Classifying graph C*-algebras

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Program

1. Preamble
2. Graph algebras
3. Ideals and $K$-theory
4. Conjecture
5. Partial verification
Finitely many ideals

Observation (cf. Jordan-Hölder)

When the $C^*$-algebra $A$ has finitely many ideals a finite decomposition series

$$0 = I_0 \triangleleft I_1 \triangleleft \cdots \triangleleft I_n = A, \quad l_j/I_{j-1} \text{ simple}$$

exists with $(l_1/I_0, l_2/l_1, \ldots, l_n/l_{n-1})$ unique up to isomorphism and permutation.

Of course, the decomposition series does \textbf{not} determine $A$. But suppose the $l_j/I_{j-1}$ are all classifiable by $K$-theory, is the same then true for $A$?
**B(H): A C*-algebra with one non-trivial ideal**

**K is AF**

The compacts form an AF algebra, i.e. for any finite set $a_1, \ldots, a_\ell$ and $\epsilon > 0$ there is a finite-dimensional algebra $F \subseteq K$ with $\|a_i - f_i\| < \epsilon$ for some $f_i \in F$.

**B(H)/K is purely infinite**

The Calkin algebra is purely infinite, i.e. for any $x, y \in B(H)/K$ with $x \neq 0$ there exist elements $a, b$ such that

$$y = axb$$
Further properties

Real rank zero

\( \mathcal{B}(H), K \) and \( \mathcal{B}(H)/K \) have real rank zero, i.e. for any self-adjoint element \( a \) and any \( \epsilon > 0 \) there is a self-adjoint element \( f \) with finite spectrum such that \( \|a - f\| < \epsilon \).

Separability and nuclearity

\( K \) is separable and nuclear. Neither of \( \mathcal{B}(H) \) and \( \mathcal{B}(H)/K \) are.
Any countable graph $G = (E^0, E^1)$ defines a $C^*$-algebra $C^*(G)$ given as a universal $C^*$-algebra by projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ subject to the Cuntz-Krieger relations:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e) = v} s_e s_e^*$ for every $v$ with $0 < |\{e | s(e) = v\}| < \infty$.

Singular vertices

When $\{e | s(e) = v\} = \emptyset$ we say that $v$ is a sink. When $|\{e | s(e) = v\}| = \infty$ we say that $v$ is an infinite emitter. In either case, we say that $v$ is singular.
For which pairs of graphs do we have

$$C^*(G) \otimes K \simeq C^*(H) \otimes K?$$
Subcase: AF

**Theorem (Kumjian-Pask-Raeburn)**

\[ C^*(G) \text{ is AF precisely when } G \text{ has no cycles, i.e. is a forest. } \]
Subsubcase: Finite forest

Theorem

The following are equivalent for finite forests $G$ and $H$

1. $C^*(G) \otimes K \simeq C^*(H) \otimes K$
2. $G$ and $H$ have the same number of leaves
Subsubcase: A matroid tree

Consider the case where $G = G[n_i]$ is given by a sequence of integers $n_i$ describing an infinite tree

$$
\bullet \xrightarrow{n_1} \bullet \xrightarrow{n_2} \bullet \xrightarrow{n_3} \bullet \xrightarrow{n_4} \ldots
$$

**Theorem**

The following are equivalent

1. $C^*(G[n_i]) \otimes \mathbb{K} \simeq C^*(G[m_i]) \otimes \mathbb{K}$
2. $\exists j : x \mid \prod_{i=1}^{j} n_i \iff \exists j : x \mid \prod_{i=1}^{j} m_i$
Subcase: Purely infinite

Theorem (Cuntz-Krieger, an Huef-Raeburn)

When $G$ is a finite and strongly connected graph then the following are equivalent

1. $C^*(G)$ has finitely many ideals
2. $C^*(G)$ is simple
3. $C^*(G)$ has real rank zero
4. $C^*(G)$ is purely infinite
5. $G$ is not a cycle
Theorem (Franks, Cuntz, Rørdam)

The relation induced on the class of finite and strongly connected graphs by stable isomorphism of the associated graph $C^*$-algebra is the smallest equivalence relation containing

<table>
<thead>
<tr>
<th>Edge expansion</th>
<th>$\bullet \rightarrow \bullet \quad \sim \rightarrow \quad \bullet \rightarrow \circ \rightarrow \bullet$</th>
</tr>
</thead>
<tbody>
<tr>
<td>State splitting</td>
<td>$\bullet \rightarrow \bullet \quad \sim \rightarrow \quad \bullet \rightarrow \circ \xrightarrow{\sim} \bullet$</td>
</tr>
<tr>
<td>Cuntz splice</td>
<td>$\bullet \quad \sim \rightarrow \quad \bullet \xleftarrow{\sim} \circ \xrightarrow{\sim} \bullet$</td>
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</table>
A graph $C^*$-algebra is separable and nuclear.

Theorem (Kumjian-Pask-Raeburn)

A simple graph $C^*$-algebra is either AF or purely infinite.

Theorem (Elliott, Kirchberg-Phillips)

$K_*(-)$ is a complete invariant for stable isomorphism of graph $C^*$-algebras which are simple, or AF.
Theorem (Hong-Szymański)

$C^*(G)$ has real rank zero precisely when no cycle in $G$ is unique.

Corollary

If $C^*(G)$ has finitely many ideals, then $C^*(G)$ has real rank zero.
Sets of vertices

**Hereditary**

\[ F^0 \subseteq E^0 \text{ is } \textbf{hereditary} \text{ when } s(e) \in F^0 \Rightarrow r(e) \in F^0 \]

**Saturated**

\[ F^0 \subseteq E^0 \text{ is } \textbf{saturated} \text{ when for any non-singular } v \not\in F^0 \text{ there is an edge } e \text{ with } r(e) = v, s(e) \not\in F^0. \]

**Breaking vertex**

An infinite emitter \( v \) is a \textbf{breaking vertex} for \( F^0 \) if

\[ 0 < \left| \{ e \in E^1 \mid r(e) = v, s(e) \not\in F^0 \} \right| < \infty \]
Ideal structure

**Theorem**

*When $C^*(G)$ has real rank zero there is a one-to-one correspondance between the ideals of $C^*(G)$ and pairs $(F^0, B^0)$ chosen such that*

- $F^0$ is hereditary
- $F^0$ is saturated
- $B^0$ is a set of breaking vertices for $F^0$

**Theorem**

*The ideal corresponding to $(F^0, \emptyset)$ is stably isomorphic to $C^*(H)$ where $H$ is the subgraph of $G$ with $F^0$ as vertex set.*
## Color coding

<table>
<thead>
<tr>
<th>$G$</th>
<th>$C^*(G)$</th>
<th>Legend</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cofinal tree</td>
<td>Simple $AF$ algebra</td>
<td><img src="Legend" alt="Red" /></td>
</tr>
<tr>
<td>Finite, strongly connected graph (not a cycle)</td>
<td>Simple Cuntz-Krieger algebra</td>
<td><img src="Legend" alt="Blue" /></td>
</tr>
<tr>
<td>Graph with a cycle, no unique cycles, and only trivial hereditary and saturated subsets</td>
<td>Simple purely infinite algebra</td>
<td><img src="Legend" alt="Blue" /></td>
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</table>
When $G$ is presented by an adjacency matrix in block form

$$\begin{bmatrix} A & \alpha \\ * & * \end{bmatrix}$$

with singular vertices in the last row and column blocks, then

$$K_0(C^*(G)) = \text{cok} \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix} \quad K_1(C^*(G)) = \text{ker} \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix}$$
The following are equivalent for finite forests $G$ and $H$

- $C^*(G) \otimes K \simeq C^*(H) \otimes K$
- $G$ and $H$ have the same number of leaves

**K-theory**

\[ K_0(C^*(G)) = \mathbb{Z}^{\#\text{leaves}} \quad K_1(C^*(G)) = 0 \]
Subsubcase: A matroid tree

Consider the case where $G = G[n_i]$ is given by a sequence of integers $n_i$ describing an infinite tree

$$
\bullet \overset{n_1}{\longrightarrow} \bullet \overset{n_2}{\longrightarrow} \bullet \overset{n_3}{\longrightarrow} \bullet \overset{n_4}{\longrightarrow} \cdots
$$

**Theorem**

- $C^*(G[n_i]) \otimes K \simeq C^*(G[m_i]) \otimes K$
- $\exists j : x | \prod_{i=1}^{j} n_i \iff \exists j : x | \prod_{i=1}^{j} m_i$

**K-theory**

$$
K_0(C^*(G[n_i])) = \lim( \mathbb{Z} \overset{n_1}{\longrightarrow} \mathbb{Z} \overset{n_2}{\longrightarrow} \mathbb{Z} \overset{n_2}{\longrightarrow} \cdots )
$$

$$
K_1(C^*(G[n_i])) = 0
$$
Theorem (Franks, Cuntz, Rørdam)

The relation induced on the class of finite and strongly connected graphs by stable isomorphism of the associated graph $C^*$-algebra is the smallest equivalence relation containing

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$K$-theory

\[
K_0(C^*(G_A)) = \text{cok}(A^t - 1) \\
K_1(C^*(G_A)) = \text{ker}(A^t - 1) = \text{cok}(A^t - 1) / \text{tor}(\text{cok}(A^t - 1))
\]
<table>
<thead>
<tr>
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<th>$K_0(G)_+$</th>
<th>Ideals</th>
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<tr>
<td>· → · → · → ·</td>
<td>$\mathbb{Z}^2$</td>
<td>${(x, y) \mid x \geq 0, y \geq 0}$</td>
<td></td>
</tr>
<tr>
<td>· → · → · → ·</td>
<td>$\mathbb{Z}^2$</td>
<td>${(x, y) \mid x + \frac{\sqrt{5}-1}{2}y \geq 0}$</td>
<td></td>
</tr>
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<td>$G$</td>
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<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td>$\mathbb{Z}^2$</td>
<td>$\mathbb{Z}^2$</td>
<td><img src="image2.png" alt="Ideal" /></td>
</tr>
<tr>
<td><img src="image3.png" alt="Graph" /></td>
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<td>$\mathbb{Z}^2$</td>
<td><img src="image4.png" alt="Ideal" /></td>
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**Preamble**

**Graph algebras**

**Ideals and $K$-theory**

**Conjecture**

**Partial verification**
Theorem (Drinen-Tomforde, Carlsen-E-Tomforde)

For $C^*(G)$ given by

\[
\begin{bmatrix}
A & \alpha & 0 & 0 \\
* & * & 0 & 0 \\
X & \xi & B & \beta \\
* & * & * & * \\
\end{bmatrix}
\]

the six-term exact sequence in $K$-theory becomes

\[
\begin{array}{cccc}
\text{cok} \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix} & \xrightarrow{I} & \text{cok} \begin{bmatrix} A^t - 1 & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - 1 \end{bmatrix} & \xrightarrow{P} & \text{cok} \begin{bmatrix} B^t - 1 \\ \beta^t \end{bmatrix} \\
\text{ker} \begin{bmatrix} B^t - 1 \\ \beta^t \end{bmatrix} & \xleftarrow{P} & \text{ker} \begin{bmatrix} A^t - 1 & X^t \\ \alpha^t & \xi^t \\ 0 & B^t - 1 \end{bmatrix} & \xleftarrow{I} & \text{ker} \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix} \\
\end{array}
\]
Filtrated $K$-theory

$\mathcal{K}(A)$:

The collection of all six term exact sequences

$$
\begin{array}{c}
K_0(J/I) \rightarrow K_0(K/I) \rightarrow K_0(K/J) \\
\uparrow \\
K_1(K/J) \leftarrow K_1(K/I) \leftarrow K_1(J/I)
\end{array}
\quad
\text{whenever } I \triangleleft J \triangleleft K \triangleleft A.
$$

Remark

Each subquotient may occur several times, in which case the $K$-groups of the various six-term exact sequences are identified. Thus the invariant is also called the “$K$-web”.
Theorem (Restorff)

When $G$ and $H$ are finite graphs with no unique cycles, no sinks, and no sources, then the following are equivalent

- $C^*(G) \otimes K \simeq C^*(H) \otimes K$
- $\mathcal{K}(C^*(G)) \simeq \mathcal{K}(C^*(H))$
**Fundamental question**

\[ \mathcal{K}(A)_+ : \]

As above, but with each \( K_0 \)-group

\[ K_0(J/I) \rightarrow K_0(K/I) \rightarrow K_0(K/J) \]

considered as an **ordered** group.

**Working conjecture**

\( \mathcal{K}(\cdot)_+ \) is a complete invariant for stable isomorphism of all graph \( C^* \)-algebras with finitely many ideals.
One ideal

Theorem (E-Tomforde)

\[ \mathcal{R}(-)_+: \]

\[ K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(A/I) \]

\[ K_1(A/I) \leftarrow K_1(A) \leftarrow K_1(I) \]

is a complete invariant up to stable isomorphism for the class of graph algebras with precisely one non-trivial ideal.
Theorem (Kirchberg)

Any $\alpha \in KK_X(A, B)^{-1}$ induces a stable isomorphism between $A$ and $B$ when these are (non-simply) purely infinite and nuclear with $\text{Prim}(A) = \text{Prim}(B) = X$.

Theorem (Meyer-Nest)

When $A, B$ are in the bootstrap class and $\text{p.dim}(\mathcal{R}(A)) \leq 1$ we have a UCT

$$0 \rightarrow \text{Ext}(\mathcal{R}(A), \mathcal{R}(B)) \rightarrow KK_X(A, B) \rightarrow \text{Hom}(\mathcal{R}(A), \mathcal{R}(B)) \rightarrow 0$$
Corollary (Meyer-Nest, Köhler-NN)

\( K(\cdot) \) is a complete invariant for purely infinite graph algebras of the form
Problem

For a certain purely infinite $C^*$-algebra $A$ with 7 ideals, $\text{p.dim}(\mathcal{R}(A)) > 1$. Consequently, $\mathcal{R}(-)$ is not a complete invariant for all nuclear, purely infinite $C^*$-algebras in the bootstrap class with real rank zero.

However, the $K$-theory of this example is not obtainable by graph algebras.
Theorem (E-Restorff-Ruiz)

\( \mathcal{R}(-)_+ \) is a complete invariant for the class of graph algebras with finite linear ideal lattices of the form:

\[
\begin{array}{cccccc}
\bullet & \leftarrow & \cdots & \leftarrow & \bullet & \leftarrow \cdots \\
\end{array}
\]

\[
\begin{array}{cccccc}
\bullet & \leftarrow & \cdots & \leftarrow & \bullet & \leftarrow \cdots \\
\end{array}
\]

Theorem (E-Restorff-Ruiz)

\( \mathcal{R}(-)_+ \) is a complete invariant for the class of graph algebras with finite linear ideal lattices when for all subquotients we have

\[
\begin{align*}
K_0(I_j/I_{j-1}) &= \mathbb{Z}^k \\
K_1(I_j/I_{j-1}) &= 0
\end{align*}
\]