Known and unknown ranges

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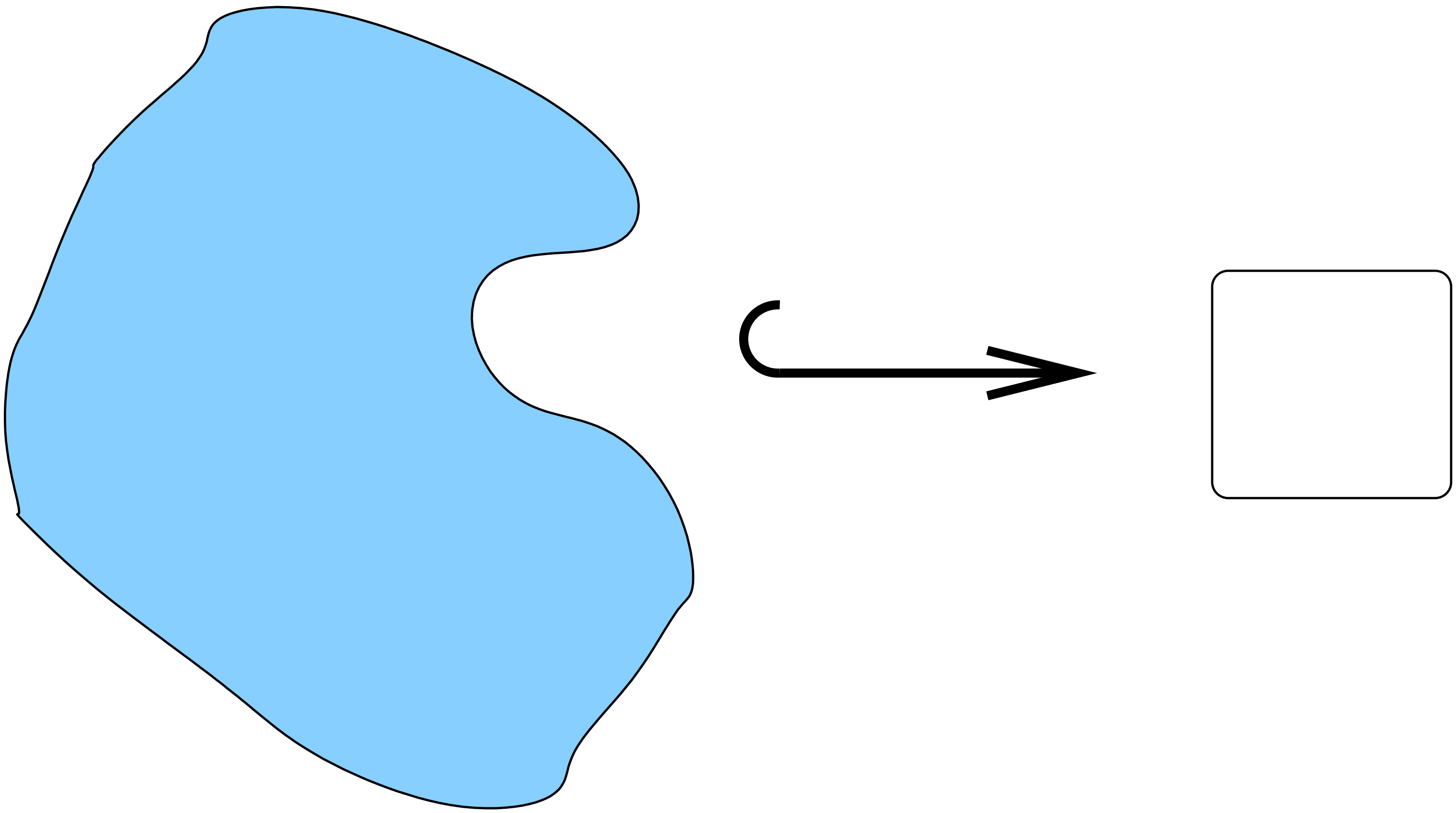
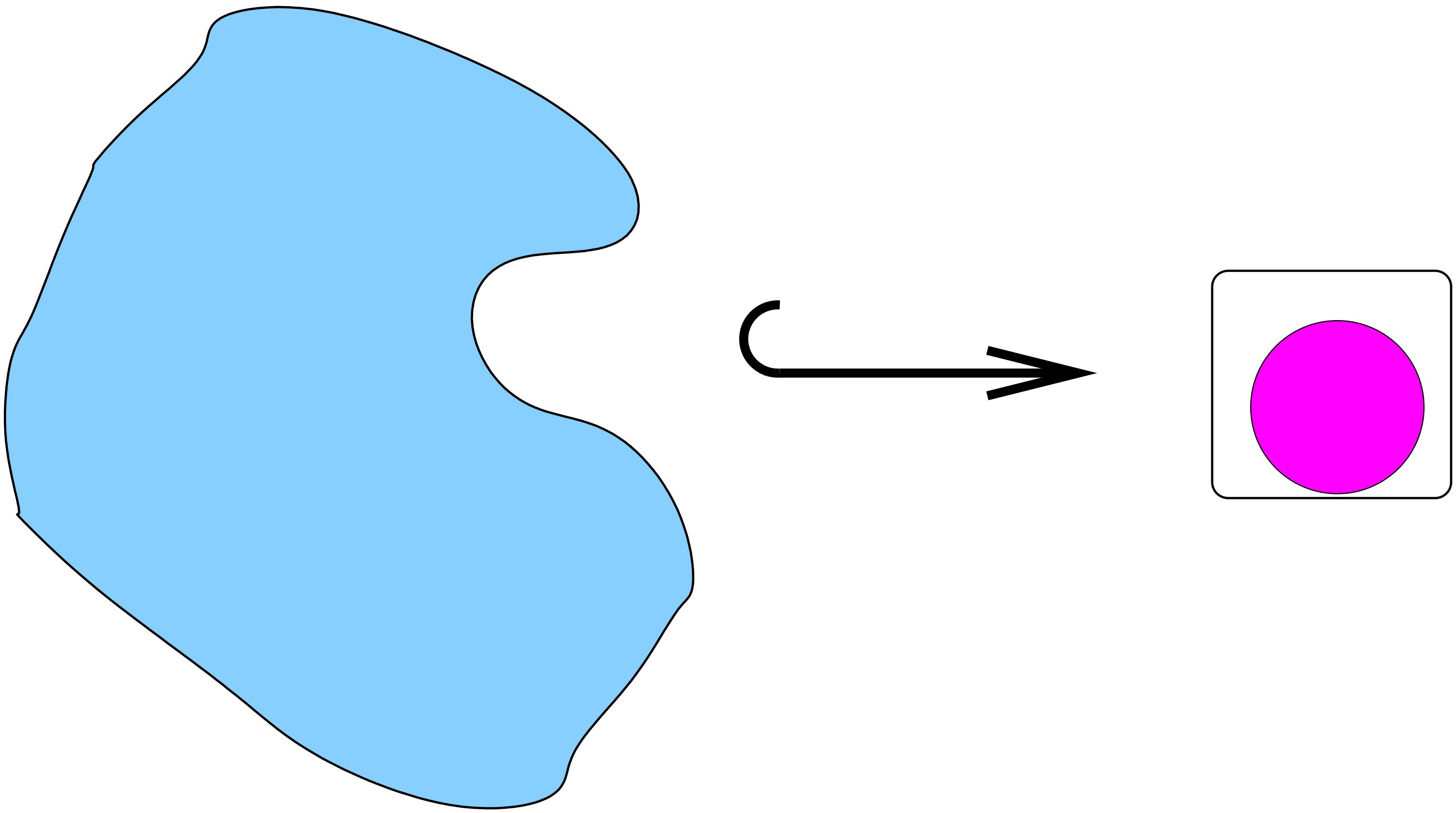
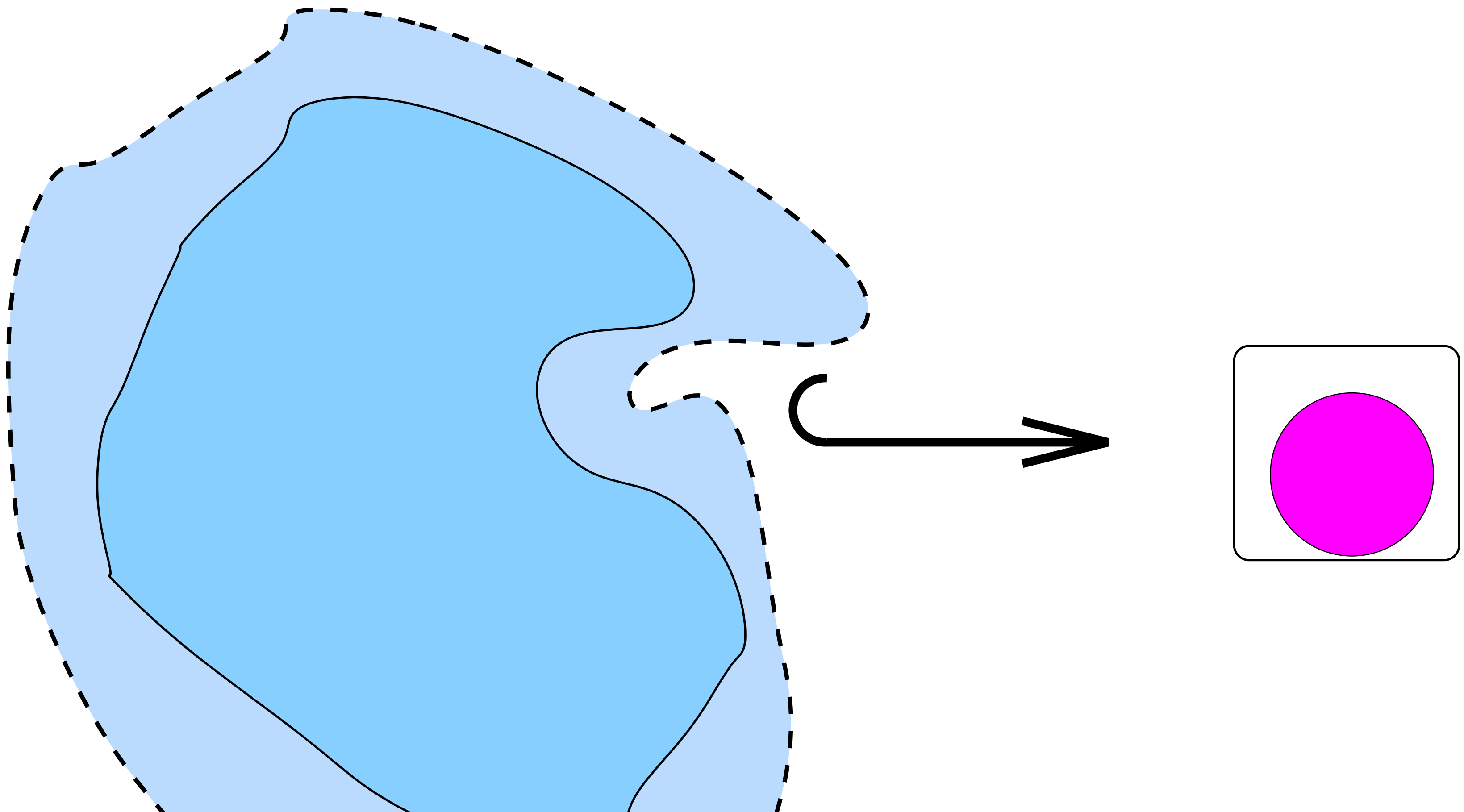
Congratulations, George – and thanks!







The class of **AF** algebras is the smallest class of C^*-algebras closed under

$$- \otimes \mathbf{M}_n, - \oplus -, \varinjlim -$$

and containing $\mathbb{C}$.

Let us abbreviate such a statement as follows:

$$\textbf{AF} = \textbf{A}\langle\langle\mathbb{C}\rangle\rangle$$

Complete invariant [Elliott] Unital **AF** algebras are classified up to isomorphism by

$$[K_0(-), K_0(-)_+, [1]]$$

Range [Effros-Handelman-Shen] The invariant ranges over all countable dimension groups with order unit.

These are the pointed ordered groups $[G, G_+, u]$ such that

G is ordered:
$G_+ \cap -G_+ = \{0\}, G_+ - G_+ = G$

G has the Riesz property:
$\forall a, b_i \in G_+ : a \leq b_1 + b_2 \exists a_i \in G_+ : a = a_1 + a_2, a_i \leq b_i$

G is unperforated:
$na \in G_+, n \in \mathbb{N} \Longrightarrow a \in G_+$

u is an order unit:
$\forall a \in G_+ \exists n \in \mathbb{N} : a \leq nu$

Complete invariant [Kirchberg-Phillips] Unital **purely infinite, simple, separable, nuclear** algebras in the bootstrap category $\mathcal{N}$ are classified up to isomorphism by

$$[K_*(-), [1]]$$

Range [Rørdam] The invariant ranges over all graded, countable pointed groups.

Complete invariant [Kirchberg] Unital **separable and nuclear** algebras A in the bootstrap category $\mathcal{N}$ with $A \simeq \mathcal{O}_2 \otimes A$ are classified up to isomorphism by

$$\mathrm{Prim}(-)$$

Range unknown!

Relevant results by Bratteli-Elliott, Kirchberg-Harnisch, Kirchberg-Rørdam.

AT = **A**$\langle\langle C(\mathbb{T})\rangle\rangle$.

Complete invariant [Elliott] Unital **AT** algebras *of real rank zero* are classified up to isomorphism by

$$[K_*(-), K_*(-)_+, [1]]$$

where $K_*(-) = K_0(-) \oplus K_1(-)$.

Range [Elliott] The invariant ranges over all countable graded dimension groups with order unit.

These are dimension groups $G_* = G_0 \oplus G_1$ such that

$$(x, y_1), (x, y_2) \in (G_*)_+ \Longrightarrow (x, y_1 \pm y_2) \in (G_*)_+$$

AH = **A**$\langle\langle C(X) \mid X$ compact Hausdorff$\rangle\rangle$.

Complete invariant [Dadarlat-Gong] Unital **AH** algebras of *real rank zero* and with *slow dimension growth* are classified up to isomorphism by

$$[\underline{\mathbf{K}}(-), \underline{\mathbf{K}}(-)_+, \Lambda, [1]]$$

Range unknown!

AH $=$ **A**$\langle\langle C(X) \mid X$ compact Hausdorff$\rangle\rangle$.

Complete invariant [Dadarlat-Gong] Unital **AH** algebras of *real rank zero* and with *slow dimension growth* are classified up to isomorphism by

$$[\underline{\mathbf{K}}(-), \underline{\mathbf{K}}(-)_+, \Lambda, [1]]$$

Here

$$\underline{\mathbf{K}}(A) = K_*(A) \oplus \bigoplus_{n \geq 2} K_*(A; \mathbb{Z}/n)$$

$$\Lambda = \{\rho_n^i, \beta_n^i, \kappa_{n,m}^i\}$$

The one thing to remember

- The invariants occurring have order induced by maps $\phi : G \longrightarrow H$ in the sense that $h \in \operatorname{im} \phi$ is positive precisely when there exists $g \in G_+$ with $\phi(g) = h$.

- This leaves a lot of freedom when ϕ is not surjective!

KK-based order:

$K_*(A) = KK(C(\mathbb{T}), A) \supseteq \{[f] \mid f$ a $*$-homomorphism$\}$

K_0-based order:

$K_*(A) = K_0(C(\mathbb{T}) \otimes A) \supseteq \{[p] \mid p$ a projection$\}$

Ideal-based order:

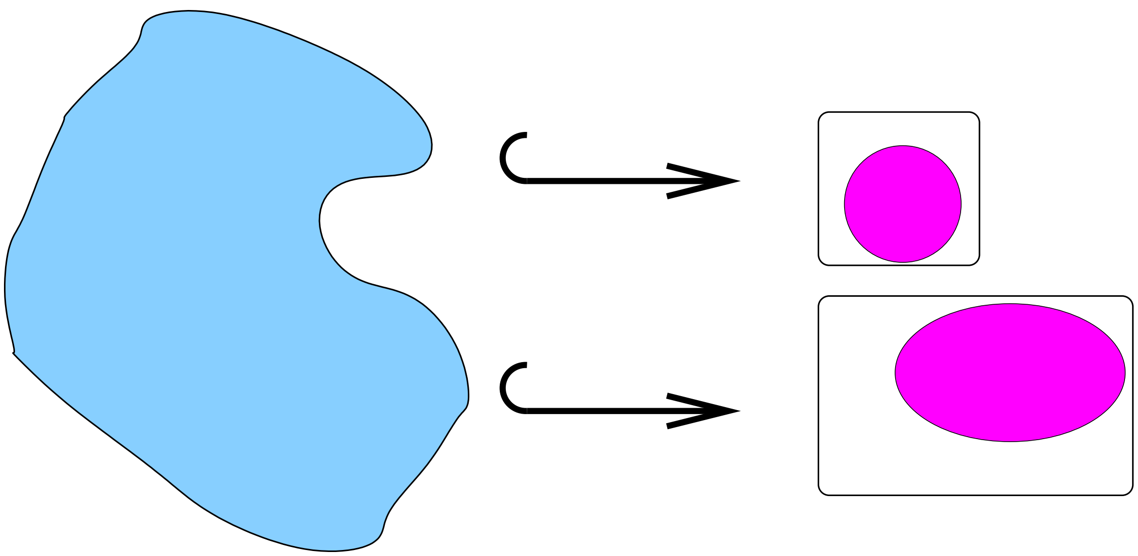
$K_*(A) \ni (x, y) \geq 0 \iff x \geq 0$ and $y \in K_1(I(X))$

All the same for an A of real rank zero. But to equip $K_0(A; \mathbb{Z}/n)$ as the odd part of a graded ordered group we need to choose the ideal-based order. Immaterial by [Dadarlat-E], so think about

$K_0(A) \oplus K_0(A; \mathbb{Z}/n) = KK(\mathbb{I}_n^{\sim}, A) \supseteq \{[f] \mid f$ a $*$-homomorphism$\}$

Assume tor $K_*(A) = 0$!

If A is an **AH** algebra of real rank zero and with slow dimension growth then A is **AT**.



Assume $\operatorname{tor} K_*(A) = 0$!

If A is an **AH** algebra of real rank zero and with slow dimension growth then A is **AT**.

ρ is surjective, so

$$\mathrm{id} \oplus \rho : K_0(A) \oplus K_0(A) \longrightarrow K_0(A) \oplus K_0(A; \mathbb{Z}/n)$$

completely determines the order on $K_0(A) \oplus K_0(A; \mathbb{Z}/n)$.

We are equipping $K_0 \oplus K_0$ as a graded ordered group!

Assume now only tor $K_0(A) = 0$!

If A is an **AH** algebra of real rank zero and with slow dimension growth then A is **AD**, where **AD** $=$ **A**$\langle\langle C(\mathbb{T}), \mathbb{I}_2^{\sim}, \mathbb{I}_3^{\sim}, \mathbb{I}_4^{\sim}, \ldots \rangle\rangle$

Complete invariant [Dadarlat-E] Unital **AD** algebras *of real rank zero* are classified up to isomorphism by

$$K_0(-) \longrightarrow K_0(-) \otimes \mathbb{Q} \longrightarrow K_0(-;\mathbb{Q}/\mathbb{Z}) \longrightarrow K_1(-)$$

Reduced invariant [E] Unital **AD** algebras *of real rank zero* are classified up to isomorphism by

$$K_0(-) \longrightarrow K_0(-;\mathbb{Z}/n) \longrightarrow K_1(-)$$

provided that n tor $K_1(-) = 0$.

Range [Elliott] $K_*($**AD** $\cap$ **RRZ**$)$ ranges over all countable graded dimension groups with torsion in the odd part.

This means that the unperforation condition is relaxed to unperforation in G_0 and <u>weak</u> unperforation in $G_0 \oplus G_1$:

$(x,my)\in(G_*)_+, \Longrightarrow y=y_1+y_2, my_1=0, (x,y_2)\in(G_*)_+$

Range [E-Toms] The reduced invariant ranges over all exact complexes

$$G_0 \xrightarrow{\times n} G_0 \xrightarrow{\rho} G_n \xrightarrow{\beta} G_1 \xrightarrow{\times n} G_1$$

where $nG_n = (0), n\,\mathrm{tor}\,G_1 = (0)$ and

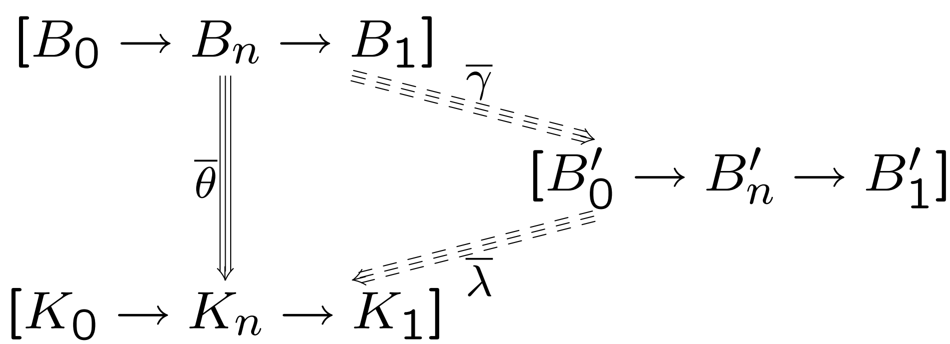
- $G_0 \oplus G_1$ is a graded dimension group with torsion

- $G_0 \oplus G_n$ is a graded ordered group

- The inherited order on $G_0 \oplus \mathrm{im}\,\rho \subseteq G_0 \oplus G_n$ equals the order induced by $\mathrm{id} \oplus \rho : G_0 \oplus G_0 \longrightarrow G_0 \oplus G_n$

- The inherited order on $G_0 \oplus \mathrm{im}\,\beta \subseteq G_0 \oplus G_1$ equals the order induced by $\mathrm{id} \oplus \beta : G_0 \oplus G_n \longrightarrow G_0 \oplus G_1$.

Proof by Shen criterion:



$\ker \overline{\gamma} = \ker \overline{\theta}$

Key to augment Elliott's proof:

[Wehrung] Suppose $a, b \in (G_0)_+$ and $a \leq nb$, G_0 an ordered group with the Riesz property.

$\exists\ b_0, \ldots, b_n \in (G_0)_+$ such that $b = \sum_{i=0}^{n} b_i$ and $a = \sum_{i=1}^{n} i b_i$.

Example by Dadarlat and Loring:

$$G_0 = \left\{(x,(y_i)) \in \mathbb{Z}[\tfrac{1}{3}] \oplus \mathbb{Z}^{\mathbb{Z}} \mid y_i - 3^{|i|}x \longrightarrow 0\right\}$$
$$G_1 = \mathbb{Z}/2$$

How many ways can we equip

$$G_0 \xrightarrow[y_i \mapsto b_i]{x \mapsto a} \left\{(a, b_i, c) \in \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^{\mathbb{Z}} \oplus \mathbb{Z}/2 \mid b_i \longrightarrow a\right\} \xrightarrow{c \mapsto z} G_1?$$

For any $(\epsilon_i) \in (\mathbb{Z}/2)^{\mathbb{Z}}$ we may take the order given by

$$((x, y_i), (a, (b_i), c)) \geq 0 \iff \begin{cases} x > 0 \text{ or } [x = 0, a = 0, c = 0] \\ y_i > 0 \text{ or } [y_i = 0, b_i + \epsilon_i c = 0] \end{cases}$$

This gives uncountably many nonisomorphic invariants!

Range [E-Toms] The complete invariant ranges over all exact complexes

$$G_0 \longrightarrow G_0 \otimes \mathbb{Q} \xrightarrow{\widetilde{\rho}} G_\infty \xrightarrow{\widetilde{\beta}} G_1$$

where G_∞ is pure torsion, $\operatorname{im} \widetilde{\beta} = \operatorname{tor} G_1$ and

- $G_0 \oplus G_1$ is a graded dimension group with torsion

- $(G_0 \otimes \mathbb{Q}) \oplus G_\infty$ is an ordered group

- The inherited order on $(G_0 \otimes \mathbb{Q}) \oplus \operatorname{im} \widetilde{\rho} \subseteq (G_0 \otimes \mathbb{Q}) \oplus G_\infty$ equals the order induced by $\operatorname{id} \oplus \widetilde{\rho}$

- The inherited order on $G_0 \oplus \operatorname{im} \widetilde{\beta} \subseteq G_0 \oplus G_1$ equals the order induced by $\operatorname{id} \oplus \widetilde{\beta}$

We are not done yet!

AH $\cap$ **RRZ** $\cap$ **SDG** $\cap \{\text{tor}\, K_1 = 0\} \subsetneq$ **AD** $\cap$ **RRZ** $\cap \{\text{tor}\, K_1 = 0\}$!

[Dadarlat-E] When A is **AH** of real rank zero we have

$$
\begin{array}{ccc}
K_0(A) \oplus K_0(A) & \xrightarrow{\ \mathrm{id} \oplus \rho\ } & K_0(A) \oplus K_0(A; \mathbb{Z}/n) \\
\Big\downarrow \mathrm{id} \oplus q & \swarrow \scriptstyle{\mathrm{id} \oplus \sigma \geq 0} & \\
K_0(A) \oplus \frac{K_0(A)}{nK_0(A) + \mathrm{Inf}\, K_0(A)}, & &
\end{array}
$$

Range unknown!

ASH $=$ **A**$\langle\langle C(X), \mathbb{I}_2^{\sim}, \mathbb{I}_3^{\sim}, \mathbb{I}_4^{\sim}, \ldots\rangle\rangle$

Complete invariant [Dadarlat-Gong] Unital **ASH** algebras of *real rank zero* and with *slow dimension growth* are classified up to isomorphism by

$$[\underline{\mathbf{K}}(-), \underline{\mathbf{K}}(-)_+, \Lambda, [1]]$$

Subrange [E-Toms] As above when tor $K_1 = 0$.

General range unknown!