Almost anticommuting matrices

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Questions (imprecise)

If two self-adjoint contractive matrices almost commute,

$$\|AB - BA\| < \delta$$

is the pair then close to a pair of (exactly) commuting self-adjoint contractive matrices,

$$\left\|A - A'\right\| < \varepsilon, \qquad \left\|B - B'\right\| < \varepsilon, \qquad A'B' = B'A'?$$

If two self-adjoint contractive matrices almost anticommute,

$$\|AB + BA\| < \delta$$

is the pair then close to a pair of (exactly) anticommuting self-adjoint contractive matrices,

$$\left\|A - A'\right\| < \varepsilon, \qquad \left\|B - B'\right\| < \varepsilon, \qquad A'B' = -B'A'?$$

POINTS: Cannot get a satisfactory answer to this using linear algebra alone. CAN get a good answer by involving C*-algebras. And up to a (unfortunately, rather hard) technical result in C*-algebra theory, the two questions are accesible in the same way. Focus now on th efirst case to explain history and role of C*-algebras.

Definitions

Matrices are elements of $\mathbf{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Distance is measured in the operator norm $\|\cdot\|$, so a matrix A is *contractive* when $\|A\| \leq 1$.

(A, B) is a δ-*commuting* pair if

$$\|AB - BA\| < \delta$$

(A, B) is ε-*close to a commuting pair* if

$$\left\|A - A'\right\| < \varepsilon \qquad \left\|B - B'\right\| < \varepsilon$$

for some (A', B') with

$$A'B' = B'A'$$

Norm choice is important, both for approach and for result. Approximations are too delicate to pass between equivalent norms.

Questions (precise)

$\boxed{\mathbf{Q1}}$ $\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists \delta > 0$: If (A, B) is a δ-commuting pair in $\mathbf{M}_n$, then (A, B) is ε-close to a commuting pair?

$\boxed{\mathbf{Q2}}$ How to choose $\delta = \delta(\varepsilon, n)$?

$\boxed{\mathbf{Q3}}$ $\forall \varepsilon > 0 \exists \delta > 0 \forall n \in \mathbb{N}$: If (A, B) is a δ-commuting pair in $\mathbf{M}_n$, then (A, B) is ε-close to a commuting pair?

$\boxed{\mathbf{Q4}}$ How to choose $\delta = \delta(\varepsilon)$?

Q3=¿Q1!. Q2 without n-dependence answers all.

Consequences, 1

An affirmative answer to $\mathbf{Q1}$ proves:

Simultaneous diagonalization

$\forall \varepsilon > 0 \forall n \in \mathbb{N} \exists \delta > 0$: If (A, B) is a δ-commuting pair in $\mathbf{M}_n$, then there exists a unitary U with

$$(UAU^*, UBU^*)$$

ε-close to a pair of diagonal matrices.

An affirmative answer to $\mathbf{Q3}$ proves:

$\forall \varepsilon > 0 \exists \delta > 0$: If (A, B) is a δ-commuting pair in $\mathcal{K}(H)$, then (A, B) is ε-close to a commuting pair.

Answers

Question first raised by Rosenthal in 69.

	Q	**A**	**W&W**
1	$\forall\varepsilon\forall n\exists\delta$?	Yes	Halmos (74)
2	$\delta(\varepsilon, n)$?	$\dfrac{2\varepsilon^2}{n-1}$	Pearcy & Shields (79)
3	$\forall\varepsilon\exists\delta\forall n$?	Yes	Lin (97) Friis & Rørdam (96)
4	$\delta(\varepsilon)$?		

- H: $(\mathbf{M}_n)_1$ is compact.

- P,S: Diagonalize A and groups its eigenvalues.

- L,F,R: C^*-algebras.

C^*-*algebras*

- $(\mathbf{M}_n(\mathbb{C}), \|\cdot\|)$ is a C^*-algebra.

- $(C(X), \|\cdot\|_u)$ is a C^*-algebra when X is a compact Hausdorff space.

- When $(A_i)_{\mathbb{N}}$ are C^*-algebras, so is

$$\bigoplus_1^\infty A_i = \{(a_i) \mid \|a_i\| \to 0\}$$

- When $(A_i)_{\mathbb{N}}$ are C^*-algebras, so is

$$\prod_1^\infty A_i = \{(a_i) \mid \|a_i\| \text{ bounded}\}$$

- When $(A_i)_{\mathbb{N}}$ are C^*-algebras, so is

$$\prod_1^\infty A_i \Big/ \bigoplus_1^\infty A_i$$

If you do not remember the axioms, don't despair. Need to introduce these, important is that they all lie in same category, complete with morphisms.

Universal C^*-algebras, 1

There exists a *universal* C^*-algebra D_1 given by the relations

$$a = a^* \qquad b = b^* \qquad ab = ba \qquad (*)$$
$$\|a\| \leq 1 \qquad \|b\| \leq 1.$$

This means that there exists $a_0, b_0 \in D_1$ so that:

1° Whenever a, b in a C^*-algebra C satisfy $(*)$.

2° There is a unique morphism $\varphi : D_1 \to C$

3° With $\varphi(a_0) = a, \varphi(b_0) = b$

Lemma $D_1 = C(\mathbb{I}^2)$, where $\mathbb{I}^2$ is the compact Hausdorff space $[-1, 1]^2$.

Overkill here. Want to emphasize this step because of what follows in the anticommuting case. Joint spectrum gives lemma.

Approximation follows from lifting

$\boxed{\textbf{Observation}}$ Knowing

$$\begin{tikzcd} & \prod_{1}^{\infty} \mathbf{M}_{n_i} \arrow[d] \\ C(\mathbb{I}^2) \arrow[r] \arrow[ur, dashed, bend left] & \prod_{1}^{\infty} \mathbf{M}_{n_i} \Big/ \bigoplus_{1}^{\infty} \mathbf{M}_{n_i} \end{tikzcd}$$

provides an affirmative answer to **Q3** .

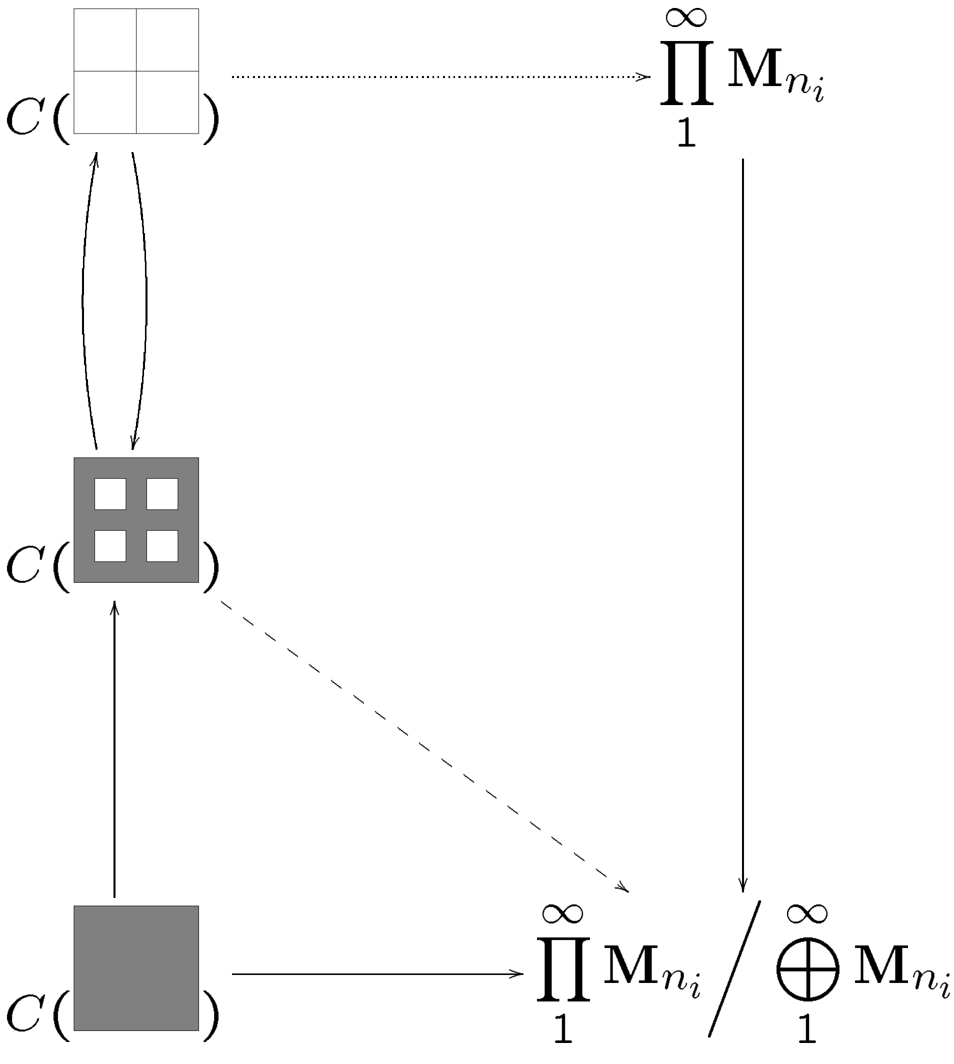
$\boxed{\textbf{Definition}}$ A C^*-algebra B is *weakly semiprojective* if, for all A_i,

$$\begin{tikzcd} & \prod_{1}^{\infty} A_i \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, bend left] & \prod_{1}^{\infty} A_i \Big/ \bigoplus_{1}^{\infty} A_i \end{tikzcd}$$

$\boxed{\textbf{Proposition}}$ [Loring] $C(X)$ is weakly semiprojective iff $\dim(X) \leq 1$.

Do sketch of observation. Note all hope is not lost becasue A_i are general, need for rather special C*-algebras.

Friis-Rørdam technique, 1



Inside the quotient we have commuting pair. Point is that we can arrange for this pair to (in a weak sense) avoid certain points of the spectrum. Result is also two-dimensional, but has holes. Now retract, this is good enough.

Q3 *for anticommuting matrices*

$\boxed{\textbf{Theorem}}$ [E-Loring-Pedersen, 98] There exists, for a given $\varepsilon > 0$, a $\delta > 0$ such that if (A, B) is a δ-anticommuting pair of contractive self-adjoint matrices, then (A, B) is ε-close to an anticommuting pair.

Consequences, 2

$\forall \varepsilon > 0 \exists \delta > 0 \forall n \in \mathbb{N}$: If (A, B) δ-anticommute in $\mathbf{M}_n$, then there exists a unitary U with (UAU^*, UBU^*) ε-close to a pair of matrices of the form

$$\begin{bmatrix} * & * & & & & & \\ * & * & & & & & \\ & & * & * & & & \\ & & * & * & & & \\ & & & & \ddots & & \\ & & & & & * & \\ & & & & & & * \end{bmatrix}$$

$\forall \varepsilon > 0 \exists \delta > 0$: If (A, B) is a δ-anticommuting pair in $\mathcal{K}(H)$, then (A, B) is ε-close to an anticommuting pair.

Universal C^*-algebras, 2

There exists a universal C^*-algebra D_{-1} given by the relations

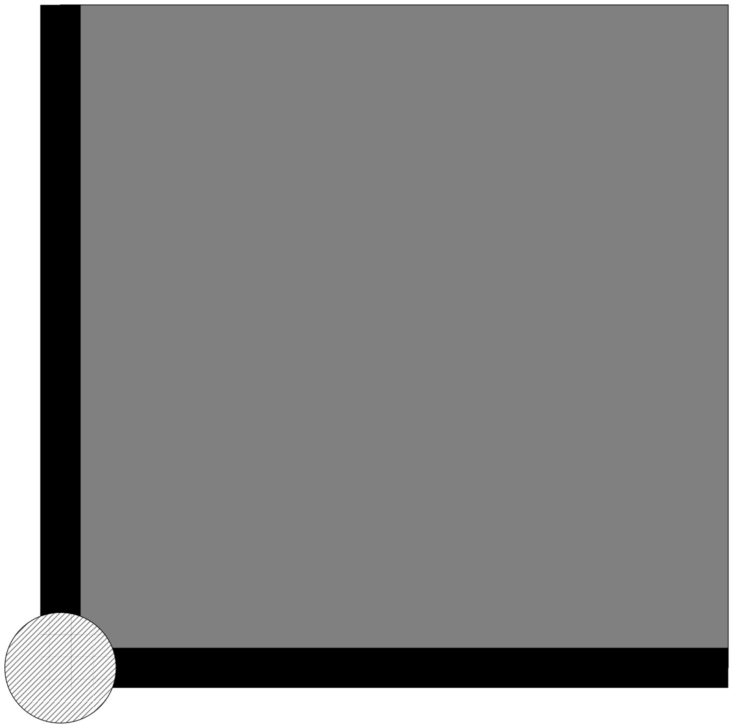
$$a = a^* \qquad b = b^* \qquad ab = -ba$$
$$\|a\| \leq 1 \qquad \|b\| \leq 1.$$

$\boxed{\textbf{Observation}}$ D_{-1} is not commutative!

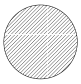
$\boxed{\textbf{Proposition}}$ D_{-1} is isomorphic to

$$\left\{ f : [0,1]^2 \to \mathbf{M}_2 \;\middle|\; \begin{array}{ll} f(0,0) & \in \mathbb{C} \subseteq \mathbf{M}_2 \\ f(0,y) & \in \mathbb{C} \oplus \mathbb{C} \subseteq \mathbf{M}_2 \\ f(x,0) & \in \mathbb{C} \oplus \mathbb{C} \subseteq \mathbf{M}_2 \end{array} \right\}$$

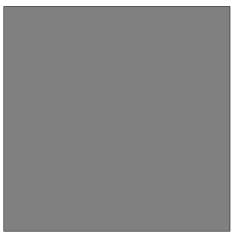
A picture of D_{-1}



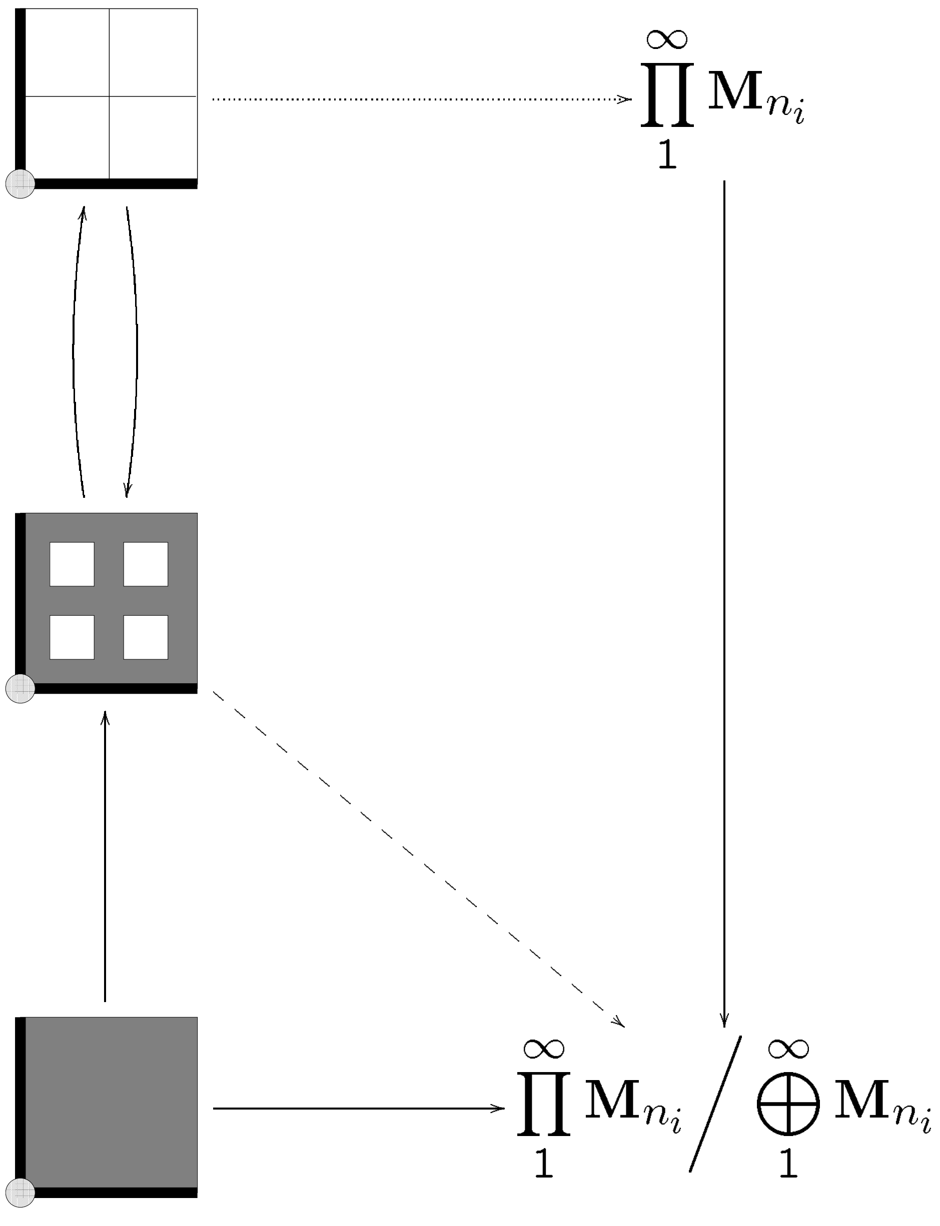
Legend

: $\mathbb{C}$

: $\mathbb{C} \oplus \mathbb{C}$

: $\mathbf{M}_2$

Friis-Rørdam technique, 2



(Weak) semiprojectivity for non-commutative C^*-algebras

$\mathbf{M}_n$ is weakly semiprojective.

$C(X, \mathbf{M}_n)$ is weakly semiprojective for $X = [0,1]$ or $X = \mathbb{T}$.

$\mathbb{I}_n^{\sim}$ given by

$$\{f : [0,1] \to \mathbf{M}_n \mid f(0), f(1) \in \mathbb{C} \subseteq \mathbf{M}_n\}$$

is weakly semiprojective.

$\boxed{\textbf{Method of proof}}$ Characterize the C^*-algebras as universal algebras. Work with relations.

$\boxed{\textbf{Definition}}$ A *finite noncommutative* CW *complex* of dimension one is a pullback over

$$
\begin{tikzcd}
 & A \arrow[d, "\gamma"] \\
C([0,1],F) \arrow[r, "\partial"'] & F \oplus F
\end{tikzcd}
$$

where $\dim(A), \dim(F) < \infty$.

Such pullbacks can be written

$$\{(f,a) \in C([0,1],F) \oplus A \mid (f(0), f(1)) = \gamma(a)\}$$

$\boxed{\textbf{Theorem}}$ Any finite noncommutative CW complex of dimension is (weakly) semiprojective.