§A. Appendix A. Some notation and prerequisites

We denote by \( \mathbb{Z} \) the integers, by \( \mathbb{N} \) the positive integers and by \( \mathbb{N}_0 \) the nonnegative integers. \( \mathbb{R} \) denotes the real numbers, \( \mathbb{R}^+ \) and \( \mathbb{R}^- \) the positive, resp. negative real numbers. \( \mathbb{R}^n \) is the \( n \)-dimensional real Euclidean space, with points \( x = (x_1, \ldots, x_n) \) and distance \( \text{dist}(x, y) = |x - y| \), where \( |x| = (x_1^2 + \cdots + x_n^2)^{1/2} \). \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_- \) denote the subsets, respectively,

\[
\mathbb{R}^n_\pm = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \},
\]

whose boundary \( \{ x \in \mathbb{R}^n \mid x_n = 0 \} \) is identified with \( \mathbb{R}^{n-1} \). The points in \( \mathbb{R}^{n-1} \) are then often denoted \( x' \),

\[
x' = (x_1, \ldots, x_{n-1}),
\]

so that \( x = (x', x_n) \).

We denote

\[
\{ t \in \mathbb{R} \mid a \leq t \leq b \} = [a, b], \quad \{ t \in \mathbb{R} \mid a < t \leq b \} = ]a, b],
\]

\[
\{ t \in \mathbb{R} \mid a \leq t < b \} = [a, b[,
\quad \{ t \in \mathbb{R} \mid a < t < b \} = ]a, b[
\]

(to avoid conflict between the use of \( (x, y) \) for an open interval and for a scalar product).

The space of complex numbers is denoted \( \mathbb{C} \); \( \mathbb{C}_\pm \) denote the complex numbers with positive resp. negative imaginary part:

\[
\mathbb{C}_\pm = \{ z \in \mathbb{C} \mid \text{Im} \, z \geq 0 \}.
\]

\( \mathbb{C}^n \) denotes the \( n \)-dimensional complex Euclidean space. The functions we consider are usually functions on (subsets of) \( \mathbb{R}^n \) taking values in \( \mathbb{C} \). (Vector valued functions, valued in \( \mathbb{C}^N \), can also occur, or we can consider real functions.)

Set inclusions are denoted by \( \subset \) (whether or not the sets are equal).

Differentiation of functions on \( \mathbb{R} \) is indicated by \( \frac{d}{dx} \), \( \partial_x \) or \( \partial \). Moreover, we write \( \frac{1}{i} \frac{d}{dx} = D_x \) or \( D \) (here \( i \) is the imaginary unit \( i = \sqrt{-1} \)); the factor
\( \frac{1}{i} \) is included for convenience in the use of the Fourier transformation later on. Partial differentiation of functions on \( \mathbb{R}^n \) is indicated by

\[
\frac{\partial}{\partial x_j} = \partial x_j \text{ or } \partial_j; \quad \frac{1}{i} \frac{\partial}{\partial x_j} = D x_j \text{ or } D_j .
\]  

(A.5)

In more complicated expressions we use multi-index notation: When \( \alpha \in \mathbb{N}_0^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), then

\[
\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n} = (-i)^{\mid \alpha \mid} \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} ,
\]  

(A.5)

here \( \mid \alpha \mid = \alpha_1 + \cdots + \alpha_n \). The notation is used for instance for functions having continuous partial derivatives up to order \( \mid \alpha \mid \), such that differentiations in different directions (up to that order) are interchangeable. Using the conventions

\[
\alpha \leq \beta \text{ means } \alpha_1 \leq \beta_1, \ldots, \alpha_n \leq \beta_n ,\\
\alpha! = \alpha_1! \cdots \alpha_n! ,\\
\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n) ,
\]  

(A.6)

we have for \( u \) and \( v \) with continuous derivatives up to order \( N \) the Leibniz formula

\[
\partial^\alpha (uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \partial^\beta u \partial^{\alpha - \beta} v , \quad \text{for } \mid \alpha \mid \leq N,
\]

\[
D^\alpha (uv) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} D^\beta u D^{\alpha - \beta} v , \quad \text{for } \mid \alpha \mid \leq N,
\]

(A.7)

and the Taylor formula

\[
u(x + y) = \sum_{\mid \alpha \mid < N} \frac{y^\alpha}{\alpha!} \partial^\alpha u(x) + \sum_{\mid \alpha \mid = N} \frac{N y^\alpha}{\alpha!} \int_0^1 (1 - \theta)^{N-1} \partial^\alpha u(x + \theta y) d\theta
\]

(A.8)

(this is an exact version from which the other well-known formulations can be deduced).

When \( x \in \mathbb{R}^n \) or \( \mathbb{C}^n \), we write

\[
x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} , \text{ and }\\
x \cdot y = x_1 y_1 + \cdots + x_n y_n , \quad \mid x \mid = (x \cdot \overline{y})^{\frac{1}{2}} .
\]

The norm \( \mid x \mid \) (the Euclidean norm) makes \( \mathbb{R}^n \) and \( \mathbb{C}^n \) Hilbert spaces over \( \mathbb{R} \) resp. \( \mathbb{C} \), with scalar product \( x \cdot \overline{y} \). (The overline indicates complex conjugation.)
We also define
\[
\langle x \rangle = \sqrt{1 + |x|^2}, \text{ which satisfies, for } m \in \mathbb{N}:
\sum_{|\alpha| \leq m} x^{2\alpha} \leq (1 + |x|^2)^m = \sum_{|\alpha| \leq m} C_{m,\alpha} x^{2\alpha};
\]  
(A.9)

here \( C_{m,\alpha} = \frac{m!}{\alpha!(m-|\alpha|)!} \), it is integer \( \geq 1 \).

When \( X \) and \( Y \) are topological spaces, \( X \times Y \) denotes the product space, consisting of pairs \( \{x, y\} \) where \( x \in X \) and \( y \in Y \), provided with the product topology (having as a subbasis the sets \( U \times V \) where \( U \) resp. \( V \) run through a subbasis of the topology of \( X \) resp. \( Y \)). When \( X \) and \( Y \) are vector spaces, \( X \times Y \) is a vector space in the obvious way. If \( X \) and \( Y \) are normed spaces, one can provide \( X \times Y \) by the norm
\[
\|\{x, y\}\|_{X \times Y} = \|x\|_X + \|y\|_Y,
\]  
(A.10)

making \( X \times Y \) a normed space. When \( X \) and \( Y \) are Hilbert spaces, it is more convenient to use the equivalent norm
\[
\|\{x, y\}\|_{X \oplus Y} = \left(\|x\|_X^2 + \|y\|_Y^2\right)^{\frac{1}{2}},
\]  
(A.11)

associated with the scalar product
\[
(\{x, y\}, \{x', y'\})_{X \oplus Y} = (x, x')_X + (y, y')_Y,
\]  
(A.12)

with which \( X \times Y \) is a Hilbert space, denoted \( X \oplus Y \). We use this notation also for the direct sum of two orthogonal closed subspaces \( X \) and \( Y \) of a Hilbert space \( H \). For \( L_p \)-spaces it can be convenient to use \( (\|x\|^p + \|y\|^p)^{\frac{1}{p}} \) as the norm on the product space.

We generally define
\[
X \pm Y = \{x \pm y \mid x \in X \text{ and } y \in Y\}
\]
\[
\Omega X = \{\alpha x \mid \alpha \in \Omega \text{ and } x \in X\}
\]  
(A.13)

when \( X \) and \( Y \) are subsets of a vector space \( V \) with scalar field \( \mathbb{L} \) (\( \mathbb{L} = \mathbb{R} \) or \( \mathbb{C} \)), and \( \Omega \subset \mathbb{L} \). In particular, we write
\[
\{x\} + Y = x + Y
\]
\[
\{\alpha\} Y = \alpha Y
\]  
(A.14)

when \( x \in X \) and \( \alpha \in \mathbb{L} \). When \( X \) and \( Y \) are subspaces of a vector space \( V \), \( X + Y \) is denoted \( X \dot{+} Y \) if \( X \) and \( Y \) are linearly independent. (There is also the notation \( X \oplus Y \) for orthogonal closed subspaces of a Hilbert space.)
A.4

When \( X \) is a closed subspace of a Hilbert space \( H \), the orthogonal complement is denoted \( H \ominus X \).

Integration by parts in one variable is generalized to functions of several variables by the Gauss and Green formulas, which we briefly recall:

Let \( \Omega \subset \mathbb{R}^n \) be an open set with \( C^1 \) boundary \( \partial \Omega \) and let \( \nu(x) \) denote the interior unit normal vector field at \( \partial \Omega \).

To explain this further: \( \Omega \) is said to have a \( C^1 \) boundary, when every boundary point has a neighborhood \( V \) such that — after a relabelling of the coordinates if necessary —

\[
\Omega \cap V = \{ (x_1, \ldots, x_n) \in V \mid x_n > f(x_1, \ldots, x_{n-1}) \},
\]

where \( f: \mathbb{R}^{n-1} \to \mathbb{R} \) is a \( C^1 \)-function. Here

\[
\partial \Omega \cap V = \{ x \in V \mid x_n = f(x_1, \ldots, x_{n-1}) \},
\]

and the interior unit normal vector at the point \( x \in \partial \Omega \cap V \) equals (with the notation (A.4))

\[
\nu(x', f(x')) = \frac{(-\partial_1 f(x'), \ldots, -\partial_{n-1} f(x'), 1)}{\sqrt{\partial_1 f(x')^2 + \cdots + (\partial_{n-1} f(x'))^2 + 1}}.
\]

For a \( C^1 \) function \( u \) defined on a neighborhood of \( \overline{\Omega} \) one has the Gauss formula (when \( u \) has compact support or the integrability is assured in some other way):

\[
\int_{\Omega} \partial_k u \, dx = -\int_{\partial \Omega^+} \nu_k(x) u(x) \, d\sigma, \quad k = 1, \ldots, n,
\]

where \( d\sigma \) is the surface measure on \( \partial \Omega \). In the situation of (A.16),

\[
d\sigma = \frac{1}{|\nu_n(x)|} \, dx = \sqrt{(\partial_1 f)^2 + \cdots + (\partial_{n-1} f)^2 + 1} \, dx_1 \ldots dx_{n-1};
\]

and the formula (A.18) is for \( k = n \) verified for functions supported in \( V \) simply by the change of coordinates \( x = (x', x_n) \mapsto (x', x_n - f(x')) \) that replaces \( \partial \Omega \cap V \) with a subset of \( \mathbb{R}^{n-1} \). From the Gauss formula one derives several other formulas, usually called Green’s formulas, when \( u \) and \( v \) are
suitably differentiable:

\[
\int_{\Omega} \partial_k u \overline{v} \, dx = - \int_{\Omega} u \partial_k v \, dx - \int_{\partial\Omega} \nu_k(x) u(x) \overline{v(x)} \, d\sigma,
\]

\[
\int_{\Omega} D_k u \overline{v} \, dx = \int_{\Omega} u \overline{D_k v} \, dx + i \int_{\partial\Omega} \nu_k(x) u(x) \overline{v(x)} \, d\sigma,
\]

\[
\int_{\Omega} (-\Delta u) \overline{v} \, dx = \sum_{k=1}^{n} \int_{\Omega} \partial_k u \overline{\partial_k v} \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma,
\]

(A.20)

\[
\int_{\Omega} (-\Delta u) \overline{v} \, dx - \int_{\Omega} u(-\Delta v) \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \overline{v} \, d\sigma - \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} \, d\sigma;
\]

where \( \frac{\partial u}{\partial \nu} = \sum_{k=1}^{n} \nu_k \partial_k u \),

the interior normal derivative. Here \( \Delta \) is the Laplace operator \( \partial^2_1 + \cdots + \partial^2_n \).

The signs are chosen with later applications in mind (it is the operator \(-\Delta\) that is “positive”).

Let \( p \in [1, \infty] \). For a Lebesgue measurable subset \( M \) of \( \mathbb{R}^n \), \( L_p(M) \) denotes the vector space of equivalence classes of measurable functions \( f : M \to \mathbb{C} \) with finite norm

\[
\| f \|_{L_p(M)} = \left( \int_M |f(x)|^p \, dx \right)^{1/p} \text{ if } p < \infty,
\]

\[
\| f \|_{L_\infty(M)} = \text{ess sup}_{M} |f| \text{ if } p = \infty .
\]

(A.21)

It is a Banach space with this norm. (The equivalence classes consist of functions that are equal almost everywhere (a.e.); we use the customary “abuse of notation” where one calls the equivalence class a function, denoting the class containing \( f \) by \( f \) again. If the class contains a continuous function — necessarily unique if \( M \) is an open set or the closure of an open set — we use the continuous function as representative. Note that \( C^0(M) \) identifies with a subset of \( L_1(M) \) when \( M \) is the closure of a bounded open set.) We recall that for a real measurable function \( u \) on \( M \),

\[
\text{ess sup}_M u = \inf \{ a \mid u(x) \leq a \text{ a.e. in } M \} .
\]

(A.22)

When \( p = 2 \) we get a Hilbert space, where the norm is associated with the scalar product

\[
(f, g)_{L_2(M)} = \int_M f(x) \overline{g(x)} \, dx .
\]

(A.23)
Hölder’s inequality

\[
\left| \int_M f(x)g(x)dx \right| \leq \|f\|_{L^p(M)} \|g\|_{L^{p'}(M)}, \quad \frac{1}{p} + \frac{1}{p'} = 1 ,
\]

(A.24)

holds for \( f \in L^p(M) \) and \( g \in L^{p'}(M) \); it is the Cauchy-Schwarz inequality in the case \( p = 2 \). Note that \( L^p(\Omega) = L^p(\overline{\Omega}) \) when for example \( \Omega \) has \( C^1 \) boundary.

When the measure of \( M \) is finite, we have an inclusion

\[
L^p(M) \subset L^q(M) \quad \text{for } 1 \leq q \leq p \leq \infty.
\]

(A.25)

Recall that the proof for \( p < \infty \) consists of observing that for \( f \in L^p(M) \) one has, with \( r = p/q, \ 1/r + 1/r' = 1 \), by the Hölder inequality:

\[
\|f\|_{L^q(M)} = \left( \int_M |f(x)|^qdx \right)^{1/q} = \left( \int_M |f(x)|^{p/r} \cdot 1dx \right)^{1/q} \\
\leq \left( \int_M |f(x)|^p \right)^{1/rq} \left( \int_M 1dx \right)^{1/r'q} \\
= \|f\|_{L^p(M)} vol(M)^{1/q-1/r}
\]

(A.26)

where \( vol(M) = \int_M 1dx \) is the volume (measure) of \( M \).

When \( M \subset V \) for some set \( V \), we denote by \( 1_M \) the function on \( V \) defined by

\[
1_M(x) = \begin{cases} 
1 & \text{for } x \in M, \\
0 & \text{for } x \in V \setminus M.
\end{cases}
\]

(A.27)

When \( f \in L^p(\mathbb{R}^n), \ g \in L^q(\mathbb{R}^n), \) and

\[
1 \leq p \leq \infty, \quad 1 \leq q \leq \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0,
\]

(A.28)

then the convolution \( (f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy \) defines a function \( f * g \) in \( L^r(\mathbb{R}^n) \), and

\[
\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}; \quad (A.29)
\]

Young’s inequality. In particular, if \( f \in L^1(\mathbb{R}^n) \) and \( g \in L^2(\mathbb{R}^n) \), then \( f * g \in L^2(\mathbb{R}^n) \), and

\[
\|f * g\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.
\]

(A.30)

When \( \Omega \) is an open subset of \( \mathbb{R}^n \), we denote by \( L^p,\text{loc}(\Omega) \) the set of funtions on \( \Omega \) whose restrictions to compact subsets \( K \subset \Omega \) are in \( L^p(K) \). In view of (A.25), one has that

\[
L^p,\text{loc}(\Omega) \subset L^q,\text{loc}(\Omega) \quad \text{for } 1 \leq q \leq p \leq \infty.
\]

(A.31)
In particular, $L_{1,\text{loc}}(\Omega)$ is the space of locally integrable functions on $\Omega$ (containing all the other spaces $L_{p,\text{loc}}(\Omega)$).

The lower index $p$ on $L_p$-spaces (instead of an upper index) reflects the fact that $p$ is placed in this way in the modern literature on function spaces, such as $L_p$-types of Sobolev spaces $H^s_p$, $B^s_p$ (and their numerous generalizations), where the upper index $s$ is reserved for the number of well-defined derivatives.
A.8

Exercises for Chapter 1.

A.1. Show the general Leibniz formulas (A.7).

A.2. (a) Let $f \in C^1(\mathbb{R}^n)$. Show for any $x, y \in \mathbb{R}^n$ that the function $g(\theta) = f(x + \theta y)$ ($\theta \in \mathbb{R}$) satisfies:

$$
\frac{d}{d\theta} g(\theta) = \sum_{j=1}^{n} \partial_j f(x + \theta y)y_j,
$$

and conclude from this that

$$
f(x + y) = f(x) + \sum_{j=1}^{n} y_j \int_{0}^{1} \partial_j f(x + \theta y) \, d\theta.
$$

(b) Show Taylor’s formula (A.8) for arbitrary $N$.

A.3. Deduce the formulas in (A.20) from (A.18).  
(Hint. Apply (A.18) to $\partial_k(u\overline{v})$.)