Matsumoto algebras for substitutional shift spaces

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Chengde 2002
Let \( a \) be a finite set of symbols, and let \( a^\# \) denote the set of finite non-empty words with letters from \( a \). A substitution is a map

\[
\tau : a \longrightarrow a^\#
\]

We write “\( w \rightarrow v \)” when \( w, v \in a^\# \) and \( w \) is a subword of \( v \).

Define

\[
\mathcal{X}_\tau = \{ (x_i) \in a^{\mathbb{Z}} \mid \forall i < j \exists n, a : x_{[i,j]} \rightarrow \tau^n(a) \}.
\]

and equip with

\[
\sigma(x_n) = (x_{n+1})
\]
Shift dynamics

Using the product topology, \((X,F,\sigma)\) is a topological dynamical system.

Applications among others:

- Automata theory
- Quasicrystals
- Recurrent sets
- Transcendence in \(\mathbb{R}\)
- Diophantine approximation

Some substitutions

\[ \tau_1(\mathbb{N}) = \mathbb{N}\mathbb{N} \quad \tau_1(\mathbb{B}) = \mathbb{B}\mathbb{B}\mathbb{B} \]

\[ \tau_2(\alpha) = \alpha\beta \quad \tau_2(\beta) = \alpha\beta\gamma\delta\epsilon \quad \tau_2(\gamma) = \alpha\beta \]
\[ \tau_2(\delta) = \gamma\delta\epsilon \quad \tau_2(\epsilon) = \alpha\beta\gamma\delta\epsilon \]

\[ \tau_3(1) = 1212345 \]
\[ \tau_3(2) = 12123451234512345 \]
\[ \tau_3(3) = 1212345 \quad \tau_3(4) = 1234512345 \]
\[ \tau_3(5) = 12123451234512345 \]

\[ \tau_4(a) = ababacb \quad \tau_4(b) = ababacbabacbabacb \]
\[ \tau_4(c) = abacbabaacb \]
**Substitution properties**

**Definition** \( \tau \) is *primitive* if

\[
\exists N \forall a, b : b \rightarrow \tau^N(a) \\
\forall a : |\tau^N(a)| \rightarrow \infty
\]

**Definition** \( \tau \) is *aperiodic* if \( |X_\tau| = \infty \).

**Observation** If \( \tau \) is primitive and aperiodic, then \( (X_\tau, \sigma) \) [with product topology] is a Cantor minimal system.
$C^*$-algebra invariants

- **Cantor minimal crossed product**
  \[ \tau \mapsto C(X_\tau) \rtimes \sigma \mathbb{Z}. \]

- **Matsumoto algebra**
  \[ \tau \mapsto O_\tau \otimes K \]

- Both!
The Matsumoto algebra

Several equivalent constructions

(i) Generators and relations

(ii) Groupoid algebra

(iii) Cuntz-Pimsner algebra

which – WARNING! – will sometimes differ from the original

(iv) Fock space algebra

cf. Carlsen/Matsumoto.
Incidence matrix

To a substitution $\tau$ one associates the $|a| \times |a|$-matrix $A_\tau$ given by

$$(A_\tau)_{a,b} = \# \text{ of occurrences of } b \text{ in } \tau(a)$$

**Theorem** [Giordano/Putnam/Skau²/Durand/Host]

When $\tau$ is aperiodic, primitive and proper*,

$$K_0(C(X_\tau) \rtimes_\sigma \mathbb{Z}) = \lim (\mathbb{Z}^{|a|}, A_\tau)$$

as ordered groups.

*No loss of generality
$\mathcal{O}_\tau$ is nonsimple, and has a maximal ideal isomorphic to $K^{n_\tau}$ for suitable $n_\tau$. Further,

$$0 \rightarrow K^{n_\tau} \rightarrow \mathcal{O}_\tau \rightarrow C(\mathcal{X}_\tau) \times_\sigma \mathbb{Z} \rightarrow 0$$

However,

$$C(\mathcal{X}_\tau) \times_\sigma \mathbb{Z} \simeq C(\mathcal{X}_\nu) \times_\sigma \mathbb{Z} \quad \text{for} \quad n_\tau = n_\nu \quad \iff \quad \mathcal{O}_\tau \sim \mathcal{O}_\nu$$
The short exact sequence induces

\[ \mathbb{Z}^{n_\tau} \rightarrow K_0(\mathcal{O}_\tau) \rightarrow K_0(C(X_\tau) \times \sigma \mathbb{Z}) \]

for suitable \( p_\tau \in \mathbb{N}^{n_\tau} \setminus \{0\} \). Consequently, \( \mathcal{O}_\tau \) has real rank zero but is not stably finite.

And again,

\[
\begin{aligned}
K_0(C(X_\tau) \times \sigma \mathbb{Z}) &\cong K_0(C(X_\upsilon) \times \sigma \mathbb{Z}) \\
&\begin{cases}
n_\tau = n_\upsilon \\
p_\tau = p_\upsilon
\end{cases}
\end{aligned}
\]

\( \not\cong \)

\[ K_0(\mathcal{O}_\tau) \cong K_0(\mathcal{O}_\upsilon) \]
**Theorem** [Carlsen/Eilers]

Let $\tau$ be a primitive, aperiodic, proper\(^{\ast}\) and injective\(^{\dagger}\) substitution of constant length\(^{\ddagger}\). For suitable $n_\tau \times |a|$-matrix $E_\tau$ we define

$$\tilde{A}_\tau = \begin{bmatrix} A_\tau & 0 \\ E_\tau & \text{Id} \end{bmatrix}$$

$$H_\tau = \mathbb{Z}^{n_\tau} / p_\tau \mathbb{Z}$$

and have

$$K_0(O_\tau) = \lim (\mathbb{Z}^{|a|} \oplus H_\tau, \tilde{A}_\tau)$$

as ordered group, where $\mathbb{Z}^{|a|} \oplus H_\tau$ is ordered by

$$(x, y) \geq 0 \iff x \geq 0$$

\(^{\ast}\)No loss of generality

\(^{\dagger}\)No loss of generality

\(^{\ddagger}\)Dispensable
What are $n_\tau, p_\tau$?

**Definition** $x \in X_\tau$ is right special if

$$\exists n : x[n, \infty[ = y[n, \infty[ \land x \neq y$$

**Theorem** [Queffélec]

If $\tau$ is a primitive and aperiodic substitution on $\alpha$, the number of orbit classes of special words is nonzero, but finite.

**Answer**

$$n_\tau = \# \{ [x]_{\text{orbit}} \mid x \text{ is right special} \}$$

Enumerate one-sided representatives of the right special orbits for $\tau$ as

$$x_1, \ldots, x_{n_\tau}.$$ 

**Answer**

$$\left( p_\tau \right)_i = \# \{ y \in X_\tau \mid y[0, \infty[ = x_i \} - 1$$
**What is $E_\tau$?**

When $\tau$ is of constant length $\ell$, the right special elements are all of the form

$$\ldots \tau^3(w)\tau^2(w)\tau(w)\tau^k(v)\tau^2(v)\tau^3(v)\ldots$$

where $w$ and $v$ are unique if $|w|, |v| < \ell$. Fix $v_i$ representing to the right orbit class $x_i$ and note that if we enumerate the corresponding $w$ as

$$w_1, \ldots, w_{m_i},$$

then $(p_\tau)_i = m_i - 1$.

**Answer** With this setup,

$$(E)_{i, b} = \sum_{i=1}^{m_i-1} \text{[# of occurrences of } b \text{ in } w_i]$$
There are efficient algorithms for computing $n_\tau, p_\tau$ and $E_\tau$:

CONCERNING $\tau$ GIVEN BY:
[a→bcada, b→bdbca, c→bccda, d→bddca]

COMPUTING $ND_\tau$:
(a,c) <-- da,+ -- (a,c)
(b,d) <-- ca,+ -- (b,d)
(c,d) <-- a,- -- (c,d)

COMPUTING CONFIGURATION DATA [Pass to $(\tau)^2$]
[0--0, 0--0, 2--2, 2--2, 4--4, 5--4, 6--6, 6--7]

COMPUTING $p$-VECTOR AND E-MATRIX [Pass to $(\tau)^2$]
Enumerating: [dabddcabcada, cabccdabcada, abcada]
$p_\tau$: [1, 1, 1]
$E_\tau$: [[2, 4, 4, 2], [2, 4, 2, 4], [3, 5, 5, 5]]

This extends to all primitive and aperiodic substitutions as regards $n_\tau$ and $p_\tau$. 

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Decidability of

\[ K_0(\mathcal{O}_\tau) \simeq K_0(\mathcal{O}_\nu) \]