§2. Function spaces and approximation

2.1. The space of test functions.

Notation and prerequisites are collected in Appendix A.

Let Ω be an open subset of $\mathbb{R}^n$. The space $C_0^\infty(\Omega)$, consisting of the C^∞ functions on Ω with compact support in Ω, is called the space of *test functions* (on Ω). The support $\operatorname{supp} u$ of a function $u \in L_{1,\mathrm{loc}}(\Omega)$ is defined as the complement of the largest open set where u vanishes; we can write it as

$$\operatorname{supp} u = \Omega \setminus \Big(\bigcup \{\, \omega \text{ open in } \Omega \mid u|_\omega = 0 \,\} \Big). \tag{2.1}$$

We show first of all that there *exist* test functions:

Lemma 2.1. 1° *Let $R > r > 0$. There is a function $\chi_{r,R}(x) \in C_0^\infty(\mathbb{R}^n)$ with the properties: $\chi_{r,R}(x) = 1$ for $|x| \le r$, $\chi_{r,R}(x) \in [0,1]$ for $r \le |x| \le R$, $\chi_{r,R}(x) = 0$ for $|x| \ge R$.*

2° *There is a function $h \in C_0^\infty(\mathbb{R}^n)$ satisfying:*

$$\operatorname{supp} h = \underline{B}(0,1), \quad h(x) > 0 \text{ for } |x| < 1, \quad \int h(x)\,dx = 1. \tag{2.2}$$

Proof. 1°. The function

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0\,, \\ 0 & \text{for } t \le 0\,, \end{cases}$$

is a C^∞ function on $\mathbb{R}$. For $t \ne 0$ this is obvious. At the point $t = 0$ we have that $f(t) \to 0$ for $t \searrow 0$, and that the derivatives of $f(t)$ for $t \ne 0$ are of the form

$$\partial_t^k f(t) = \begin{cases} p_k(1/t)e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t < 0, \end{cases}$$

for certain polynomials p_k, $k = 0, 1, 2 \ldots$. Since any polynomial p satisfies $p(1/t)e^{-1/t} \to 0$ for $t \searrow 0$, f and its derivatives are differentiable at 0.

From f we construct the functions (see the figure)

$$f_1(t) = f(t-r)f(R-t)\,, \quad f_2(t) = \int_t^\infty f_1(s)\,ds\,.$$

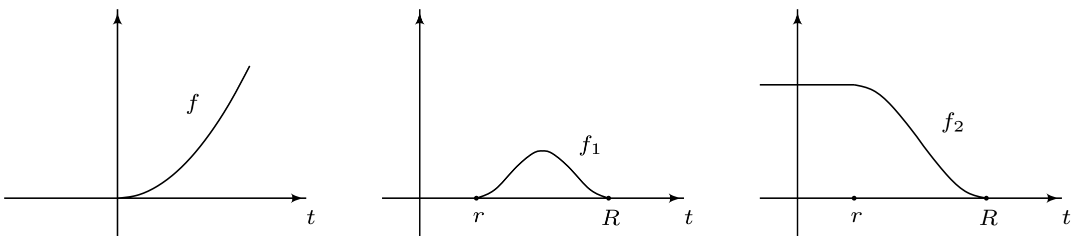
Here we see that $f_2(x) \ge 0$ for all x, equals 0 for $t \ge R$ and equals

$$C = \int_r^R f_1(s)\,ds > 0$$

for $t \le r$. The function

$$\chi_{r,R}(x) = \frac{1}{C} f_2(|x|)\,,\; x \in \mathbb{R}^n\;,$$

then has the desired properties.



2°. Here one can for example take

$$h(x) = \frac{\chi_{\frac{1}{2},1}(x)}{\int \chi_{\frac{1}{2},1}(x)\,dx}\;. \qquad \square$$

Note that *analytic functions* (functions defined by a converging Taylor expansion) cannot be in $C_0^\infty(\mathbb{R})$ without being identically zero! So we have to go outside the elementary functions (such as $\cos t$, e^t, e^{-t^2}, etc.) to find nontrivial C_0^∞ functions. The construction in Lemma 2.1 can be viewed from a "plumber's point of view": We want a C^∞ function that is 0 on a certain interval and takes a certain positive value on another; we can get it by twisting the graph suitably. But analyticity is lost then.

For later reference we shall from now on denote by χ a function in $C_0^\infty(\mathbb{R}^n)$ satisfying

$$\chi(x) \begin{cases} = 1 & \text{for } |x| \le 1 \\ \in [0,1] & \text{for } 1 \le |x| \le 2\,, \\ = 0 & \text{for } |x| \ge 2\,, \end{cases} \tag{2.3}$$

one can for example take $\chi_{1,2}$ constructed in Lemma 2.1. A C_0^∞ function that is 1 on a given set and vanishes outside some larger given set is often called a *cut-off function.* Of course we get some other cut-off functions by translating the functions $\chi_{r,R}$ around. More refined examples will be constructed later by convolution, see e.g. Theorem 2.13. These functions are all examples of test functions, when their support is compact.

We use throughout the following convention (of "extension by zero") for test functions: If $\varphi \in C_0^\infty(\Omega)$, Ω open $\subset \mathbb{R}^n$, we also denote the function obtained by extending by zero on $\mathbb{R}^n \setminus \Omega$ by φ; it is in $C_0^\infty(\mathbb{R}^n)$. When $\varphi \in C_0^\infty(\mathbb{R}^n)$ and its support is contained in Ω, we can regard it as an element of $C_0^\infty(\Omega)$ and again denote it φ. Similarly, we can view a C^∞ function φ with compact support in $\Omega \cap \Omega'$ (Ω and Ω' open) as an element of $C_0^\infty(\Omega)$ or $C_0^\infty(\Omega')$, whatever is convenient.

Before we describe the topology of the space $C_0^\infty(\Omega)$ we recall how some other useful spaces are topologized. The reader can find the necessary information on topological vector spaces in Appendix B and its problem session.

When we consider an open subset Ω of $\mathbb{R}^n$, the compact subsets play an important role.

Lemma 2.2. *Let Ω be a nonempty open subset of $\mathbb{R}^n$. There exists a sequence of compact subsets $(K_j)_{j\in\mathbb{N}}$ such that*

$$K_1 \subset K_2^\circ \subset K_2 \subset \dots \subset K_j^\circ \subset K_j \subset \dots$$
$$\bigcup_{j\in\mathbb{N}} K_j^\circ = \Omega \,. \tag{2.4}$$

Proof. We can for example take

$$K_j = \{\, x \in \Omega \mid |x| \le j \;\text{ and }\; \operatorname{dist}(x, \complement\Omega) \ge \tfrac{1}{j} \,\}; \tag{2.5}$$

the interior of this set is defined by the formula with $\le$ and $\ge$ replaced by $<$ and $>$. (If $\complement\Omega = \emptyset$, the condition $\operatorname{dist}(x, \complement\Omega) \ge \frac{1}{j}$ is left out.) If necessary, we can omit the first, at most finitely many, sets with $K_j^\circ = \emptyset$ and modify the indexation. ☐

[A drawing is needed here.]

When K is a compact subset of Ω, it is covered by the system of open sets $\{K_j^\circ\}_{j\in\mathbb{N}}$ and hence by a finite sub-system, say with $j \le j_0$. Then $K \subset K_j$ for all $j \ge j_0$.

Recall that when $[a,b]$ is a compact interval of $\mathbb{R}$, $C^k([a,b])$ (in one of the versions $C^k([a,b],\mathbb{C})$ or $C^k([a,b],\mathbb{R})$) is defined as the Banach space of complex resp. real functions having continuous derivatives up to order k, provided with a norm

$$\|f\|'_{C^k} = \sum_{0\le j\le k} \sup_x |f^{(j)}(x)|, \text{ or the equivalent norm}$$
$$\|f\|_{C^k} = \sup\{\, |f^{(j)}(x)| \mid x \in [a,b], 0 \le j \le k \,\} \,. \tag{2.6}$$

In the proof that these normed spaces are complete one uses the well-known theorem that when f_l is a sequence of C^1 functions such that f_l and f_l' converge uniformly to f resp. g for $l \to \infty$, then f is C^1 with derivative $f' = g$. There is a similar result for functions of several variables:

Lemma 2.3. *Let $J = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed box in $\mathbb{R}^n$ and let f_l be a sequence of functions in $C^1(J)$ such that $f_l \to f$ and $\partial_j f_l \to g_j$ uniformly on J for $j = 1, \dots, n$. Then $f \in C^1(J)$ with $\partial_j f = g_j$ for each j.*

Proof. For each j we use the above mentioned theorem in situations where all but one coordinate are fixed. This shows that f has continuous partial derivatives $\partial_j f = g_j$ at each point of J. $\square$

So $C^k(J)$ is a Banach space with the norm

$$\|u\|_{C^k(J)} = \sup\{\, |\partial^\alpha u(x)| \mid x \in J,\, |\alpha| \le k \,\}. \tag{2.7}$$

We define

$$C^\infty(J) = \bigcap_{k \in \mathbb{N}_0} C^k(J). \tag{2.8}$$

This is no longer a Banach space, but can be shown to be a Fréchet space with the family of (semi)norms $p_k(f) = \|f\|_{C^k(J)}$, $k \in \mathbb{N}_0$, by arguments as in Lemma 2.4 below. (See Appendix B for details on Fréchet spaces.)

For spaces of differentiable functions over *open* sets, the full sup-norms are unsatisfactory since the functions and their derivatives need not be bounded. We here use sup-norms over compact subsets to define a Fréchet topology. Let Ω be open and let K_j be an increasing sequence of compact subsets as in Lemma 2.2. Define the system of seminorms

$$p_{k,j}(f) = \sup\{\, |\partial^\alpha f(x)| \mid |\alpha| \le k\ ,\ x \in K_j \,\}, \quad \text{for } j \in \mathbb{N}. \tag{2.9}$$

Lemma 2.4. 1° *For each $k \in \mathbb{N}_0$, $C^k(\Omega)$ is a Fréchet space when provided with the family of seminorms $\{p_{k,j}\}_{j \in \mathbb{N}}$.*

2° *The space $C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}_0} C^k(\Omega)$ is a Fréchet space when provided with the family of seminorms $\{p_{k,j}\}_{k \in \mathbb{N}_0, j \in \mathbb{N}}$.*

Proof. 1°. The family $\{p_{k,j}\}_{j \in \mathbb{N}}$ is separating, for when $f \in C^k(\Omega)$ is $\neq 0$, then there is a point x_0 where $f(x_0) \neq 0$, and $x_0 \in K_j$ for j sufficiently large; for such j, $p_{k,j}(f) > 0$. The seminorms then define a translation invariant metric by Theorem B.8. We just have to show that the space is complete under this metric. Here we use Lemma 2.3:

Let $(f_l)_{l \in \mathbb{N}}$ be a Cauchy sequence in $C^k(\Omega)$. Let $x_0 = \{x_{01}, \dots, x_{0n}\} \in \Omega$ and consider a box $J = J_{x_0,\delta} = \{\, x \mid |x_m - x_{0m}| \le \delta,\, m = 1, \dots, n \,\}$ around x_0, with δ taken so small that $J \subset \Omega$. Since $J \subset K_{j_0}$ for a certain j_0, the Cauchy sequence property implies that f_l defines a Cauchy sequence in $C^k(J)$, i.e., f_l and its derivatives up to order k are Cauchy sequences with respect to uniform convergence on J. So there is a limit f_J in $C^k(J)$. We use similar arguments for other boxes J' in Ω and find that the limits $f_{J'}$

and f_J are the same on the overlap of J and J'. In this way we can define a C^k-function f that is the limit of the sequence in $C^k(J)$ for all boxes $J \subset \Omega$. Finally, $p_{j,k}(f_l - f) \to 0$ for all j, since each K_j can be covered by a finite number of box-interiors J°. Then f_l has the limit f in the Fréchet topology of $C^k(\Omega)$.

2°. The proof in this case is a variant of the preceding proof, where we now investigate $p_{j,k}$ for all k also. □

The family (2.9) has the max-property (see Remark B.6), so the sets

$$V(p_{k,j}, \varepsilon) = \{ f \in C^\infty(\Omega) \mid |\partial^\alpha f(x)| < \varepsilon \text{ for } |\alpha| \le k,\, x \in K_j \} \tag{2.10}$$

constitute a local basis for the system of neighborhoods at 0. One could in fact make do with the sequence of seminorms $\{p_{k,k}\}_{k\in\mathbb{N}}$, which increase with k.

For any compact subset K of Ω we define

$$C_K^\infty(\Omega) = \{ u \in C^\infty(\Omega) \mid \operatorname{supp} u \subset K \}\ , \tag{2.11}$$

the space of C^∞-functions with support in K (cf. (2.1)); this space is provided with the topology inherited from $C^\infty(\Omega)$.

The space $C_K^\infty(\Omega)$ is a closed subspace of $C^\infty(\Omega)$ (so it is a Fréchet space). The topology is for example defined by the family of seminorms $\{p_{k,j_0}\}_{k\in\mathbb{N}_0}$ (cf. (2.9)) with j_0 taken so large that $K \subset K_{j_0}$. This family has the max-property.

In the theory of distributions we need not only the Fréchet spaces $C^\infty(\Omega)$ and $C_K^\infty(\Omega)$ but also the space

$$C_0^\infty(\Omega) = \{ \varphi \in C^\infty(\Omega) \mid \operatorname{supp}\varphi \text{ is compact in } \Omega \}\ . \tag{2.12}$$

As already mentioned, it is called the space of *test functions*, and it is also denoted $\mathcal{D}(\Omega)$.

If we provide this space with the topology inherited from $C^\infty(\Omega)$, we get an incomplete metric space. For example, if Ω is the interval $I =]0,3[$ and $\varphi(x)$ is a C^∞ function on I with $\operatorname{supp}\varphi = [1,2]$, then $\varphi_l(x) = \varphi(x - 1 + \frac{1}{l})$, $l \in \mathbb{N}$, is a sequence of functions in $C_0^\infty(I)$ which converges in $C^\infty(I)$ to the function $\varphi(x-1) \in C^\infty(I) \setminus C_0^\infty(I)$.

We prefer to provide $C_0^\infty(\Omega)$ with a stronger and somewhat more complicated vector space topology that makes it a *sequentially complete* (but not metric) space. More precisely, we regard $C_0^\infty(\Omega)$ as

$$C_0^\infty(\Omega) = \bigcup_{j=1}^{\infty} C_{K_j}^\infty(\Omega)\ , \tag{2.13}$$

where K_j is an increasing sequence of compact subsets as in (2.4) and the topology is the *inductive limit topology*, cf. Theorem B.16 (also called the $\mathcal{LF}$-topology). The spaces $C^\infty_{K_j}(\Omega)$ are provided with Fréchet space topologies by families of seminorms (2.9).

The properties of this space that we shall need, are summed up in the following theorem, which just specifies the general properties given in Appendix B (Theorem B.17 and Corollary B.18):

Theorem 2.5. *The topology on $C_0^\infty(\Omega)$ has the following properties:*

(a) *A sequence $(\varphi_l)_{l\in\mathbb{N}}$ of test functions converges to φ_0 in $C_0^\infty(\Omega)$ if and only if there is a $j \in \mathbb{N}$ such that* $\operatorname{supp}\varphi_l \subset K_j$ *for all $l \in \mathbb{N}_0$, and $\varphi_l \to \varphi_0$ in $C^\infty_{K_j}(\Omega)$:*

$$\sup_{x\in K_j} |\partial^\alpha \varphi_l(x) - \partial^\alpha \varphi_0(x)| \to 0 \quad \text{for } l \to \infty, \tag{2.14}$$

for all $\alpha \in \mathbb{N}_0^n$.

(b) *A set $E \subset C_0^\infty(\Omega)$ is bounded if and only if there exists a $j \in \mathbb{N}$ such that E is a bounded subset of $C^\infty_{K_j}(\Omega)$. In particular, if $(\varphi_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $C_0^\infty(\Omega)$, then there is a j such that* $\operatorname{supp}\varphi_l \subset K_j$ *for all l, and φ_l is convergent in $C^\infty_{K_j}(\Omega)$ (and then also in $C_0^\infty(\Omega)$).*

(c) *Let Y be a locally convex topological vector space. A mapping T from $C_0^\infty(\Omega)$ to Y is continuous if and only if $T\colon C^\infty_{K_j}(\Omega) \to Y$ is continuous for each $j \in \mathbb{N}$.*

(d) *A linear functional $\Lambda\colon C_0^\infty(\Omega) \to \mathbb{C}$ is continuous if and only if there is an $N_j \in \mathbb{N}_0$ and a $c_j > 0$ for any $j \in \mathbb{N}$, such that*

$$|\Lambda(\varphi)| \le c_j \sup\{\, |\partial^\alpha \varphi(x)| \mid x \in K_j\,,\ |\alpha| \le N_j \,\} \tag{2.15}$$

for all $\varphi \in C^\infty_{K_j}(\Omega)$.

Note that (a) is a very *strong* assumption on the sequence φ_l. Convergence in $C_0^\infty(\Omega)$ implies convergence in practically all the other spaces we shall meet. On the other hand, (d) is a very *mild* assumption on the functional Λ; practically all the functionals that we shall meet will have this property.

We underline that a sequence can only be a Cauchy sequence when there is a j such that all functions in the sequence have support in K_j and the sequence is Cauchy in $C^\infty_{K_j}(\Omega)$. Then the sequence converges because of the completeness of the Fréchet space $C^\infty_{K_j}(\Omega)$. In the example mentioned above, the sequence $\varphi_l(x) = \varphi(x - 1 + \frac{1}{l})$ in $C_0^\infty(\,]0,3[\,)$ is clearly not a Cauchy sequence with respect to this topology on $C_0^\infty(\,]0,3[\,)$.

It is not hard to show that when $(K'_j)_{j\in\mathbb{N}}$ is another sequence of compact subsets as in (2.4), the topology on $C_0^\infty(\Omega)$ defined by use of this sequence is the same as that based on the first one (Exercise 2.1).

We now consider two important operators on these spaces. One is *differentiation*, the other is *multiplication* (by a C^∞ function f); both will be shown to be continuous. The operators are denoted ∂^α (with $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$) resp. M_f. We also write $M_f\varphi$ as $f\varphi$ — and the same notation will be used later for generalizations of these operators.

Theorem 2.6. 1° *The mapping* $\partial^\alpha\colon \varphi \mapsto \partial^\alpha\varphi$ *is a continuous linear operator in* $C_0^\infty(\Omega)$. *The same holds for* D^α.

2° *For any* $f \in C^\infty(\Omega)$, *the mapping* $M_f\colon \varphi \mapsto f\varphi$ *is a continuous linear operator in* $C_0^\infty(\Omega)$.

Proof. Clearly, ∂^α and M_f are linear operators from $C_0^\infty(\Omega)$ to itself.

As for the continuity it suffices, according to Theorem 2.5 (c), to show that ∂^α resp. M_f is continuous from $C_{K_j}^\infty(\Omega)$ to $C_0^\infty(\Omega)$ for each j. Since the operators satisfy:

$$\operatorname{supp}\partial^\alpha\varphi \subset \operatorname{supp}\varphi\,, \quad \operatorname{supp} M_f\varphi \subset \operatorname{supp}\varphi \tag{2.16}$$

for all φ, the range space can for each j be replaced by $C_{K_j}^\infty(\Omega)$. Here we have for each k:

$$\begin{aligned} p_{k,j}(\partial^\alpha\varphi) &= \sup\{\, |\partial^\beta\partial^\alpha\varphi| \mid |\beta| \le k\,,\, x \in K_j\,\} \\ &\le \sup\{\, |\partial^\gamma\varphi| \mid |\gamma| \le k + |\alpha|\,,\, x \in K_j\,\} = p_{k+|\alpha|,j}(\varphi)\,, \end{aligned} \tag{2.17}$$

which shows the continuity of ∂^α. The result extends immediately to D^α.

By the Leibniz rule (A.7) we have for each k:

$$\begin{aligned} p_{k,j}(f\varphi) &= \sup\{\, |\partial^\alpha(f\varphi)| \mid |\alpha| \le k\,,\, x \in K_j\,\} \\ &\le \sup\{\, \textstyle\sum_{\beta\le\alpha} |c_{\alpha,\beta}\partial^\beta f \partial^{\alpha-\beta}\varphi| \mid |\alpha| \le k\,,\, x \in K_j\,\} \\ &\le C_k p_{k,j}(f) p_{k,j}(\varphi) \end{aligned} \tag{2.18}$$

for a suitably large constant C_k; this shows the continuity of M_f. □

2.2 Some other function spaces.

$C_0^\infty(\Omega)$ is contained in practically every other space defined in connection with Ω, that we shall meet. For example,

$$\begin{aligned} &C_0^\infty(\Omega) \subset L_p(\Omega) \quad \text{ for } p \in [1,\infty]\,, \\ &C_0^\infty(\Omega) \subset C^\infty(\Omega)\,, \end{aligned} \tag{2.19}$$

and these injections are continuous: According to Theorem 2.5 (c) it suffices to show that the corresponding injections of $C_{K_j}^\infty(\Omega)$ are continuous, for each j (with K_j as in (2.4)). For $\varphi \in C_{K_j}^\infty(\Omega)$ we have when $p < \infty$:

$$\|\varphi\|_{L_p} \le \sup_{x\in K_j} |\varphi(x)| \operatorname{vol}(K_j)^{1/p}, \tag{2.20}$$

which shows that the injection $J\colon C^\infty_{K_j}(\Omega)\to L_p(\Omega)$ maps the basic neighborhood

$$V(p_{0,j},\varepsilon)=\{\,\varphi\mid \sup_x|\varphi(x)|<\varepsilon\,\}$$

into the ball $B(0,r)$ in $L_p(\Omega)$ with $r=\varepsilon\,\mathrm{vol}(K_j)^{1/p}$. The continuity of the other injection in (2.19) follows from the fact that $C^\infty_{K_j}(\Omega)$ had the inherited topology as a subspace of $C^\infty(\Omega)$.

We now introduce some further spaces of functions.

It is typical for the space $C^\infty(\Omega)$ that it gives no restriction on the *global* behavior of the elements (their behavior on Ω as a whole): from the knowledge of the behavior of a function on the compact subsets of Ω one can determine whether it belongs to $C^\infty(\Omega)$. This does not hold for $L_p(\Omega)$ where a certain globally defined number (the norm) must be finite in order for the function to belong to $L_p(\Omega)$. Sometimes one needs the following "locally" defined variant of $L_p(\Omega)$:

$$L_{p,\mathrm{loc}}(\Omega)=\{\,u\text{ measurable }\mid u|_K\in L_p(K)\text{ when }K\text{ compact }\subset\Omega\,\},\tag{2.21}$$

with the usual identification of functions that are equal almost everywhere.

$L_{p,\mathrm{loc}}(\Omega)$ is provided with the Fréchet space topology defined from the family of seminorms

$$p_j(u)=\|1_{K_j}u\|_{L_p(\Omega)}\,,\ j=1,2,\dots$$

where K_j is as in (2.4). For K compact $\subset\Omega$, we can identify $L_p(K)$ with

$$L_{p,K}(\Omega)=\{\,u\in L_p(\Omega)\mid \operatorname{supp}u\subset K\,\},\tag{2.22}$$

by extension by zero in $\Omega\setminus K$; it is a closed subspace of $L_p(\Omega)$. The completeness of the spaces $L_{p,\mathrm{loc}}(\Omega)$ can be deduced from the completeness of the spaces $L_{p,K_j}(\Omega)$.

In analogy with the subspace $C_0^\infty(\Omega)$ of $C^\infty(\Omega)$ (with a stronger topology) we define the subspace $L_{p,\mathrm{comp}}(\Omega)$ of $L_{p,\mathrm{loc}}(\Omega)$ (and of $L_p(\Omega)$) by

$$L_{p,\mathrm{comp}}(\Omega)=\{\,u\in L_p(\Omega)\mid \operatorname{supp}u\text{ compact }\subset\Omega\,\}.\tag{2.23}$$

It is provided with the inductive limit topology, when written as

$$L_{p,\mathrm{comp}}(\Omega)=\bigcup_{j=1}^\infty L_{p,K_j}(\Omega).\tag{2.24}$$

In this way, $L_{p,\mathrm{comp}}(\Omega)$ is an $\mathcal{LF}$ space (cf. Appendix B), with a stronger topology than the one inherited from $L_p(\Omega)$.

Remark 2.7. The above choices of topology assure for example that $L_{2,\mathrm{loc}}(\Omega)$ and $L_{2,\mathrm{comp}}(\Omega)$ may be identified with the dual space of one another, in such a way that the duality is a generalization of the integral $\int_\Omega u\bar{v}\,dx$ (Exercises 2.3 and 2.7).

It is not hard to show (cf. (A.21), (A.31)) that

$$\begin{aligned} &C^\infty(\Omega) \subset L_{p,\mathrm{loc}}(\Omega) \subset L_{q,\mathrm{loc}}(\Omega)\ , \\ &C_0^\infty(\Omega) \subset L_{p,\mathrm{comp}}(\Omega) \subset L_{q,\mathrm{comp}}(\Omega)\ , \quad \text{ for } p > q\ , \end{aligned} \tag{2.25}$$

with continuous injections.

More function spaces are defined in Appendix C. The reader can bypass them until needed in the text. There is a large number of spaces that one can define for various purposes, and rather than learning all these spaces by heart, the reader should strive to be able to introduce the appropriate space with the appropriate topology ("do-it-yourself") when needed.

2.3. Approximation theorems.

From the test functions constructed in Lemma 2.1 one can construct a wealth of other test functions by convolution. Recall that when f and g are measurable functions on $\mathbb{R}^n$ and the product $f(y)g(x-y)$ is an integrable function of y for a fixed x, then the convolution product $(f * g)(x)$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy\ . \tag{2.26}$$

Note that $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = (g*f)(x)$. (2.26) is for example defined when $f \in L_{1,\mathrm{loc}}$ and g is bounded and one of them has compact support (or both have support in a conical set such as for example $\{\, x \mid x_1 \geq 0, \dots, x_n \geq 0 \,\}$). (2.26) is also well-defined when $f \in L_p$ and $g \in L_q$ with $1/p + 1/q = 1$.

It will be convenient to use the following general convergence principles that follow from the theorem of Lebesgue:

Lemma 2.8. *Let $M \subset \mathbb{R}^n$ be measurable and let I be an interval of $\mathbb{R}$. Let $f(x,a)$ be a family of functions of $x \in M$ depending on the parameter $a \in I$, such that for each $a \in I$, $f(x,a) \in L_1(M)$. Consider the function F on I defined by*

$$F(a) = \int_M f(x,a)\,dx \text{ for } a \in I. \tag{2.27}$$

1° *Assume that for each $x \in M$, $f(x,a)$ is a continuous function of a at the point $a_0 \in I$, and that there is a function $g(x) \in L^1(M)$ such that $|f(x,a)| \leq g(x)$ for all $(x,a) \in M \times I$. Then $F(a)$ is continuous at the point a_0.*

A similar statement holds when the parameter a runs in a ball $B(a_0, r)$ in $\mathbb{R}^k$.

2° *Assume that $\frac{\partial}{\partial a} f(x,a)$ exists for all $(x,a) \in M \times I$ and that there is a function $g(x) \in L^1(M)$ such that*

$$|\frac{\partial}{\partial a} f(x,a)| \leq g(x) \text{ for all } (x,a) \in M \times I. \tag{2.28}$$

Then $F(a)$ is a differentiable function of $a \in I$, and

$$\frac{d}{da} F(a) = \int_M \frac{\partial}{\partial a} f(x,a)\, dx. \tag{2.29}$$

Let $h(x)$ be a function with the properties:

$$h \in C_0^\infty(\mathbb{R}^n)\,,\ h \geq 0\,,\ \int_{\mathbb{R}^n} h(x)\, dx = 1\,,\ \operatorname{supp} h \subset \underline{B}(0,1)\,. \tag{2.30}$$

Such functions exist according to Lemma 2.1. For $j \in \mathbb{N}$ we set

$$h_j(x) = j^n h(jx)\,; \tag{2.31}$$

then we have for each j,

$$h_j \in C_0^\infty(\mathbb{R}^n)\,,\ h_j \geq 0\,,\ \int h_j(x)\, dx = 1\,,\ \operatorname{supp} h_j \subset \underline{B}(0, \tfrac{1}{j})\,. \tag{2.32}$$

The sequence $(h_j)_{j \in \mathbb{N}}$ is often called an *approximate unit.* This refers to the approximation property shown in Theorem 2.10 below, generalized to distributions in Chapter 3.

We shall study convolutions with the functions h_j. Let $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and consider $h_j * u$,

$$(h_j * u)(x) = \int_{B(0,\frac{1}{j})} h_j(y) u(x-y)\, dy = \int_{B(x,\frac{1}{j})} h_j(x-y) u(y)\, dy\,. \tag{2.33}$$

Concerning supports of these functions it is clear that if $\operatorname{dist}(x, \operatorname{supp} u) > \frac{1}{j}$, then $(h_j * u)(x) = 0$. Thus $\operatorname{supp}(h_j * u)$ is contained in the closed set

$$\operatorname{supp}(h_j * u) \subset \operatorname{supp} u + \underline{B}(0, \tfrac{1}{j})\,. \tag{2.35}$$

In particular, if u has compact support then $h_j * u$ has a slightly larger compact support.

Lemma 2.9. *When $u \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, then $h_j * u \in C^\infty(\mathbb{R}^n)$, and*

$$\partial^\alpha (h_j * u) = (\partial^\alpha h_j) * u \quad \text{for all} \quad \alpha \in \mathbb{N}_0^n \, . \tag{2.36}$$

Proof. Let x_0 be an arbitrary point of $\mathbb{R}^n$; we shall show that $h_j * u$ is C^∞ on a neighborhood of the point and satisfies (2.36) there. When $x \in B(x_0, 1)$, then $h_j(x-y)$ vanishes for $y \notin B(x_0, 2)$, so we may write

$$(h_j * u)(x) = \int_{B(x_0,2)} h_j(x-y) u(y)\, dy \, , \tag{2.37}$$

for such x. Note that

$$\partial_x^\alpha h_j(x) = j^{n+|\alpha|} \partial_y^\alpha h(y)|_{y=jx}, \text{ so } \sup_x |\partial_x^\alpha h_j(x)| = j^{n+|\alpha|} \sup_x |\partial_x^\alpha h(x)|, \tag{2.38}$$

and hence the x-dependent family of functions $h_j(x-y)u(y)$ and its x-derivatives are bounded by multiples of $|u(y)|$:

$$|h_j(x-y)u(y)| \le C_j |u(y)|, \; |\partial_x^\alpha h_j(x-y) u(y)| \le C_{\alpha,j} |u(y)|. \tag{2.39}$$

Since u is integrable on $B(x_0, 2)$, we can first use Lemma 2.8 1° (with $k = n$) to see that $(h_j * u)(x)$ is continuous at the points $x \in B(x_0, 1)$. Next, we can use Lemma 2.8 2° for each of the partial derivatives $\frac{\partial}{\partial x_k}$, $k = 1, \dots, n$, where $x \in B(x_0, 1)$, keeping all but one coordinate fixed. This gives that $\frac{\partial}{\partial x_k}(h_j * u)(x)$ exists and equals the continuous function

$$\tfrac{\partial}{\partial x_k}(h_j * u)(x) = \int_{B(x_0,2)} \tfrac{\partial}{\partial x_k} h_j(x-y) u(y)\, dy, \; k = 1, \dots, n,$$

for $x \in B(x_0, 1)$. Here is a new formula where we can apply the argument again, showing that $\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k}(h_j * u)(x)$ exists and equals $((\frac{\partial}{\partial x_l}\frac{\partial}{\partial x_k} h_j) * u)(x)$. By induction we include all derivatives and obtain (2.36). □

One can place the differentiations on u, to the extent that u has well-defined partial derivatives, by a variant af the above arguments (Exercise 2.6).

Theorem 2.10.

1° *When v is continuous and has compact support in $\mathbb{R}^n$, i.e., $v \in C_0^0(\mathbb{R}^n)$ (cf. (C.7)), then $h_j * v \to v$ for $j \to \infty$ uniformly, hence also in $L_p(\mathbb{R}^n)$ for any $p \in [1, \infty]$ (in particular in $C^0_{L_\infty}(\mathbb{R}^n)$, cf. (C.11)).*

2° *For any* $p \in [1, \infty]$ *one has that*

$$\|h_j * u\|_{L_p} \le \|u\|_{L_p} \text{ for } u \in L_p(\mathbb{R}^n) . \tag{2.40}$$

3° *When* $p \in [1, \infty[$ *and* $u \in L_p(\mathbb{R}^n)$, *then* $h_j * u \to u$ *in* $L_p(\mathbb{R}^n)$ *for* $j \to \infty$. *Moreover,* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $L_p(\mathbb{R}^n)$.

Proof. 1°. When v is continuous with compact support, then v is *uniformly continuous* and one has for $x \in \mathbb{R}^n$:

$$\begin{aligned} |(h_j * v)(x) - v(x)| &= \Big| \int_{B(0,\frac{1}{j})} v(x-y)h_j(y)dy - \int_{B(0,\frac{1}{j})} v(x)h_j(y)dy \Big| \\ &\le \sup_{y \in B(0,\frac{1}{j})} |v(x-y) - v(x)| \le \varepsilon_j , \end{aligned} \tag{2.41}$$

where $\varepsilon_j \to 0$ for $j \to \infty$, *independently of* x. It follows immediately that $h_j * v \to v$ pointwise and in sup-norm, and one finds by integration over the compact set $\operatorname{supp} v + \underline{B}(0,1)$ that $h_j * v \to v$ in L_p for $p \in [1, \infty]$.

2°. The inequality is for $1 < p < \infty$ a consequence of Hölder's inequality (A.24), where we set $f(y) = h_j(x-y)^{1/p}u(y)$ and $g(y) = h_j(x-y)^{1/p'}$:

$$\begin{aligned} \|h_j * u\|_{L_p}^p &= \int \Big| \int h_j(x-y)u(y)dy \Big|^p \, dx \\ &\le \int \Big(\int h_j(x-y)|u(y)|^p dy \Big) \Big(\int h_j(x-y)dy \Big)^{p/p'} dx \\ &= \iint h_j(x-y)|u(y)|^p dy \, dx = \|u\|_{L_p}^p , \end{aligned} \tag{2.42}$$

using (2.32) and the Fubini theorem. In the cases $p = 1$ and $p = \infty$ one uses suitable variants of this argument.

3°. We here use the result known from measure theory that when $p < \infty$, the functions in $L_p(\mathbb{R}^n)$ may be approximated in L_p norm by continuous functions with compact support. Let $u \in L_p(\mathbb{R}^n)$, let $\varepsilon > 0$ and let $v \in C_0^0(\mathbb{R}^n)$ with $\|u - v\|_{L_p} \le \varepsilon/3$. By 1°, j_0 can be chosen so large that

$$\|h_j * v - v\|_{L_p} \le \varepsilon/3 \text{ for } j \ge j_0 .$$

Then by (2.40),

$$\begin{aligned} \|h_j * u - u\|_{L_p} &\le \|h_j * (u - v)\|_{L_p} + \|h_j * v - v\|_{L_p} + \|v - u\|_{L_p} \\ &\le 2\|v - u\| + \varepsilon/3 \le \varepsilon , \quad \text{for } j \ge j_0 , \end{aligned}$$

which shows that $h_j * u \to u$ in L_p for $j \to \infty$. The last assertion is seen from the fact that in this construction, $h_j * v$ approximates u. □

The theorem shows how sequences of smooth functions $h_j * u$ approximate u in a number of different spaces. We can extend this to still other spaces.

Lemma 2.11. *For every* $p \in [1, \infty[$, $C_0^\infty(\mathbb{R}^n)$ *is dense in* $L_{p,\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Note first that $\chi(x/N)u \to u$ in $L_{p,\mathrm{loc}}$ for $N \to \infty$ (cf. (2.3)). For, $\chi(x/N) = 1$ for $|x| \le N$, and hence for any j,

$$p_j(\chi(x/N)u - u) \equiv \int_{B(0,j)} |\chi(x/N)u - u|^p \, dx = 0 \ \text{ for } N \ge j \ , \tag{2.43}$$

so $p_j(\chi(x/N)u - u) \to 0$ for $N \to \infty$, any j, whereby $\chi(x/N)u - u \to 0$ in $L_{p,\mathrm{loc}}(\mathbb{R}^n)$. The convergence of course holds in any metric defining the topology. Now $\chi(x/N)u \in L_p(\mathbb{R}^n)$, and $h_l * (\chi(x/N)u) \to \chi(x/N)u$ in $L_p(\mathbb{R}^n)$ by Theorem 2.10, with $h_l * (\chi(x/N)u) \in C_0^\infty(\mathbb{R}^n)$ supported in $\underline{B}(0, 2N + \frac{1}{l})$; the convergence also holds in $L_{p,\mathrm{loc}}(\mathbb{R}^n)$. We conclude that the functions in $L_{p,\mathrm{loc}}(\mathbb{R}^n)$ may be approximated by testfunctions, with respect to the topology of $L_{p,\mathrm{loc}}(\mathbb{R}^n)$. $\square$

When we consider $u \in L_{p,\mathrm{loc}}(\Omega)$ for an open subset Ω of $\mathbb{R}^n$, the expression $(h_j * u)(x)$ is usually not well-defined for x close to the boundary. But one does have the following result:

Lemma 2.12. *Let* $u \in L_{p,\mathrm{loc}}(\Omega)$ *for some* $p \in [1, \infty[$ *and let* $\varepsilon > 0$. *When* $j > 1/\varepsilon$, *then*

$$v_j(x) = (h_j * u)(x) = \int_{B(0,\frac{1}{j})} h_j(y) u(x - y) dy \tag{2.44}$$

is defined for x *in the set*

$$\Omega_\varepsilon = \{\, x \in \Omega \mid \operatorname{dist}(x, \complement\Omega) > \varepsilon \,\} \ , \tag{2.45}$$

and one has for any $R > 0$

$$\Big(\int_{\Omega_\varepsilon \cap B(0,R)} |u(x) - v_j(x)|^p dx\Big)^{1/p} \to 0 \ \text{ for } j \to \infty \ . \tag{2.46}$$

Proof. Let $j > 1/\varepsilon$, then $v_j(x)$ is defined for $x \in \Omega_\varepsilon$. In the calculation of the integral (2.46), when $j > 2/\varepsilon$ one only uses the values of u on $K_{\varepsilon,R} = \overline{\Omega}_\varepsilon \cap \underline{B}(0, R) + \underline{B}(0, \varepsilon/2)$, which is a compact subset of Ω. We can then replace u by

$$u_1(x) = \begin{cases} u(x) & \text{for } x \in K_{\varepsilon,R} \\ 0 & \text{otherwise.} \end{cases} \tag{2.47}$$

Here $u_1 \in L_p(\mathbb{R}^n)$, whereby $v_j = h_j * u_1$ on $\Omega_\varepsilon \cap B(0, R)$ and the result follows from Theorem 2.10. $\square$

Other types of approximation results in $L_{p,\mathrm{loc}}(\Omega)$ can be obtained by use of more refined cut-off functions than those in Lemma 2.1.

Theorem 2.13. *Let M be a subset of $\mathbb{R}^n$, let $\varepsilon > 0$, and set $M_{k\varepsilon} = \overline{M} + \underline{B}(0, k\varepsilon)$ for $k > 0$. There exists a function $\eta \in C^\infty(\mathbb{R}^n)$ which is 1 on M_ε and is supported in $M_{3\varepsilon}$, and which satisfies $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}^n$.*

Proof. The function

$$\psi(x) = \begin{cases} 1 & \text{on } M_{2\varepsilon}, \\ 0 & \text{on } \mathbb{R}^n \setminus M_{2\varepsilon}, \end{cases} \tag{2.48}$$

is in $L_{1,\text{loc}}(\mathbb{R}^n)$, and for $j \ge 1/\varepsilon$, the function $h_j * \psi$ is nonnegative and C^∞ with support in

$$M_{2\varepsilon} + \underline{B}(0, \tfrac{1}{j}) \subset M_{3\varepsilon}.$$

When $x \in M_\varepsilon$, we have that $\psi = 1$ on the ball $\underline{B}(x, \varepsilon)$, and hence $(h_j * \psi)(x) = \int_{B(0,\frac{1}{j})} h_j(y)\psi\ (x - y)dy = 1$ when $j \ge 1/\varepsilon$. The function takes values in $[0, 1]$ elsewhere. Thus, as the function η we can use $h_j * \psi$ for $j \ge 1/\varepsilon$. □

Observe that η in Theorem 2.13 has compact support when $\overline{M}$ is compact.

One often needs the following special cases:

Corollary 2.14. 1° *Let Ω be open and let K be compact $\subset \Omega$. There is a function $\eta \in C_0^\infty(\Omega)$ taking values in $[0, 1]$ such that $\eta = 1$ on a neighborhood of K.*

2° *Let K_j, $j \in \mathbb{N}$, be a sequence of compact sets as in (2.4). There is a sequence of functions $\eta_j \in C_0^\infty(\Omega)$ taking values in $[0, 1]$ such that $\eta_j = 1$ on a neighborhood of K_j and $\operatorname{supp} \eta_j \subset K_{j+1}^\circ$.*

Proof. We use Theorem 2.13, noting that $\operatorname{dist}(K, \complement\Omega) > 0$ and that for all j, $\operatorname{dist}(K_j, \complement K_{j+1}) > 0$. □

Using these functions we can moreover show:

Theorem 2.15. *Let Ω be open $\subset \mathbb{R}^n$.*

1° *$C_0^\infty(\Omega)$ is dense in $C^\infty(\Omega)$.*

2° *$C_0^\infty(\Omega)$ is dense in $L_{p,\text{loc}}(\Omega)$ for all $p \in [1, \infty[$.*

3° *$C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$ for all $p \in [1, \infty[$.*

Proof. 1°. Let $u \in C^\infty(\Omega)$. Choosing η_j as in Corollary 2.14 one has that $\eta_j u \in C_0^\infty(\Omega)$ and $\eta_j u \to u$ in $C^\infty(\Omega)$ for $j \to \infty$ (since $\eta_l u = u$ on K_j for $l \ge j$).

2°. Let $u \in L_{p,\text{loc}}(\Omega)$. Now $\eta_l u \in L_p(\Omega)$ with support in K_{l+1}, and $\eta_l u \to u$ in $L_{p,\text{loc}}(\Omega)$ for $l \to \infty$, since $\eta_l u = u$ on K_j for $l \ge j$. Next, $h_k * \eta_l u \to \eta_l u$ in $L_p(\mathbb{R}^n)$ for $k \to \infty$ by Theorem 2.10. Since $\operatorname{supp}(h_k * \eta_l u) \subset K_{l+2}$ for k sufficiently large, this is also a convergence in $L_{p,\text{loc}}(\Omega)$.

3°. Let $u \in L_p(\Omega)$. Again, $\eta_l u \in L_p(\Omega)$, and now $\eta_l u \to u$ in $L_p(\Omega)$ by the theorem of Lebesgue (since $0 \le \eta_l \le 1$). The proof is completed as under 2°. □

2.4. Partitions of unity.

The special test functions from Corollary 2.14 are also used in the construction of a so-called "partition of unity", i.e. a system of smooth functions with supports in given sets and sum 1 on a given set.

We shall show two versions, in Theorems 2.16 and 2.17; it is the latter that is most often used. In this text we shall not need it until Theorem 3.14.

For Theorem 2.16 we fill out an open set Ω with a countable family of bounded open subsets V_j that is *locally finite* in Ω, i.e., each compact subset of Ω has nonempty intersection with only a finite number of the V_j's. Moreover, we require that the V_j can be shrunk slightly to open sets V_j' with $\overline{V_j'} \subset V_j$ such that the union of the V_j' still covers Ω. As an example of this situation, take the sets

$$\begin{aligned} V_0 &= K_4^\circ, \quad V_j = K_{j+4}^\circ \setminus K_j \text{ for } j \in \mathbb{N}, \\ V_0' &= K_3^\circ, \quad V_j' = K_{j+3}^\circ \setminus K_{j+1} \text{ for } j \in \mathbb{N}, \end{aligned} \tag{2.49}$$

where the K_j are as in (2.4). This system is locally finite since every compact subset of Ω is contained in some K_{j_0}, hence does not meet the V_j with $j \ge j_0$.

Theorem 2.16. *Let the open set Ω be a union of bounded open sets V_j with $\overline{V}_j \subset \Omega$, $j \in \mathbb{N}_0$, for which there are open subsets V_j' such that $\overline{V_j'} \subset V_j$ and we still have $\bigcup_{j\in\mathbb{N}_0} V_j' = \Omega$. Assume moreover, that the cover $\{V_j\}_{j\in\mathbb{N}_0}$ is locally finite in Ω. Then there is a family of functions $\psi_j \in C_0^\infty(V_j)$ taking values in $[0,1]$ such that*

$$\textstyle\sum_{j\in\mathbb{N}_0} \psi_j(x) = 1 \text{ for all } x \in \Omega. \tag{2.50}$$

Proof. Since $\overline{V_j'}$ is a compact subset of V_j, we can for each j choose a function $\zeta_j \in C_0^\infty(V_j)$ that is 1 on V_j' and takes values in $[0,1]$, by Corollary 2.14. Now

$$\Psi(x) = \textstyle\sum_{j\in\mathbb{N}_0} \zeta_j(x)$$

is a well-defined C^∞-function on Ω, since any point $x \in \Omega$ has a compact neighborhood in Ω where only finitely many of the functions ζ_j are nonzero. Moreover, $\Psi(x) \ge 1$ at all $x \in \Omega$, since each x is in V_j' for some j. Then let

$$\psi_j(x) = \frac{\zeta_j(x)}{\Psi(x)} \text{ for } x \in \Omega,\ j \in \mathbb{N}_0.$$

The system $\{\psi_j\}_{j\in\mathbb{N}_0}$ has the desired properties. □

We say that $\{\psi_j\}_{j\in\mathbb{N}_0}$ is a *partition of unity* (cf. (2.50)) for Ω subordinate to the cover $\{V_j\}_{j\in\mathbb{N}_0}$.

The other partition of unity version we need is as follows:

Theorem 2.17. *Let K be a compact subset of $\mathbb{R}^n$, and let $\{V_j\}_{j=0}^N$ be a bounded open cover of K (i.e., the V_j are bounded and open in $\mathbb{R}^n$, and $K \subset \bigcup_{j=0}^N V_j$). There exists a family of functions $\psi_j \in C_0^\infty(V_j)$ taking values in $[0,1]$ such that*

$$\textstyle\sum_{j=0}^N \psi_j(x) = 1 \textit{ for } x \in K. \tag{2.51}$$

Proof. Let us first show that there exist open sets $V_j' \subset V_j$, still forming a cover $\{V_j'\}_{j=0}^N$ of K, such that $\overline{V_j'}$ is a compact subset of V_j for each j. For this, let $V_{jl} = \{\, x \in V_j \mid \operatorname{dist}(x, \partial V_j) > \frac{1}{l} \,\}$, then the family of sets $\{V_{jl}\}_{j=0,\dots,N; l\in\mathbb{N}}$ forms an open cover of K. Since K is compact, there is a finite subfamily that still covers K; here since $V_{jl} \subset V_{jl'}$ for $l < l'$, we can reduce to a system where there is at most one l for each j. Use these V_{jl} as V_j', and supplement by $V_j' = V_{j1}$ for each of those values of j that are not represented in the system.

Now use Corollary 2.14 for each j to choose $\zeta_j \in C_0^\infty(V_j)$, equal to 1 on V_j' and taking values in $[0,1]$. Then

$$\Psi(x) = \textstyle\sum_{j=0}^N \zeta_j(x) \geq 1 \text{ for } x \in \bigcup_{j=0}^N V_j' \supset K. \tag{2.52}$$

Since $\bigcup_{j=0}^N V_j'$ is an open set containing K, we can use Corollary 2.14 once more to find a function $\varphi \in C_0^\infty(\bigcup_{j=0}^N V_j')$ that is 1 on K and takes values in $[0,1]$. Now set

$$\psi_j(x) = \begin{cases} \zeta_j(x)\frac{\varphi(x)}{\Psi(x)} & \text{on } \bigcup_{j=0}^N V_j', \\ 0 & \text{elsewhere}; \end{cases} \tag{2.53}$$

it is a well-defined C^∞ function supported in a compact subset of V_j and taking values in $[0,1]$, and the family of functions ψ_j clearly satisfies (2.51). $\square$

In this case we say that $\{\psi_j\}_{i=0}^N$ is a partition of unity for K subordinate to the cover $\{V_j\}_{i=0}^N$.

Exercises for Chapter 2

2.1. Show that the topology on $C_0^\infty(\Omega)$ is independent of which system of compact subsets satisfying (2.4) is used. (This amounts to a comparison of the corresponding systems of seminorms.)

2.2. Show that convergence of a sequence in $C_0^\infty(\Omega)$ implies convergence of the sequence in $C^\infty(\Omega)$, in $L_p(\Omega)$ and in $L_{p,\mathrm{loc}}(\Omega)$ (for $p \in [1,\infty]$).

2.3. Show that $L_{2,\mathrm{comp}}(\Omega)$ can be identified with the dual space $(L_{2,\mathrm{loc}}(\Omega))^*$ of $L_{2,\mathrm{loc}}(\Omega)$ (the space of continuous linear functionals on $L_{2,\mathrm{loc}}(\Omega)$) in such a way that the element $v \in L_{2,\mathrm{comp}}(\Omega)$ corresponds to the functional

$$u \mapsto \int u(x)\overline{v}(x)\,dx\ , \quad u \in L_{2,\mathrm{loc}}(\Omega)\ .$$

One can use Lemma B.7. (You are just asked to establish the identification for each element.)

2.4. Show that when $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\varphi \subset \underline{B}(0,R)$, then

$$\sup|\varphi(x)| \le 2R\sup|\partial_{x_1}\varphi(x)|\ .$$

(*Hint.* Express φ as an integral of $\partial_{x_1}\varphi$.)

2.5. Show that when $u \in C^1(\mathbb{R}^n)$, then

$$\partial_k(h_j * u) = h_j * \partial_k u\ .$$

(Even if u is not assumed to have compact support, it is only the behavior of u on a compact set that is used when one investigates the derivative at a point.)

2.6. (a) Show that $C_0^\infty(\Omega)$ is dense in $C^k(\Omega)$ for each $k \in \mathbb{N}_0$.
(b) Find out whether $C_0^\infty(\Omega)$ is dense in $C^k(\overline{\Omega})$, in the case $\Omega =]0,1[\subset \mathbb{R}$, for $k \in \mathbb{N}_0$.

2.7. Show that $L_{2,\mathrm{loc}}(\Omega)$ can be identified with $(L_{2,\mathrm{comp}}(\Omega))^*$ in such a way that the element $v \in L_{2,\mathrm{loc}}(\Omega)$ corresponds to the functional

$$u \mapsto \int u(x)\overline{v}(x)\,dx\ ,\ u \in L_{2,\mathrm{comp}}(\Omega)\ .$$