MOMENT FORMULAS FOR
THE QUASI-NILPOTENT DT-OPERATOR

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ABSTRACT. Let $T$ be the quasi-nilpotent DT-operator. By use of Voiculescu’s amalgamated $R$-transform we compute the moments of $(T - \lambda I)(T - \lambda I)$ where $\lambda \in \mathbb{C}$, and the Brown-measure of $T + \sqrt{\epsilon} Y$, where $Y$ is a circular element $*$-free from $T$ for $\epsilon > 0$. Moreover we give a new proof of Śniady’s formula for the moments $\tau((T^*)^k T^k)^{\otimes n}$ for $k, n \in \mathbb{N}$.

1. INTRODUCTION

The quasi-nilpotent DT-operator $T$ was introduced by Dykema and the second author in [4]. It can be described as the limit in $*$-moments for $n \to \infty$, of random matrices of the form

$$ T^{(n)} = \begin{pmatrix}
0 & t_{1,2} & \cdots & t_{1,n} \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & t_{n-1,n} \\
0 & \cdots & 0 & 0
\end{pmatrix} 
$$

where $\{\Re(t_{ij}), \Im(t_{ij})\}_{1 \leq i < j \leq n}$ is a set of $n(n-1)$ independent identically distributed Gaussian random variables with mean 0 and variance $\frac{1}{2n}$. More precisely, $T$ is an element in a finite von Neumann algebra, $M$, with a faithful normal tracial state, $\tau$, such that for all $s_1, s_2, \ldots, s_k \in \{1, *\}$,

$$ \tau(T^{s_1} T^{s_2} \cdots T^{s_k}) = \lim_{n \to \infty} \mathbb{E}[\text{tr}_n((T^{(n)})^{s_1} (T^{(n)})^{s_2} \cdots (T^{(n)})^{s_k})], $$

where $\text{tr}_n$ is the normalized trace on $M_n(\mathbb{C})$. Moreover the pair $(T, W^*(T))$ is uniquely determined up to $*$-isomorphism by (1.1). The quasi-nilpotent DT-operator can be realized as an element in the free group factor, $L(F_2)$, in the following way (cf. [4, Sect. 4]): Let $(D_0, X)$ be a pair of free selfadjoint elements in a tracial $W^*$-probability space

† The second named author is affiliated with MaPhySto - A network in Mathematical Physics and Stochastics, which is funded by a grant from the Danish National Research Foundation.
(M, τ), such that \( d\mu_{D_0}(t) = 1_{[0,1]}(t)dt \) and \( X \) is semi-circular distributed, i.e. \( d\mu_X(t) = \frac{1}{\pi} \sqrt{4 - t^2}1_{[-2,2]}(t)dt \). Then \( W^*(D_0, X) \simeq W^*(D_0) \ast W^*(X) \simeq L(F_2) \). Put

\[
T_N = \sum_{j=1}^{2^N} p_{N,j}Xq_{N,j}
\]

for \( N = 1, 2, \ldots \), where

\[
p_{N,j} = 1_{[\frac{j-1}{2^N}, \frac{j}{2^N}]}(D_0), \quad q_{N,j} = 1_{[\frac{j-1}{2^N}, \frac{j}{2^N}]}(D_0),
\]

for \( j = 1, 2, \ldots, 2^N \). Then \( (T_N)_{N=1}^{\infty} \) converges in norm to an operator \( T \in W^*(D_0, X) \), and the \( * \)-moments of \( T \) are given by (1.1), i.e. \( T \) is a realization of the quasi-nilpotent DT-operator. In the notation of [4, Sect. 4], \( T = \mathcal{U}(X, \lambda) \), where \( \lambda : L^\infty[0,1] \to W^*(D_0) \) is the \( * \)-isomorphism given by \( \lambda(f) = t(D_0) \) for \( f \in L^\infty([0,1]) \). In the following we put \( D = W^*(D_0) \simeq L^\infty([0,1]) \) and let \( E_D \) denote the trace-preserving conditional expectation of \( W^*(D_0, X) \) onto \( D \).

In this paper we apply Voiculescu’s \( \mathcal{R} \)-transform with amalgamation to compute various \( * \)-moments of \( T \) and of operators closely related to \( T \). First we compute in section 3 moments and the scalar valued \( \mathcal{R} \)-transform of \( (T - \lambda 1)^{*}(T - \lambda 1) \) for \( \lambda \in \mathbb{C} \). The specialized case of \( \lambda = 0 \) was treated in [4] by more complicated methods. In section 4 we consider the operator

\[
T + \sqrt{\epsilon}Y,
\]

where \( Y \) is a circular operator \( * \)-free from \( T \) and \( \epsilon > 0 \). By random matrix considerations it is easily seen, that if \( T_1 \) and \( T_2 \) are two quasi-nilpotent DT-operators, which are \( * \)-free with respect to amalgamation over the same diagonal, \( \mathcal{D} \), then \( T + \sqrt{\epsilon}Y \) has the same \( * \)-distribution as \( S = \sqrt{\alpha}T_1 + \sqrt{\beta}T_2 \), when \( a = 1 + \epsilon \) and \( b = \epsilon \) (cf. [1]). We use this fact to prove, that the Brown measure of \( T + \sqrt{\epsilon}Y \) is equal to the uniform distribution on the closed disc \( \overline{B}(0, \log(1 + \frac{1}{2})^{-\frac{1}{2}}) \) in the complex plane. Moreover we show, that the spectrum of \( T + \sqrt{\epsilon}Y \) is not a DT-operator for any \( \epsilon > 0 \).

In [4] it was conjectured, that

\[
\tau((T^*)^kT^*)^n) = \frac{n^n}{(nk + 1)!}
\]

for \( n,k \in \mathbb{N} \). This formula was proved by Śniady in [9]. Śniady’s proof of (1.2) is based on Speicher’s combinatorial approach to free probability with amalgamation from [11]. The key step in the proof of
(1.2) was to establish a recursion formula for the $\mathcal{D}$-valued moments,
\begin{equation}
E_{\mathcal{D}}\left(\left(T^*kT^k\right)^n\right)
\end{equation}
for each fixed $k \in \mathbb{N}$. Šniady’s recursion formula for the $\mathcal{D}$-valued moments (1.3), was later used by Dykema and the second author to prove, that
\begin{equation}
W^*(T) = W^*(D_0, X) \simeq L(\mathbb{F}_2)
\end{equation}
and that $T$ admits a one parameter family of non-trivial hyperinvariant subspaces (cf. [5]). In section 5 and section 6 of this paper we give a new proof of Šniady’s recursion formula for the $\mathcal{D}$-valued moments (1.3), which at the same time gives a new proof of (1.2). The new proof is based on Voiculescu’s $\mathcal{R}$-transform with respect to amalgamation over $M_{2k}(\mathcal{D})$, the algebra of $2k \times 2k$ matrices over $\mathcal{D}$.

2. Preliminaries

In this section we give a few preliminaries on amalgamated probability theory. Let $\mathcal{A}$ be a unital Banach algebra, and let $\mathcal{B}$ be a Banach-sub-algebra containing the unit of $\mathcal{A}$. Then a map, $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$, is a conditional expectation if
(a) $E_{\mathcal{B}}$ is linear,
(b) $E_{\mathcal{B}}$ preserves the unit i.e. $E_{\mathcal{B}}(1) = 1$
(c) and $E_{\mathcal{B}}$ has the $\mathcal{B}$, $\mathcal{B}$ bi-module property i.e. $E_{\mathcal{B}}(b_1ab_2) = b_1ab_2$
for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

If $\mathcal{B}$, $\mathcal{A}$ and $E_{\mathcal{B}}$ are as above we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a $\mathcal{B}$-probability space. If $\phi : \mathcal{A} \to \mathcal{C}$ is a state on $\mathcal{A}$ which respects $E_{\mathcal{B}}$, i.e. $\tau = \tau \circ E_{\mathcal{B}}$, we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is compatible to the (non-amalgamated) free probability space $(\mathcal{A}, \phi)$.

If $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a $\mathcal{B}$-probability space and $a \in \mathcal{A}$ is a fixed variable, we define the amalgamated Cauchy transform of $a$ by
\begin{equation}
G_a(b) = E_{\mathcal{B}}((b - a)^{-1}).
\end{equation}
for $b \in \mathcal{B}$ and $b - a \in \mathcal{B}_{inv}$. The Cauchy transform is 1-1 in 
\{ $b \in \mathcal{B}_{inv} ||b^{-1}|| < \epsilon$ \} for $\epsilon$ sufficiently small and Voiculescu’s amalgamated $\mathcal{R}$-transform [13] is now defined for $a \in \mathcal{A}$ by
\begin{equation}
\mathcal{R}_a(b) = G_a^{(-1)}(b) - b^{-1},
\end{equation}
for $b$ being an invertible element of $\mathcal{B}$ suitably close to zero. It turns out that this definition coincides on invertible element with Speicher’s definition of the amalgamated $\mathcal{R}$-transform (cf. [11, Th. 4.1.2] and [2]);
\begin{equation}
\mathcal{R}_a(b) = \sum_{n=1}^{\infty} \kappa_n^B(a \otimes_B b a \otimes_B \cdots \otimes_B b a).
\end{equation}
We will need the following useful lemma for solving equations involving the amalgamated \(R\)-transform and Cauchy-transform.

**Lemma 2.1.** Let \((B \subset A, E_B)\) be a \(B\)-probability space, and let \(a \in A\). Then there exists \(\delta > 0\) such that if \(b \in B\) is invertible, \(\|b\| < \delta, \|\mu\| > \frac{1}{\delta}\) and
\[
R^B_a\(b\) + b^{-1} = \mu 1_A
\]
then \(b = G^B_a(\mu 1_A)\).

**Proof.** Let \(\delta = \frac{1}{11|a|}\) and define \(g_a(b) = G^B_a(b^{-1})\). By [2, Prop. 2.3] we know that \(g_a\) maps \(B(0, \frac{1}{11|a|})\) bijectively onto a neighborhood of zero containing \(B(0, \frac{1}{11|a|})\) and furthermore that
\[
g_a^{-1}(B(0, \frac{1}{11|a|})_{inv}) \subseteq B(0, \frac{2}{11|a|})_{inv}.
\]
By definition we know that
\[
R^B_a(b) = G^B_a(g_a^{-1}(b)) + b^{-1} = (g_a^{-1}(b))^{-1} + b^{-1}
\]
so if \(R_a(b) + b^{-1} = \mu 1_A\) then
\[
\mu 1_A = g_a^{-1}(b) - b^{-1} + b^{-1} = (g_a^{-1}(b))^{-1}
\]
and thus
\[
g_a^{-1}(b) = \frac{1}{\mu} 1_A.
\]
If \(|\mu| > \frac{1}{\delta}\) then especially \(\frac{1}{|\mu|} < \frac{1}{11|a|}\) so \(\frac{1}{\mu} 1_A\) is in the bijective domain of \(g_a\), so applying \(g_a\) on both sides of (2.3) we get exactly
\[
G^B_a(\mu 1_A) = g_a(\frac{1}{\mu} 1_A) = b
\]
since also \(\|b\| < \frac{1}{11|a|}\). \(\square\)

If \(a \in A\) is a random variable in the \(B\)-probability space \((B \subset A, E_B)\), then following Speicher we define \(a\) to be \(B\)-Gaussian [11, Def 4.2.3] if only \(B\)-cumulants of length 2 survive. From (2.2) it follows that in this case the \(R\)-transform has a particularly simple form, namely,

\[
R_a(b) = \kappa^B_2(a \otimes_B ba) = E_B(aba).
\]

In the following theorem (which is probably not a new one we just could not find a proper reference) concerning cumulants we have adopted the notation of Speicher from [11].

**Lemma 2.2.** Let \(N \in \mathbb{N}\) and let \((B \subset A, E_B)\) be a \(B\)-probability space. Then \((M_N(B) \subset M_N(A), E_{M_N(B)})\) is a \(M_N(B)\)-probability space with
cumulants determined by the following formula:

\[ \kappa_n^{M_n(B)}((m_1 \otimes a_1) \otimes M_n(B) \otimes \cdots \otimes M_n(B) (m_n \otimes a_n)) = (m_1 \cdots m_n) \otimes \kappa_n^{B}(a_1 \otimes_B \cdots \otimes_B a_n) \]

when \( m_1, \ldots, m_n \in M_n(\mathbb{C}) \) and \( a_1, \ldots, a_n \in A \).

We have of course made the identification \( M_n(A) \cong M_n(\mathbb{C}) \otimes A \).

Proof. Since \( M_n(\mathbb{C}) \subset M_n(B) \) we observe that

\[ \kappa_n^{M_n(B)}((m_1 \otimes a_1) \otimes M_n(B) \otimes \cdots \otimes M_n(B) (m_n \otimes a_n)) = ((m_1 \cdots m_n) \otimes 1) \cdot \kappa_n^{M_n(B)}((1 \otimes a_1) \otimes M_n(B) \otimes \cdots \otimes M_n(B) (1 \otimes a_n)). \]

To finish the proof we claim that

\[ (2.5) \quad \kappa_n^{M_n(B)}((1 \otimes a_1) \otimes M_n(B) \otimes \cdots \otimes M_n(B) (1 \otimes a_n)) = 1 \otimes \kappa_n^{B}(a_1 \otimes_B \cdots \otimes_B a_n). \]

The case \( n = 1 \) is obvious since

\[ 1_N \otimes \kappa_1^B(a_1) = 1_N \otimes E_B(a_1) = E_{M_n(B)}(1 \otimes a_1) = \kappa_1^{M_n(B)}(1 \otimes a_1). \]

Now assume that the claim is true for \( 1, 2, \ldots, n - 1 \). Then (2.5) has an obvious extension to noncrossing partitions of length less than or equal to \( n - 1 \). Hence

\[ 1_N \otimes \kappa_n^{B}(a_1 \otimes_B \cdots \otimes_B a_n) = 1_N \otimes E_B(a_1 \cdots a_n) - \sum_{\pi \in NC([n], \pi \neq 1_n)} 1 \otimes \kappa_n^{B}(a_1 \otimes_B \cdots \otimes_B a_n) = E_{M_n(B)}(1 \otimes M_n(B) a_1 \cdots a_n) \]

\[ - \sum_{\pi \in NC([n], \pi \neq 1_n)} \kappa_n^{M_n(B)}((1 \otimes a_1) \otimes M_n(B) \cdots \otimes M_n(B) (1 \otimes a_n)) = \kappa_n^{M_n(B)}((1 \otimes a_1) \otimes M_n(B) \cdots \otimes M_n(B) (1 \otimes a_n)). \]

By induction this proves the lemma. \( \square \)

Assume that \( \mathcal{M} \) contains a pair \( (D_0, X) \) of \( \tau \)-free selfadjoint elements such that \( d \mu_{D_0}(t) = 1_{[0,1]}(t) dt \) and \( X \) is a semicircular distributed. Put \( \mathcal{D} = W^*(D_0) \). Then \( \lambda : L^\infty([0,1]) \to \mathcal{D} \) given by

\[ \lambda(f) = f(D_0), \]
for \( f \in L^\infty([0,1]) \) is a \( * \)-isomorphism of \( L^\infty([0,1]) \) onto \( \mathcal{D} \) and
\[
\tau \circ \lambda(f) = \int_0^1 f(t) dt, \quad f \in L^\infty([0,1]).
\]
We will identify \( \mathcal{D} \) with \( L^\infty([0,1]) \) and thus consider elements of \( \mathcal{D} \) as functions. As explained in the introduction, we can realize the quasi-nilpotent DT-operator as the operator \( T = \mathfrak{T}(X, \lambda) \) in \( \mathcal{W}^*(D_0, X) \cong L(\mathbb{F}_2) \).

Define for \( f \in \mathcal{D} \cong L^\infty([0,1]) \)
\[
(2.6) \quad (L^*(f))(x) := \int_0^x f(t) dt \quad \text{and} \quad (L(f))(x) := \int_x^1 f(t) dt.
\]
From the appendix of [5] it follows that \((T, T^*)\) is a \( \mathcal{D} \)-Gaussian pair and that the covariances of \((T, T^*)\) are given by the following lemma

**Lemma 2.3.** [5, Appendix] Let \( f \in \mathcal{D} \). Then
\[
E_\mathcal{D}(TfT^*) = L(f) \quad \text{and} \quad E_\mathcal{D}(T^*fT) = L^*(f)
\]
and \( E_\mathcal{D}(TfT) = E_\mathcal{D}(T^*fT^*) = 0 \).

3. **Moments and \( \mathcal{R} \)-transform of \((T - \lambda 1)^*(T - \lambda 1)\)**

Let \( T \) be the quasi-diagonal DT-operator and define
\[
\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.
\]
Since \((T, T^*)\) is a \( \mathcal{D} \)-Gaussian pair, it follows from lemma 2.2, that cumulants of the form
\[
\kappa^{M_2(\mathcal{D})}_n(((m_1 \otimes a_1) \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} (m_n \otimes a_n)))
\]
vanishes when \( n \neq 2, \ m_1, m_2, \ldots, m_n \in M_2(\mathcal{C}) \) and \( a_1, a_2, \ldots, a_n \in \{T, T^*\} \). Hence by the linearity of \( \kappa^{M_2(\mathcal{D})}_n \),
\[
\kappa^{M_2(\mathcal{D})}_n(\tilde{T} \otimes_{M_2(\mathcal{D})} \tilde{T} \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} \tilde{T}) = 0
\]
when \( n \neq 2 \), i.e, \( \tilde{T} \) is a \( M_2(\mathcal{D}) \)-Gaussian element in \( M_2(\mathcal{M}) \) under the conditional expectation \( E_{M_2(\mathcal{D})} : M_2(\mathcal{M}) \to M_2(\mathcal{D}) \) given by
\[
E_{M_2(\mathcal{D})} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E_{\mathcal{D}}(a_{11}) & E_{\mathcal{D}}(a_{12}) \\ E_{\mathcal{D}}(a_{21}) & E_{\mathcal{D}}(a_{22}) \end{pmatrix}.
\]
Since $\tilde{T}$ is $\mathcal{M}_2(\mathcal{D})$-Gaussian the $\mathcal{R}$-transform of $\tilde{T}$ is by (2.4) the linear mapping $\mathcal{M}_2(\mathcal{D}) \to \mathcal{M}_2(\mathcal{D})$ given by

$$
\mathcal{R}_{\tilde{T}}^{\mathcal{M}_2(\mathcal{D})}(z) = E_{\mathcal{M}_2(\mathcal{D})}(\tilde{T}z\tilde{T})
$$

$$
= E_{\mathcal{M}_2(\mathcal{D})} \left( \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \right)
$$

$$
= E_{\mathcal{M}_2(\mathcal{D})} (T^*z_{22}T 0 \\ 0 Tz_{11}T^*)
$$

$$
= \left( \begin{array}{cc}
E_D(T^*z_{22}T) & 0 \\
0 & E_D(Tz_{11}T^*)
\end{array} \right)
$$

$$
= \left( \begin{array}{cc}
L^*(z_{22}) & 0 \\
0 & L(z_{11})
\end{array} \right).
$$

For $\lambda \in \mathbb{C}$, we put $T_\lambda = T\lambda 1$ and define

$$
\tilde{T}_\lambda = \begin{pmatrix} 0 & T_\lambda \\ T_\lambda & 0 \end{pmatrix} = \tilde{T} - \begin{pmatrix} 0 & \bar{\lambda} \\ \lambda & 0 \end{pmatrix}
$$

Since $\begin{pmatrix} 0 & \bar{\lambda} \\ \lambda & 0 \end{pmatrix} \in \mathcal{M}_2(\mathcal{D})$ we have by $\mathcal{M}_2(\mathcal{D})$-freeness that the $\mathcal{R}$-transform is additive [11, Th. 4.1.22] i.e.

$$
\mathcal{R}_{\tilde{T}_\lambda}^{\mathcal{M}_2(\mathcal{D})}(z) = \mathcal{R}_{\tilde{T}}^{\mathcal{M}_2(\mathcal{D})} - \begin{pmatrix} 0 & \bar{\lambda} \\ \lambda & 0 \end{pmatrix} = \begin{pmatrix} L^*(z_{22}) & -\lambda \\ -\lambda & L(z_{11}) \end{pmatrix}.
$$

One easily checks, that if $\delta \in \mathbb{C}$, $\delta \neq 0$, $\delta \neq -\frac{1}{|\lambda|^2}$ and $\mu \in \mathbb{C}$ is one of the two solutions to

$$
\mu^2 = \frac{e^\sigma}{\sigma} (1 + |\lambda|^2 \sigma),
$$

then

$$
\begin{align*}
z_{11} &= \mu \sigma e^\sigma (x^{-1}) \\
z_{12} &= -\bar{\lambda} \sigma \\
z_{21} &= -\lambda \sigma \\
z_{22} &= \mu \sigma e^{-\sigma x}
\end{align*}
$$

is a solution to

$$
\mathcal{R}_{\tilde{T}_\lambda}^{\mathcal{M}_2(\mathcal{D})}(z) + z^{-1} = \mu 1_2.
$$

Here $x$ is the variable for the function in $\mathcal{D} = L^\infty([0, 1])$. In particular $z_{12}$ and $z_{21}$ are constant operators. If $\sigma \to 0$ then $|\mu| \to \infty$ and $\|z\| \to 0$, so by lemma 2.1 there exists $\rho > 0$ such that $|\sigma| < \rho$ implies

$$
G_{\tilde{T}_\lambda}^{\mathcal{M}_2(\mathcal{D})}(\mu 1_2) = \begin{pmatrix} z_{11} & z_{12} \\
z_{21} & z_{22} \end{pmatrix},
$$
where \((z_{ij})_{i,j \in \{1,2\}}\) is given by (3.1) and

\[
\mu = \pm \sqrt{\frac{\sigma}{\sigma^2(1 + |\lambda|^2 \sigma)}).
\]

On the other hand the Cauchy-transform of \(T\) in \(\mu_2\) is

\[
\begin{pmatrix}
  z_{11} \\
  z_{12} \\
  z_{21} \\
  z_{22}
\end{pmatrix}
= G_{T,\lambda}^{M_2}(\mu_2)
= E_{M_2}(\mu_2)
\left(
\begin{pmatrix}
  \mu_1 & 0 \\
  0 & \mu_1
\end{pmatrix}
- \begin{pmatrix}
  T_\lambda^* \\
  T_\lambda
\end{pmatrix}^{-1}
\right)
\]

\[
= E_{M_2}(\mu_2)
\left(
\begin{pmatrix}
  \mu_1 & -T_\lambda^* \\
  -T_\lambda & \mu_1
\end{pmatrix}^{-1}
\right)
\]

\[
= E_{M_2}(\mu_2)
\left(
\begin{pmatrix}
  \mu_1(\mu_2^2 - T_\lambda^* T_\lambda) & -T_\lambda^* (\mu_2^2 - T_\lambda^* T_\lambda) \\
  T_\lambda(\mu_2^2 - T_\lambda^* T_\lambda) & \mu_1(\mu_2^2 - T_\lambda^* T_\lambda)
\end{pmatrix}
\right) .
\]

Thus

\[
\begin{pmatrix}
  z_{11} \\
  z_{12} \\
  z_{21} \\
  z_{22}
\end{pmatrix}
= \begin{pmatrix}
  \mu E_D((\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) \\
  E_D(T_\lambda(\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) \\
  E_D((\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) \\
  \mu E_D((\mu_2^2 - T_\lambda^* T_\lambda)^{-1})
\end{pmatrix}
\]

(3.2)

Combining (3.1) and (3.2) we have

\[
\begin{pmatrix}
  \mu E_D((\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) = \sigma e^{\sigma(x - 1)} \\
  E_D(T_\lambda(\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) = -\lambda \sigma \\
  E_D((\mu_2^2 - T_\lambda^* T_\lambda)^{-1}) = \sigma e^{-\sigma x}
\end{pmatrix}.
\]

(3.3)

We can now compute the \(\mathcal{R}\)-transform of \(T_\lambda^* T_\lambda\) (wrt. \(\mathbb{C}\)) from (3.3) and the defining equality for \(\mu_2^2\).

\[
\text{tr}\left(\frac{e^{\sigma}}{\sigma}(1 + |\lambda|^2 \sigma)^{-1} - T_\lambda^* T_\lambda\right) = \int_0^1 \sigma e^{\sigma(x - 1)} dx
\]

\[
= [e^{\sigma(x - 1)}]_0^1 = 1 - e^{-\sigma}.
\]

Thus

\[
G_{T_\lambda^* T_\lambda}^\mathbb{C}\left(\frac{e^{\sigma}}{\sigma}(1 + |\lambda|^2 \sigma)^{-1}\right) = 1 - e^{-\sigma}
\]

i.e.

\[
\mathcal{R}_{T_\lambda^* T_\lambda}(1 - e^{-\sigma}) = \frac{e^{\sigma}}{\sigma}(1 + |\lambda|^2 \sigma) - \frac{1}{1 - e^{-\sigma}}
\]
for \( \sigma \) in a neighborhood of zero. Substituting \( z = 1 - e^{-\sigma} \) we get 
\[
\Re_{T_\lambda}^C(z) = -\frac{1}{(1 - z) \log(1 - z)} \left( 1 - |\lambda|^2 \log(1 - z) \right) - \frac{1}{z}.
\]
Hence we have proved the following extension of [4, Theorem 8.7(b)]:

**Theorem 3.1.** Let \( T \) be the quasi-nilpotent DT-operator. Let \( \lambda \in \mathbb{C} \) and put \( T_\lambda = T - \lambda 1 \). Then
\[
\Re_{T_\lambda}^C(z) = -\frac{1}{(1 - z) \log(1 - z)} - \frac{1}{z} + \frac{|\lambda|^2}{1 - z}
\]
for \( z \) in some neighborhood of 0.

We next determine the \( D \)-valued (resp. \( \mathbb{C} \)-valued) moments of \( T_\lambda^*T_\lambda \) for all \( \lambda \in \mathbb{C} \). The special case \( \lambda = 0 \) was treated in [9, Theorem 5] (resp. [4, Theorem 8.7(a)]) by different methods.

**Theorem 3.2.** Let \( \lambda \in \mathbb{C} \) and let \( T, T_\lambda \) be as in theorem 3.1

(a) Let \( Q_n \) be the sequence of polynomials on \( \mathbb{R} \) uniquely determined by the following recursion formula
\[
Q_0(x) = 1, \\
Q_{n+1}(x) = |\lambda|^2 Q_n(x) + \int_0^x Q_n(y + 1) dy \quad \text{for } n \geq 1.
\]
Then
\[
E_D((T_\lambda^*T_\lambda)^n)(x) = Q_n(x), \quad x \in [0,1], \quad n \in \mathbb{N}.
\]

(b) 
\[
\tau((T_\lambda^*T_\lambda)^n) = \sum_{k=0}^n \frac{n^k}{(k+1)!} \left( \frac{n}{k} \right) |\lambda|^{2n-2k}, \quad n \in \mathbb{N}.
\]

**Proof.** By (3.3), we have
\[
E_D((\frac{\sigma}{\tau}(1 + |\lambda|^2 \sigma) 1 - T_\lambda^*T_\lambda)^{-1}) = \sigma e^{\psi(x)}
\]
for \( \sigma \in B(0, \rho) \setminus \{0\} \) for some \( \rho > 0 \). Put
\[
\psi(\sigma) = \frac{\sigma}{e^{\sigma(1 + |\lambda|^2 \sigma)}}, \quad \sigma \in \mathbb{C} \setminus \{-\frac{1}{|\lambda|^2}\}.
\]
Since \( \psi(0) = 0 \) and \( \psi'(0) = 1 \), \( \psi \) has an analytic invers \( \psi^{(-1)} \) defined in a neighborhood \( B(0, \delta) \) of 0, and we can choose \( \delta > 0 \), such that \( \psi^{(-1)}(B(0, \delta)) \subset B(0, \rho) \). By (3.5)
\[
E_D((\frac{1}{t} 1 - T_\lambda^*T_\lambda)^{-1}) = \psi^{(-1)}(t) e^{\psi^{(-1)}(t)}(x-1)
\]
for \( t \in B(0, \delta) \setminus \{0\} \). By power series expansion of the left hand side, we get

\[
\sum_{n=0}^{\infty} t^{n+1} E_D((T^n_\lambda T_\lambda)^n) = \psi^{-1}(t) e^{\psi^{-1}(t)(x-1)}
\]

for \( t \in B(0, \delta') \), where \( 0 < \delta' \leq \delta \) and where the LHS of (3.6) is absolutely convergent in the Banach space \( L^\infty([0, 1]) \). Hence by Cauchy’s integral formulas

\[
E_D((T^n_\lambda T_\lambda)^n) = \frac{1}{2\pi i} \int_C \frac{\psi^{-1}(t) e^{\psi^{-1}(t)(x-1)}}{t^{n+2}} dt
\]

as a Banach space integral in \( L^\infty([0, 1]) \), where \( C = \partial B(0, r) \) with positive orientation and \( 0 < r < \delta' \). For each fixed \( x \in \mathbb{R} \)

\[
t \mapsto \psi^{-1}(t) e^{\psi^{-1}(t)(x-1)}
\]

is an analytic function in \( B(0, \delta') \) which is 0 for \( t = 0 \). Hence the function has a power series expansion of the form

\[
\psi^{-1}(t) e^{\psi^{-1}(t)(x-1)} = \sum_{n=0}^{\infty} Q_n(x) t^{n+1}
\]

for \( t \in B(0, \delta') \), where the numbers \((Q_n(x))_{n=0}^{\infty}\) are given by

\[
Q_n(x) = \frac{1}{2\pi i} \int_C \frac{\psi^{-1}(t) e^{\psi^{-1}(t)(x-1)}}{t^{n+2}} dt.
\]

In particular the \( Q_n \)'s are continuous functions of \( x \in \mathbb{R} \). Substituting \( \sigma = \psi(t) \) in (3.8) we get

\[
\sum_{n=0}^{\infty} Q_n(x) \psi(\sigma)^{n+1} = \sigma e^\sigma(x-1)
\]

for \( \sigma \in B(0, \rho') \), where \( \rho' \in (0, \rho) \). Put

\[
\begin{cases}
R_0(x) = 0 \\
R_{n+1}(x) = |\lambda|^2 Q_n(x + 1) + \int_0^x Q_n(x) dy, \quad n \geq 0.
\end{cases}
\]
Then
\[
\sum_{n=0}^{\infty} R_n(x) \psi(\sigma)^{n+1} = \psi(\sigma) \left( 1 + \sum_{n=0}^{\infty} R_{n+1}(x) \psi(\sigma)^{n+1} \right)
\]
\[
= \psi(\sigma) \left( 1 + |\lambda|^2 \left( \sum_{n=0}^{\infty} Q_n(x + 1) \right) + \int_0^x \left( \sum_{n=0}^{\infty} Q_n(y + 1) \right) \, dy \right)
\]
\[
= \psi(\sigma) \left( 1 + |\lambda|^2 \sigma e^{\sigma x} + \int_0^x \sigma e^{\sigma y} \, dy \right)
\]
\[
= \psi(\sigma)(|\lambda|^2 \sigma + 1)e^{\sigma x} = \sigma e^{\sigma(x-1)} = \sum_{n=0}^{\infty} Q_n(x) \psi(\sigma)^{n+1}
\]
for all \(\sigma \in B(0, \rho')\). Since \(\psi(B(0, \rho'))\) is an open neighborhood of 0 in \(\mathbb{C}\), it follows that \(R_n(x) = Q_n(x)\) for all \(n \in \mathbb{N}\) and all \(x \in \mathbb{R}\).

Hence \((Q_n(x))_{n=0}^{\infty}\) is the sequence of polynomials given by the recursive formula (3.4). Moreover by (3.7) and (3.9), \(E_D((T^*_\lambda T^*_\lambda)^n) = Q_n\) as functions in \(L^\infty([0, 1])\). This proves (a).

(b) By (3.7), we have
\[
\tau((T^*_\lambda T^*_\lambda)^n) = \int_0^1 E_D((T^*_\lambda T^*_\lambda)^n) \, dx = \frac{1}{2\pi i} \int_C \frac{1 - e^{-\psi(t+1)}}{t^{n+2}} \, dt.
\]
Note that \(C' = \psi(C)\) is a positively oriented simple path around 0. Hence by the substitution \(t = \psi(\sigma)\), we get
\[
\tau((T^*_\lambda T^*_\lambda)^n) = \frac{1}{2\pi i} \int_{C'} \frac{\psi'(\sigma)}{\psi(\sigma)^{n+2}} (1 - e^{-\sigma}) \, d\sigma
\]
\[
= \frac{1}{2\pi i} \int_{C'} \frac{\psi'(\sigma)}{\psi(\sigma)^{n+1}} \frac{1}{n + 1 - \psi(\sigma)^{n+1}} (1 - e^{-\sigma}) \, d\sigma
\]
\[
= \frac{1}{2\pi i} \int_{C'} \frac{e^{-\sigma}}{\psi(\sigma)^{n+1}} \, d\sigma
\]
\[
= \frac{1}{n + 1} \left( \frac{1}{2\pi i} \int_{C'} e^{\sigma} (1 + |\lambda|^2 \sigma)^{n+1} \, d\sigma \right)
\]
\[
= \frac{1}{n + 1} \text{Res} \left( \frac{e^{\sigma} (1 + |\lambda|^2 \sigma)^{n+1}}{\sigma^{n+1}}, 0 \right)
\]
where the second equation is obtained by partial integration and the last equality is obtained by the Residue theorem,
The above Residue is equal to the coefficient of $\sigma^n$ in the Power series expansion of
\[
e^{n\sigma}(1 + |\lambda|^2\sigma)^{-1} = \left(\sum_{k=0}^{\infty} \frac{(n\sigma)^k}{k!}\right) \left(\sum_{i=1}^{n+1} \frac{1}{i} (|\lambda|^2\sigma)^i\right).
\]
Hence
\[
\tau((T^\ast T)^n) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{n^k}{k!} \binom{n+1}{n-k} |\lambda|^{2(n-k)}
= \frac{1}{n+1} \sum_{k=0}^{n} \frac{n^k}{(k+1)!} \binom{n}{k} |\lambda|^{2n-2k}.
\]

4. Spectrum and Brown-measure of $T + \sqrt{\varepsilon}Y$

Let $T$ be the quasinilpotent DT-operator and let $Y$ be a circular operator *-free from $T$. In this section we will show, that
\[
\sigma(T + \sqrt{\varepsilon}Y) = B\left(0, \frac{1}{\sqrt{\log(1 + \varepsilon^{-1})}}\right)
\]
and that the Brown-measure $\mu_{T + \sqrt{\varepsilon}Y}$ is equal to the uniform distribution on $B\left(0, \frac{1}{\sqrt{\log(1 + \varepsilon^{-1})}}\right)$, i.e. it has constant density w.r.t. the Lebesgue measure on this disk.

**Theorem 4.1.** For every $\varepsilon > 0$
\[(4.1) \quad \sigma(T + \sqrt{\varepsilon}Y) = B\left(0, \frac{1}{\sqrt{\log(1 + \varepsilon^{-1})}}\right).\]

*Proof.* The result can be obtained by the method of Biane and Lehner [3, Section 5]. Let $a \in \mathbb{C} \setminus \{0\}$. Since $\sigma(T) = \{0\}$ we can write
\[
a1 - (T + \sqrt{\varepsilon}Y) = \sqrt{\varepsilon}\left(\frac{1}{\sqrt{\varepsilon}}1 - Y(a1 - T)^{-1}\right)(a1 - T).
\]
Hence
\[(4.2) \quad a \notin \sigma(T + \sqrt{\varepsilon}Y) \quad \text{iff} \quad \frac{1}{\sqrt{\varepsilon}} \notin \sigma(Y(a1 - T)^{-1}).\]

Let $Y = UH$ be the polar decomposition of $Y$. Then $Y(a1 - T)^{-1} = UH(a1 - T)^{-1}$, where $U$ is *-free from $H(a1 - T)^{-1}$. Hence $Y(a1 - T)^{-1}$ is $R$-diagonal. Moreover, since $0 \notin \sigma(Y)$, $Y(a1 - T)^{-1}$ is not invertible, so by [7, Prop. 4.6.(ii)]
\[(4.3) \quad \sigma(Y(a1 - T)^{-1}) = B(0, \|Y(a1 - T)^{-1}\|_2).\]
By \( +\)-freeness of \( Y \) and \( (aI - T)^{-1} \) we have
\[
(4.4) \quad \| Y(aI - T)^{-1} \|_2^2 = \| Y \|_2^2 \| (aI - T)^{-1} \|_2^2 = \| (aI - T)^{-1} \|_2^2 = \left\| \sum_{n=0}^{\infty} \frac{T^n}{a^n + 1} \right\|_2^2.
\]

Applying now [4, lemma 7.2] to \( D = 1 \) and \( \lambda = \frac{1}{a} \) and \( \mu = \delta_0 \), we get
\[
\left\| \sum_{n=0}^{\infty} \frac{T^n}{a^n} \right\|_2^2 = |a|^2 \left( \exp \left( \frac{1}{|a|^2} \right) - 1 \right)
\]

Hence by (4.4)
\[
\| Y(aI - T)^{-1} \|_2^2 = \exp \left( \frac{1}{|a|^2} \right) - 1.
\]

Thus for \( a \in \mathbb{C} \setminus \{0\} \) we get by (4.2) and (4.3)
\[
a \notin \sigma(T + \sqrt{\epsilon} Y) \Leftrightarrow \frac{1}{\sqrt{\epsilon}} \notin \sigma(Y(aI - T)^{-1})
\]
\[
\Leftrightarrow \frac{1}{\sqrt{\epsilon}} > \exp \left( \frac{1}{|a|^2} \right) - 1 \Leftrightarrow |a| > \frac{1}{\sqrt{\log(1 + \epsilon)}}.
\]

Hence \( \sigma(T + \sqrt{\epsilon} Y) \cup \{0\} = \overline{B} \left( 0, \frac{1}{\sqrt{\log(1 + \epsilon)}} \right) \). Since \( \sigma(T + \sqrt{\epsilon} Y) \) is closed it follows that \( \sigma(T + \sqrt{\epsilon} Y) = \overline{B} \left( 0, \frac{1}{\sqrt{\log(1 + \epsilon)}} \right) \). \( \square \)

In order to compute the Brown measure of \( T + \sqrt{\epsilon} Y \), we first observe that \( T + \sqrt{\epsilon} Y \) has the same \( +\)-distribution as
\[
S = \sqrt{\alpha T_1} + \sqrt{\beta T_2}
\]
when \( T_1 \) and \( T_2 \) are two \( \mathcal{D}\)-free quasidiagonal operators and \( a = 1 + \epsilon \) and \( b = \epsilon [1] \). We next compute the Brown measure of \( S \) for all values of \( a, b \in (0, \infty) \).

**Lemma 4.2.** Let \( \mu_Q \) be the Brown measure of an operator \( Q \) in a tracial \( W^* \)-probability space \( (M, \text{tr}) \). Let \( r > 0 \) and assume that \( \mu_Q(\partial B(0, r)) = 0 \). Then
\[
\mu_Q(B(0, r)) = -\frac{1}{2\pi} \lim_{\alpha \to 0^+} \Im \left( \int_{\partial B(0, r)} \text{tr}((Q_\lambda Q_\lambda + \alpha I)^{-1} Q_\lambda) d\lambda \right)
\]
where \( Q_\lambda = Q - \lambda I \) for \( \lambda \in \mathbb{C} \).
Proof. Let $\Delta : M \to [0, \infty)$ be the Fuglede-Kadison determinant on $M$, and put $L(\lambda) = \log \Delta(Q_\lambda)$ and
\[
L_\alpha(\lambda) = \log \Delta((Q_\lambda^* Q_\lambda + \alpha 1)^{1/2}) = \frac{1}{2} \text{tr}(\log(Q_\lambda^* Q_\lambda + \alpha 1))
\]
for $\lambda \in \mathbb{C}$.

Put $\lambda_1 = \Re \lambda, \lambda_2 = \Im \lambda$ and let $\nabla^2 = \frac{\partial^2}{\partial \lambda_1^2} + \frac{\partial^2}{\partial \lambda_2^2}$ denote the Laplace operator on $\mathbb{C}$. Then by [6, Section 2] $\nabla^2 L_\alpha \geq 0$ and for each $\alpha > 0$, the measure
\[
\mu_\alpha = \frac{1}{2\pi} \nabla^2 L_\alpha(\lambda) d\lambda_1 d\lambda_2
\]
is a probability measure on $\mathbb{C}$. Moreover
\[
\lim_{\alpha \to 0} \mu_\alpha = \mu
\]
in the weak* topology on $\text{Prob}(\mathbb{C})$. Also from [6, Section 2] the gradient $(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2})$ of $L_\alpha$ is given by
\[
\frac{\partial}{\partial \lambda_1} L_\alpha(\lambda) = -\Re \left( \text{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}) \right)
\]
and
\[
\frac{\partial}{\partial \lambda_2} L_\alpha(\lambda) = -\Im \left( \text{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}) \right)
\]
By (4.6)
\[
\lim_{\alpha \to 0} \int_C \phi d\mu_\alpha = \int_C \phi d\mu
\]
for all $\phi \in C_0(\mathbb{C})$. Since $1_{B(0, r)}$ is the limit of an increasing sequence $(\phi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$-functions with $0 \leq \phi_n \leq 1$ for all $n \in \mathbb{N}$ it follows that
\[
\mu_Q(B(0, r)) = \lim_{n \to \infty} \int_C \phi_n d\mu_Q
\]
\[
= \lim_{n \to \infty} \left( \lim_{\alpha \to 0} \int_C \phi_n d\mu_\alpha \right) \leq \lim_{\alpha \to 0} \left( \lim_{n \to \infty} \int_C 1_{B(0, r)} d\mu_\alpha \right) = \lim inf_{\alpha \to 0} \mu_\alpha(B(0, r))
\]
Writing $1_{\partial B(0, r)}$ as the limit of a decreasing sequence $(\psi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$-functions, with $0 \leq \psi_n \leq 1$, one gets in the same way
\[
\mu_Q(\partial B(0, r)) \geq \lim sup_{\alpha \to 0} \mu_\alpha(B(0, r))
\]
Hence if $\mu_Q(\partial B(0, r)) = 0$ we have
\[
\lim sup_{\alpha \to 0} \mu_\alpha(B(0, r)) \leq \mu_Q(B(0, r)) \leq \lim inf_{\alpha \to 0} \mu_\alpha(B(0, r))
\]
and therefore
\[ \mu_Q(B(0, r)) = \lim_{\alpha \to 0} \mu_\alpha(B(0, r)). \]

Using (4.5) together with Green’s theorem applied to the vector-field \((P_\alpha, Q_\alpha) = (\frac{\partial P_\alpha}{\partial \lambda_2}, \frac{\partial P_\alpha}{\partial \lambda_1})\) we get

\[
\mu_\alpha(B(0, r)) = \frac{1}{2\pi} \int_{B(0, r)} \nabla^2 L_\alpha(\lambda) d\lambda_1 d\lambda_2
\]
\[
= \frac{1}{2\pi} \int_{B(0, r)} \left( \frac{\partial Q_\alpha}{\partial \lambda_1} - \frac{\partial P_\alpha}{\partial \lambda_2} \right) d\lambda_1 d\lambda_2
\]
\[
= \frac{1}{2\pi} \int_{\partial B(0, r)} P_\alpha d\lambda_2 + Q_\alpha d\lambda_1
\]
\[
= \frac{1}{2\pi} \int_{\partial B(0, r)} \left( \frac{\partial L_\alpha}{\partial \lambda_2} - \frac{\partial L_\alpha}{\partial \lambda_1} \right) d\lambda_2 + \frac{1}{2\pi} \int_{\partial B(0, r)} \left( \frac{\partial L_\alpha}{\partial \lambda_1} - i \frac{\partial L_\alpha}{\partial \lambda_2} \right) (d\lambda_1 + i d\lambda_2)
\]

By (4.7) and (4.8)
\[
\frac{\partial L_\alpha}{\partial \lambda_1} - i \frac{\partial L_\alpha}{\partial \lambda_2} = -\text{tr}(Q_\alpha(Q_\alpha^* Q_\alpha + \alpha 1)^{-1}) = -\text{tr}((Q_\alpha^* Q_\alpha + \alpha 1)^{-1} Q_\alpha^*)
\]
Hence
\[
\mu_\alpha(B(0, r)) = -\Re \left( \frac{1}{2\pi} \int_{\partial B(0, r)} \text{tr}((Q_\alpha^* Q_\alpha + \alpha 1)^{-1} Q_\alpha^*) d\lambda \right)
\]
which completes the proof of the lemma.

Let \( S = \sqrt{a} T_1 + \sqrt{b} T_2^* \) with \( 0 < b < a \). Since \( cS \) and \( S \) have the same \( \ast \)-distribution for all \( c \in \mathbb{T} \), the Brown measure \( \mu_S \) of \( S \) is rotation invariant (i.e., invariant under the transformation \( z \mapsto cz, \ z \in \mathbb{C} \) when \( |c| = 1 \)). Hence by lemma 4.2 we can compute \( \mu_S \), if we can determine
\[
\text{tr}((S_\lambda^* S_\lambda + \alpha 1)^{-1} S_\lambda^*)
\]
for all \( \lambda \in \mathbb{C} \), where \( S_\lambda = S - \lambda 1 \), and for all \( \alpha \) in some interval of the form \((0, \alpha_0)\). This can be done by minor modifications of the methods used in section 3.

Put
\[
\tilde{S}_\lambda = \begin{pmatrix} 0 & S_\lambda^* \\ S_\lambda & 0 \end{pmatrix}.
\]
Then there exists a \( \delta > 0 \) (depending on \( a, b \) and \( \gamma \)) such that when \( ||z|| \leq \delta \) and \( |\mu| > \frac{1}{\delta} \) the equality
\[
(4.9) \quad \mathcal{R}^{M_2(D)}_{S_\lambda}(z) + z^{-1} = \mu 1_2
\]
implies that

\[(4.10) \quad z = G_{\mathcal{X}}^{M_{2}(\mathcal{D})}(\mu_{12}) = (\text{id} \otimes E_{\mathcal{D}}) \left( \frac{\mu(\mu^{2}S_{\lambda} - S_{\lambda}^{*}S_{\lambda}^{-1})}{S_{\lambda}^{*}(\mu^{2}S_{\lambda} - S_{\lambda}^{*}S_{\lambda}^{-1})} \frac{S_{\lambda}^{*}(\mu^{2}S_{\lambda}^{-1} - S_{\lambda})}{\mu(\mu^{2}S_{\lambda}^{-1} - S_{\lambda})} \right). \]

Moreover, \( \tilde{S} = \begin{pmatrix} 0 \\ S^{*} \\ 0 \end{pmatrix} \) is \( M_{2}(\mathcal{D}) \)-Gaussian by lemma 2.2 since \((T_{1}, T_{2}^{*}, T_{2}, T_{2}^{*})\) is a \( \mathcal{D} \)-Gaussian set. Hence for \( z = (z_{ij})^{2}_{i,j=1} \in M_{2}(\mathcal{D}) \),

\[ R_{\tilde{S}}^{M_{2}(\mathcal{D})}(z) = E_{M_{2}(\mathcal{D})}(\tilde{S}z\tilde{S}) = \begin{pmatrix} E_{\mathcal{D}}(S^{*}\tilde{z}_{22}\tilde{S}) & 0 \\ 0 & E_{\mathcal{D}}(\tilde{z}_{11}S^{*}) \end{pmatrix}. \]

Using that \((T_{1}, T_{1}^{*})\) and \((T_{2}, T_{2}^{*})\) have the same \( \mathcal{D} \)-distribution as \((T, T^{*})\) and that \((T_{1}, T_{1}^{*})\) and \((T_{2}, T_{2}^{*})\) are two \( \mathcal{D} \)-free sets, we get

\[ E_{\mathcal{D}}(S^{*}\tilde{z}_{22}\tilde{S}) = (aL^{*} + bL)(\tilde{z}_{22}), \quad E_{\mathcal{D}}(\tilde{z}_{11}S^{*}) = (aL + bL^{*})(\tilde{z}_{11}), \]

where \( L(f) : x \mapsto \int_{x}^{1} f(y)dy \) and \( L^{*}(f) : x \mapsto \int_{0}^{x} f(y)dy \) for \( f \in \mathcal{D} \).

Since \( \tilde{S}_{\lambda} = \tilde{S} - \begin{pmatrix} 0 \\ \lambda \overline{1} \\ 0 \end{pmatrix} \) it follows that

\[ R_{\tilde{S}}^{M_{2}(\mathcal{D})}(z) = \begin{pmatrix} (aL + bL^{*})(\tilde{z}_{22}) & -\lambda \overline{1} \\ \lambda \overline{1} & (aL^{*} + bL)(\tilde{z}_{11}) \end{pmatrix}. \]

Thus (4.10) becomes

\[(4.11) \quad \begin{pmatrix} \mu_{1} \\ 0 \\ \mu_{1} \end{pmatrix} = \begin{pmatrix} (aL + bL^{*})(\tilde{z}_{22}) & -\lambda \overline{1} \\ \lambda \overline{1} & (aL^{*} + bL)(\tilde{z}_{11}) \end{pmatrix} + \frac{1}{\det(z)} \begin{pmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11} \end{pmatrix}. \]

In analogy with section 3, we look for solutions \( z_{ij} \in \mathcal{D} = L^{\infty}[0, 1] \) of the form

\[(4.12) \quad \begin{pmatrix} \tilde{z}_{11} & \tilde{z}_{12} \\ \tilde{z}_{21} & \tilde{z}_{22} \end{pmatrix} = \begin{pmatrix} c_{11} \exp(\sigma x) & c_{12} \\ c_{21} & c_{22} \exp(-\sigma x) \end{pmatrix}, \]
where \( \sigma \in \mathbb{C} \) and 
\[
c = \begin{pmatrix}
    c_{11} & c_{12} \\
    c_{21} & c_{22}
\end{pmatrix} \in \text{GL}(2, \mathbb{C}).
\]
It is easy to check that (4.12) is a solution to (4.11) if the following 5 conditions are fulfilled:
\[
\begin{align*}
\det(c) &= \frac{\sigma}{a - b} \\
c_{11} &= \frac{\sigma \mu}{ae^\sigma - b} \\
c_{12} &= -\frac{\sigma \lambda}{a - b} \\
c_{21} &= -\frac{\sigma \lambda}{a - b} \\
c_{22} &= \frac{\sigma \mu}{a - be^{-\sigma}}
\end{align*}
\]

The first of these conditions is consistent with the remaining 4 if and only if
\[
\frac{(\sigma \mu)^2}{(ae^\sigma - b)(a - be^{-\sigma})} - \frac{\sigma^2 |\lambda|^2}{(a - b)^2} = \frac{\sigma}{a - b}
\]
which is equivalent to
\[
(4.13) \quad \mu^2 = \frac{(ae^\sigma - b)(a - be^{-\sigma})(a - b + \sigma |\lambda|^2)}{\sigma(a - b)^2}.
\]

Put
\[
\sigma_0 := -\min \left\{ \frac{a - b}{|\lambda|^2}, \log \left( \frac{a}{b} \right) \right\}.
\]

Then for \( \sigma_0 < \sigma < 0 \), the right hand side of (4.13) is negative. Let in this case \( \mu(\sigma) \) denote the solution to (4.13) with positive imaginary part, i.e.
\[
(4.14) \quad \mu(\sigma) = \frac{ae^\sigma / 2 - be^{-\sigma} / 2}{|\sigma|^{1/2}(a - b)} \sqrt{a - b + \sigma |\lambda|^2}
\]
for \( \sigma_0 < \sigma < 0 \). Then with
\[
\begin{align*}
c_{11} &= \frac{\sigma \mu(\sigma)}{ae^\sigma - b} \\
c_{12} &= -\frac{\sigma \lambda}{a - b} \\
c_{21} &= -\frac{\sigma \lambda}{a - b} \\
c_{22} &= \frac{\sigma \mu(\sigma)}{a - be^{-\sigma}}
\end{align*}
\]
the matrix \( z(\sigma) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \) given by (4.12) is a solution to
\[
\mathcal{R}_{S_\lambda}^{M_2(2)}(z(\sigma)) + z(\sigma)^{-1} = \mu \mathbf{1}_2.
\]
By (4.14) \( \lim_{\sigma \to 0^-} |\mu(\sigma)| = \infty \) and \( \lim_{\sigma \to 0^-} |\sigma \mu(\sigma)| = 0 \) and therefore \( \lim_{\sigma \to 0^-} \|z(\sigma)\| = 0 \).

Hence for some \( \sigma_1 \in (\sigma_0, 0) \) we have \( |\mu(\sigma)| > \frac{1}{\sigma} \) and \( \|z(\sigma)\| > \delta \) when \( \sigma \in (\sigma_1, 0) \) where \( \delta > 0 \) is the number described in connection with (4.9). Thus

\[
(4.15) \quad z(\sigma) = G_{\tilde{S}_\lambda}^{M_2(\mathbb{D})}(\mu(\sigma)1_2)
\]

for \( \sigma \in (\sigma_1, 0) \). But since both \( \sigma \mapsto z(\sigma) \) and \( \sigma \mapsto \mu(\sigma) \) are analytic functions (of the real variable \( \sigma \)) it follows that (4.15) holds for all \( \sigma \in (\sigma_0, 0) \). Note that \( \sigma \mapsto -i\mu(\sigma) \) is a continuous strictly positive function on \( (\sigma_0, 0) \), and

\[
\lim_{\sigma \to 0^-} (-i\mu(\sigma)) = +\infty \quad \lim_{\sigma \to \sigma_0^+} (-i\mu(\sigma)) = 0.
\]

Hence for every fixed real number \( \alpha > 0 \) we can chose \( \sigma \in (\sigma_0, 0) \), such that

\[-i\mu(\sigma) = \sqrt{\alpha}.
\]

Thus by (4.10) and (4.15)

\[
E_D(S^{*}_\lambda(-\alpha 1 - S^{*}_\lambda S^{*}_\lambda)^{-1}) = z(\sigma)_{12} = -\frac{\sigma\lambda}{a - b}
\]

which is a constant function in \( L^\infty[0, 1] \). Hence

\[
\text{tr}(S^{*}_\lambda(S^{*}_\lambda S^{*}_\lambda + \alpha 1)^{-1}) = \frac{\sigma\lambda}{a - b}
\]

from which

\[
\int_{\partial H(0, r)} \text{tr}(S^{*}_\lambda(S^{*}_\lambda S^{*}_\lambda + \alpha 1)^{-1})d\lambda = 2\pi i \frac{\sigma\lambda^2}{a - b}
\]

when \( \sigma_0 < \sigma < 0 \), where as before \( \sigma_0 = -\min \left\{ \frac{a-b}{2\pi r}, \log\left(\frac{a}{b}\right) \right\} \).

Now \( \alpha \to 0^+ \) corresponds to \( \sigma \to \sigma_0^+ \). Hence

\[
\lim_{\alpha \to 0^+} \left(-\frac{1}{2\pi^3} \int_{\partial H(0, r)} \text{tr}(S^{*}_\lambda(S^{*}_\lambda S^{*}_\lambda + \alpha 1)^{-1})d\lambda \right)
\]

\[
= -\frac{\sigma_0 r^2}{a - b} = +\min \left\{ 1, r^2 \log\left(\frac{a}{b}\right) \right\}.
\]
Observe that $S^*_\lambda (S_{ \lambda} S_{ \lambda} + \alpha 1)^{-1} = (S_{ \lambda} S_{ \lambda} + \alpha 1)^{-1} S^*_\lambda$. Thus by lemma 4.2 we have for all but countably many $r > 0$, that

$$
\mu_S(B(0,r)) = \min \left\{ 1, r^2 \log \left( \frac{\pi}{a-b} \right) \right\} = \begin{cases} 
\frac{r^2 \log \left( \frac{\pi}{a-b} \right)}{a-b}, & r \leq \sqrt{\frac{a-b}{\log \left( \frac{\pi}{a-b} \right)}}. \\
1, & r > \sqrt{\frac{a-b}{\log \left( \frac{\pi}{a-b} \right)}}.
\end{cases}
$$

Since the right hand side is a continuous function of $r$, the formula actually holds for all $r > 0$. This together with the rotation invariance of $\mu_S$ shows, that $\mu_S$ is equal to the uniform distribution on $B \left( 0, \sqrt{\frac{a-b}{\log \left( \frac{\pi}{a-b} \right)}} \right)$, i.e. has constant density $\frac{1}{\pi} \log \left( \frac{\pi}{a-b} \right)$ on this ball, and vanishes outside the ball. Putting $a = 1 + \epsilon$ and $b = \epsilon$ we get in particular

**Theorem 4.3.** The Brown measure of $T + \sqrt{\epsilon} Y$ is equal to the uniform distribution on $B \left( 0, \sqrt{\frac{1}{\log (1+\epsilon^{-1})}} \right)$.

The Brown measure of $T + \sqrt{\epsilon} Y$ can be used to give an upper bound of the microstate entropy of $T + \sqrt{\epsilon} Y$. By [8] we have for $S \in \mathcal{M}$

$$
\chi(S) \leq \int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| d\mu_S(z_1) d\mu_S(z_2) + \frac{5}{4} + \log (\pi \sqrt{20d_S})
$$

where $\mu_S$ is the Brown measure of $S$ on $\mathbb{C}$ and $d_S$ is the off-diagonality of $S$ defined by

$$
(4.17) \quad d_S := \tau(SS^*) - \int_{\mathbb{C}} |z|^2 d\mu_S(z).
$$

**Lemma 4.4.** For $R > 0$ we have

$$
I := \int_{B(0,R)} \int_{B(0,R)} \log |z_1 - z_2| dz_1 dz_2 = \pi^2 (R^2 \log R - \frac{1}{4})
$$

**Proof.** Polar substitution in $I$ gives

$$
I := 4\pi^2 \int_0^R \int_0^R \left( \frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}s| d\theta \right) rdr ds.
$$

Let $0 < s < r$, $z \mapsto \log |r - zs|$ is the real value of the complex holomorphic function $z \mapsto \text{Log}(r - zs)$, where Log is the principal branch of the complex logarithm, so $z \mapsto \text{Log}|r - zs|$ is a harmonic function in $B(0, \frac{r}{s})$. By the mean value property of harmonic functions

$$
\frac{1}{2\pi} \int_0^{2\pi} \log |r - e^{i\theta}s| d\theta = \log(r),
$$
so symmetry in $r$ and $s$ reduces $I$ to

\[
I := 4\pi^2 \int_0^R \int_0^R \max\{\log(r), \log(s)\} r dr ds \\
= 8\pi^2 \int_0^R \left( \int_0^r \log(r) ds \right) dr \\
= 4\pi^2 \int_0^R r^3 \log(r) dr = \pi^2 R^4 (\log(R) - \frac{1}{4}).
\]

\[\square\]

**Theorem 4.5.**

(4.18) $\chi(T + \sqrt{\epsilon} Y) \leq -\frac{1}{2} \log(\log(1 + \epsilon^{-1})) - \frac{1}{4} + \log \pi$

\[
+ \frac{1}{2} \log \left( 1 + 2\epsilon - \frac{1}{\log(1 + \epsilon^{-1})} \right).
\]

**Proof.** Let $\nu_R$ be the uniform distribution on $\overline{B}(0, R)$. Since $\nu_R$ has constant density $(\pi R^2)^{-1}$ on $\overline{B}(0, R)$, we have by lemma 4.4

\[
\int_{\mathbb{C}} \int_{\mathbb{C}} \log |z_1 - z_2| d\nu_R(z_1) d\nu_R(z_2) = \log R - \frac{1}{4}.
\]

The Brown measure of $S = T + \sqrt{\epsilon} Y$ is $\mu_S = \nu_R$ with $R = \log(1 + \epsilon^{-1})^{-\frac{1}{2}}$, and

\[
od_s = \frac{1}{2} + \epsilon - \int_{\mathbb{C}} |z|^2 d\nu_R = \frac{1}{2} + \epsilon - \frac{R^2}{2}.
\]

Hence by (4.16)

\[
\chi(T + \sqrt{\epsilon} Y) \leq \log R - \frac{1}{4} + \log \pi + \frac{1}{2} \log(1 + 2\epsilon - R^2).
\]

This proves (4.18). \[\square\]

In [1] the first author proved that the microstate-free analog, $\delta^*_0(T)$, of the free entropy dimension is equal to 2. From Theorem 4.5 one gets only the trivial estimate of the free entropy dimension $\delta_0(T)$, namely

(4.19) $\delta_0(T) \leq 2 + \lim_{\delta \to 0^+} \frac{\chi(T + \sqrt{2\delta} Y)}{|\log \delta|} = 2.$

If $T + \sqrt{\epsilon} Y$ was a DT-operator for all $\epsilon > 0$ then by [8] equality would hold in (4.18), and hence also in (4.19). In the rest of this section, we prove that unfortunately $T + \sqrt{\epsilon} Y$ is not a DT-operator for any $\epsilon > 0$. 
If $R = D + T$ is a DT($\mu, 1$) operator it follows from [4, lemma 7.2] that for $|\lambda| < \|R\|^{-1}$,

$$\left\| \sum_{n=0}^{\infty} \lambda^n R^n \right\|_2^2 = \frac{1}{|\lambda|^2} \left( \exp \left( \sum_{k,l=1}^{\infty} \lambda^{k+1} M_\mu(k, l) - 1 \right) \right),$$

where $M_\mu(k, l) = \int_{\sigma(R)} z^{k+l} d\mu_R(z)$.

If thus $\mu_D$ is the uniform distribution on a disk with radius $d$ then

$$M_{\mu_D}(k, l) = 0$$

when $k \neq l$ and

$$M_{\mu_D}(k, k) = \frac{1}{\pi d^2} \int_{B(0, d)} |z|^{2k} d\mu_D(z)
= \frac{2\pi}{\pi d^2} \int_0^d r^{2k+1} dr = \frac{2}{d^2} \left[ \frac{r^{2k+2}}{2k+2} \right]_0^d = \frac{d^{2k}}{k+1}$$

for $k \in \mathbb{N}$. Thus

$$\left(4.20\right) \left\| \sum_{n=0}^{\infty} \lambda^n (D + T)^n \right\|_2^2 = \frac{1}{|\lambda|^2} \left( \exp \left( \sum_{k=0}^{\infty} |\lambda|^{2(k+1)} \frac{d^{2k}}{k+1} \right) - 1 \right)$$

$$= \frac{1}{|\lambda|^2} \exp \left( \frac{1}{d^2} \left( - \log(1 - d^2|\lambda|^2) \right) \right)$$

$$= \frac{1}{|\lambda|^2} \left[ (1 - d^2|\lambda|^2)^{-\frac{1}{d^2}} - 1 \right].$$

If instead $D + cT$ is a DT($\mu_D, c$) operator with $\mu_D$ being the uniform distribution on a disc of radius $d$ then

$$D + cT = c(D' + T)$$

where $D'$ now has the uniform distribution on $B(0, \frac{d}{c})$, so from (4.20) we obtain

$$\left(4.21\right) \left\| \sum_{n=0}^{\infty} \lambda^n (D + cT)^n \right\|_2^2 = \left\| \sum_{n=0}^{\infty} (c\lambda)^n (D' + T)^n \right\|_2^2 = \frac{1}{c^2|\lambda|^2} \left[ (1 - c^2|\lambda|^2)^{-\frac{2}{c^2}} - 1 \right].$$
Lemma 4.6. Let \( a > b > 0 \) and let \( S = \sqrt{a} T_1 + \sqrt{b} T_2 \) where \( T_1 \) and \( T_2 \) are two \( \mathcal{D} \)-free quasidiagonal \( \mathcal{D} \)-operators. Then

\[
\left\| \sum_{n=0}^{\infty} \lambda^n S^n \right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{(a-b)|\lambda|^2} - 1}{a - b e^{(a-b)|\lambda|^2}}, \quad |\lambda| < \frac{1}{\|S\|}.
\]

Proof. Let \( F_n(x) = E_{\mathcal{D}}((S^*)^n S^n) \) for \( n \in \mathbb{N} \) and \( x \in [0,1] \). For \( t < \frac{1}{\|S\|^2} \) define the \( \mathcal{D} \)-valued function

\[
(4.22) \quad F(t,x) = \sum_{n=0}^{\infty} F_n(x) t^n.
\]

By Speicher's cumulant formula we have by \( \mathcal{D} \)-Gaussianity of \( S \) that

\[
F_n = E_{\mathcal{D}}((S^*)^n S^n) = \sum_{\pi \in \text{NC}(2n)} \kappa_\pi^D ((S^*)^\otimes 2^n \otimes_{\mathcal{B}} S^\otimes 2^n)
\]

\[
= \kappa_2^D (S^* \otimes_{\mathcal{B}} E_{\mathcal{D}}((S^*)^{n-1} S^{n-1})) S
\]

\[
= (aL^* + bL)(E_{\mathcal{D}}((S^*)^{n-1} S^{n-1})) = (aL^* + bL)(F_{n-1}),
\]

so we get the following recursive algorithm for determining the \( F_n \)'s,

\[
\begin{cases}
F_0(x) = 1 \\
F_n(x) = aL^*(F_{n-1})(x) + bL(F_{n-1})(x), \quad x \in [0,1]
\end{cases}
\]

where \( L^*(f) : x \mapsto \int_0^x f(y) dy \) and \( L(f) : x \mapsto \int_x^1 f(y) dy \). Observe that

\[
\frac{d}{dx} L(f)(x) = -f(x) \quad \text{and} \quad \frac{d}{dx} L^*(f)(x) = f(x),
\]

and that

\[
F_n(0) = aL^*(F_{n-1})(0) + bL(F_{n-1})(0) = b \int_0^1 F_{n-1}(x) dx = b \tau(F_{n-1})
\]

for \( n \geq 1 \). Using (4.22) we have the following differential equation and initial condition in \( x \)

\[
\begin{cases}
\frac{d}{dx} F(t,x) = (a - b)tF(t,x), \quad x \in [0,1] \\
F(t,0) = f(t),
\end{cases}
\]
where the function $f$ is given by

$$f(t) = F(t, 0) = \sum_{n=0}^{\infty} F_n(0) t^n$$

$$= 1 + \sum_{n=1}^{\infty} (aL^*(F_{n-1})(0) + bL(F_{n-1})(0)) t^n$$

$$= 1 + b \sum_{n=1}^{\infty} \left( \int_{0}^{1} F_{n-1}(x) \, dx \right) t^n$$

$$= 1 + bt \int_{0}^{1} \left( \sum_{n=1}^{\infty} F_{n-1}(x) t^{n-1} \right) \, dx$$

$$= 1 + bt \tau(F(t, \cdot))$$

We thus have the unique solution

$$F(t, x) = f(t)e^{(a-b)t}x,$$

where we can now use (4.23) and the initial condition to find the function $f$.

$$f(t) = 1 + bt \int_{0}^{1} F(t, x) \, dx$$

$$= 1 + bt \left[ \frac{f(t) - e^{(a-b)t}}{(a-b)t} \right]_{0}^{1} = 1 + bf(t) \frac{e^{(a-b)t} - 1}{a - b}.$$ 

Hence

$$f(t) = \frac{a - b}{a - be^{(a-b)t}}$$

so that

$$F(t, x) = \frac{(a - b)e^{(a-b)t}x}{a - be^{(a-b)t}}.$$ 

Now observe that

$$\left\| \sum_{n=0}^{\infty} \lambda^n S^n \right\|_2^2 = \tau \left( F(|\lambda|^2, x) \right)$$

$$= \int_{0}^{1} F(|\lambda|^2, x) \, dx = \frac{1}{|\lambda|^2} \frac{e^{(a-b)|\lambda|^2} - 1}{a - be^{(a-b)|\lambda|^2}} \frac{1}{|\lambda|^2}.$$

\textbf{Theorem 4.7.} The operator $T + \sqrt{e}Y$ is not a DT-operator.
Proof. By substituting \( a = 1 + \epsilon \) and \( b = \epsilon \) in lemma 4.6 we have

\[
(4.24) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon} Y)^n \right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{|\lambda|^2} - 1}{1 + \epsilon - \epsilon e^{|\lambda|^2}}
\]

for all \( \lambda \) in a neighborhood of 0. If \( T + \sqrt{\epsilon} Y \) is a DT-operator, then by Theorem 4.3 and (4.21), there exists a \( c > 0 \), such that when \( d = \log(1 + \frac{1}{\epsilon})^{-\frac{1}{c}} \)

\[
(4.25) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon} Y)^n \right\|_2^2 = \frac{1}{c^2 |\lambda|^2} \left( (1 - d^2 |\lambda|^2)^{-\frac{2}{c^2}} - 1 \right)
\]

for all \( \lambda \) in a neighborhood of 0. Consider the two analytic functions,

\[
f(s) = \frac{e^s - 1}{1 + \epsilon - \epsilon e^s}, \quad g(s) = \frac{1}{c^2} \left( (1 - d^2 s)^{-\frac{2}{c^2}} - 1 \right)
\]

which are both defined in the complex disc \( U = B(0, \log(1 + \frac{1}{\epsilon})^{-\frac{1}{c}}) \).

By (4.24) and (4.25) \( f(s) = g(s) \) for \( s \) in some real interval of the form \((0, \delta)\) and hence \( f(s) = g(s) \) for all \( s \in U \). Moreover \( f \) has a meromorphic extension to the full complex plane with a simple pole at \( s_0 = \log(1 + \frac{1}{\epsilon}) \). Hence \( g \) also has a meromorphic extension to the full complex plane with a simple pole at \( \log(1 + \frac{1}{\epsilon}) = d^{-2} \). This implies \( c = d \). In this case

\[
g(s) = \frac{1}{d^2} \left( (1 - d^2 s)^{-1} - 1 \right)
\]

which is analytic in \( \mathbb{C} \setminus \{s_0\} \). However \( f \) has infinitely many poles, namely

\[
s_p = \log \left( 1 + \frac{1}{\epsilon} \right) + p2\pi, \quad p \in \mathbb{Z}.
\]

Since the meromorphic extensions of \( f \) and \( g \) must coincide, we have reached a contradiction. Therefore \( T + \sqrt{\epsilon} Y \) is not a DT-operator. \( \square \)

5. Śniady’s moment formulas. The case \( k = 2 \).

Let \( k \in \mathbb{N} \) be fixed, and let \( (P_{k,n})_{n=0}^{\infty} \) be the sequence of polynomials defined recursively by

\[
P_{k,n}(x) = 1, \quad P_{k,0}(x) = P_{k,n-1}(x + 1), \quad n = 1, 2, \ldots, \quad P_{k,n}(0) = P_{k,n}^{(1)}(0) = \cdots P_{k,n}^{(k-1)}(0) = 0, \quad n = 1, 2, \ldots
\]
where \( P^{(l)}_{k,n} \) denotes the \( l \)th derivative of \( P_{k,n} \). As in the previous sections, \( T \) denotes the quasinilpotent \( DT \) operator. Šniady’s main results from [9] are:

**Theorem 5.1.** [9, Theorem 5 and Theorem 7]

(a) For all \( k, n \in \mathbb{N} \):

\[
E_D \left( \left( (T^*)^k T^k \right)^n \right) (x) = P_{k,n}(x), \quad x \in [0, 1].
\]

(b) For all \( k, n \in \mathbb{N} \):

\[
\tau \left( \left( (T^*)^k T^k \right)^n \right) = \frac{n^{nk}}{(nk + 1)!}
\]

Actually Šniady considers \( E_D \left( (T^k(T^*)^k)^n \right) \) instead of \( E_D \left( \left( (T^*)^k T^k \right)^n \right) \), but it is easily seen, that Theorem 5.1 (a) is equivalent to [9, Theorem 5], by the simple change of variable \( x \mapsto 1 - x \).

Šniady’s proof of Theorem 5.1 is a very technical combinatorial proof. In this and the following section we will give an analytical proof of Theorem 5.1 based on Voiculescu’s \( \mathcal{R} \)-transform with amalgamation.

As in [5, (2.11)] we put

\[
\rho(z) = -W_0(-z), \quad z \in \mathbb{C} \setminus \left[ \frac{1}{e}, \infty \right),
\]

where \( W_0 \) is the principal branch of Lambert’s W-function. Then \( \rho \) is the principal branch of the inverse function of \( z \mapsto z e^{-z} \). We shall need the following result from [5, Prop. 4.2].

**Lemma 5.2.** [5, Prop. 4.2] Let \( (P_{k,n})_{n=0}^{\infty} \) be a sequence of polynomials given by (5.1). Put for \( s \in \mathbb{C}, |s| < \frac{1}{e} \) and \( j = 1, \ldots, k \)

\[
\alpha_j(s) = \rho \left( s e^{\frac{2\pi j}{k}} \right),
\]

\[
\gamma_j(s) = \begin{cases} 
\prod_{l \neq j} \frac{\alpha_l(s)}{\alpha_l(s) - \alpha_j(s)}, & 0 < |s| < \frac{1}{e} \\
\frac{1}{\bar{s}}, & s = 0.
\end{cases}
\]

Then

\[
\sum_{n=0}^{\infty} (ks)^n P_{k,n}(x) = \sum_{j=1}^{k} \gamma_j(s) e^{\lambda_j(s)} x
\]

for all \( x \in \mathbb{R} \) and all \( s \in B(0, \frac{1}{e}) \).

The case \( k = 1 \) of theorem 5.1 is the special case \( \lambda = 0 \) of theorem 3.2. To illustrate our method of proof of theorem 5.1 for \( k \geq 2 \), we first consider the case \( k = 2 \).
Define \( \tilde{T} \in M_4(\mathcal{A}) \) by

\[
\tilde{T} = \begin{pmatrix}
0 & 0 & 0 & T^* \\
T & 0 & 0 & 0 \\
0 & T & 0 & 0 \\
0 & 0 & T^* & 0
\end{pmatrix}.
\]

Then \( \|\tilde{T}\| = \|T\| = \sqrt{\mathcal{E}} \). (cf. [4, Corollary 8.11]) For \( \mu \in \mathbb{C}, \ |\mu| < \frac{1}{e} \) we let \( z = z(\mu) \), denote the Cauchy transform of \( \tilde{T} \) at \( \tilde{\mu} = \mu 1_{M_4(\mathcal{A})} \) wrt. amalgamation over \( M_4(\mathcal{D}) \) i.e.

\[
z = E_D \left( (\tilde{\mu} - \tilde{T})^{-1} \right).
\]

Clearly

\[(5.7) \quad (\tilde{\mu} - \tilde{T})^{-1} = \sum_{n=0}^{\infty} \mu^{-n-1} \tilde{T}^n = \left( \sum_{n=0}^{3} \mu^{-n-1} \tilde{T}^n \right) \left( \sum_{n=0}^{\infty} \mu^{-4n} \tilde{T}^{4n} \right).\]

By direct computation

\[
\tilde{T}^2 = \begin{pmatrix}
0 & 0 & (T^*)^2 T & 0 \\
0 & 0 & 0 & TT^* \\
T^2 & 0 & 0 & 0 \\
0 & T^* T & 0 & 0
\end{pmatrix},
\]

\[
\tilde{T}^3 = \begin{pmatrix}
0 & (T^*)^2 T & 0 & 0 \\
0 & 0 & T(T^*)^2 & 0 \\
0 & 0 & 0 & T^2 T^* \\
T^* T^2 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
\tilde{T}^4 = \begin{pmatrix}
(T^*)^2 T^2 & 0 & 0 & 0 \\
0 & T(T^*)^2 T & 0 & 0 \\
0 & 0 & T^2(T^*)^2 & 0 \\
0 & 0 & 0 & T^* T^2 T^*
\end{pmatrix}.
\]

Hence using the fact that the expectation \( E_D \) of a monomial in \( T \) and \( T^* \) vanishes unless \( T \) and \( T^* \) occur the same number of times, we get from (5.7) that \( z \) is of the form

\[(5.8) \quad z = \begin{pmatrix}
z_{11} & 0 & 0 & 0 \\
0 & z_{22} & 0 & z_{24} \\
0 & 0 & z_{33} & 0 \\
0 & z_{42} & 0 & z_{44}
\end{pmatrix} \]
where $z_{11}, z_{22}, z_{33}, z_{42}, z_{44} \in \mathcal{D}$ are given by

$$
\begin{align*}
    z_{11} &= \mu^{-1}E_\mathcal{D}((1 - \mu^{-4}(T^*)^2T^2)^{-1}), \\
    z_{22} &= \mu^{-1}E_\mathcal{D}((1 - \mu^{-4}T(T^*)^2T)^{-1}), \\
    z_{33} &= \mu^{-1}E_\mathcal{D}((1 - \mu^{-4}T^2(T^*)^2)^{-1}), \\
    z_{44} &= \mu^{-1}E_\mathcal{D}((1 - \mu^{-4}T^*T^2T^*)^{-1}), \\
    z_{24} &= \mu^{-3}E_\mathcal{D}(T(1 - \mu^{-4}(T^*)^2T^2)^{-1}T^*), \\
    z_{42} &= \mu^{-3}E_\mathcal{D}(T^*(1 - \mu^{-4}T^2(T^*)^2)^{-1}T^*). 
\end{align*}
$$

For the last 2 identities, we have used, that

$$
A(1 - \eta BA)^{-1} = (1 - \eta AB)^{-1}A
$$

for $A, B \in \mathcal{A}$ and $\eta \in \mathbb{C}$ whenever both sides of this equality are well-defined.

By lemma 2.1, we know, that there exists a $\delta > 0$ such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $||w|| < \delta, |\mu| > \frac{1}{2}$ and

$$(5, 9) \quad \mathfrak{p}_{\mathcal{F}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{A})}$$

then $w = E_{M_4(\mathcal{D})}((\bar{\mu} - \bar{T})^{-1}) = z$. In particular

$$
    w_{11} = z_{11} = \mu^{-1}((1 - \mu^{-4}(T^*)^2T^2)^{-1}),
$$

Hence, if we can find a suitable solution to (5,8) for all $\mu \in \mathbb{C}$ in a neighborhood of $\infty$, we can find $E_\mathcal{D}((T^*)^2T^2)^n$ for $n = 1, 2, \ldots$ by determining the power series expansion of $w_{11}$ as a function of $\mu^{-1}$.

Since $(T, T^*)$ is a $\mathcal{D}$-Gaussian pair by [5, Appendix] it follows from lemma 2.2 that

$$
\kappa_n^{M_4(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} (m_n \otimes a_n)) = 0
$$

when $n \neq 2, m_1, m_2, \ldots, m_n \in M_4(\mathbb{C})$ and $a_1, a_2, \ldots, a_n \in \{ T, T^* \}$. By definition

$$
\bar{T} = (e_{21} + e_{32}) \otimes T + (e_{43} + e_{14}) \otimes T^*
$$

so by linearity of $\kappa_n^{M_4(\mathcal{D})}$, it follows that

$$
\kappa_n^{M_4(\mathcal{D})}(\bar{T} \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} \bar{T}) = 0
$$

when $n \neq 2$ i.e. $\bar{T}$ is $M_4(\mathcal{D})$-Gaussian.
Hence using (2.4) we get
\[
\mathcal{R}^{M_4(\mathcal{D})}_T (w) = \kappa_2^{M_4(\mathcal{D})} (\mathcal{T} \otimes M_4(\mathcal{D}) \ T) = E_{M_4(\mathcal{D})} \left( \mathcal{T} \mathcal{wT} \right) \\
= E_{M_4(\mathcal{D})} \left( \begin{pmatrix}
T^* w_{42} T & 0 & T^* w_{44} T & 0 \\
0 & 0 & 0 & T w_{11} T^*
\end{pmatrix} \right)
\]
for \( w = (w_{ij})_{i,j=1,...,4} \in M_4(\mathcal{D}). \)
Since \( E_M(T f T) = E_M(T^* f T^*) = 0 \), and \( E_M(T^* f T) = L(f) \) for \( f \in L^\infty([0,1]) \), we have:
\[
\mathcal{R}^{M_4(\mathcal{D})}_T (w) = \begin{pmatrix}
L^*(w_{42}) & 0 & 0 & 0 \\
0 & 0 & 0 & L(w_{11}) \\
0 & 0 & L(w_{24}) & 0 \\
0 & L^*(w_{33}) & 0 & 0
\end{pmatrix}
\]
for \( w \in M_4(\mathcal{D}) \). By (5.8) we only have to consider \( w \) of the form
\[
w = \begin{pmatrix}
w_{11} & 0 & 0 & 0 \\
w_{22} & 0 & w_{24} \\
w_{33} & 0 & w_{33} \\
w_{42} & 0 & w_{44}
\end{pmatrix}.
\]
For \( w \in M_4(\mathcal{D})_{\text{inv}} \) of the form (5.10), (5.9) reduces to the three equations
\[
\left\{ \begin{array}{l}
L^*(w_{42}) + \frac{1}{w_{21}} = \mu 1_{\mathcal{D}} \\
L^*(w_{33}) + \left( \begin{array}{cc}
w_{22} & w_{24} \\
w_{42} & w_{44}
\end{array} \right)^{-1} = \mu 1_{M_4(\mathcal{D})} \\
L(w_{24}) + \frac{1}{w_{33}} = \mu 1_{\mathcal{D}}
\end{array} \right.
\]
\[
(5.11)
\]
**Definition 5.3.** Let \( f \in C([0,1]) \). We call \( (f^{(-n)})_{n=1} \) for the successive antiderivatives of \( f \) if
\[
\frac{d}{dx} (f^{(-n)}) = f^{(-n)} \text{ for } n = 2, 3, \ldots, l
\]
and
\[
\frac{d}{dx} (f^{(-1)}) = f.
\]
**Lemma 5.4.** Let \( f \in C^2([0,1]) \) and let \( f^{(-1)} \) and \( f^{(-2)} \) be the successive antiderivatives of \( f \) for which
(i) \( f^{(-1)}(1) = 0, \quad f^{(-2)}(1) = \mu^3 \).
Assume further, that
(ii) $f(0) = \mu^{-1}$ and $f^{[1]}(0) = 0$.

(iii) For all $x \in [0, 1],$

$$f(x) \neq 0$$

$$f^{(-1)}(x) f(x) 
\begin{vmatrix} f(x) \\ f(x) \end{vmatrix} \neq 0$$

while

$$f^{(-2)}(x) f^{(-1)}(x) f(x) 
\begin{vmatrix} f(x) \\ f(x) \end{vmatrix} = 0$$

Then $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42} \in C([0, 1])$ given by

$$w_{11} = f$$
$$w_{22} = w_{44} = -\frac{1}{\mu} \begin{vmatrix} f^{(-1)} & f \\ f & f^{[1]} \end{vmatrix}$$
$$w_{24} = \frac{1}{\mu^2} \begin{vmatrix} f^{(-1)} & f \\ f & f^{[1]} \end{vmatrix}$$
$$w_{42} = f^{[1]}$$
$$w_{33} = \mu^2 \begin{vmatrix} f & f^{[1]} \\ f^{[1]} & f^{[2]} \end{vmatrix}$$

(5.12)

is a solution to (5.11). Moreover

$$w_{22} \ w_{24} 
\begin{vmatrix} w_{24} \\ w_{42} \end{vmatrix} = -\frac{1}{\mu^2} \begin{vmatrix} f^{(-1)} & f \\ f & f^{[1]} \end{vmatrix}$$

(5.13)
and
\[
\begin{align*}
L(w_{11}) &= -f^{(-1)} \\
L(w_{24}) &= \mu - \frac{1}{\mu^2} \begin{bmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{bmatrix} \\
L^*(w_{42}) &= \mu - \frac{1}{f} \\
L^*(w_{33}) &= -\mu^2 \begin{bmatrix} f^{(1)} \\ f \\ f^{(1)} \end{bmatrix}
\end{align*}
\tag{5.14}
\]

Proof. Assume \(w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}\) is given by (5.12). Then (5.13)
follows immediately. Note that for \(f \in C([0, 1])\), the functions \(g = L(f)\) and \(h = L^*(f)\) are
categorized by
\[
\begin{align*}
g^{(1)} &= -f \quad \text{and} \quad g(1) = 0 \\
h^{(1)} &= f \quad \text{and} \quad h(0) = 0.
\end{align*}
\]

Hence (5.14) is equivalent to (5.15) and (5.16) below.
\[
\begin{align*}
\frac{d}{dx} f^{(-1)} &= w_{11} \\
\frac{d}{dx} \left( \frac{1}{\mu^2} \begin{bmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{bmatrix} \right) &= w_{24} \\
\frac{d}{dx} \left( \frac{1}{f} \right) &= w_{42} \\
\frac{d}{dx} \left( -\mu^2 \begin{bmatrix} f^{(1)} \\ f \\ f^{(1)} \end{bmatrix} \right) &= w_{33}
\end{align*}
\tag{5.15}
\]
\[
\begin{align*}
\begin{bmatrix} f^{(-1)}(1) \\ f^{(1)}(1) \\ f^{(1)}(0) \end{bmatrix} &= 0, \\
\frac{1}{f^{(0)}} &= \mu,
\end{align*}
\tag{5.16}
\]
Now, (5.16) is trivial from (i) and (ii). Next we prove (5.15): Clearly
\[
\frac{d}{dx} f^{(-1)} = f = w_{11} \quad \text{and} \quad \frac{d}{dx} \left( \frac{1}{f} \right) = \frac{f^{(1)}}{f^2} = w_{42}.
\]
Moreover
\[
(5.17) \quad \frac{d}{dx} \left( \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix} \right)
= \left( f f^{(1)} - f^{(1)} f \right) \frac{f^{(-1)}}{f^2} \frac{f^{(-1)}}{f} \frac{f^{(-1)}}{f} = \mu^2 w_{24}
\]
and
\[
\frac{d}{dx} \left( \begin{vmatrix} f^{(-1)} & f^{(2)} \\ f & f^{(1)} \end{vmatrix} \right)
= \left( f f^{(1)} - f^{(1)} f \right) \frac{f^{(-1)}}{f^2} \frac{f^{(-1)}}{f} = - \frac{1}{\mu^2} w_{33}.
\]
Hence (5.15) holds. It remains to be proved that \( w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42} \)
is a solution to (5.11). By (5.12) and (5.14), we have
\[
L^*(w_{42}) + \frac{1}{w_{11}} = \left( \mu - \frac{1}{f} \right) + \frac{1}{f} = \mu.
\]
Moreover by (5.12) and (5.13)
\[
\begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \frac{1}{w_{22} w_{44} - w_{24} w_{42}} \begin{pmatrix} w_{44} & -w_{24} \\ -w_{42} & w_{22} \end{pmatrix} \frac{f^{(-1)}}{f} \frac{f^{(-1)}}{f}
= \begin{pmatrix} \mu & f^{(-1)} \\ f & f^{(1)} \end{pmatrix}
\]
which proves that the first and the second inequality in (5.11).
By (5.12) and (5.14),

\[ w_{33}(\mu - L(w_{24})) = \begin{vmatrix} f & f^{[1]} & f^{[-2]} & f^{[-1]} \\ f^{[1]} & f^{[2]} & f^{[-1]} & f \\ f^{[-1]} & f & f^{[1]} & f \\ f & f^{[1]} & f & f \end{vmatrix} = 1 + \begin{vmatrix} f^{[-1]} & f^{[-1]} \\ f & f^{[1]} \end{vmatrix}^2 \]

where

\[ \sigma = \begin{vmatrix} f & f^{[1]} \\ f^{[1]} & f^{[2]} \end{vmatrix} \begin{vmatrix} f^{[-1]} & f^{[-1]} \\ f & f^{[1]} \end{vmatrix} - \begin{vmatrix} f^{[-1]} & f^{[-1]} \\ f & f^{[1]} \end{vmatrix}^2 = \begin{vmatrix} f^{[-2]} & f^{[-1]} \\ f^{[-1]} & f^{[1]} \end{vmatrix} \begin{vmatrix} f^{[-2]} & f^{[-1]} \\ f^{[-1]} & f^{[1]} \end{vmatrix}. \]

Hence by (iii), \( \sigma = 0 \). Therefore \( w_{33}(x) \neq 0 \) for all \( x \in [0,1] \) and \( w_{33}^{-1} = \mu - L(w_{24}) \), proving the last equality in (5.11).

**Lemma 5.5.** Let \( \alpha_j(s), \gamma_j(s) \) for \( j = 1, 2 \) be as in lemma 5.2 for \( k = 2 \), i.e. \( \alpha_1(0) = \alpha_2(0) = 0, \gamma_1(0) = \gamma_2(0) = \frac{1}{2} \) and for \( 0 < |s| < \varepsilon^{-1} : \)

\[ \begin{align*}
\alpha_1(s) &= \rho(s), & \alpha_2(s) &= \rho(-s), \\
\gamma_1(s) &= \frac{\alpha_1(s)}{\alpha_1(s) - \alpha_2(s)}, & \gamma_2(s) &= \frac{\alpha_2(s)}{\alpha_2(s) - \alpha_1(s)}. 
\end{align*} \]

Let \( \mu \in \mathbb{C}, |\mu| > \sqrt{e} \), put \( s = \frac{1}{2} \mu^{-2} \) and

\[
(5.18) \quad f(x) = \frac{1}{\mu} \left( \sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)|x|} \right), \quad x \in \mathbb{R} 
\]

\[
(5.19) \quad f^{-1}(x) = \frac{1}{2\mu} \left( \sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)|x|} \right), \quad x \in \mathbb{R} 
\]

\[
(5.20) \quad f^{-2}(x) = \frac{1}{4\mu} \left( \sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)|x|} \right), \quad x \in \mathbb{R} 
\]

Then

(i) \( f^{-1}, f^{-2} \) are sucessively antiderivatives of \( f \),

\[
(5.21) \quad f^{-1}(1) = 0, \quad f^{-2}(1) = \mu^3
\]

and

\[
(5.22) \quad f(0) = \mu^{-1}, \quad f^{(1)}(0) = 0.
\]
(ii) The following asymptotic formulas holds for $|\mu| \to \infty$:

$$
\begin{align*}
  f^{(-2)}(x) &= \mu^2 + \mathcal{O}(\mu^{-1}) \\
  f^{(-1)}(x) &= (x - 1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\
  f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\
  f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}) \\
  f^{(2)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9})
\end{align*}
$$

where the error estimates holds uniformly in $x$ on a compact subset in $\mathbb{R}$.

(iii) There exists $\mu_0 \geq \sqrt{e}$ such that the restriction of $f$ to $[0, 1]$ satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$.

Proof. Clearly $f^{(-1)}$ and $f^{(-2)}$ are successively antiderivatives of $f$ and

$$
  f(0) = \frac{1}{\mu} \sum_{j=1}^{2} \gamma_j(s) = \frac{1}{\mu}
$$

$$
  f^{(1)}(0) = \frac{2}{\mu} \sum_{j=1}^{2} \alpha_j(s) \gamma_j(s) = 0.
$$

To prove (5.21), note first, that since $\rho : \mathbb{C} \setminus [\frac{1}{\mu}, \infty) \to \mathbb{C}$ is a branch of the inverse function of $z \mapsto ze^{-z}$, we have

$$
  \rho(w)e^{-\rho(w)} = w, \ |w| < \frac{1}{e}
$$

and therefore

$$
  e^{2\alpha_j(s)} = \frac{\alpha_j(s)^2}{s^2}, \ j = 1, 2.
$$

Since $s^2 = \frac{1}{\mu} \mu^{-1}$, it follows that

$$
\begin{align*}
  f^{(-2)}(x + 1) &= \mu^4 f(x), \ x \in \mathbb{R} \\
  f^{(-1)}(x + 1) &= \mu^4 f^{(1)}(x), \ x \in \mathbb{R} \\
  f(x + 1) &= \mu^4 f^{(2)}(x), \ x \in \mathbb{R}.
\end{align*}
$$

In particular

$$
\begin{align*}
  f^{(-2)}(1) &= \mu^4 f(0) = \mu^2 \\
  f^{(-1)}(1) &= \mu^4 f^{(1)}(0) = 0.
\end{align*}
$$

By the proof of [5, Prop. 4.2], $\alpha_j(s)$ and $\rho_j(s)$ are continuous functions of $s \in B(0, \frac{1}{e})$. Hence, regarding $f$ as a function of $\mu$,

$$
\lim_{|\mu| \to \infty} (\mu f(x)) = \sum_{j=1}^{2} \gamma_j(0)e^{2\alpha_j(0)x} = 1
$$
where the limit holds uniformly in $x$ on compact subsets of $\mathbb{R}$. Hence by (5.25) $f^{(2)}(x) = \mathcal{O}(\mu^{-5})$ as $|\mu| \to \infty$ uniformly in $x$ on compact subsets of $\mathbb{R}$. By (5.22),

\begin{align}
(5.26) 
    f^{(1)}(x) &= \int_0^x f^{(2)}(t) \, dt \\
(5.27) 
    f(x) &= \mu^{-1} + \int_0^x f^{(1)}(t) \, dt
\end{align}

which implies, that $f^{(1)}(x) = \mathcal{O}(\mu^{-5})$ and

\begin{align}
(5.28) 
    f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5})
\end{align}

uniformly in $x$ on compact subsets of $\mathbb{R}$.

Using again (5.25), (5.26) and (5.27), we get

\begin{align*}
    f^{(2)}(x) &= \mu^{-5} + \mathcal{O}(\mu^{-9}) \\
    f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}).
\end{align*}

By (5.21)

\begin{align*}
    f^{(-1)}(x) &= \int_1^x f(t) \, dt \\
    f^{(-2)}(x) &= \mu^3 + \int_1^x f^{(-1)}(t) \, dt.
\end{align*}

Hence by (5.28),

\begin{align*}
    f^{(-1)}(x) &= (x - 1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\
    f^{(-2)}(x) &= \mu^3 + \mathcal{O}(\mu^{-1})
\end{align*}

where all estimates holds uniformly on compact subsets of $\mathbb{R}$. This proves (ii).

By (i), $f^{(-1)}, f^{(-2)}$ coincide with the successive antiderivatives of $f$ considered in lemma 5.4 and $f(0) = \mu^{-1}, f^{(1)}(0) = 0$.

Moreover, by (ii),

\begin{align*}
    f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\
    \begin{vmatrix}
        f^{(-1)}(x) & f(x) \\
        f(x) & f^{(1)}(x)
    \end{vmatrix} &= \mu^{-2} + \mathcal{O}(\mu^{-6})
\end{align*}

where the error terms holds uniformly in $x \in [0,1]$. Hence there exists $\mu_0 \geq \sqrt{e}$, such that

\begin{align*}
    f(x) \neq 0 \text{ and } \begin{vmatrix}
        f^{(-1)}(x) & f(x) \\
        f(x) & f^{(1)}(x)
    \end{vmatrix} \neq 0
\end{align*}
for all \( x \in [0,1] \). Moreover by the matrix factorization

\[
(5.29) \begin{pmatrix}
 f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\
 f^{(-1)}(x) & f(x) & f^{(1)}(x) \\
 f(x) & f^{(1)}(x) & f^{(2)}(x)
\end{pmatrix}
= \begin{pmatrix}
 \frac{1}{2a_1(s)} & \frac{1}{2a_2(s)} & 0 \\
 \frac{1}{4a_1(s)^2} & \frac{1}{4a_2(s)^2} & 0 \\
 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
 \frac{\gamma_1(s)}{2a_1(s)^2} e^{2a_1(s)x} \\
 \frac{\gamma_2(s)}{2a_2(s)^2} e^{2a_2(s)x} \\
 1
\end{pmatrix}
\begin{pmatrix}
 f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\
 f^{(-1)}(x) & f(x) & f^{(1)}(x) \\
 f(x) & f^{(1)}(x) & f^{(2)}(x)
\end{pmatrix}
\]

it follows, that the matrix on the left hand side has rank less than or equal to 2, i.e.,

\[
\begin{pmatrix}
 f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\
 f^{(-1)}(x) & f(x) & f^{(1)}(x) \\
 f(x) & f^{(1)}(x) & f^{(2)}(x)
\end{pmatrix} = 0
\]

for \( x \in [0,1] \). Hence \( f \) satisfies all the conditions in lemma 5.4, when \( |\mu| > \mu_0 \).

\[\square\]

Proof of Theorem 5.1 in the case \( k = 2 \): By lemma 2.1 there exists a \( \delta > 0 \), such that when \( w \in M_4(\mathcal{D})_{\text{inv}} \) and \( \mu \in \mathbb{C} \) satisfies \( ||w|| < \delta, |\mu| > \frac{1}{\delta} \) and

\[
(5.30) \quad \mathcal{R}^T_{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{D})}
\]

then \( w = E_{\mathcal{D}}((\tilde{\mu} - \tilde{T})^{-1}) \). In particular

\[
(5.31) \quad w_{11} = \mu^{-1}E_{\mathcal{D}}((1 - \mu^{-4}(T^*T)^2)^{-1}).
\]

Let \( \mu \in \mathbb{C}, |\mu| > \sqrt{e} \), put \( s = \frac{1}{2}\mu^{-2} \) and

\[
 f(x) = \frac{1}{\mu} \left( \sum_{j=1}^{2} \gamma_j(s)e^{2a_j(s)x} \right)
\]

for \( x \in [0,1] \) as in lemma 5.5. By lemma 5.5 (iii) there exists a \( \mu_0 > \sqrt{e} \), such that when \( |\mu| > \mu_0 \), then \( f \) satisfies all the requirements of lemma 5.4. Hence by lemma 5.4, the matrix \( w \in M_4(\mathcal{D}) \) given by (5.10) and (5.12) is a solution to (5.30). Moreover by the asymptotic formulas in lemma 5.5 (ii),

\[
\begin{pmatrix}
 f^{(-2)}(x) & f^{(-1)}(x) \\
 f^{(-1)}(x) & f(x)
\end{pmatrix} = \mu^2 + \mathcal{O}(\mu^{-2})
\]

\[
\begin{pmatrix}
 f^{(-1)}(x) & f(x) \\
 f(x) & f'(x)
\end{pmatrix} = -\mu^{-2} + \mathcal{O}(\mu^{-6})
\]

\[
\begin{pmatrix}
 f(x) & f'(x) \\
 f'(x) & f''(x)
\end{pmatrix} = \mu^{-6} + \mathcal{O}(\mu^{-10}).
\]
Hence by (5.12) and the asymptotic formulas for $f^{(-1)}, f$ and $f'$, we have

\begin{align*}
    w_{11} &= \mu^{-1} + O(\mu^{-5}), \\
    w_{22} &= w_{44} = \mu^{-1} + O(\mu^{-5}), \\
    w_{24} &= (1 - x)\mu^{-3} + O(\mu^{-3}), \\
    w_{42} &= x\mu^{-3} + O(\mu^{-3}), \\
    w_{33} &= \mu^{-1} + O(\mu^{-5}),
\end{align*}

where all the error estimates holds uniformly in $x \in [0, 1]$. Hence, there exists $\mu_1 \geq \max\{\mu_0, \frac{1}{2}\}$, such that when $|\mu| > \mu_1$ then $\|w\| < \delta$, and hence

\[w = E_{M_{\text{d}}}(\mu - \bar{T})^{-1}.\]

By (5.12), $w_{11} = f$. Hence by (5.31) and (5.18)

\[E_D((1 - \mu^{-4}(T^*)^2T^2)^{-1})(x) = \mu f(x) = \sum_{j=1}^{2} \gamma_j(s)e^{2\alpha_j(s)x}\]

where $s = \frac{1}{2}\mu^{-2}$, i.e. for $|s| < \frac{1}{2}\mu_1^{-2}$,

\[E_D((1 - (2s)^2(T^*)^2T^2)^{-1})(x) = \sum_{j=1}^{2} \gamma_j(s)e^{2\alpha_j|s|x}\]

and therefore

\[\sum_{j=0}^{\infty} (2s)^{2n} E_D(((T^*)^2T^2)^n)(x) = \sum_{j=1}^{2} \gamma_j(s)e^{2\alpha_j|s|x}.\]

Hence by lemma 5.2 and by the uniqueness of the power series expansions of analytic functions, we have

\[E_D(((T^*)^2T^2)^n)(x) = P_{2,n}(x)\]

for $n \in \mathbb{N}$ and $x \in [0, 1]$. This proves theorem 5.1(a) in the case $k = 2$. Theorem 5.1 (b) also follows from (5.32) by integrating the right hand side of (5.32) from 0 to 1 with respect to $x$ (cf. [5, remark 4.3]).

6. Śniady’s moment formulas. The general case.

The above proof of Theorem 5.1 in the case $k = 2$ can fairly easily be generalized to all $k \geq 2$ (Recall that the case $k = 1$ is contained in theorem 3.2).
Let $k \geq 2$ and define $\tilde{T} \in M_{2k}(A)$ by

$$\tilde{T} = \sum_{j=1}^{k} (T \otimes e_{j+1,j} + T^* \otimes e_{k+j+1,k+j})$$

where the indices are computed modulo $2k$, such that $e_{2k+1,2k} = e_{1,2k}$.

For $\mu \in \mathbb{C}, |\mu| < \frac{1}{\sqrt{e}},$ we put $\tilde{\mu} = \mu 1_{2k}$ and

$$z = z(\mu) = E_{M_{2k}(D)}((\tilde{\mu} - \tilde{T})^{-1}).$$

Then only the diagonal entries $z_{11}, \ldots, z_{2k,2k}$ and the off-diagonal entries $z_{2,2k}, z_{3,2k-1}, \ldots, z_{2k,2}$ can be non-zero. Moreover,

$$z_{11} = \mu^{-1}E_{D}((1 - \mu^{-2k}(T^*)^{k}T^{-k})^{-1}).$$

The operator $\tilde{T}$ is $M_{2k}(D)$-Gaussian, and repeating the arguments for $k = 2$, we get that for $w \in M_{2k}(D)$, the matrix

$$u = R_{F}^{M_{2k}(D)}(w)$$

can have at most $2k$ non-zero entries, namely the entries

$$u_{11} = L^*(w_{2k,2})$$
$$u_{2k,2} = L^*(w_{2k-1,3})$$
$$\vdots$$

(6.2)

$$u_{k+2,k} = L^*(w_{k+1,k+1})$$
$$u_{k+1,k+1} = L(w_{k,k+2})$$
$$u_{k,k+2} = L(w_{k-1,k+3})$$
$$\vdots$$

$$u_{2,2k} = L(w_{1,1}).$$

By lemma 2.1 there exists a $\delta > 0$ (depending on $k$), such that if $w \in M_{2k}(D)_{\text{inv}}, \|w\| < \delta, \mu \in \mathbb{C}, |\mu| > \frac{1}{\delta}$ and

$$R_{F}^{M_{2k}(D)}(w) + w^{-1} = \mu 1_{M_{2k}(D)},$$

then

$$w = z = E_{M_{2k}(D)}((\tilde{\mu} - \tilde{T})^{-1}).$$

In particular

$$w_{11} = \mu^{-1}E_{D}((1 - \mu^{-2k}(T^*)^{k}T^{-k})^{-1}).$$

Next we construct an explicit solution to (6.3). By the above remarks on $z$, it is sufficient to consider those $w \in M_{2k}(D)_{\text{inv}}$ for which only the
entries $z_{11}, \ldots, z_{2k,2k}$ and $z_{2,2k}, z_{3,2k-1}, \ldots, z_{2k,2}$ can be non-zero. For such $w$, (6.3) can by (6.1) and (6.2) be reduced to the $k + 1$ identities:

\[
\begin{align*}
L^*(w_{2k,2}) + \frac{1}{w_{11}} &= \mu_1 D, \\
\left( L^*(w_{2k-1-j,j+3}) \right) + \begin{pmatrix} w_{2k,j+2+j} & w_{2k-2+j,j+3} \\ w_{2k-2+j,j+3} & w_{2k-2+j,j+3} \end{pmatrix}^{-1} &= \mu_1 \text{M}_2(D), \\
L(w_{k,k+2}) + \frac{1}{w_{k+1,k+1}} &= \mu_1 D.
\end{align*}
\]

**Definition 6.1.** For $j \in \mathbb{N} \cup \{0\}$ and $g \in C^{2j+2}$, we let $\Delta_j(g)$ denote the determinant

\[
\Delta_j(g) = \begin{vmatrix} g & g^{(1)} & \cdots & g^{(j)} \\
g^{(1)} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
g^{(j)} & \cdots & g^{(j-1)} & g^{(j)} \end{vmatrix}.
\]

In particular $\Delta_0(g) = g$.

**Lemma 6.2.** Let $g \in C^{2j+2}(\mathbb{R})$ and $j \in \mathbb{N}$. Then

\[
\Delta_j(g^{(2)})\Delta_j(g) - \Delta_j(g^{(1)})^2 = \Delta_{j-1}(g^{(2)})\Delta_{j+1}(g)
\]

and

\[
\Delta_{j-1}(g^{(2)}) \frac{d}{dx} \left( \Delta_j(g) \right) - \Delta_j(g) \frac{d}{dx} \left( \Delta_{j-1}(g^{(2)}) \right) = \Delta_{j-1}(g^{(1)})\Delta_j(g^{(1)}).
\]

The proof of lemma 6.2 relies on elementary matrix manipulations and is contained in lemma A.1 of appendix A. More specifically (6.6) is a direct consequence of (a) from lemma A.1, and (6.7) follows from (b) of lemma A.1 by using the elementary fact that:

\[
\frac{d}{dx} \left( \Delta_j(g) \right) = \begin{vmatrix} g & g^{(1)} & \cdots & g^{(j)} \\
g^{(1)} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
g^{(j)} & \cdots & g^{(j-1)} & g^{(j)} \end{vmatrix},
\]

that is, differentiating (6.5) is the same as differentiating the last row of (6.5).

The next two lemmas are the generalizations of lemma 5.4 and lemma 5.5 to arbitrary $k \geq 2$. 
Lemma 6.3. Let \( f \in C^k([0,1]) \) and let \((f^{(-j)})_j^{k=1}\) be the antiderivatives of \( f \) for which,

(i) \[ f^{(-j)}(1) = \begin{cases} 
0, & 1 \leq j \leq k - 1, \\
\mu^{2k-1}, & j = k.
\end{cases} \]

(ii) Assume further that \( f(0) = \mu^{-1} \) and \( f^{(-j)}(0) = 0 \) for \( 1 \leq j \leq k - 1 \).

(iii) For all \( x \in [0,1] \),

\[ \Delta_j(f^{(-j)})(x) \neq 0, \text{ for } j = 0 \ldots, k - 1 \]

and

\[ \Delta_k(f^{(-k)})(x) = 0 \]

Then the set of \( 4k - 2 \) functions listed in (6.8), (6.9) and (6.10) below is a solution to (6.4).

\[
\begin{align*}
&w_{11} = f \\
&w_{22} = w_{2k,2k} = -\frac{1}{\mu} \Delta_1(f^{(-1)}) \\
&w_{2,2k} = \frac{1}{\mu^2} f^{(-1)} \Delta_1(f^{(-1)}) \\
&w_{2k,2} = \frac{f^{(1)}}{f}
\end{align*}
\]

(6.8)

For \( j = 1, \ldots, k - 2 \)

\[
\begin{align*}
&w_{j+2,j+2} = w_{2k-j,2k-j} = -\frac{1}{\mu} \Delta_j-1(f^{(1-j)}) \Delta_{j+1}(f^{(-1-j)}) \\
&w_{j+2,2k-j} = \frac{1}{\mu^{j+2}} \Delta_j(f^{(-1-j)}) \frac{\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\
&w_{2k-j,j+2} = \mu^j \frac{\Delta_{j+1}(f^{(1-j)}) \Delta_{j}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2}
\end{align*}
\]

(6.9)

\[ w_{k+1,k+1} = \mu^{2k+2} \frac{\Delta_{k-2}(f^{(2-k)}) \Delta_{k-1}(f^{(2-k)})}{\Delta_{k-1}(f^{(1-k)})^2} \]

(6.10)

Moreover for \( j = 0, \ldots, k - 2 \)

\[
\begin{align*}
&w_{j+2,j+2} w_{2k-j,2k-j} = \frac{1}{\mu} w_{j+2,j+2}
\end{align*}
\]

(6.11)
and

\[
\begin{align*}
L(w_{11}) &= -f^{(-1)} \\
L(w_{j+2,k-\ell}) &= -\frac{1}{\mu^{j+2}} \frac{\Delta_{i+1}(f^{(-2-j)})}{\Delta_{i}(f^{(-j)})}, \quad 0 \leq j \leq k - 3 \\
L(w_{k,k+2}) &= \mu - \frac{1}{\mu^{k+2}} \frac{\Delta_{k-1}(f^{(-k)})}{\Delta_{k-2}(f^{(-k)})}
\end{align*}
\]

(6.12)

\[
\begin{align*}
L'(w_{2k,2}) &= \mu - \frac{1}{\mu} \\
L'(w_{2k-j,2+\ell}) &= -\mu^{2j} \frac{\Delta_{j-1}(f^{(-2-j)})}{\Delta_{j}(f^{(-j)})}, \quad 1 \leq j \leq k - 2 \\
L'(w_{k+1,k+1}) &= -\mu^{2k-2} \frac{\Delta_{k-2}(f^{(-3-k)})}{\Delta_{k-1}(f^{(-k)})}
\end{align*}
\]

(6.13)

**Proof.** Let \( w_{11}, w_{22}, \ldots, w_{kk}, w_{2k}, w_{3,2k-1}, \ldots, w_{2k,2} \) be given by (6.8), (6.9) and (6.10). Then for \( 1 \leq j \leq k - 2 \) the left hand side of (6.11) is equal to

\[
-\frac{1}{\mu^2} \frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(1-j)}) A}{\Delta_{j}(f^{(-j)})^4}.
\]

where \( A = \Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(1-j)}) - \Delta_{j}(f^{(1-j)})\Delta_{j}(f^{(1-j)}) \).

By applying (6.6) to \( g = f^{(-1-j)} \) it follows that \( A = -\Delta_{j}(f^{(-j)})^2 \), which proves (6.11) for \( 1 \leq j \leq k - 2 \). The case \( j = 0 \) of (6.11) follows immediately from (6.8).

The proofs of (6.12) and 6.13 can be obtained exactly as in the case \( k = 2 \) provided the following two identities holds: For \( j = 0, \ldots, k - 2 \):

\[
\frac{d}{dx} \left( \frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_{j}(f^{(-j)})} \right) = \frac{\Delta_{j}(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_{j}(f^{(-j)})^2}
\]

(6.14)

For \( j = 1, \ldots, k - 1 \):

\[
\frac{d}{dx} \left( \frac{\Delta_{j-1}(f^{(2-j)})}{\Delta_{j}(f^{(-j)})} \right) = \frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j}(f^{(1-j)})}{\Delta_{j}(f^{(-j)})^2}
\]

(6.15)

However (6.14) follows from (6.7) with \( g = f^{(-2-j)} \) after changing \( j \) in (6.7) to \( j + 1 \). In the same way (6.15) follows from (6.7) with \( g = f^{(-j)} \) and \( j \) unchanged. It remains to be proved, that \( w_{11}, \ldots, w_{kk}, w_{2k}, \ldots, w_{2k,2} \) form a solution to (6.4). The proof of the first 2 identities in (6.4) is exactly the same as in the case \( k = 2 \). Let us check the next \( k - 2 \)
identities in (6.4) i.e.

\[
(6.16) \quad \left( \begin{array}{cc}
0 & L(w_{j+1,2k+1-j}) \\
L^*(w_{2k-1-j, j+3}) & 0
\end{array} \right)
\]

\[+ \left( \begin{array}{cc}
w_{2+j,2+j} & w_{2+j,2k-j} \\
w_{2k-j,2+j} & w_{2k-j,2k-j}
\end{array} \right)^{-1} = \mu 1_{M_0(\mathbb{D})}
\]

for \( j = 1, \ldots, k - 2 \). By (6.11) and the fact that \( w_{2+j,2+j} = w_{2k-j,2k-j} \) (cf. (6.8)) we have

\[
\left( \begin{array}{cc}
w_{2+j,2+j} & w_{2+j,2k-j} \\
w_{2k-j,2+j} & w_{2k-j,2k-j}
\end{array} \right)^{-1} = \left( \begin{array}{cc}
\mu 1_{\mathbb{D}} & \beta \\
\gamma & \mu 1_{\mathbb{D}}
\end{array} \right),
\]

where

\[
\beta = -\mu \frac{w_{2+j,2k-j}}{w_{2+j,2+j}} = \frac{1}{\mu 2^j} \frac{\Delta_j(f^{(1-j)})}{\Delta_j(f^{(1-j)})}
\]

and

\[
\gamma = -\mu \frac{w_{2k-j,2+j}}{w_{2+j,2+j}} = \mu 2^{j+2} \frac{\Delta_j(f^{(1-j)})}{\Delta_{j+1}(f^{(1-j)})}.
\]

Hence by (6.12) and (6.13)

\[
\beta = -L(w_{j+1,2k-j+1}) \quad \text{and} \quad \gamma = -L^*(w_{2k-1-j, j+3})
\]

for \( j = 1, \ldots, k - 2 \). This proves (6.16). Observe next that by (6.10) and (6.12)

\[
w_{k+1,k+1}(\mu - L(w_{k,k+2})) = \frac{\Delta_{k-1}(f^{(2-k)}) \Delta_{k-1}(f^{(1-k)})}{\Delta_{k-1}(f^{(1-k)})^2}
\]

\[= 1 + \frac{\sigma}{\Delta_{k-1}(f^{(1-k)})^2},\]

where

\[
\sigma = \Delta_{k-1}(f^{(2-k)}) \Delta_{k-1}(f^{(-k)}) - \Delta_{k-1}(f^{(1-k)})^2.
\]

By (6.6) and the assumptions (iii) in lemma 6.3

\[
\sigma = \Delta_{k-2}(f^{(2-k)}) \Delta_{k}(f^{(-k)}) = 0.
\]

Hence \( w_{k+1,k+1}(\mu - L(w_{k,k+2})) = 1 \), which proves the last equality in (6.4). This completes the proof of lemma 6.3. \( \square \)

**Lemma 6.4.** Let \( k \in \mathbb{N}, k \geq 2 \) and let \( \alpha_j(s), \gamma_j(s) \) for \( j = 1, \ldots, k \) and \( 0 < |s| < \frac{1}{\alpha} \) be as in lemma 5.2. Let \( \mu \in \mathbb{C}, |\mu| > \sqrt{e} \), put \( s = \frac{1}{k^2} k^2 \) and

\[
(6.17) \quad \begin{cases}
f(x) = \frac{1}{\mu} \left( \sum_{\nu=1}^{k} \gamma_{\nu}(s) e^{\mu \alpha_{\nu}(s)x} \right), & x \in \mathbb{R} \\
f(-j)(x) = \frac{1}{\mu k^2} \left( \sum_{\nu=1}^{k} \frac{\gamma_{\nu}(s)}{\alpha_{\nu}(s)} e^{\mu \alpha_{\nu}(s)x} \right), & x \in \mathbb{R}, j = 1, \ldots, k.
\end{cases}
\]
Then

(i) \((f^{(-j)})^j_{j=1} \) are successive antiderivatives of \( f \). Moreover

\[
\begin{aligned}
& f^{(-j)}(1) = 0, \quad 1 \leq j \leq k - 1 \\
& f^{(-k)}(1) = \mu^{2k-1}
\end{aligned}
\]  

(6.18)

and

\[
\begin{aligned}
& f(0) = \mu^{-1} \\
& f^{(j)}(0) = 0, \quad 1 \leq j \leq k - 1 .
\end{aligned}
\]  

(6.19)

(ii) The following asymptotic formulas holds for \(|\mu| \to \infty\)

\[
\begin{aligned}
& f^{(-k)}(x) = \mu^{2k-1} + O(\mu^{-1}) \\
& f^{(-j)}(x) = \frac{1}{j!}(x-1)^j \mu^{-1} + O(\mu^{-2k-1}), \quad 1 \leq j \leq k - 1 \\
& f(x) = \mu^{-1} + O(\mu^{-2k-1}) \\
& f^{(j)}(x) = \frac{1}{j!}x^j \mu^{-2k-1} + O(\mu^{-4k-1}), \quad 1 \leq j \leq k - 1 \\
& f^{(k)}(x) = \mu^{-2k-1} + O(\mu^{-4k-1})
\end{aligned}
\]  

(6.20)

where the error estimates holds uniformly in \( x \) on compact subsets of \( \mathbb{R} \).

(iii) There exists a \( \mu_0 \geq \sqrt{e} \), such that the restriction of \( f \) to \([0,1]\) satisfies all the conditions in lemma 6.3, when \(|\mu| > \mu_0\).

Proof. From the proof of [5, Prop. 4.2], we know that \( \alpha_j(s) \) and \( \gamma_j(s) \) are analytic functions of \( s \in B(0, \frac{1}{c}) \). Moreover by [4, Prop. 4.1]

\[
\begin{aligned}
& \sum_{\nu=1}^{k} \gamma_{\nu}(s) = 1 \\
& \sum_{\nu=1}^{k} \gamma_{\nu}(s) \alpha_{\nu}(s) = 1, \quad j = 1, \ldots, k - 1.
\end{aligned}
\]  

(6.21)

Moreover, since \( \alpha_j(s) = \rho(e^{2\pi j/\nu}) \), where \( \rho \) satisfies

\[
\rho(w)e^{-\rho(w)} = w \text{ for } |w| < \frac{1}{e}
\]

we have \( (\alpha_{\nu}(s)e^{-\alpha_{\nu}(s)})^k = s^k \) and therefore

\[
e^{k\alpha_{\nu}(s)} = \frac{s^k}{(\alpha_{\nu}(s))^k}
\]  

(6.22)

for \( \nu = 1, \ldots, k \). Having (6.21) and (6.22) in mind, the proof of (i) and (ii) in lemma 6.4 is now a routine generalization of the proof of lemma...
5.5. Concerning (iii) in lemma 6.4, we have
\[
\begin{align*}
\Delta_j(f^{(-j)}) &= \sigma(j)\mu^{-j-1} + O(\mu^{-2k-j-1}), \quad 0, \ldots, k - 1, \\
\text{where } \sigma(j) &= 1 \text{ for } j = 0, 3 \pmod{4}, \\
\text{and } \sigma(j) &= -1 \text{ for } j = 1, 2 \pmod{4}
\end{align*}
\]  
(6.23)
because the leading term in the determinant \(\Delta_j(f^{(-j)})\) comes from the antidiagonal, i.e.,
\[
\Delta_j(f^{(-j)}) = \begin{vmatrix}
0 & \cdots & 0 & f \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
f & 0 & \cdots & 0
\end{vmatrix} + O(\mu^{-2k-j-1}) = \sigma(j)f^{j+1} + O(\mu^{-2k-j-1})
\]
since the matrix in question has size \(j + 1\). Hence \(\Delta_j(f^{(-j)})(x) \neq 0\) for \(x \in [0, 1]\) and \(0 \leq j \leq k - 1\), when \(|\mu|\) is sufficiently large. Moreover \(\Delta_k(f^{(-k)})(x) = 0\) for \(x \in [0, 1]\), because in analogy with (5.29), \(\Delta_k(f^{(-k)})(x)\) is the determinant of the \((k + 1) \times (k + 1)\) matrix
\[
F = (f^{(i+j-k)})_{i,j=0,\ldots,k}
\]
which has the factorization \(F = ADA^t\), where \(A\) is the \((k + 1) \times k\) matrix with entries
\[
a_{il} = (k\alpha_l(s))^i, \quad i = 0, \ldots, k, \quad l = 1, \ldots, k
\]
and \(D\) is the \(k \times k\) diagonal matrix, with diagonal entries
\[
d_{ll} = \frac{\gamma_l(s)}{(k\alpha_l(s))^{k-1}}, \quad l = 1, \ldots, k.
\]

Proof of Theorem 5.1 in the general case. Let \(\mu_0\) be as in lemma 6.4, let \(\mu \in \mathbb{C}, |\mu| > \mu_0\) and put \(s = \frac{1}{\mu^2}\). Put as before
\[
f(x) = \frac{1}{\mu} \left( \sum_{\nu=1}^{k} \gamma_\nu(s) e^{k\alpha_\nu(x)} \right)
\]
for \(x \in [0, 1]\), and define \(w_1, w_{22}, \ldots, w_{k,k}, w_{2,2k}, w_{3,2k-1}, \ldots, w_{2k,2}\) by (6.8), (6.9) and (6.10), and put all other entries of \(w \in M_{2k}(\mathbb{D})\) equal to 0. Then by lemma 6.4, (6.4) holds, and therefore
\[
\mathcal{R}^{M_{2k}(\mathbb{D})}f(w) + w^{-1} = \mu 1_{M_{2k}(\mathbb{D})}.
\]
Let \(\delta > 0\) be chosen according to lemma 2.1. If we can find a \(\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}\), such that
\[
|\mu| \geq \mu_1 \Rightarrow \|w\| < \delta
\]
(6.24)
then \( w = E_{M_{2k}(D)}((\tilde{\mu} - \tilde{T})^{-1}) \). In particular

\[
f = w_{11} = \mu^{-1}E_D((1 - \mu^{-2k}(T^*)^kT^k)^{-1}),
\]

and the proof of theorem 5.1 for \( k \geq 2 \) can be completed exactly as in the case \( k = 2 \). By (6.23)

\[
\begin{align*}
\Delta_j(f^{(j)}) &= \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k - 1 \\
\Delta_{k-1}(f^{(k)}) &= \mathcal{O}(\mu^k), & 0 \leq j \leq k - 1 \\
\Delta_j(f^{(1-j)}) &= \mathcal{O}(\mu^{-j-2k-1}), & 0 \leq j \leq k - 2 \\
\Delta_{k-1}(f^{(2-k)}) &= \mathcal{O}(\mu^{-3k})
\end{align*}
\]

uniformly in \( x \in [0,1] \) for \( |\mu| \to \infty \). We claim that

\[
\begin{align*}
\Delta_j(f^{(j)}) &= \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k - 2 \\
\Delta_{k-1}(f^{(k)}) &= \mathcal{O}(\mu^k) \\
\Delta_j(f^{(1-j)}) &= \mathcal{O}(\mu^{-j-2k-1}), & 0 \leq j \leq k - 2 \\
\Delta_{k-1}(f^{(2-k)}) &= \mathcal{O}(\mu^{-3k})
\end{align*}
\]

Recall by definition 6.1 that

\[
\Delta_j(g) = \det \left( (g^{(k+l)})_{k,l=0,\ldots,j} \right).
\]

Hence for \( 0 \leq j \leq k - 2 \), \( \Delta_j(f^{(j-1)}) \) is the determinant of a \((j+1) \times (j+1)\) matrix, where each entry is equal to one of the functions \( f^{(j-1)}, f^{(j)}, \ldots, f^{(j)} \). By (6.20) all these functions are of order \( \mathcal{O}(\mu^{-1}) \) as \( |\mu| \to \infty \). Hence \( \Delta_j(f^{(j-1)}) = \mathcal{O}(\mu^{-j-1}) \) proving the first estimate in (6.27). By the same argument, \( \Delta_{k-1}(f^{(k)}) \) is the determinant of a \( k \times k \) matrix for which the upper left entry is of the order \( \mathcal{O}(\mu^{2k-1}) \) and all the other entries are of order \( \mathcal{O}(\mu^{-1}) \). Hence \( \Delta_{k-1}(f^{(k)}) = \mathcal{O}(\mu^{2k-1}(\mu^{-1})^{k-1}) = \mathcal{O}(\mu^k) \). Let \( 0 \leq j \leq k - 1 \). Then \( \Delta_j(f^{(1-j)}) \) is by (6.20) a determinant of a \((j+1) \times (j+1)\) matrix \( M = (m_{k,l})_{k,l=0,\ldots,j} \) for which

\[
\begin{align*}
m_{k,l} &= \mathcal{O}(\mu^{-1}) \quad \text{when } k + l < 0 \\
m_{k,l} &= \mathcal{O}(\mu^{-2k-1}) \quad \text{when } k + l \geq 0
\end{align*}
\]

Hence for any permutation \( \pi \) of \( \{0,1,\ldots,k\} \) the product

\[
m_{0\pi(0)}m_{1\pi(1)} \cdots m_{j\pi(j)}
\]

contains at least one factor of order \( \mathcal{O}(\mu^{-2k-1}) \). Therefore

\[
\Delta_j(f^{(1-j)}) = \det(M) = \sum_{\pi \in S_{j+1}} (-1)^{\text{sign}(\pi)} m_{0\pi(0)}m_{1\pi(1)} \cdots m_{k\pi(k)}
\]

is of order \( \mathcal{O}(\mu^{-2k-1}(\mu^{-1})^j) = \mathcal{O}(\mu^{-2k-j-1}) \). This proves the last two estimates in (6.27). Clearly all estimates holds uniformly in \( x \in [0,1] \).
Combining (6.8), (6.9), (6.10) and (6.27), we get
\[
\begin{align*}
  w_{l,l} &= O(\mu^{-1}), & 1 \leq l \leq 2k \\
  w_{j+2,2k-j} &= O(\mu^{-2j-3}), & 0 \leq j \leq k-2 \\
  w_{2k-j,j+2} &= O(\mu^{2j+1-2k}), & 0 \leq j \leq k-2
\end{align*}
\]

In particular all the entries of \( w \) are of size \( O(\mu^{-1}) \) as \( |\mu| \to \infty \) uniformly in \( x \in [0,1] \). Hence there exists \( \mu_1 \geq \max \{ \mu_0, \frac{1}{2} \} \) such that (6.24) holds. Hence by (6.25) we have for \( |s| < \frac{1}{k\mu_1^{-2}} \),
\[
\sum_{k=0}^{\infty} (ks)^n E_2((T^*)^{k}T^k)^n(x) = \sum_{\nu=1}^{\infty} \gamma_j(s) e^{k\nu(s)x}, \quad x \in [0,1].
\]

Now Theorem 5.1 follows from lemma 5.2 and [5, remark 4.3] as in the case \( k = 2 \). \( \square \)

**APPENDIX A. DETERMINANT-IDENTITIES ON HANKEL-MATRICES**

We need the following lemma on Hankel-determinants.

**Lemma A.1.** Let \( a_{-(n-1)}, a_{-(n-2)}, \ldots, a_{n-1}, a_n \in \mathbb{C} \) for some \( n \in \mathbb{N} \). Then

\[(a)\]

| \( a_{-(n-3)} \) | \( a_{-(n-4)} \) | \( \cdots \) | \( a_0 \) |
|-----------------|-----------------|-----------------|
| \( a_{-(n-4)} \) | \( a_{-(n-5)} \) | \( \cdots \) | \( a_{n-4} \) |
| \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) |
| \( a_0 \) | \( a_{n-4} \) | \( a_{n-3} \) | \( a_{n-2} \) |

\[
= \begin{vmatrix}
  a_{-(n-1)} & a_{-(n-2)} & \cdots & a_{n-1} \\
  a_{-(n-2)} & \cdots & \cdots & a_{n-2} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{-(n-3)} & a_{-(n-4)} & \cdots & a_1 \\
  a_{-(n-4)} & \cdots & \cdots & a_2 \\
  \cdots & \cdots & \cdots & \cdots \\
  a_1 & a_{n-4} & \cdots & a_{n-3} \\
  a_{n-4} & \cdots & \cdots & a_{n-2} \\
  a_{n-3} & \cdots & \cdots & a_{n-1}
\end{vmatrix}^2,
\]
(b) 

\[
\begin{pmatrix}
\alpha_{i}(n-2) & \alpha_{i}(n-3) & \cdots & \alpha_{i} \\
\alpha_{i}(n-3) & \cdots & \cdots & \alpha_{i} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{i} & \cdots & \cdots & \cdots \\
\end{pmatrix}
= \\
\begin{pmatrix}
\alpha_{i}(n-1) & \alpha_{i}(n-2) & \cdots & \alpha_{i} \\
\alpha_{i}(n-2) & \cdots & \cdots & \alpha_{i} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{i} & \cdots & \cdots & \cdots \\
\end{pmatrix}
- \\
\begin{pmatrix}
\alpha_{i}(n-1) & \alpha_{i}(n-2) & \cdots & \alpha_{i} \\
\alpha_{i}(n-2) & \cdots & \cdots & \alpha_{i} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{i} & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Proof. To prove (a) we actually prove the more general equation

(A.1) 

\[
\begin{pmatrix}
\alpha_{12} & \cdots & \alpha_{1,n-1} \\
\alpha_{13} & \cdots & \alpha_{3,n-1} \\
\vdots & \vdots & \vdots \\
\alpha_{n-1,1} & \cdots & \alpha_{n-1,n-1} \\
\end{pmatrix}
= \\
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,n} \\
\alpha_{13} & \alpha_{12} & \cdots & \alpha_{3,n} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \\
\end{pmatrix}
- \\
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,n} \\
\alpha_{13} & \alpha_{12} & \cdots & \alpha_{3,n} \\
\vdots & \vdots & \vdots & \vdots \\
\alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \\
\end{pmatrix}
\]

for \( a_{ij} \in \mathbb{C} \) and \( i, j \in \{1, \ldots, n\} \).

We first add some zero terms to the left-hand side (LHS) of (A.1).

\[
\text{LHS} = \\
\begin{pmatrix}
\alpha_{22} & \cdots & \alpha_{2,n-1} \\
\vdots & \vdots & \vdots \\
\alpha_{n-1,2} & \cdots & \alpha_{n-1,n-1} \\
\end{pmatrix}
+ \sum_{k=2}^{n-1} \\
\begin{pmatrix}
\alpha_{21} & \cdots & \alpha_{2,k-1} & \alpha_{2,k+1} & \cdots & \alpha_{2,n-1} \\
\alpha_{31} & \cdots & \alpha_{3,k-1} & \alpha_{3,k+1} & \cdots & \alpha_{3,n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-1,1} & \cdots & \alpha_{n-1,k-1} & \alpha_{n-1,k+1} & \cdots & \alpha_{n-1,n-1} \\
\end{pmatrix}
\]

We note that the last matrix in the sum is zero because column \( k-1 \) and \( k \) are equal. Now we expand LHS after the \( k \)th coloumn of the
second matrix in the \(k\)'th addent. We get

\[
\text{LHS} = \sum_{j=1}^{n} (-1)^{1+j} a_{j,1} \begin{vmatrix} a_{12} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}
\begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix}
\]

\[+ \sum_{k=2}^{n-1} \sum_{j=1}^{n} (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{12} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}
\begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix}
\]

where \(j = 1\) and \(j = n\) means leave out row 1 and \(n\) respectively. Switching the indices we have

\[
(A.2) \quad \text{LHS} = \sum_{j=1}^{n} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix}
\begin{vmatrix} a_{12} & \cdots & a_{2,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}
\begin{vmatrix} (-1)^{1+j} a_{j,1} \\ \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix}
\begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-2,2} & \cdots & a_{n-2,n} \end{vmatrix}
\]

But the parenthesis on the right-hand side is exactly expansion along the \(j\)'th row of the following determinants

\[
(A.3) \quad \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}, \quad j = 1
\]

\[
\begin{vmatrix} a_{j,1} & \cdots & a_{j,n-1} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix} = 0, \quad 2 \leq j \leq n - 1
\]

\[
\begin{vmatrix} a_{n,1} & \cdots & a_{n,n-1} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix}, \quad j = n.
\]

Combining (A.2) and (A.3) we obtain the right-hand side of (A.1) and thus also the proof of (a).
To prove (b) we prove the more general equation

\[(A.4)\]

\[
\begin{bmatrix}
  a_{21} & a_{22} & \cdots & a_{2,n} \\
  a_{31} & a_{32} & \cdots & a_{3,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\
  a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{bmatrix}
= \begin{bmatrix}
  g_{11} & g_{12} & \cdots & g_{1,n} \\
  g_{21} & g_{22} & \cdots & g_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{n-1,1} & g_{n-1,2} & \cdots & g_{n-1,n} \\
  g_{n+1,1} & g_{n+1,2} & \cdots & g_{n+1,n}
\end{bmatrix}
- \begin{bmatrix}
  g_{11} & g_{12} & \cdots & g_{1,n} \\
  g_{21} & g_{22} & \cdots & g_{2,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_{n-1,1} & g_{n-1,2} & \cdots & g_{n-1,n} \\
  g_{n+1,1} & g_{n+1,2} & \cdots & g_{n+1,n}
\end{bmatrix}
\]

for \(a_{ij} \in \mathbb{C}, i \in \{1, \ldots, n+1\} \) and \(j \in \{1, \ldots, n\}\). We remark that Hankel-matrices are symmetric and for these \((A.4)\) reduces to \((b)\). Observe that for \(k \in \{2, \ldots, n\}\) we have

\[
0 = (-1)^k \begin{bmatrix}
  a_{1,k} & a_{11} & a_{12} & \cdots & a_{1,n} \\
  a_{2,k} & a_{12} & a_{12} & \cdots & a_{2,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{n,k} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \\
  a_{n+1,k} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n}
\end{bmatrix}
= (-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{bmatrix}
  a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\
  a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{bmatrix}
\]

where the \(j = 1\) and \(j = n + 1\) are interpreted as remove the 1st and \((n + 1)^{th}\) column respectively. Thus also

\[
0 = \sum_{k=2}^{n} \begin{bmatrix}
  a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\
  a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n}
\end{bmatrix}
\cdot \left( (-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1} \begin{bmatrix}
  a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\
  a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{bmatrix} \right)
\]
Switching the indices we have

\[
0 = \sum_{j=1}^{n+1} \begin{vmatrix}
\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{j-1,1} & \alpha_{j-1,2} & \cdots & \alpha_{j-1,n} \\
\alpha_{j+1,1} & \alpha_{j+1,2} & \cdots & \alpha_{j+1,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{n+1,1} & \alpha_{n+1,2} & \cdots & \alpha_{n+1,n}
\end{vmatrix}
\cdot \left( \sum_{k=2}^{n} (-1)^{k+j-1} a_{j,k} \begin{vmatrix}
\alpha_{22} & \cdots & \alpha_{2,k-1} & \cdots & \alpha_{2,n} \\
\alpha_{3,2} & \cdots & \alpha_{3,k-1} & \cdots & \alpha_{3,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{n-1,2} & \cdots & \alpha_{n-1,k-1} & \cdots & \alpha_{n-1,n} \\
\alpha_{n,2} & \cdots & \alpha_{n,n} \\
\end{vmatrix} \right)
\]

The parenthesis of (A.5) is the expansion along the \( j \)th row of the following expression except for \( j = n + 1 \) where we expand along the \( n \)th row.

\[
\begin{cases}
\begin{vmatrix}
\alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{n,2} & \alpha_{n,3} & \cdots & \alpha_{n,n} \\
\end{vmatrix}, & j = 1 \\
0, & j \in \{2, \ldots, n - 1\} \\
\begin{vmatrix}
\alpha_{22} & \cdots & \alpha_{2,k-1} & \cdots & \alpha_{2,n} \\
\alpha_{3,2} & \cdots & \alpha_{3,k-1} & \cdots & \alpha_{3,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{n-1,2} & \cdots & \alpha_{n-1,k-1} & \cdots & \alpha_{n-1,n} \\
\alpha_{n,2} & \cdots & \alpha_{n,n} \\
\end{vmatrix}, & j = n \\
-\begin{vmatrix}
\alpha_{n-1,2} & \alpha_{n-1,3} & \cdots & \alpha_{n-1,n} \\
\alpha_{n,2} & \alpha_{n,3} & \cdots & \alpha_{n,n} \\
\vdots & \vdots & & \vdots \\
\alpha_{n-1,2} & \alpha_{n-1,3} & \cdots & \alpha_{n-1,n} \\
\alpha_{n,2} & \alpha_{n,3} & \cdots & \alpha_{n,n} \\
\end{vmatrix}, & j = n + 1.
\end{cases}
\]

Combining (A.5) and (A.6) we obtain (A.4) and this finishes the proof of (b).

\[\square\]

REFERENCES


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