1.1 Motivation and notation

1.1. Introduction.

In the study of ordinary differential equations one can get very far by using just the classical concept of differentiability, working with spaces of continuously differentiable functions on an interval $I \subset \mathbb{R}$:

$$C^m(I) = \{ u: I \to \mathbb{C} \mid \frac{d^j}{dx^j} u \text{ exists and is continuous on } I \text{ for } 0 \leq j \leq m \}.$$  \hfill (1.1)

The need for more general concepts comes up for example in the study of eigenvalue problems for second order operators on an interval $[a, b]$ with boundary conditions at the endpoints $a, b$, by Hilbert space methods. But here it usually suffices to extend the notions to absolutely continuous functions, i.e., functions $u(x)$ of the form

$$u(x) = \int_{x_0}^x v(y) dy + c, \quad v \in L^1_{\text{loc}}(I).$$  \hfill (1.2)

Here $v$ is regarded as the derivative $\frac{d}{dx} u$ of $u$, and the fundamental formula

$$u(x) = u(x_0) + \int_{x_0}^x \frac{d}{dy} u(y) dy$$  \hfill (1.3)

still holds.

But for partial differential equations one finds when using methods from functional analysis that the spaces $C^m$ are inadequate, and there is no good concept of absolute continuity in the case of functions of several real variables. One can get some ways by using the concept of weak derivatives: When $u$ and $v$ are in $L^1_{\text{loc}}(\Omega)$, $\Omega$ an open subset of $\mathbb{R}^n$, we say that $v = \frac{\partial}{\partial x_j} u$ in the weak sense, when

$$- \int_{\Omega} u \frac{\partial}{\partial x_j} \varphi \, dx = \int_{\Omega} \varphi \, v \, dx, \quad \text{for all } \varphi \in C^\infty_0(\Omega);$$  \hfill (1.4)

here $C^\infty_0(\Omega)$ denotes the space of $C^\infty$ functions on $\Omega$ with compact support in $\Omega$. (The support $\text{supp } f$ of a function $f$ is the complement of the largest
1.2

open set where the function is zero.) This criterion is modeled after the fact that the formula (1.4) holds when \( u \in C^1(\Omega) \), with \( v = \frac{\partial}{\partial x_j} u \).

Sometimes even the concept of weak derivatives is not sufficient, and the need arises to define derivatives that are not \( L_{1,\text{loc}} \)-functions but are more general objects. Some measures and derivatives of measures will enter. For example, there is the Dirac measure \( \delta_0 \) that assigns 1 to every Lebesgue measurable set in \( \mathbb{R}^n \) containing \( \{0\} \), and 0 to any Lebesgue measurable set not containing \( \{0\} \). For \( n = 1 \), \( \delta_0 \) is the derivative of the Heaviside function defined in (1.8) below. In the book of Laurent Schwartz [S 1961] there is also a description of the derivative of \( \delta_0 \) (on \( \mathbb{R} \)) — which is not even a measure — as a “dipole”, with some kind of physical explanation.

For the purpose of setting up the rules for a general theory of differentiation where classical differentiability fails, Schwartz brought forward around 1950 the concept of distributions: a class of objects containing the locally integrable functions and allowing differentiations of any order.

This book gives an introduction to distribution theory, based on the work of Schwartz and of many other people. Our aim is also to show how the theory is combined with the study of operators in Hilbert space by methods of functional analysis, with applications to ordinary and partial differential equations. In some chapters of a more advanced character, we show how the distribution theory is used to define pseudodifferential operators and how they are applied in the discussion of solvability of PDE, with or without boundary conditions. A bibliography of relevant books and papers is collected at the end.

**Plan.**

Part I gives an introduction to distributions.

In the rest of Chapter 1 we begin the discussion of taking derivatives in the distribution sense, motivating the study of function spaces in the following chapter. Moreover, we recall some basic rules of calculus and set up the notation.

Chapter 2 studies the spaces of \( C^\infty \) functions (and \( C^k \) functions) needed in the theory, and their relations to \( L_p \) spaces.

The relevant topological considerations are collected in Appendix A.

In Chapter 3 we introduce distributions in full generality and show the most prominent rules of calculus for them.

Part II connects the distribution concept with differential equations and Fourier transformation.

Chapter 4 is aimed at linking distribution theory to the treatment of partial differential equations (PDE) by Hilbert space methods. Here we
introduce Sobolev spaces and realizations of differential operators, both in the (relatively simple) one-dimensional case and in $n$-space, and study some applications.

Here we use some of the basic results on unbounded operators in Hilbert space that are collected in Chapter 11.

In Chapter 5, we study the Fourier transformation in the framework of temperate distributions.

Chapter 6 gives a further development of Sobolev spaces as well as applications to PDE by use of Fourier theory, and shows the fundamental result on the structure of distributions.

Part III contains more advanced material, primarily on pseudodifferential operators.

Chapter 7 gives the basic ingredients of the local calculus of pseudodifferential operators.

Chapter 8 shows how to define ps.d.o.s on manifolds, and how they in the elliptic case define Fredholm operators, with solvability properties modulo finite-dimensional spaces. (An introduction to Fredholm operators is included.)

Chapter 9 takes up the study of boundary value problems by use of Fourier transformation. The main effort is spent on an important constant-coefficient case which, as an example, shows how Sobolev spaces of noninteger and negative order can enter. Also, a connection is made to the abstract theory of Chapter 12. This chapter can be read directly after Parts I and II.

In Chapter 10 we present the basic ingredients in a pseudodifferential theory of boundary value problems; this builds on the methods of Chapters 7 and 8.

Part IV is a supplement, giving some “abstract” formulations in functional analysis.

Chapter 11, based on the knowledge of bounded linear operators in Hilbert spaces, shows some basic results for unbounded operators that are needed in the study of PDE.

Chapter 12 gives a thorough presentation of certain families of extensions of closed operators, of interest for the study of boundary value problems for elliptic PDE and their positivity properties.

Chapter 13 establishes some basic results on semigroups of operators, relevant for parabolic PDE (problems with a time-parameter), and appealing to positivity and variationality properties discussed in earlier chapters.
Appendix A gives some elements of the theory of topological vector spaces, that can be invoked when one wants the correct topological formulation of the properties of distributions.

1.2 On the definition of distributions.

The definition of a weak derivative $\partial_j u$ was mentioned in (1.4) above. Here both $u$ and its weak derivative $v$ are in $L_{1,\text{loc}}(\Omega)$. Observe that the right hand side is a linear functional on $C^\infty_0(\Omega)$, i.e., a linear mapping $\Lambda_v$ of $C^\infty_0(\Omega)$ into $\mathbb{C}$, here defined by

$$\Lambda_v : \varphi \mapsto \Lambda_v(\varphi) = \int_{\Omega} v\varphi \, dx.$$  \hfill (1.5)

The idea of Distribution Theory is to allow much more general functionals than this. In fact, when $\Lambda$ is any linear functional on $C^\infty_0(\Omega)$ such that

$$-\int_{\Omega} u\partial_j \varphi \, dx = \Lambda(\varphi) \text{ for all } \varphi \in C^\infty_0(\Omega);$$  \hfill (1.6)

we shall say that

$$\partial_j u = \Lambda \text{ in the distribution sense,}$$  \hfill (1.7)

even if there is no function $v \in L_{1,\text{loc}}(\Omega)$ such that $\Lambda$ can be defined from it as in (1.5).

**Example 1.1.** Here is the most famous example in the theory: Let $\Omega = \mathbb{R}$ and consider the Heaviside function $H(x)$; it is defined by

$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$  \hfill (1.8)

It is in $L_{1,\text{loc}}(\mathbb{R})$. But there is no function $v \in L_{1,\text{loc}}(\mathbb{R})$ such that (1.4) holds with $u = H$:

$$-\int_{\mathbb{R}} H \frac{d}{dx} \varphi \, dx = \int v\varphi \, dx, \text{ for all } \varphi \in C^\infty_0(\mathbb{R}).$$  \hfill (1.9)

For, assume that $v$ were such a function, and let $\varphi \in C^\infty_0(\mathbb{R})$ with $\varphi(0) = 1$ and set $\varphi_N(x) = \varphi(Nx)$. Note that $\max |\varphi(x)| = \max |\varphi_N(x)|$ for all $N$, and that when $\varphi$ is supported in $[-R, R]$, $\varphi_N$ is supported in $[-R/N, R/N]$. Thus by the theorem of Lebesgue,

$$\int_{\mathbb{R}} v\varphi_N \, dx \to 0 \text{ for } N \to \infty,$$  \hfill (1.10)
but on the other hand,

\[- \int_{\mathbb{R}} H \frac{d}{dx} \varphi_N \, dx = - \int_{0}^{\infty} N \varphi'(Nx) \, dx = - \int_{0}^{\infty} \varphi'(y) \, dy = \varphi(0) = 1.\]  

(1.11)

So (1.9) cannot hold for this sequence of functions \( \varphi_N \), and we conclude that a locally integrable function \( v \) for which (1.9) holds for all \( \varphi \in C^\infty_0(\mathbb{R}) \) cannot exist.

A linear functional that does match \( H \) in a formula (1.6) is the following one:

\[ \Lambda : \varphi \to \varphi(0) \]  

(1.12)

(as seen by a calculation as in (1.11)). This is the famous delta-distribution, usually denoted \( \delta_0 \). (It identifies with the delta-measure mentioned earlier.)

There are some technical things that have to be cleared up before we can define distributions in a proper way.

For one thing, we have to look more carefully at the elements of \( C^\infty_0(\Omega) \). We must demonstrate that such functions really do exist, and we need to show that there are elements with convenient properties (such as having the support in a prescribed set and being 1 on a smaller prescribed set).

Moreover, we have to describe what is meant by convergence in \( C^\infty_0(\Omega) \), and provide it with a topology. There are also some other spaces of \( C^\infty \) or \( C^k \) functions with suitable support or integrability properties that we need to introduce.

These preparatory steps will take some time, before we begin to introduce distributions in full generality. (The theories that go into giving \( C^\infty_0(\Omega) \) a good topology are quite advanced, and will partly be relegated to Appendix A. In fact, the urge to do this in all details has been something of an obstacle to making the tool of distributions available to everybody working with PDE — so we shall here take the point of view of giving full details of how one operates with distributions, but tone down the topological discussion to some statements one can use without necessarily checking all proofs.)

1.3. Some notation and prerequisites.

We denote by \( \mathbb{Z} \) the integers, by \( \mathbb{N} \) the positive integers and by \( \mathbb{N}_0 \) the nonnegative integers. \( \mathbb{R} \) denotes the real numbers, \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) the positive, resp. negative real numbers. \( \mathbb{R}^n \) is the \( n \)-dimensional real Euclidean space, with points \( x = (x_1, \ldots, x_n) \) and distance \( \text{dist}(x, y) = |x - y| \), where \( |x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} \). \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_- \) denote the subsets, respectively,

\[ \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}, \]  

(1.13)
whose boundary \( \{ x \in \mathbb{R}^n \mid x_n = 0 \} \) is identified with \( \mathbb{R}^{n-1} \). The points in \( \mathbb{R}^{n-1} \) are then often denoted \( x' \),

\[
    x' = (x_1, \ldots, x_{n-1}),
\]

so that \( x = (x', x_n) \).

The space of complex numbers is denoted \( \mathbb{C} \); \( \mathbb{C}_\pm \) denote the complex numbers with positive resp. negative imaginary part. \( \mathbb{C}^n \) denotes the \( n \)-dimensional complex Euclidean space. The functions we consider are usually functions on (subsets of) \( \mathbb{R}^n \) taking values in \( \mathbb{C} \) (vector valued functions, valued in \( \mathbb{C}^N \), can also occur, or we can consider real functions).

Set inclusions are denoted by \( \subset \) (whether or not the sets are equal).

Differentiation of functions on \( \mathbb{R} \) is indicated by \( \frac{d}{dx} \), \( \partial_x \) or \( \partial \). Moreover, we write \( \frac{1}{i} \frac{d}{dx} = D_x \) or \( D \) (here \( i \) is the imaginary unit \( i = \sqrt{-1} \)); the factor \( \frac{1}{i} \) is included for convenience in the use of the Fourier transformation later on. Partial differentiation of functions on \( \mathbb{R}^n \) is indicated by

\[
    \frac{\partial}{\partial x_j} = \partial_{x_j} \text{ or } \partial_j ; \quad \frac{1}{i} \frac{\partial}{\partial x_j} = D_{x_j} \text{ or } D_j .
\]

In more complicated expressions we use multi-index notation: When \( \alpha \in \mathbb{N}_0^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), then

\[
    \partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} , \quad D^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} = (-i)^{|\alpha|} \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} , \tag{1.15}
\]

here \( |\alpha| = \alpha_1 + \cdots + \alpha_n \). The notation is used for instance for functions having continuous partial derivatives up to order \( |\alpha| \), such that differentiations in different directions (up to that order) are interchangeable. Using the conventions

\[
    \alpha \leq \beta \text{ means } \alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n , \quad \alpha! = \alpha_1! \ldots \alpha_n! , \\
    \alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n) , \tag{1.16}
\]

we have for \( u \) and \( v \) with continuous derivatives up to order \( N \) the Leibniz formula

\[
    \partial^\alpha (uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \partial^\beta u \partial^{\alpha - \beta} v , \quad \text{for } |\alpha| \leq N , \tag{1.17}
\]

\[
    D^\alpha (uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta u D^{\alpha - \beta} v , \quad \text{for } |\alpha| \leq N ,
\]
and the *Taylor formula*

\[ u(x+y) = \sum_{|\alpha|<N} \frac{y^\alpha}{\alpha!} \partial^\alpha u(x) + \sum_{|\alpha|=N} \frac{N y^\alpha}{\alpha!} \int_0^1 (1-\theta)^{N-1} \partial^\alpha u(x+\theta y) d\theta \quad (1.18) \]

(this is an exact version from which the other well-known formulations can be deducted).

When \( x \in \mathbb{R}^n \) or \( \mathbb{C}^n \), we write

\[ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} , \quad x \cdot y = x_1 y_1 + \cdots + x_n y_n , \quad |x| = (x \cdot \overline{x})^{\frac{1}{2}} . \]

The norm \(|x|\) makes \( \mathbb{R}^n \) and \( \mathbb{C}^n \) Hilbert spaces over \( \mathbb{R} \) resp. \( \mathbb{C} \), with scalar product \( x \cdot \overline{y} \).

We also define

\[ \langle x \rangle = \sqrt{1 + |x|^2} , \text{ which satisfies, for } m \in \mathbb{N} : \]

\[ \sum_{|\alpha| \leq m} x^{2\alpha} \leq (1 + |x|^2)^m = \sum_{|\alpha| \leq m} C_{m,\alpha} x^{2\alpha} ; \]  \hspace{1cm} (1.19)

here \( C_{m,\alpha} = \frac{m!}{\alpha!(m-|\alpha|)!} \), it is integer \( \geq 1 \).

When \( X \) and \( Y \) are topological spaces, \( X \times Y \) denotes the product space, consisting of pairs \( \{x, y\} \) where \( x \in X \) and \( y \in Y \), provided with the product topology (having as a subbasis the sets \( U \times V \) where \( U \) resp. \( V \) run through a subbasis of the topology of \( X \) resp. \( Y \)). When \( X \) and \( Y \) are vector spaces, \( X \times Y \) is a vector space in the obvious way. If \( X \) and \( Y \) are normed spaces, one can provide \( X \times Y \) by the norm

\[ \|\{x, y\}\|_{X \times Y} = \|x\|_X + \|y\|_Y , \]  \hspace{1cm} (1.20)

making \( X \times Y \) a normed space. When \( X \) and \( Y \) are Hilbert spaces, it is more convenient to use the equivalent norm

\[ \|\{x, y\}\|_{X \oplus Y} = (\|x\|^2_X + \|y\|^2_Y)^{\frac{1}{2}} , \]  \hspace{1cm} (1.21)

associated with the scalar product

\[ \langle \{x, y\}, \{x', y'\} \rangle_{X \oplus Y} = \langle x, x' \rangle_X + \langle y, y' \rangle_Y , \]  \hspace{1cm} (1.22)

with which \( X \times Y \) is a Hilbert space, denoted \( X \oplus Y \). We use this notation also for the direct sum of two orthogonal closed subspaces \( X \) and \( Y \) of a
Hilbert space $H$. For $L_p$-spaces it can be convenient to use $(\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ as the norm on the product space.

We generally define

$$X \pm Y = \{x \pm y \mid x \in X \text{ and } y \in Y\}$$
$$\Omega X = \{\alpha x \mid \alpha \in \Omega \text{ and } x \in X\}$$

(1.23)

when $X$ and $Y$ are subsets of a vector space $V$ with scalar field $\mathbb{L}$ ($\mathbb{L} = \mathbb{R}$ or $\mathbb{C}$), and $\Omega \subset \mathbb{L}$. In particular, we write

$$\{x\} + Y = x + Y$$
$$\{\alpha\}Y = \alpha Y$$

(1.24)

when $x \in X$ and $\alpha \in \mathbb{L}$. When $X$ and $Y$ are subspaces of a vector space $V$, $X + Y$ is denoted $X \dot{+} Y$ if $X$ and $Y$ are linearly independent. (There is also the notation $X \oplus Y$ for orthogonal closed subspaces of a Hilbert space.)

When $X$ is a closed subspace of a Hilbert space $H$, the orthogonal complement is denoted $H \ominus X$.

Integration by parts in one variable is generalized to functions of several variables by the Gauss and Green formulas, which we briefly recall:

Let $\Omega \subset \mathbb{R}^n$ be an open set with $C^1$ boundary $\partial \Omega$ and let $\nu(x)$ denote the interior unit normal vector field at $\partial \Omega$.

To explain this further: $\Omega$ is said to have a $C^1$ boundary, when every boundary point has a neighborhood $V$ such that — after a relabelling of the coordinates if necessary —

$$\Omega \cap V = \{(x_1, \ldots, x_n) \in V \mid x_n > f(x_1, \ldots, x_{n-1})\},$$

(1.25)

where $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is a $C^1$-function. Here

$$\partial \Omega \cap V = \{x \in V \mid x_n = f(x_1, \ldots, x_{n-1})\},$$

(1.26)

and the interior unit normal vector at the point $x \in \partial \Omega \cap V$ equals (with the notation (1.14))

$$\nu(x', f(x')) = \frac{(-\partial_1 f(x'), \ldots, -\partial_{n-1} f(x'), 1)}{\sqrt{(\partial_1 f(x'))^2 + \cdots + (\partial_{n-1} f(x'))^2 + 1}}.$$  

(1.27)

For a $C^1$ function $u$ defined on a neighborhood of $\overline{\Omega}$ one has the Gauss formula (when $u$ has compact support or the integrability is assured in some other way):

$$\int_{\Omega} \partial_k u \, dx = - \int_{\partial \Omega} \nu_k(x) u(x) \, d\sigma, \quad k = 1, \ldots, n,$$ 

(1.28)
where $d\sigma$ is the surface measure on $\partial\Omega$. In the situation of (1.26),
\[
d\sigma = \frac{1}{|\nu_n(x)|} \, dx = \sqrt{(\partial_1 f)^2 + \cdots + (\partial_{n-1} f)^2 + 1} \, dx_1 \ldots dx_{n-1};
\]
and the formula (1.28) is for $k = n$ verified for functions supported in $V$ simply by the change of coordinates $x = (x', x_n) \mapsto (x', x_n - f(x'))$ that replaces $\partial\Omega \cap V$ with a subset of $\mathbb{R}^{n-1}$. From the Gauss formula one derives several other formulas, usually called Green’s formulas, when $u$ and $v$ are suitably differentiable:
\[
\int_{\Omega} \partial_k u \, \overline{v} \, dx = - \int_{\Omega} u \, \overline{\partial_k v} \, dx - \int_{\partial\Omega} \nu_k(x) u(x) \overline{v(x)} \, d\sigma,
\]
\[
\int_{\Omega} D_k u \, \overline{v} \, dx = \int_{\Omega} u \, \overline{D_k v} \, dx + i \int_{\partial\Omega} \nu_k(x) u(x) \overline{v(x)} \, d\sigma,
\]
\[
\int_{\Omega} (-\Delta u) \, \overline{v} \, dx = \sum_{k=1}^{n} \int_{\Omega} \partial_k u \, \overline{\partial_k v} \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma,
\]
\[
\int_{\Omega} (-\Delta u) \, \overline{v} \, dx = \int_{\Omega} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma - \int_{\partial\Omega} u \, \overline{\partial v} \, d\sigma;
\]
where $\frac{\partial u}{\partial \nu} = \sum_{k=1}^{n} \nu_k \partial_k u$,
the interior normal derivative.

The signs are chosen with later applications in mind (it is the operator $-\Delta$ that is “positive”).

Let $p \in [1, \infty]$. For a Lebesgue measurable subset $M$ of $\mathbb{R}^n$, $L_p(M)$ denotes the vector space of equivalence classes of measurable functions $f : M \to \mathbb{C}$ with finite norm
\[
\|f\|_{L_p(M)} = \left( \int_M |f(x)|^p \, dx \right)^{1/p} \quad \text{if } p < \infty,
\]
\[
\|f\|_{L_\infty(M)} = \text{ess sup}_M |f| \quad \text{if } p = \infty.
\]
It is a Banach space with this norm. (The equivalence classes consist of functions that are equal almost everywhere (a.e.); we use the customary “abuse of notation” where one calls the equivalence class a function, denoting the class containing $f$ by $f$ again. If the class contains a continuous function — necessarily unique — we use this as representative. In this way, $C^0(\Omega)$ identifies with a subset of $L_{1,\text{loc}}(\Omega)$ when $\Omega$ is an open set.) We recall that for a real measurable function $u$ on $M$,
\[
\text{ess sup}_M u = \inf \{ a \mid u(x) \leq a \text{ a.e. in } M \}.
\]
When $p = 2$ we get a Hilbert space, where the norm is associated with the scalar product

$$(f, g)_{L^2(M)} = \int_M f(x)\overline{g}(x)\,dx.$$  \hspace{1cm} (1.33)

**Hölder’s inequality**

$$\left| \int_M f(x)g(x)\,dx \right| \leq \|f\|_{L^p(M)} \|g\|_{L^{p'}(M)}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$  \hspace{1cm} (1.34)

holds for $f \in L^p(M)$ and $g \in L^{p'}(M)$; it is the Cauchy-Schwarz inequality in the case $p = 2$. Note that $L^p(\Omega) = L^p(\overline{\Omega})$ when for example $\Omega$ has $C^1$ boundary.

When the measure of $M$ is finite, we have an inclusion

$$L^p(M) \subset L^q(M) \quad \text{for } 1 \leq q \leq p \leq \infty.$$  \hspace{1cm} (1.35)

Recall that the proof for $p < \infty$ consists of observing that for $f \in L^p(M)$ one has, with $r = p/q$, $1/r + 1/r' = 1$, by the Hölder inequality:

$$\|f\|_{L^q(M)} = \left( \int_M |f(x)|^q\,dx \right)^{1/q} = \left( \int_M |f(x)|^{p/r} \cdot 1\,dx \right)^{1/q} \leq \left( \int_M |f(x)|^p \right)^{1/rq} \left( \int_M 1\,dx \right)^{1/r'q} = \|f\|_{L^p(M)} \text{vol}(M)^{1/q-1/p}$$  \hspace{1cm} (1.36)

where $\text{vol}(M) = \int_M 1\,dx$ is the volume (measure) of $M$.

When $M \subset V$ for some set $V$, we denote by $1_M$ the function on $V$ defined by

$$1_M(x) = \begin{cases} 1 & \text{for } x \in M, \\ 0 & \text{for } x \in V \setminus M. \end{cases}$$  \hspace{1cm} (1.37)

When $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, and

$$1 \leq p \leq \infty, \quad 1 \leq q \leq \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0,$$  \hspace{1cm} (1.38)

then the convolution $(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$ defines a function $f * g$ in $L^r(\mathbb{R}^n)$, and

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)};$$  \hspace{1cm} (1.39)

**Young’s inequality.** In particular, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$, then $f * g \in L^2(\mathbb{R}^n)$, and

$$\|f * g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$  \hspace{1cm} (1.40)
Exercises for Chapter 1.

1.1. Show the general Leibniz formulas (1.17).

1.2. (a) Let \( f \in C^1(\mathbb{R}^n) \). Show for any \( x, y \in \mathbb{R}^n \) that the function \( g(\theta) = f(x + \theta y) (\theta \in \mathbb{R}) \) satisfies:

\[
\frac{d}{d\theta} g(\theta) = \sum_{j=1}^{n} \partial_j f(x + \theta y) y_j,
\]

and conclude from this that

\[
f(x + y) = f(x) + \sum_{j=1}^{n} y_j \int_{0}^{1} \partial_j f(x + \theta y) d\theta.
\]

(b) Show Taylor’s formula (1.18) for arbitrary \( N \).

1.3. Deduce the formulas in (1.30) from (1.28).

(Hint. Apply (1.28) to \( \partial_k(uv) \).)