	ERS	ERR13			
4		ARR14	ERS	ERRS			
n		ERRS	ERS	ERRS			

Xx: Kirchberg, Köhler, Phillips, Rørdam.

Y: Arklint, Bentmann, Restorff, Ruiz, Sørensen, Tomforde.

Moves

Move **(S)**

Remove a regular source, as

Move **(R)**

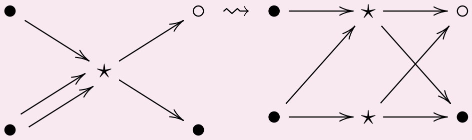
Reduce a configuration with a transitional regular vertex, as

or

Moves

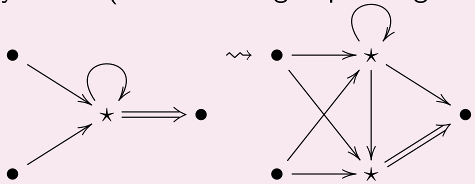
Move (I)

Insplit at regular vertex



Move (O)

Outsplit at any vertex (at most one group of edges infinite)



Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Definition

$E \sim_M F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,**(C)**,

and their inverses, leading from E to F.

Theorem (E-Restorff-Ruiz-Sørensen)

Let $C^*(E)$ and $C^*(F)$ be unital graph algebras with real rank zero. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_M F$

(iii) $FK^{\gamma,+}(C^*(E)) \simeq FK^{\gamma,+}(C^*(F))$

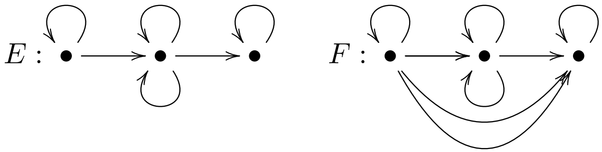
Theorem (E-Ruiz-Sørensen)

Let E and F be finite graphs with heredity of negative temperatures. Then the following are equivalent

(i) $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

(ii) $E \sim_M F$

(iii) $FK^{\gamma,+}(C^*(E)) \simeq FK^{\gamma,+}(C^*(F))$



Example (E-Ruiz-Sørensen)

$E \not\sim_M F$, yet

$$C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$$

Theorem [Hong-Szymanski]

$C(L_q(r;\underline{m}))$ is a graph algebra given by a graph with an $n \times n$ adjacency matrix on the form

$$\mathsf{A}_{r,\underline{m}} = \begin{bmatrix} 1 & r & * & * & \cdots & * \\ & 1 & r & * & & * \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & r & * \\ & & & & 1 & r \\ & & & & & 1 \end{bmatrix}$$

Hence we are in the resolved unital case $n/\square$. In fact, we can do better:

Theorem (E-Ruiz-Sørensen)

The following are equivalent

- $C(L_q(r;\underline{m})) \simeq C(L_q(r;\underline{m}'))$
- $C(L_q(r;\underline{m})) \otimes \mathbb{K} \simeq C(L_q(r;\underline{m}')) \otimes \mathbb{K}$
- *There exist integer matrices* U, V *on the form*

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & * \\ & & \ddots & \ddots & \\ & & & 1 & * \\ & & & & 1 \end{bmatrix}$$

so that

$$U(1 - \mathsf{A}_{r,\underline{m}})V = (1 - \mathsf{A}_{r,\underline{m}'})$$

Corollary

$C(L_q(r;(1,1,1,1))) \simeq C(L_q(r;(1,-1,1,1))$ if and only if $3|r$

Set

$$\varphi(r) = \min\{n \in \mathbb{N} \mid \exists \underline{m} \in \mathbb{N}^n : C(L_q(r;\underline{m})) \not\simeq C(L_q(r;\underline{1}))\}$$

then computer experiments give

r	2	3	4	5	6	7	8	9	10	11	12	13
$\varphi(r)$	∞	4	6	6	4	8	6	4	6	12	4	14

Theorem (Jensen-Klausen-Rasmussen)

$$\varphi(r) = \min\{2n : 2n > a > 2, a \mid r\}$$

Observation

K-theory shows that not every $C^*(A_\Gamma^+)$ is a graph algebra.

Theorem (E, Katsura, Restorff, Ruiz, Tomforde, West)

Suppose $C^*(E)$ is simple. Then in any extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathfrak{A} \longrightarrow C^*(E) \longrightarrow 0$$

$\mathfrak{A}$ is isomorphic to a graph algebra.

Thus we are in the resolved 2/■/■ case when Γ is co-irreducible.

Theorem (E-Li-Ruiz)

Suppose Γ, Γ' are both co-irreducible with $|\Gamma_0|, |\Gamma_0'| > 1$. Then

$$C^*(A_\Gamma^+) \simeq C^*(A_{\Gamma'}^+) \Longleftrightarrow \chi(\Gamma) = \chi(\Gamma')$$

The general case

Definition

When $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$, define

$$t(\Gamma) = \#\{i \mid |\Gamma_i| = 1\}$$
$$N_k(\Gamma) = \#\{i \mid \chi(\Gamma_i) = k\}$$

Theorem (E–Li–Ruiz)

For general graphs Γ, Γ' *we have*

$$C^*(A^+_\Gamma) \simeq C^*(A^+_{\Gamma'})$$

precisely when

1. $t(\Gamma) = t(\Gamma')$
2. $N_k(\Gamma) + N_{-k}(\Gamma) = N_k(\Gamma') + N_{-k}(\Gamma')$ *for all* k
3. $N_0(\Gamma) > 0$ *or* $\sum_{k>0} N_k(\Gamma) \equiv \sum_{k>0} N_k(\Gamma') \mod 2$