Semiprojectivity and classification of (unital) graph C^*-algebras

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Program

Outline

Coworkers

Hawaii	-10	Efren Ruiz
Houston	-5	Mark Tomforde
	-5	James West
Faroe Islands	0	Gunnar Restorff
Copenhagen	+2	Sara Arklint
	+2	Rasmus Bentmann
	+2	Adam Sørensen
Trondheim	+2	Toke Carlsen
Tokyo	+9	Takeshi Katsura

Leitmotif

We are mainly going to work with non-simple C^*-algebras which are **separable**, **nuclear** and of **real rank zero**:

Definition

$\mathfrak{A}$ has **real rank zero** when the invertible elements are dense in $\mathfrak{A}_{\mathsf{sa}}$.

Permanence problems

Let an extension of C^*-algebras

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}/\mathfrak{I} \longrightarrow 0$$

be given. Suppose $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ have a certain desirable property. Can we explain when the same is true for $\mathfrak{A}$?

We shall often require the number of ideals of $\mathfrak{A}$ to be finite.

We extensively use the *six-term exact sequence*:

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{A}) & \longrightarrow & K_0(\mathfrak{A}/\mathfrak{I}) \\ \uparrow \partial_1 & & & & \downarrow \partial_0 \\ K_1(\mathfrak{A}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{A}) & \longleftarrow & K_1(\mathfrak{I}) \end{array}$$

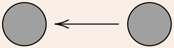
Theorem [Brown-Pedersen 91]

Suppose the real rank of $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ is zero. Then

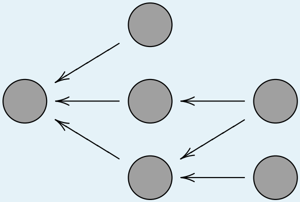
$$\mathfrak{A} \text{ is of real rank zero} \iff \partial_0 = 0$$

The simple case

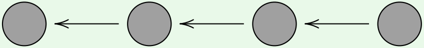
The unique ideal case



The general case



The linear case



Recurrent subtheme

The projections p,q are *Murray-von Neumann equivalent*, written $p \sim q$, when there exists v with $p = vv^*$, $q = v^*v$.

Definition

1. p is **finite** when $p' \sim p$ with $p' \leq p$ implies $p' = p$.
2. p is **properly infinite** when there exist orthogonal projections $p', p'' \leq p$ with $p \sim p' \sim p''$.

All known simple C^*-algebras of real rank zero are of one of the types

	$\mathfrak{A}$ stably finite	All projections in $\mathfrak{A} \otimes \mathbb{K}$ are finite
	$\mathfrak{A}$ purely infinite	All projections in $\mathfrak{A}$ are properly infinite

Of course this (possible) dichotomy breaks down even for C^*-algebras with a unique non-trivial ideal. There are (at least) four cases

	$\mathfrak{I}$ stably finite	$\mathfrak{A}/\mathfrak{I}$ stably finite
	$\mathfrak{I}$ purely infinite	$\mathfrak{A}/\mathfrak{I}$ purely infinite
	$\mathfrak{I}$ stably finite	$\mathfrak{A}/\mathfrak{I}$ purely infinite
	$\mathfrak{I}$ purely infinite	$\mathfrak{A}/\mathfrak{I}$ stably finite

Postulate

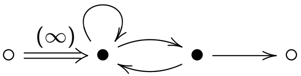
Interesting and rather unexplored phenomena occur in the two latter **mixed** cases!

Graph algebras

Graph algebras

Any countable graph $E = (E^0, E^1)$ defines a C^*-algebra $C^*(E)$ given as a universal C^*-algebra by **projections** $\{p_v : v \in E^0\}$ and **partial isometries** $\{s_e : e \in E^1\}$ subject to the *Cuntz-Krieger relations*:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every v with $0 < |\{e \mid s(e) = v\}| < \infty$.



Graph algebra need-to-know

There is a huge body of knowledge about graph algebras. Of prime importance here is

Theorem

Graph algebras with finitely many ideals have real rank zero.

Theorem

Ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \Longrightarrow r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \Longrightarrow [v \in V \text{ or } v \text{ is singular } (\circ)]$

and in many cases, all ideals arise this way.

Theorem

The dichotomy of simple graph C^*-algebras:

	E has no loops	*$C^*(E)$ is AF*
	All vertices in E can reach at least two loops	*$C^*(E)$ is purely infinite*

The unital case

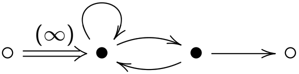
Observation

$C^*(E)$ is unital $\iff$ E_0 is finite

In this case we get a finite presentation, e.g.

$$\begin{bmatrix} 0 & \infty & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for



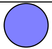
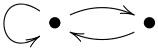
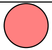
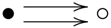
Key subclasses

Definition

A graph C^*-algebra given by a finite graph with no sinks or sources is called a **Cuntz-Krieger** algebra and denoted $\mathcal{O}_A$ with A the adjacency matrix.

Lemma

The only unital and simple graph C^*-algebras which are AF are $M_n(\mathbb{C})$.

	$\mathcal{O}_{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}$	
	$M_3(\mathbb{C})$	

Outline

Task

Find conditions on the K-theory of A and B in some class of C^*-algebra to ensure that

$$A \otimes \mathbb{K} \simeq B \otimes \mathbb{K},$$

in other words: that A and B are Morita equivalent.

This is a natural place to start since $K_*(A) = K_*(A \otimes \mathbb{K})$.

The simple case

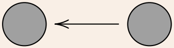
Theorem (Elliott 76, Kirchberg/Phillips 00)

$C^*(E)$ is determined up to stable isomorphism by

$$[K_0(C^*(E)), K_0(C^*(E))_+, K_1(C^*(E))]$$

in the class of simple graph C^*-algebras.

The unique ideal case



Cuntz-Krieger example

$$A_1 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 0 & 1 & 4 \\ 0 & 0 & 2 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 4 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Challenge

$\mathcal{O}_{A_i}$ has a unique ideal $\mathfrak{I}_i$ for which

$$K_*(\mathfrak{I}_1) = [\mathbb{Z} \oplus \mathbb{Z}/2] \oplus \mathbb{Z} = K_*(\mathfrak{I}_2)$$
$$K_*(\mathcal{O}_{A_1}) = [\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4] \oplus \mathbb{Z} = K_*(\mathcal{O}_{A_2})$$
$$K_*(\mathcal{O}_{A_1}/\mathfrak{I}_1) = [\mathbb{Z}/2 \oplus \mathbb{Z}/4] \oplus 0 = K_*(\mathcal{O}_{A_2}/\mathfrak{I}_2)$$

But $\mathcal{O}_{A_1} \otimes \mathbb{K} \not\simeq \mathcal{O}_{A_2} \otimes \mathbb{K}$.

The six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{I}_i) & \longrightarrow & K_0(\mathcal{O}_{A_i}) & \longrightarrow & K_0(\mathcal{O}_{A_i}/\mathfrak{I}_i) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(\mathcal{O}_{A_i}/\mathfrak{I}_i) & \longleftarrow & K_1(\mathcal{O}_{A_i}) & \longleftarrow & K_1(\mathfrak{I}_i) \end{array}$$

here becomes

$$\begin{array}{ccccc} \mathbb{Z} \oplus \mathbb{Z}/2 & \xrightarrow{\chi_i} & \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/4 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & \mathbb{Z} & \longleftarrow & \mathbb{Z} \end{array}$$

with

$$\chi_1(0,1) = (0,1,0) \qquad \chi_2(0,1) = (0,0,2)$$

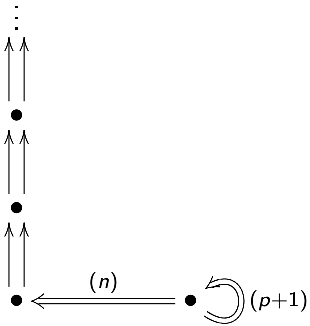
Complete classification result

Theorem (Elliott 76, Rørdam 97, E-Tomforde 10)

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(C^*(E)) & \longrightarrow & K_0(C^*(E)/\mathfrak{I}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(E)/\mathfrak{I}) & \longleftarrow & K_1(C^*(E)) & \longleftarrow & K_1(\mathfrak{I}) \end{array}$$

determines $C^*(E)$ up to stable isomorphism among all graph C^*-algebras with a unique nontrivial ideal.
Here, all K_0-groups are to be considered as ordered groups.
However, the order of $K_0(C^*(E))$ is redundant unless $C^*(E)$ is AF.

Consider the class of graphs E_p^n for p an odd prime and $n > 0$.

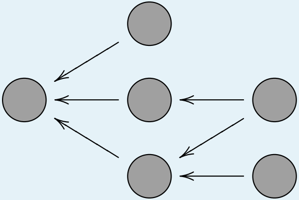


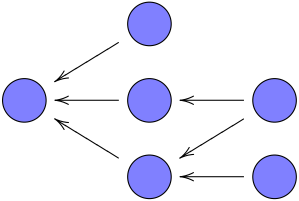
Corollary

The following are equivalent

1. $C^*(E_{p_1}^{n_1}) \otimes \mathbb{K} \simeq C^*(E_{p_2}^{n_2}) \otimes \mathbb{K}$
2. $p_1 = p_2$ and $[p_1 \mid n_1 \iff p_2 \mid n_2]$

The general case





Theorem (Restorff 04)

The collection of **all** sequences

$$\begin{array}{ccccc}
K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\
\uparrow & & & & \downarrow \\
K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I})
\end{array}$$

with $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathcal{O}_A$ determines $\mathcal{O}_A$ up to stable isomorphism among all Cuntz-Krieger algebras with finitely many ideals.

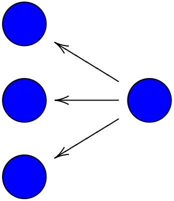
Definition

We denote this the *filtered K-theory* of $\mathfrak{A}$ and denote it $FK(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, filtered K-theory* $FK^+(\mathfrak{A})$.

Conjecture (E-Restorff-Ruiz 09)

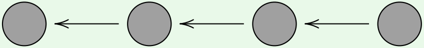
$FK^+(C^*(E))$ determines up to stable isomorphism $C^*(E)$ among all graph algebras with finitely many ideals. (At least when $FK(C^*(E))$ is finitely generated).

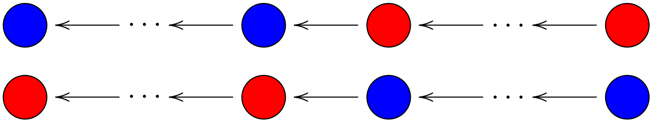
Complications



1. $FK(\mathfrak{A})$ does **not** determine $\mathfrak{A}$ up to stable isomorphism among all purely infinite C^*-algebras with this ideal lattice [Meyer-Nest 08].
2. $FK(\mathfrak{A})$ **does** determine $\mathfrak{A}$ up to stable isomorphism among all purely infinite C^*-algebras **of real rank zero** and this ideal lattice [Arklint-Restorff-Ruiz 10]
3. $FK(\mathfrak{A})$ **does not** determine $\mathfrak{A}$ up to stable isomorphism among all purely infinite C^*-algebras of real rank zero with a finite ideal lattice [Arklint-Bentmann 10]

The linear case





Theorem (Meyer-Nest 09)

$FK(\mathfrak{A})$ determines $\mathfrak{A}$ up to stable isomorphism among all purely infinite C^*-algebras with linear ideal lattice.

Theorem (E-Restorff-Ruiz 10)

$FK^+(C^*(E))$ determines $C^*(E)$ up to stable isomorphism among graph C^*-algebras with linear ideal lattice when $\mathfrak{I}$ satisfies either

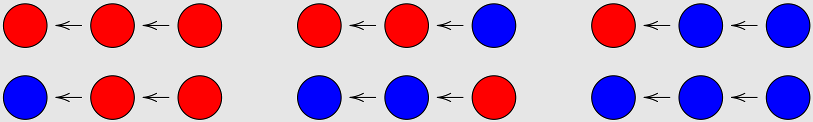
- $\mathfrak{I}$ is AF and $C^*(E)/\mathfrak{I}$ is purely infinite
- $\mathfrak{I}$ is purely infinite and $C^*(E)/\mathfrak{I}$ is AF

Theorem (E-Restorff-Ruiz 11)

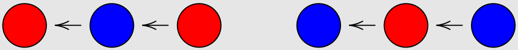
When $FK(C^*(E))$ is finitely generated, $FK^+(C^*(E))$ determines $C^*(E)$ up to stable isomorphism among all graph C^*-algebras in even more cases of linear ideal lattices.

Status quo

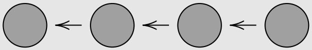
Stably classified



Stably classified when $FK(C^*(E))$ is finitely generated



Stably classified when $C^*(E)$ is unital



Outline

Definition

$\mathfrak{A}$ is **semiprojective** when

$$
\begin{array}{ccccccccc}
 & & & & & & & & \mathfrak{A} \\
 & & & & & \swarrow\!\!{\scriptstyle -\,-\,-\,-} & & & \downarrow \\
\mathfrak{B}/\mathfrak{I}_1 & \longrightarrow & \mathfrak{B}/\mathfrak{I}_2 & \longrightarrow \cdots \longrightarrow & \mathfrak{B}/\mathfrak{I}_{n_0} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{B}/\overline{\bigcup \mathfrak{I}_n}
\end{array}
$$

This notion due to Blackadar is a key concept in all C^*-algebra theory, including classification.

Task

Determine conditions on $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ such that

$$\mathfrak{I}, \mathfrak{A}/\mathfrak{I} \text{ semiprojective } \Longrightarrow \mathfrak{A} \text{ semiprojective}$$

The commutative case

Theorem (Sørensen-Thiel 10)

$C_0(X)$ *is semiprojective* $\iff \dim(X) \leq 1$ *and* X *is an ANR*

Note that for

$$0 \longrightarrow C_0(Y) \longrightarrow C(X) \longrightarrow C(X/Y) \longrightarrow 0$$

we get

$$C_0(Y), C(X/Y) \text{ semiprojective } \not\Longrightarrow C(X) \text{ semiprojective}$$

Theorem (Sørensen-Thiel 10)

When X/Y *is a finite set, we have*

$$C_0(Y) \textit{ semiprojective } \iff C(X) \textit{ semiprojective}$$

Long-standing open question

Question [Blackadar-Loring c. 94]

Suppose an extension

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{F} \longrightarrow 0$$

is given with $\dim \mathfrak{F} < \infty$. Will then

$$\mathfrak{I} \text{ semiprojective } \iff \mathfrak{A} \text{ semiprojective?}$$

Partial results

1. $\mathfrak{I}$ semiprojective $\iff \mathfrak{I}^{\sim}$ semiprojective [Blackadar 85]
2. $\mathfrak{A}$ semiprojective $\Longrightarrow \mathfrak{I}$ semiprojective [Enders 10]

The simple case

Theorem (Spielberg 09)

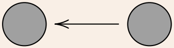
The following are equivalent for a simple graph C^*-algebra $C^*(E)$

- $C^*(E)$ is semiprojective
- $C^*(E) = M_n(\mathbb{C})$ or $C^*(E)$ is purely infinite with $K_*(C^*(E))$ finitely generated

Corollary (Szymanski 00)

Any simple and unital graph C^*-algebra is semiprojective.

The unique ideal case



Theorem (E-Katsura)

The only unital graph C^*-algebras with precisely one ideal that are **not** *semiprojective are*

$$\mathbb{K}^\sim, M_2(\mathbb{K}^\sim), M_3(\mathbb{K}^\sim), \dots$$

Corollary

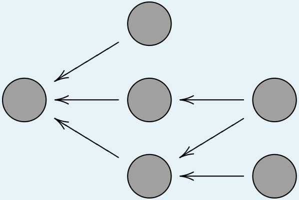
When $C^*(F)$ is a unital graph algebra and $\mathfrak{I}$ is simple with

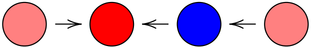
$$0 \longrightarrow \mathfrak{I} \longrightarrow C^*(E) \longrightarrow M_n(\mathbb{C}) \longrightarrow 0.$$

we have

$$\mathfrak{I} \text{ semiprojective } \iff C^*(E) \text{ semiprojective}$$

The general case





Example (E-Katsura 11)

There exists a unital graph C^*-algebra $C^*(E)$ with

$$0 \longrightarrow \mathfrak{I} \longrightarrow C^*(E) \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

such that $\mathfrak{I}$ is semiprojective, but $C^*(E)$ is not.

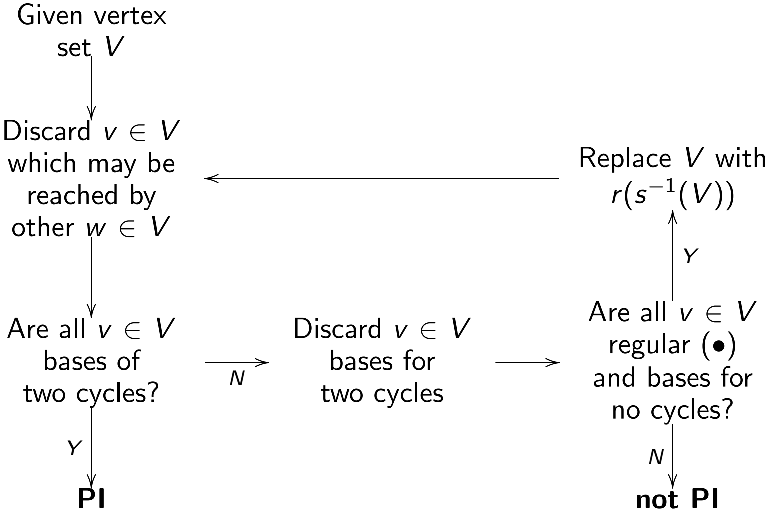
Deciding proper infiniteness in $C^*(E)$

For a finite set of vertices V, consider

$$p_V = \sum_{v \in V} p_v$$

Theorem (E-Katsura 10)

There is an algorithm for deciding when p_V is properly infinite.


Given vertex set V
Discard $v \in V$ which may be reached by other $w \in V$
Replace V with $r(s^{-1}(V))$
Y
Are all $v \in V$ bases of two cycles?
N
Discard $v \in V$ bases for two cycles
Are all $v \in V$ regular ($\bullet$) and bases for no cycles?
Y
N
PI
not PI

Definition

With v a vertex, define

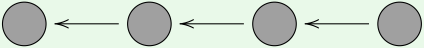
$$\Omega_v = \{w \in E^0 \mid v \xrightarrow{(\infty)} w\}$$

Theorem (E-Katsura 10)

Let $C^*(E)$ be a unital graph algebra. The following are equivalent

- *$C^*(E)$ is semiprojective*
- *$C^*(E)$ is weakly semiprojective*
- *For each vertex v of E, p_{Ω_v} is properly infinite*

The linear case



Blackadar's conjecture

Blackadar 99

If $\mathfrak{I}$ in

$$0 \longrightarrow \mathfrak{I} \longrightarrow \mathfrak{A} \overset{\leftarrow\cdots}{\longrightarrow} \mathbb{C} \longrightarrow 0$$

is semiprojective, then so is $\mathfrak{A}$.

Theorem (E-Katsura 11)

A corner of the graph C^*-algebra just considered provides a counterexample. It is mixing of type

Outline

The simple case

Scale

Definition

When $\mathfrak{A}$ is not unital, the *scale* Σ of $K_0(\mathfrak{A})$ is given by

$$\{[p] \mid p \in \mathfrak{A}\}$$

When $\mathfrak{A}$ is unital, the scale is just the singleton

$$\{[1]\}$$

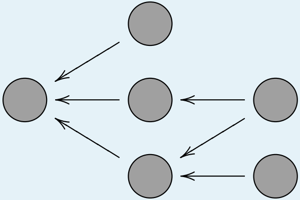
Theorem (Elliott 76, Kirchberg/Phillips 00)

The scaled ordered group

$$[K_0(C^*(E)), K_0(C^*(E))_{+,\Sigma}, K_1(C^*(E))]$$

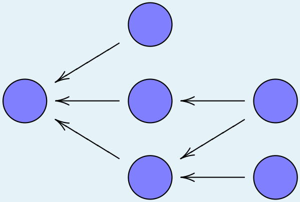
determines $C^*(E)$ up to isomorphism among all simple graph C^*-algebras.

The general case



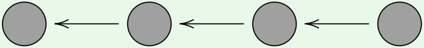
Example

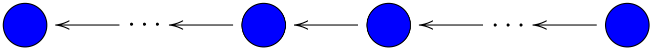
We do not have exact classification for Cuntz-Krieger algebras of the form



Some other ideal lattices have exact classification by Arklint-Restorff-Ruiz 10. If we can prove it for Cuntz-Krieger algebras we can also prove it in the general purely infinite graph C^*-algebra case.

The linear case

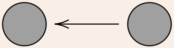




Theorem

$FK^{+,\Sigma}(C^*(E))$ determines $C^*(E)$ up to isomorphism among all unital purely infinite graph C^*-algebras with a linear ideal lattice.

The unique ideal case

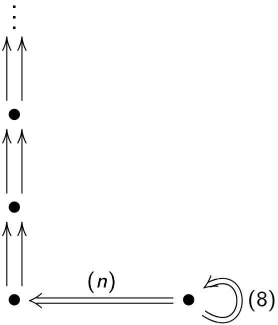


Theorem (E-Restorff-Ruiz 11)

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \longrightarrow & K_0(C^*(E)) & \longrightarrow & K_0(C^*(E)/\mathfrak{I}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(E)/\mathfrak{I}) & \longleftarrow & K_1(C^*(E)) & \longleftarrow & K_1(\mathfrak{I}) \end{array}$$

determines $C^*(E)$ up to isomorphism among all graph C^*-algebras with a unique nontrivial ideal when all K_0-groups are considered as ordered groups, and $K_0(\mathfrak{A})$ and $K_0(\mathfrak{A}/\mathfrak{I})$ are further considered as scaled groups.

However, the order and scale of $K_0(C^*(E))$ is redundant unless $C^*(E)$ is AF or unital.



Stable classification:

n	1	2	3	4	5	6	7

Exact classification:

n	1	2	4	3	6	5	7